Random Obtuse Triangles and Convex Quadrilaterals

by

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Abstract

We intend to discuss in detail two well known geometrical probability problems. The first one deals with finding the probability that a random triangle is obtuse in nature. We initially discuss the various ways of choosing a random triangle. The problem is at first analyzed based on random angles (adding to 180 degrees) and random sides (obeying the triangle inequality) which is a direct modification of the Broken Stick Problem. We then study the effect of shape on the probability that when three random points are chosen inside a figure of that shape they will form an obtuse triangle. Literature survey reveals the existence of the analytical formulae only in the cases of square, circle and rectangle. We used Monte Carlo simulation to solve this problem in various shapes. We intend to show by means of simulation that the given probabilatity will reach its minimum value when the random points are taken inside a circle.

We then introduce the concept of Random Walk in Triangles and show that the probability that a triangle formed during the process is obtuse is itself random. We also propose the idea of Differential Equation in Triangle Space and study the variation of angles during this dynamic process.

We then propose to extend this to the problem of calculating the probability of the quadrilateral formed by four random points is convex. The effects of shape are distinctly different than those obtained in the random triangle problem. The effect of true random numbers and normally generated pseudorandom numbers are also compared for both the problems considered.

Thesis Supervisor: Gilbert Strang
Title: Professor of Mathematics

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Thank you to all my friends who have made my stay at MIT memorable. I will always look back to my experience here with fond memories. I especially thank my friends in the CDO office. Thank you Rupa,George, Jie and Lisa.

Finally I dedicate this work to my parents who had been a constant source of support throughout my career.

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Chapter 1

Introduction

1.1 Problem and Overview

Geometric Probability consists of the application of probability principles to various geometric problems. They deal with random elements which are not quantities but geometrical objects such as points,lines and rotations [8]. In this thesis we look at the solutions of two such problems in geometric probability. They involve finding the probability that a random triangle is obtuse and a random quadrilateral is convex. Though analytical solutions of these problems exist in some special cases, it becomes increasingly difficult to solve them analytically for complex cases. Hence we use Monte-Carlo simulations in solving these problems. Monte-Carlo method relies on repeated random sampling to compute the probabilities and is most suitable in cases (such as this) where it is difficult to solve by a deterministic algorithm.

We start off with a review of the analytical solutions obtained to special cases of both the problems. We then discuss the various methods of random number generation. As the entire work is based on sampling random triangles and quadrilaterals, it is very important that we choose the best algorithms to sample points so that they are truly random in nature.

Our initial aim in this thesis is to study the effect of shape on the probability that when three random points are chosen inside a figure of that shape they will form an obtuse triangle. This will also be extended to sampling points from three dimensional shapes. We also intend to experimentally show that the given probability will reach its minimum value when the random points are taken inside a circle. We then consider some very interesting applications of the obtuse triangle problem by considering random walk of triangles and the concept of differential equations in triangle space. Further we propose to extend this to the problem of calculating the probability that the quadrilateral formed by four random points is convex. In each case we shall consider the effects of true random numbers as well as normally distributed ones.

The entire work has been done using MATLAB R2008a. All the animations and source codes used to produce the images in this thesis are available online at `http://web.mit.edu/~nirjhar/`

1.2 Literature Review

1.2.1 Analytical Solution for the Triangle Problem

We wish to find out the probability that a random triangle chosen from a rectangle of dimension $1 \times L$ is obtuse. We give a summary of the anaytical solution [9].
We consider a triangle ABC. Let the sides be denoted as a,b and c opposite to corner angles A,B and C. Then using the cosine rule we have,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1}$$

Let the vertices have co ordinates X_i, Y_i. As ABC is a random triangle, X_i are random numbers uniformly distributed between 0 and 1; Y_i are random numbers uniformly distributed between 0 and L. If $A > 90°$ then $\cos A < 0$ and we can rewrite Eqn. 1.1 as

$$(Y_3-Y_1)^2 + (X_3-X_1)^2 + (Y_2-Y_1)^2 + (X_2-X_1)^2 - (Y_3-Y_2)^2 - (X_3-X_2)^2 < 0 \tag{1.2}$$

This can further be reduced to

$$X + Y \ < \ 0 \tag{1.3}$$

where $X = (X_2 - X_1)(X_3 - X_1)$ and $Y = (Y_2 - Y_1)(Y_3 - Y_1)$. A triangle can only possibly have one of its angles as obtuse. Hence

$$P(\text{triangle is obtuse}) \ = \ 3P(X + Y < 0) \tag{1.4}$$

where P denotes the probability of the corresponding event. As the probability depends on L we will denote it as $P(L)$. As the probability will not depend on whether we choose a rectangle of dimensions $1 \times L$ or $L \times 1$, we have,

$$P(L) = P(\frac{1}{L}) \tag{1.5}$$

Let $F(x)$ be the CDF (Cumulative Distribution Function) of X and $G(y)$ be that of Y. We initially need to find a relation between $F(x)$ and $G(y)$. We rewrite Y as:

$$\frac{Y}{L^2} \ = \ (\frac{Y_2}{L} - \frac{Y_1}{L})(\frac{Y_3}{L} - \frac{Y_1}{L})$$

We then observe that the random variable $\frac{Y_i}{L}$ is uniformly distributed between 0 and 1 and hence has the same distribution as X_i. This also implies that $\frac{Y}{L^2}$ has the same distribution as X. Hence we can conclude that:

$$G(y) = F(\frac{y}{L^2}). \tag{1.6}$$

Also as X depends only on X_is and Y depends only on Y_is, X and Y are independent random variables. Using Equation 1.4, the required probability $P(L)$ can be written as:

$$P(L) \ = \ 3P(\frac{X}{L^2} + \frac{Y}{L^2} \ < \ 0) \tag{1.7}$$

$$= \ 3P(\frac{Y}{L^2} \ < \ \frac{-X}{L^2}) \tag{1.8}$$

Hence $P(L)$ can be expressed by the following Riemann-Stieltjes integral:

$$P(L) = 3\int_{-\infty}^{\infty} F(-\frac{x}{L^2})dF(x) \tag{1.9}$$

$F(x)$ is computed and the integration is carried out to get a final expression for P(L). A detailed derivation of this portion can be obtained in [10].

1.2.2 Analytical Solution for the Quadrilateral Problem

The Sylvester's Four Point Problem [18] is defined as:

> *The problem of finding out the probability that four points chosen at random in a planar region R have a convex hull which is a quadrilateral.*

The problem can also be reframed as follows:

> *What is the probability that four points A,B,C and D taken at random inside a convex domain, form a convex quadrilateral,i.e. none of the points is inside the triangle formed by the other three.*

The problem was solved analytically by Kendall and Moran [8]. A brief outline of the proof is given here.

We consider a convex domain of area S. The complementary probability is that the random points will not form a convex quadrilateral. This will happen when any of the points will lie inside the triangle formed by the other three. If we consider the mean area of the triangle thus formed to be T, then the probability that the points do not form a convex quadrilateral is equal to the proportion of area contained by the four triangles which is equal to $4\frac{T}{S}$. Therefore,

$$P(convex) = 1 - 4\frac{T}{S} \tag{1.10}$$

To proceed further we need to know the Crofton's Formula ([3] and [17]). It states that:

Let N points $\xi_1, \ldots, \xi_n$ be randomly distributed on a domain S, and let H be some event that depends on the positions of the N points. Let S' be a domain slightly smaller than S but contained within it, and let δS be the part of S not in S'. Let $P[H]$ be the probability of event H, s be the measure of S, and δs the measure of δS, then Crofton's formula states that

$$\delta P[H] = n(P[H\xi_1 \,\varepsilon\, \delta S] - P[H])s^{-1}\delta s \tag{1.11}$$

In our case P (from Eqn. 1.10) is unaffected by the scale of the domain in which the four points lie and hence we can imagine that the points are included in a larger domain of the same shape. This will also be shown by means of simulation in Section 3.9. Hence $\delta P[H] = 0$ and using Eqn. 1.11 we get $P_1 = P$ where $P_1 = P[H\xi_1 \,\varepsilon\, \delta S]$. P_1 actually refers to the probability of the quadrilateral to be convex when one of its points is constrained to lie at random in the added infinitesimal part of the domain. To calculate P_1 we need to consider the triangles whose one vertex is fixed (constrained to lie in the infinitesimal part). Hence if T_1 denotes the average area of a triangle one of whose points lies on the boundary (averaged over all possible positions of the boundary point), we get,

$$P_1 = 1 - 3T_1S^{-1} \tag{1.12}$$

The basic approach to solve this problem for any domain involves expressing T_1 in terms of S and using Eqn. 1.12. Next we consider the case of a convex polygon. To calculate P_1 we constrain one of the vertices of the convex polygon. Let this constrained vertex be A as shown in Figure 1-1. Let there be n vertices of the polygon. Hence the number of triangles formed with vertex A as one of the three vertices is $n-1$. The areas of these triangles are denoted as $S_1, \ldots, S_{n-1}$ (Figure 1-1). As T_1 is the average of the areas of these triangles , we can derive the expression,

$$(\sum_{i=1}^{n-1} S_i)^2 T_1 = \sum_{i=1}^{n-1} S_i^2 T_{ii} + \sum_{i,j=1,i\neq j}^{n-1} S_i S_j T_{ij} \tag{1.13}$$

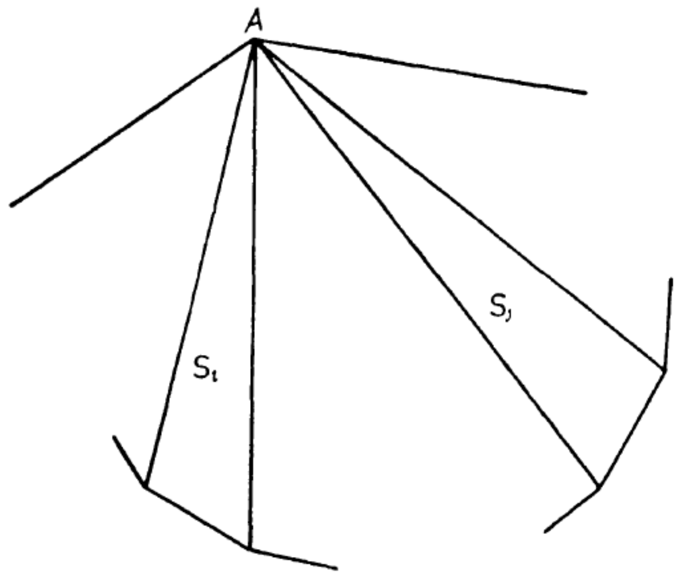


Figure 1-1: Discrete Case of the Derivation in a Convex Region [8]

In this equation T_{ij} is the average area of a triangle one of whose vertices is A and the other two points are taken at random in the triangles i and j (including both the cases of $i = j$ or $i \neq j$).

Let G_i be the centre of gravity of the triangle i. Let B be a random point in triangle S_i and C be a random point in triangle S_j. Keeping A and B fixed, mean area of triangle ABC is area of triangle ABG_j from the definition of center of gravity. Hence now varying B , we can conclude that,

$$T_{ij} = \Delta(AG_iG_j) \qquad \text{for all } i \neq j \tag{1.14}$$

When $i = j$ we are talking about picking random points in the same triangle. This triangle can be transformed into any other triangle by projection followed by scaling. Hence we can argue that T_{ii} is a scalar multiple of S_i.

$$T_{ij} = \lambda S_i \qquad \text{for all } i = j \tag{1.15}$$

As an example we consider the triangular domain (Figure 1-2). We sample points from the triangle AXY.The change in domain as explained is created by moving

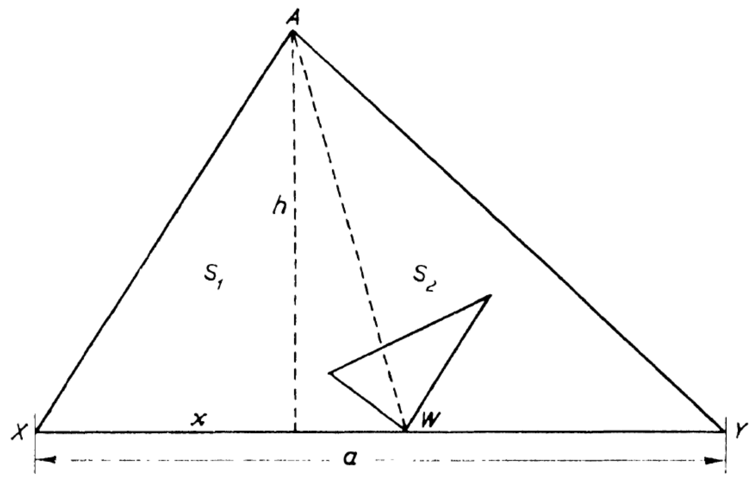


Figure 1-2: Derivation inside a Triangular Domain [8]

XY parallel to itself. If W is a point on XY at a distance x from X, T is first evaluated keeping W as the fixed vertex of the random triangle. Then it is averaged by integrating T over the length of XY to obtain an expression of T_1. Let S_1 and S_2 be the areas of the triangles AXW and AWY. Let S be the area of AXY. Hence $S = S_1 + S_2$. Let the height of the triangle be h and XY be equal to a. Hence we get,

$$S = \frac{1}{2}ah \tag{1.16}$$

$$S_1 = \frac{1}{2}xh \tag{1.17}$$

$$S_2 = \frac{1}{2}(a-x)h \tag{1.18}$$

Substituting these equations in Eqn. 1.13 we get,

$$S^2T = S_1^2T_{11} + +S_2^2T_{22} + 2S_1S_2T_{12} \tag{1.19}$$

We can show for a triangular domain from Eqns 1.13, 1.14 and 1.15,

$$T_{ii} = \frac{4}{27}S_i \tag{1.20}$$

Substituting Eqns 1.16, 1.17 and 1.18 in Eqn. 1.19 we get,

$$T = \frac{h}{2a^2}\{\frac{4}{27}x^3 + \frac{4}{27}(a-x)^3 + \frac{2}{9}ax(a-x)\} \tag{1.21}$$

T_1 can hence be calculated as,

$$T_1 = h(2a^2)^{-1}\int_0^a\{\frac{4}{27}x^3 + \frac{4}{27}(a-x)^3 + \frac{2}{9}ax(a-x)\}dx \tag{1.22}$$

On solving we get $T_1 = \frac{1}{9}S$. Hence we get,

$$P = 1 - 3T_1S^{-1} = \frac{2}{3} = 0.6667 \tag{1.23}$$

This can be expanded to find out the probability when points are sampled from square, regular hexagon etc.

For the continuous case , Equation 1.13 can be modified into an integral for any convex region as shown in Figure 1-3. Let A be a point on the boundary and let $p(\theta)$ be the distance of the boundary from A along a line making an angle θ with a tangent at A. Hence in this continuous case, similar arguments can be used to modify Equation 1.13 into Equation 1.24.

$$S^2T_1 = \frac{1}{18}\int_0^{\pi}\int_0^{\pi} p(\theta)^3p(\phi)^3 sin|\theta-\phi|d\theta d\phi \tag{1.24}$$

The case of sampling random points from a circle can be solved using Equation 1.24 to yield a value of 0.7045.

Another much easier way to solve the same problem is to use Alikoski's formula [1].This method is also known as 'triangle picking' method. As was explained, the problem involves finding out the area in random triangles with each of the four vertices of the quadrilateral as a vertex of the triangle and then averaging them. Hence when we sample points from a polygon to check the probability that a random quadrilateral is convex, we 'pick' triangles from that polygon. We refer to that as 'polygon triangle picking'. The mean area of a triangle which is randomly picked from a regular polygon

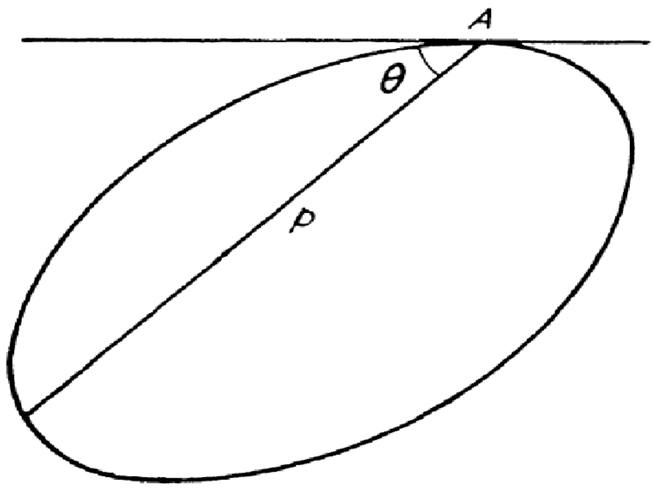


Figure 1-3: Continuous Case of the Derivation in a Convex Region [8]

of unit area and 'n' sides is given by Alikoski's formula as:

$$A_n = \frac{9cos^2\omega \ + \ 52cos\omega \ + \ 44}{36n^2sin^2\omega} \tag{1.25}$$

where $\omega = \frac{2\pi}{n}$. Comparing Equations 1.25 and 1.10 we get,

$$P(convex) = 1 - 4A_n \tag{1.26}$$

Probability values along with the A_n for various shapes from which random points are sampled are listed in Table 1.1. In case of a circle the mean area of the randomly

Table 1.1: Probabilities of a random quadrilateral being convex using Alikoski Formula when points are sampled inside a regular polygon of 'n' sides.

n	A_n	**P(convex)**
3	$\frac{1}{12}$	0.6667
4	$\frac{11}{144}$	0.6944
5	$\frac{1}{180}(9+2\sqrt{5})$	0.7006
6	$\frac{289}{3888}$	0.7027

picked triangle is found out by 'disk triangle picking'. Let the three random points be denoted by $P(x_1, y_1)$, $Q = (x_2, y_2)$ and $R = (x_3, y_3)$. The area of this triangle PQR

is given by:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{1.27}$$

They are distributed independently and uniformly in the interior of the unit circle. Then the average area of the triangle will be given by:

$$\bar{A} = \frac{\iint\limits_{P\in K} \iint\limits_{Q\in K} \iint\limits_{R\in K} \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} dy_3\, dy_2\, dy_1\, dx_3\, dx_2\, dx_1}{\iint\limits_{P\in K} \iint\limits_{Q\in K} \iint\limits_{R\in K} dy_3\, dy_2\, dy_1\, dx_3\, dx_2\, dx_1} \tag{1.28}$$

This messy integration can be simplified using the trigonometric substitution :

$$x_i = \sqrt{r}\cos\theta \qquad y_i = \sqrt{r}\sin\theta \quad \forall x \in \{1, 3\}$$

The integration reduces to:

$$\bar{A} = \frac{1}{2\pi^3}\int_0^1\int_0^1\int_0^1\int_0^{\pi}\int_0^{2\pi} |A|\, d\theta_3\, d\theta_2\, du_1\, du_2\, du_3 \tag{1.29}$$

where

$A = \frac{1}{2}(\sqrt{u_1 u_2}\sin\theta_2 - \sqrt{u_2 u_3}\cos\theta_3\, \sin\theta_2 - \sqrt{u_1 u_3}\sin\theta_3 + \sqrt{u_2 u_3}\cos\theta_2\, \sin\theta_3)$

This was solved to yield:

$$\bar{A} = \frac{35}{48\pi^2} \tag{1.30}$$

Substituing Eqn. 1.30 in Eqn. 1.26 we get the the probability as 0.70448. Also the value is the same for any ellipse as the 'disk triangle picking' method remains the same for the ellipse. It can also be shown([2], [14]) that the probability $P(convex)$ follows the following inequality for sampling points in two-dimensional shapes:

$$\frac{2}{3} \leq P(convex) \leq 1 - \frac{35}{12\pi^2} \tag{1.31}$$

Hence the circle/ellipse case is the limiting case of this probability. However this method is not as general as the one described earlier. The reason is when we sample points from other non-regular shapes the triangle picking problem becomes more difficult to solve.

Chapter 2

Random Number Generation

In this chapter we shall discuss the various methods of generating random numbers within a given shape. Initially we shall describe the details of two such methods and also explain the advantage of one over the other. Then we shall consider the generation of normally distributed pseudorandom numbers.

2.1 First Method

The shapes considered in the thesis in which random numbers will be generated are the regular ones such as square,circle,triangle and hexagon. In order to generate a random point within such a figure it is necessary to generate random numbers for each of the x and the y coordinates. One of the methods that can be used consists of the following steps:

- The random number corresponding to the x coordinate is generated (by scaling the 'rand' command in MATLAB according to the range the x coordinate can take).
- The random number corresponding to the y coordinate is generated based on the x coordinate in the range as permitted by the geometry of the shape.

Let us take an equilateral triangle as an example. Hence the problem is to generate random points within an equilateral triangle of unit area. We know the area of an equilateral triangle is given by the expression:

$$Area = \frac{\sqrt{3}}{4} side^2$$

where 'side' is the length of a side of the triangle. Hence the equilateral triangle of unit area will have side(a) = 1.5197. According to this method the random numbers are generated as follows:

1) At first X is randomly chosen from 0 to a.

2) We note that the equations of the three lines of the triangle are

$$x = 0 \qquad y = \sqrt{3}x \qquad \text{and} \qquad y = \sqrt{3}(a - x)$$

3) Hence Y is randomly chosen from 0 to $\sqrt{3}(a - X)$ for $x > (a/2)$ and for $x < (a/2)$ it is randomly chosen from 0 to $\sqrt{3}X$

The random points corresponding to the vertices of a triangle are shown in Figure 2-1. The number of triangles considered in this figure is 10,000 and hence there are 30,000 points in the figure.

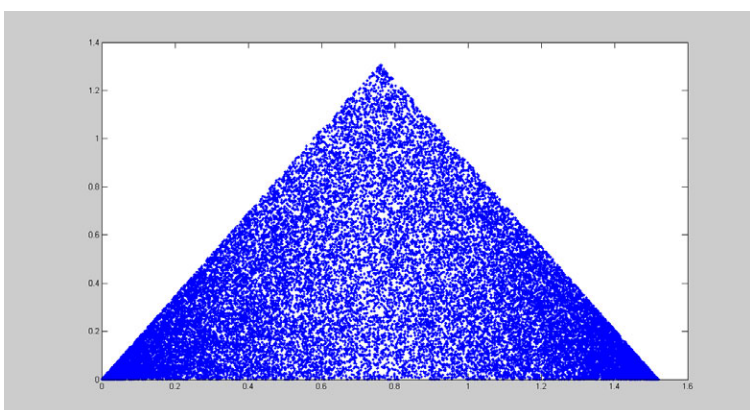


Figure 2-1: Random Points sampled in an equilateral triangle of unit area

As it can be seen from Figure 2-1 the y points are not uniformly distributed as they are dependent on the x coordinates. This means more points are chosen near the edges of the triangle than in the centre region. This is not a problem for shapes such as rectangle or a square but becomes a problem in case of other shapes (such as the triangle or a circle) where the bordering lines are not parallel to the x and the y axes.

We further explain the disadvantage of this method by considering sampling random points from a circle of unit area.In Figure 2-2 as X is chosen randomly from [-1,1] there is an equal probability of selection of a and b. However once b is chosen, we choose y on the line passing through b as shown. Similar is the case when a is chosen. As the line passing through a is longer than the one passing through b, we might argue that the probability of a point being chosen in the 'b' line is greater than any point on the 'a' line. Hence the random points generated by this method are not truly random.

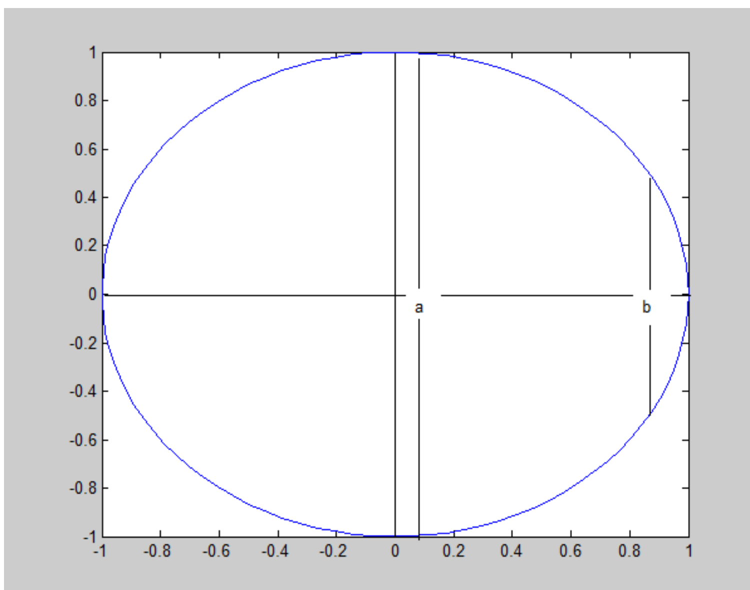


Figure 2-2: Diagram showing Random Points selected by method 1 may be dependent. There is an equal probability of choosing a and b. However probability of choosing a point on $x = b$ line is more than one on $x = a$ line.

2.2 Second Method

This method generates random points according to the following steps:

1) Initially a random point is chosen in the smallest (area wise) rectangle covering

the area of interest.

2) Then it is checked whether the particular point lies inside or outside the area of interest.

3) The point is considered as a candidate if it lies inside the figure. Otherwise steps 1-3 are repeated.

In case of an equilateral triangle Figure 2-3 shows the rectangle (in bold) of smallest area covering the triangle. Points are initially sampled from this rectangle. We only consider the point if it lies inside the triangle. This method generates points

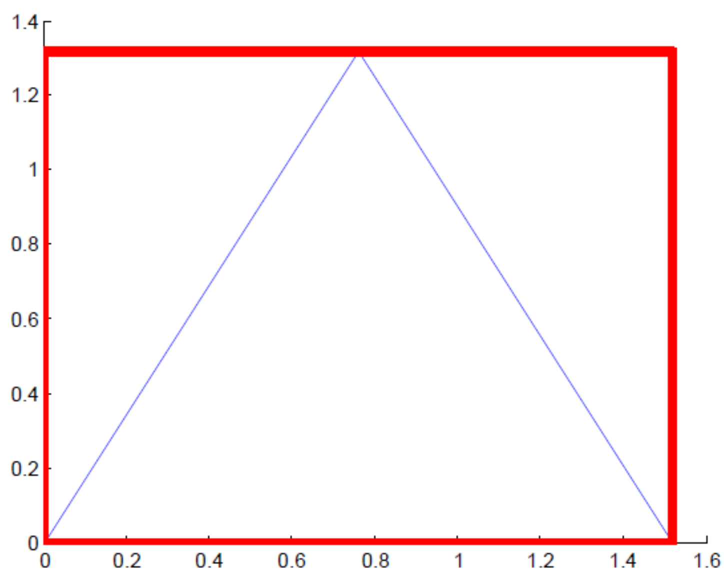
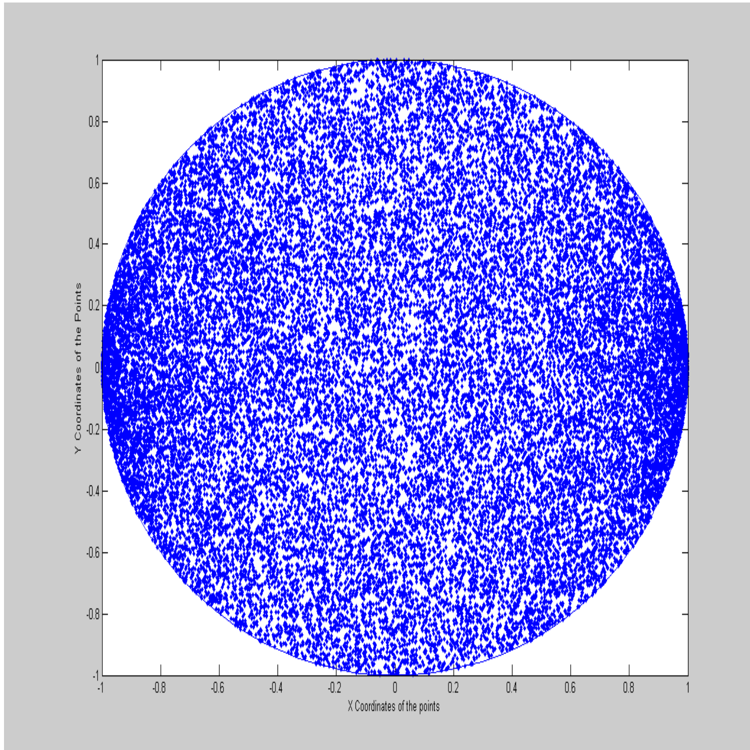
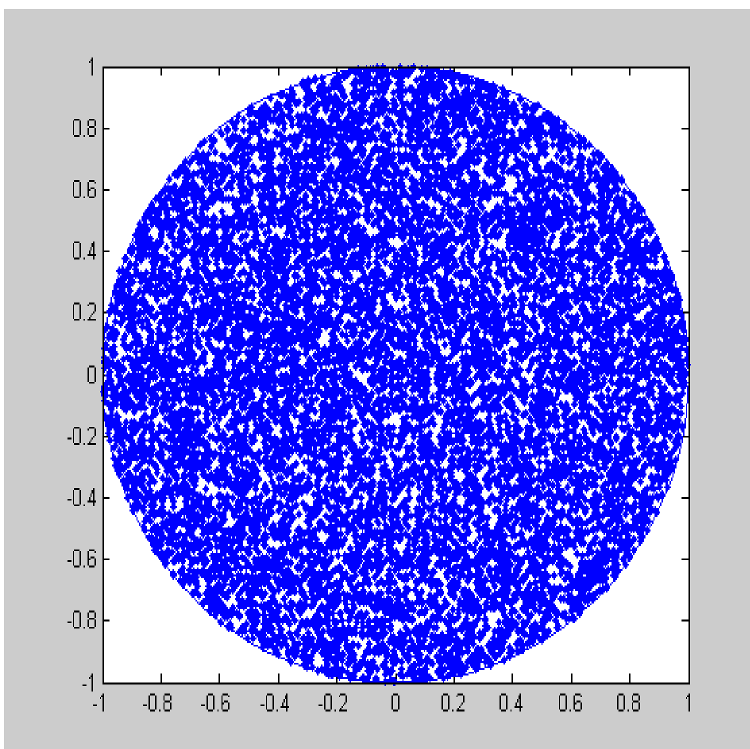


Figure 2-3: The Rectangle of Minimum area covering the triangle

which are truly random. Figure 2-4 shows random points generated in a circle using both the methods. As can be seen the points generated by this method are not concentrated along the sides as is the case with ones generated using method 1. This serves as a visual confirmation that this method generates true random numbers unlike the previous one.



(a) Random Points in a Circle generated by Method 1



(b) Random Points in a Circle generated by Method 2

Figure 2-4: Random Point Generation

2.3 Normally Generated PseudoRandom Numbers

Normally generated pseudorandom numbers are not truly random. They are scalar values drawn from a normal distribution of mean 0 and standard deviation 1. They are generated using the 'randn' command in MATLAB. Almost all algorithms for generating normally distributed random numbers are based on transformations of uniform distributions [12]. The simplest way to generate an m-by-n matrix with approximately normally distributed elements is to use the expression

$$sum(rand(m, n, 12), 3) - 6$$

This works because $R = rand(m, n, p)$ generates a three-dimensional uniformly distributed array and $sum(R, 3)$ sums along the third dimension. The result is a two-dimensional array with elements drawn from a distribution with mean $p/2$ and variance $p/12$ that approaches a normal distribution as p increases. If we take p = 12, we get a pretty good approximation to the normal distribution and we get the variance to be equal to one without any additional scaling. The two disadvantages with this approach are:

1) It requires twelve uniforms to generate one normal, so it is slow.

2) The finite p approximation causes it to have poor behavior in the tails of the distribution.

Beginning with Matlab 5, the normal random number generator 'randn' uses a modified version of Ziggurat Algorithm [11]. A simpler version of the one used in MATLAB is described below.

The pdf (probability density function) of the normal distribution is the bell-shaped curve given by

$$f(x) = ce^{-x^2/2}$$

where $c = 1/(2\pi)^{1/2}$ is a normalizing constant and can be ignored. The method involves generating random points (x, y), uniformly distributed in the plane and rejecting any of them that do not fall under this curve.The remaining x's form our desired normal distribution.The ziggurat algorithm covers the area under the pdf by a slightly larger area with n sections. Figure 2-5 has $n = 8$; actual code might use $n = 128$. The top $n-1$ sections are rectangles. The bottom section is a rectangle together with an infinite tail under the graph of $f(x)$. The right-hand edges of the rectangles are at the points z_k, $k = 2, \ldots, n$, shown with circles in the Figure. For a specified value of n it is possible to solve for the z_ks.We define the *core* of the ziggurat as the ratio $\sigma_k = z_{k-1}/z_k$. The code [12] to find random points is then given by:

```
j = ceil(128*rand);
u = 2*rand-1;
if abs(u) < sigma(j)
r = u*z(j);
return
end
```

Core is basically the fraction of each section that lies underneath the section above it. As its value is nearly 1 almost always (see Figure 2-5) the test is true almost 97% of the time [11].

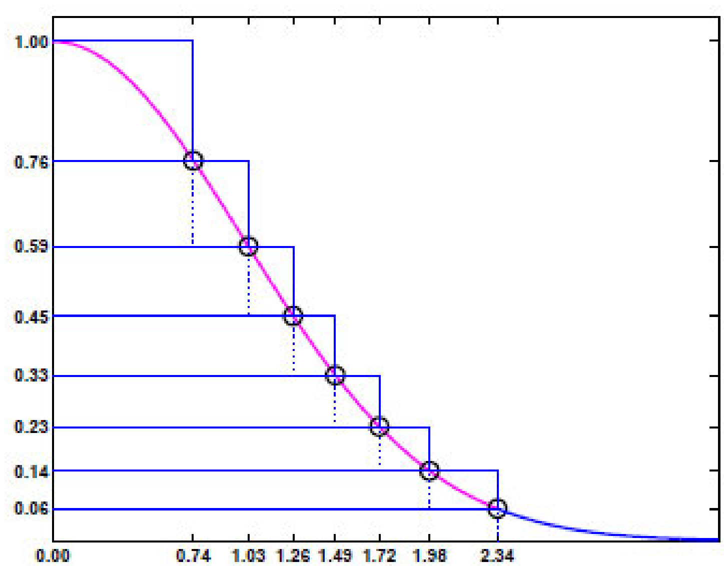


Figure 2-5: The Ziggurat Algorithm [12]

Chapter 3

Random Triangles

This section mainly deals with the probability that a random triangle will be obtuse. We consider several shapes to sample the random points and perform Monte-Carlo Simulation with points from each such shape. We shall discuss the various algorithms followed to check whether a triangle is obtuse. In some cases simulation results have been matched with analytical ones (where they are available). We also discuss two simulation based experiments to prove that this probability decreases as a figure becomes circular in nature. Finally we consider random walk of triangles and the concept of differential equations in triangle space.

3.1 Langford's Algorithm

Langford [9] in 1969 came up with two empirical formulae for finding the probability of an obtuse triangle. The formulae are for three points chosen at random in a rectangle with dimensions $1 \times L$. According to Langford for $1 \leq L \leq 2$,

$$P(L) = \frac{1}{3} + \frac{47}{300}(L^2 + \frac{1}{L^2}) + \frac{\pi}{80}(L^3 + \frac{1}{L^3}) - \frac{logL}{5}(L^2 - \frac{1}{L^2}) \qquad (3.1)$$

For higher values of L ($L \geq 2$)the expression is given by,

$$\begin{aligned} P(L) = \frac{1}{3} + \frac{1}{L^2}(\frac{\pi}{80L} + \frac{47}{300} + \frac{logL}{5}) + (\frac{L^2}{10} - \frac{3}{5L^2})log(\frac{L+\sqrt{L^2-4}}{L-\sqrt{L^2-4}}) \\ + \frac{L^3}{40}\arcsin\frac{2}{L} - \frac{L^2 logL}{5} + \frac{47L^2}{300} + \frac{L\sqrt{L^2-4}}{150}(-31 + \frac{63}{L^2} + \frac{64}{L^4}) \end{aligned} \tag{3.2}$$

In the special case when the figure is a square ($L = 1$), we get from 3.1

$$P(1) = 0.72520648$$

Using this algorithm the probabilities are calculated for values of L from 1 (square) to 50. These values are plotted in Figure 3-1.

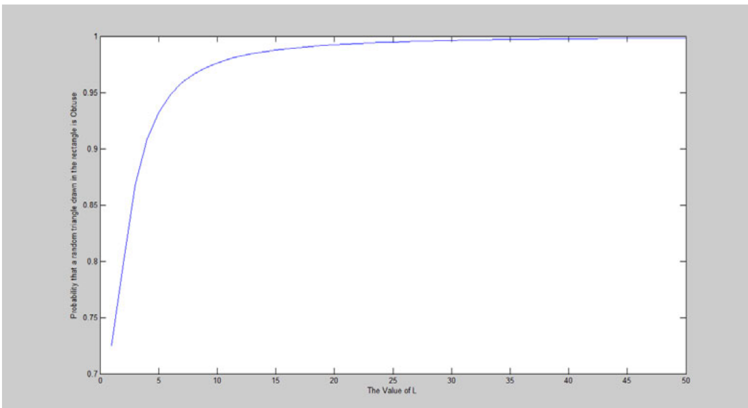


Figure 3-1: Probabilities plotted with varying values of L (from 1 to 50)

3.2 Obtuse Triangle Criteria

There are several tests cited in literature to check whether a triangle is obtuse. The one used in this report is: A triangle is obtuse if the square of the length of the longest side is greater than the sum of the squares of the other two sides. Hence if a,b and c are the lengths of the sides of the triangle, the triangle is obtuse if and only if,

$$a^2 > b^2 + c^2 \quad \text{or} \quad b^2 > a^2 + c^2 \quad \text{or} \quad c^2 > a^2 + b^2 \tag{3.3}$$

For acute angled triangles this is not true and hence a triangle is acute if and only if:

$$a^2 < b^2 + c^2 \quad \text{and} \quad b^2 < a^2 + c^2 \quad \text{and} \quad c^2 < a^2 + b^2 \tag{3.4}$$

3.3 Random Angle and Random Side Approach

In this section we consider the most basic and probably intuitive way of approaching the random triangle problem. These methods correspond to choosing the angles randomly or the sides randomly.

3.3.1 Random Angles

This method corresponds to chosing the angles randomly. However the angles must sum up to 180 degrees for the triangle equality to hold true. Figure 3-2 depicts the

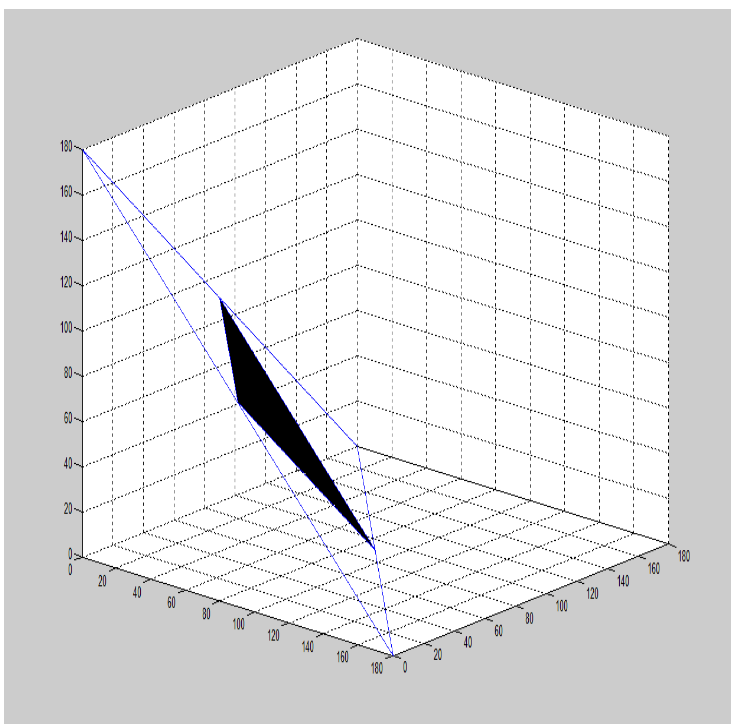


Figure 3-2: Random Angle Approach. The big triangle correponds to the area where points will result in a triangle. The inner small triangle corresponds to the area where points will result in an acute triangle.

entire problem. In this figure each axis corresponds to one of the angles in the triangle. Hence the axes range from 0 to 180. The big triangle in the figure correponds to the area where the angles actually form a triangle. All other points in the cube will not result in a triangle. This is because the equation of the plane corresponding to the big triangle is:

$$x + y + z = 180$$

Next we consider the area inside the plane that will result in an obtuse triangle when points are sampled from the same. Because each of the angles (when greater than 90 degrees) can lead to obtuseness, it is quite evident that there will be three distinct areas inside the plane which will result in an obtuse triangle. These areas correspond to the three outer small triangles whereas the inner triangle (shown in black) corresponds to the one which will lead to an acute triangle when points are sampled from it. As the four triangles are congruent to each other, the area corresponding to obtuse is three times as big as the one corresponding to acute. Hence the probability that an obtuse triangle will be formed when random points are sampled from the plane is 0.75.

3.3.2 Random Sides

In this method we chose three random numbers corresponding to the three sides of a triangle. For this initially three numbers are chosen randomly between 0 and 1. Then we check whether they form a triangle or not based on the three triangle inequalities. Hence if the numbers are a,b and c , they will form a triangle if and only if,

$$a < b + c, \quad b < a + c \quad \text{and} \quad c < a + b$$

We considered only those numbers which obeyed the above condition. It was observed that 50% of times the sides resulted in triangles. This can also be verified analytically as half the volume of the unit cube formed will satisy the inequalities. We then checked whether they form an obtuse triangle based on the criteria given by Eqn. 3.3. This

Table 3.1: Simulation Results of the random sides problem.

Simulation	**Probability**	**Mean**	**Variance**
1	0.5742		
2	0.5694		
3	0.5723		
4	0.5684		
5	0.5715	0.5714	3.30E-06
6	0.5697		
7	0.5714		
8	0.5715		
9	0.5735		
10	0.5724		

simulation was iterated 10 times with $100,000$ triangles considered in each simulation. The mean proportion of obtuse triangles found was 0.5714. Table 3.1 enlists the simulation results.

3.4 Broken Stick Problem

In its classical form the Broken Stick Problem [5] refers to the following:

> Given a stick of say unit length we cut it at any two points.What is the probability that the resulting three pieces will form a triangle.

Let the cuts be at the points x_1 and x_2 such that x_1 is greater than x_2. Therefore the three pieces are

$$x_2, \qquad x_1 - x_2, \qquad 1 - x_1$$

The condition for forming a triangle is that the largest side should be less than half of the sum of the three sides and hence in this case the largest side should be less than 0.5. Hence the conditions are:

$$x_2 < \frac{1}{2}, \qquad x_1 - x_2 < \frac{1}{2}, \qquad x_1 > \frac{1}{2} \tag{3.5}$$

The sample space corresponding to the problem (case $x_1 > x_2$) is shown in Figure 3-3 The area of the triangle thus created will be equal to the probability that the pieces

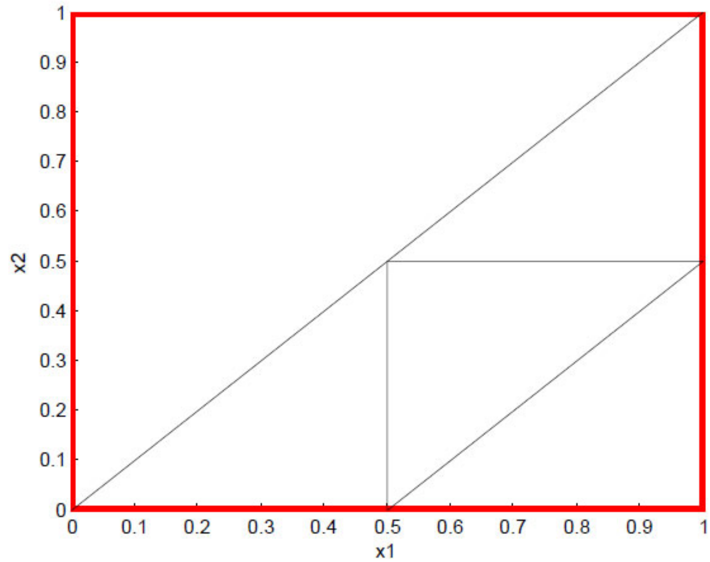


Figure 3-3: The Broken Stick Problem

will form a triangle if $x_1 > x_2$. The area is equal to $\frac{1}{8}$. Hence the total probability will be two times this area (including the case of $x_1 < x_2$). Hence:

P(triangle will be formed with the three pieces) $= \frac{1}{4}$

This problem can be modified to find out the probability that the pieces will form an obtuse triangle. Similarly initially we take $x_1 > x_2$ and the sides are hence $x_2, x_1 - x_2$, and $1 - x_1$. The sides should follow the criteria as mentioned by Equations 3.5 and 3.3. Equation 3.3 is applied to the sides to obtain:

$$\begin{aligned} 2x_1^2 - 2x_1x_2 - 2x_1 + 1 &< 0 \\ 2x_1 - 2x_1x_2 - 1 &> 0 \\ 2x_2^2 - 2x_1x_2 + 2x_1 - 1 &< 0 \end{aligned} \tag{3.6}$$

The joint sample space corresponds to Figure 3-4. The shaded region corresponds to the favourable area in the sample space where the sides will lead to an obtuse triangle. The area of the shaded region is 0.085. Also considering the case $x_1 < x_2$ we get

Probabilty of forming an obtuse triangle by the three broken pieces $=$ 0.17.

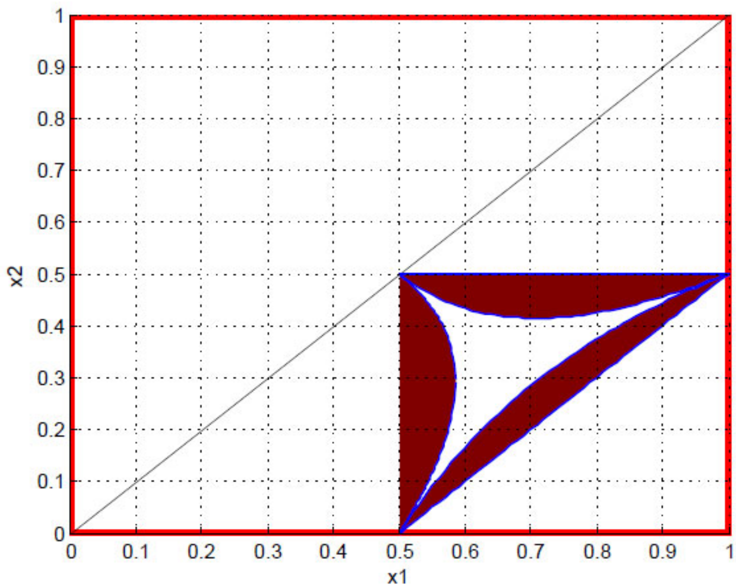


Figure 3-4: The Broken Stick Problem as modified for the Obtuse Triangle Case

Hence we can conclude that 0.68 (obtained by $\frac{0.17}{0.25}$) is the probability that the three pieces will form an obtuse triangle given that they form a triangle in the first place.

3.5 Monte-Carlo Simulation in Two Dimensional Shapes

This section deals with Monte-Carlo Simulations in various regular shapes to find out the probability that a random triangle is obtuse when points are sampled from that particular shape. The random numbers are generated as described in Section 2.2. The shapes considered are:

- **Circle**: Random points are generated inside a circle centered at the origin and of unit radius.
- **Square**: Random points are generated inside a square with unit side.
- **Rectangle**: Random points are generated inside a rectangle of unit area. The sides of the rectangle are in the ration 1:L. Various values of L have been chosen for simulation from 2 to 20.
- **Equilateral Triangle**: Random points are generated within an equilateral

triangle of unit area (side 1.5197).

- **Hexagon**: Random points are generated within a hexagon of unit area centered at the origin (side 0.6204).
- **Normal Distribution**: Random points are generated inside the normal distribution function from $x = -3$ to $x = 3$.
- **randn**: Normally distributed pseudorandom numbers are generated as described in Section 2.3

A small modification was done in the generation of random numbers. The algorithm described in Section 2.2 was modified to include the following line every time the code was run

```
rand('twister',sum(100*clock))
```

This actually assured that the random numbers generated in every simulation were truly random (assigned to a different state). In each of the above cases 10 simulations were executed each consisting of 100,000 random triangles (hence a total of 300,000 points for each simulation). Because of the large number of random points considered, the points eventually are uniformly distributed in the region from which they are sampled. This is the visual proof that the points chosen are actually random in nature. In case of normally generated pseudorandom numbers they are not confined to any particular shape as is seen in Figure 3-5. This figure plots 30,000 (corresponding to 10,000 triangles) points generated by the MATLAB command 'randn'. Table 3.2 enlists the results corresponding to each of the simulations for the various shapes.

Table 3.2: Simulation Results

Circle	**Square**	**randn()**	**Rectangle (1:2)**
0.71913	0.72535	0.75002	0.79786
0.72059	0.72704	0.75008	0.7975
0.72171	0.7243	0.75109	0.79939
0.7194	0.72442	0.75004	0.79786
0.71935	0.72658	0.75005	0.79958
0.71919	0.72347	0.74949	0.79808
0.72027	0.72749	0.74823	0.79904
0.72054	0.72693	0.74751	0.79999
0.72102	0.72513	0.74855	0.79964
0.72026	0.72597	0.75217	0.79806

Rectangle(1:3)	**Rectangle (1:4)**	**Rectangle (1:5)**	**Rectangle(1:15)**
0.86729	0.90625	0.93426	0.98762
0.86654	0.9074	0.9329	0.98768
0.8653	0.90756	0.93255	0.988
0.86656	0.90733	0.93279	0.98775
0.86468	0.90748	0.93353	0.9878
0.86832	0.9101	0.93148	0.98737
0.86568	0.90523	0.93304	0.98767
0.86859	0.90913	0.93238	0.98744
0.86602	0.90786	0.93311	0.98762
0.86894	0.90821	0.93275	0.98782

Rectangle(1:20)	**Equilateral Triangle**	**Hexagon**	**Normal Distribution**
0.99231	0.74712	0.72102	0.97466
0.99255	0.74822	0.71919	0.97418
0.99208	0.74904	0.72084	0.97371
0.99253	0.75022	0.72111	0.97481
0.99225	0.74382	0.71916	0.97347
0.99222	0.7492	0.72126	0.97501
0.99246	0.75028	0.72301	0.97434
0.99207	0.74568	0.72015	0.97497
0.99217	0.74926	0.71884	0.97463
0.99212	0.74939	0.71915	0.97527

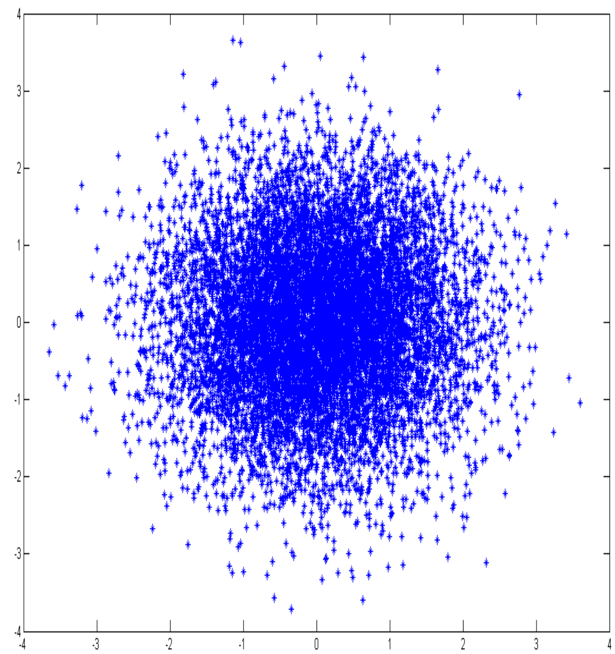


Figure 3-5: Normally distributed pseudorandom numbers

The mean p values for each of the shapes are listed in Table 3.3 along with the standard error and the reference value if obtained. The reference values are obtained using the Langford Algorithm [9] as described in Section 3.1 and [4]. The mean

Table 3.3: Mean p values for all shapes considered

	p Value	**Standard Error**	**P Value in Reference**
Circle	0.720146	0.000865	0.7201 [4]
Square	0.725668	0.001349	0.725206483006412 [9]
randn()	0.749723	0.001368	
Rectangle (1:2)	0.7987	0.000916	0.798374285126921 [9]
Rectangle(1:3)	0.866792	0.001455	0.867735019414964 [9]
Rectangle (1:4)	0.907655	0.001357	0.908010796936007 [9]
Rectangle (1:5)	0.932879	0.000726	0.932296731136648 [9]
Rectangle(1:15)	0.987677	0.000183	0.987601721549503 [9]
Rectangle(1:20)	0.992276	0.000181	0.992306053810452 [9]
Equilateral Triangle	0.748223	0.002091	0.7484 [4]
Hexagon	0.720373	0.00132	
Normal Distribution	0.974505	0.00058	

values are then plotted after sorting in Figure 3-6. It can be seen from Figure 3-6 the

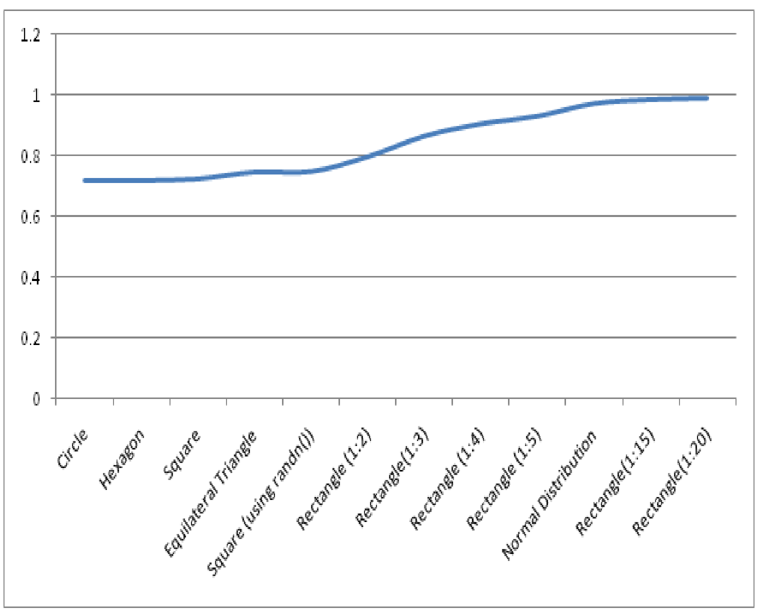


Figure 3-6: Probability in various shapes

required probability is smallest in case of a circle and highest in case of a rectangle (1:20) where it almost approaches 1. The rectangle appraches a line in this case.

Asymptotically it can be concluded that if we choose three points randomly on a straight line they will always form an obtuse triangle. A straight line is a triangle with angles 0,0 and 180 degrees and hence this conclusion is validated.

3.6 Monte-Carlo Simulation inside a sphere

The problem is then extended by sampling points from a sphere.Points are randomly taken inside a sphere of unit radius as shown in Figure 3-7. The probabilities of

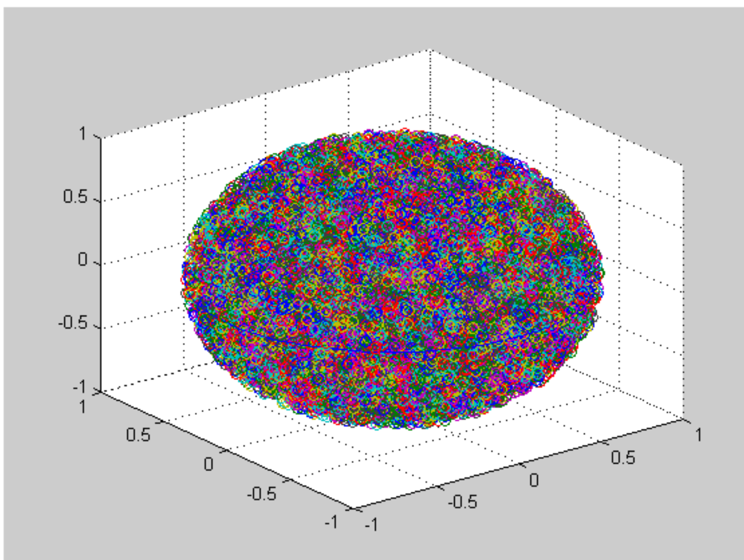


Figure 3-7: Random Points sampled inside a sphere of unit radius

forming an obtuse triangle as obtained in various simulations are listed in Table 3.4 and the mean was found out to be 0.5281. This is significantly less than those obtained when points are sampled from any of the regular two dimensional shapes.

Table 3.4: Probabilities obtained when random points are sampled from a sphere

Simulation	**Probability**
1	0.5302
2	0.5269
3	0.5276
4	0.5269
5	0.5278
6	0.5283
7	0.5279
8	0.5279
9	0.5307
10	0.5264

3.7 Square with Rounded Corners

In this section we carry out a simulation based experiment to study the change in the probability of forming an obtuse triangle when the geometrical shape from which the random points are sampled becomes more and more circular in nature. More specifically we want to study the effect of shape by considering figures which are intermediate between a square and a circle. This led to the concept of square with round corners. Let us consider a square of unit area. Let us make small squares of side r at the corners of the bigger square where $r < 0.5$. Then we draw semicircles taking the inner edges of the smaller squares as centres. Finally the new 'intermediate' figure is obtained by the semicircles along with the inside region. Further the radii of the semicircles is varied from 0.05 to 0.5 with a step of 0.05 to get all the intermediate figures. The last figure is hence a circle.Sampling of random points from each of these figures is shown in Figure 3-8. The average probability (p) values obtained by 10 runs of each simulation (one simulation used random points corresponding to 100,000 triangles) is listed in Table 3.5 with the standard errors. The graph of the p values with the varying r values is shown in Figure 3-9. $r = 0$ corresponds to square whereas $r = 0.5$ corresponds to a circle. The graph shows an almost continuous decrease in mean probability values as we approach from a square towards a circle. The abrupt change in the graph at r=0.45 can be taken to be an intrinsic error of simulation. This experiment hence suggests that the probability of forming an obtuse triangle

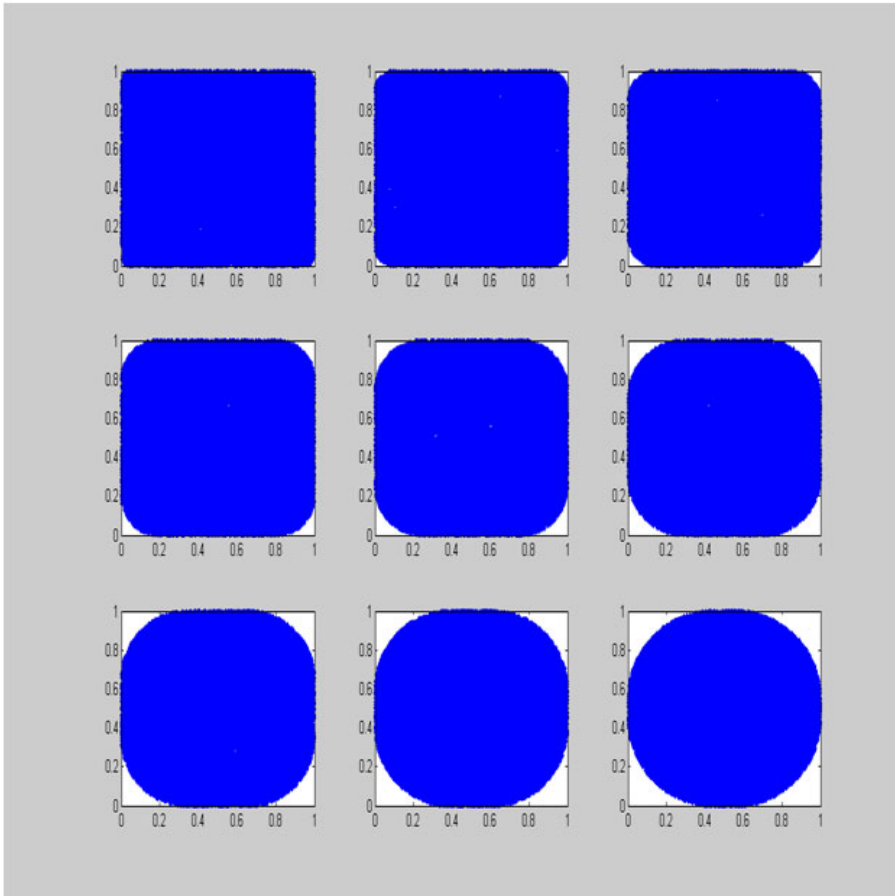


Figure 3-8: Square with rounded corners

Table 3.5: Simulation in Squares with Rounded Corners

Value of r	**P value**	**Standard Deviation**
0 (Square)	0.72555	0.001833
0.05	0.72456	0.001514
0.1	0.72393	0.001305
0.15	0.72339	0.00109
0.2	0.72307	0.001374
0.25	0.72279	0.001189
0.3	0.72207	0.00136
0.35	0.72113	0.001859
0.4	0.71939	0.001241
0.45	0.72039	9.94E-04
0.5(Circle)	0.71923	0.001236

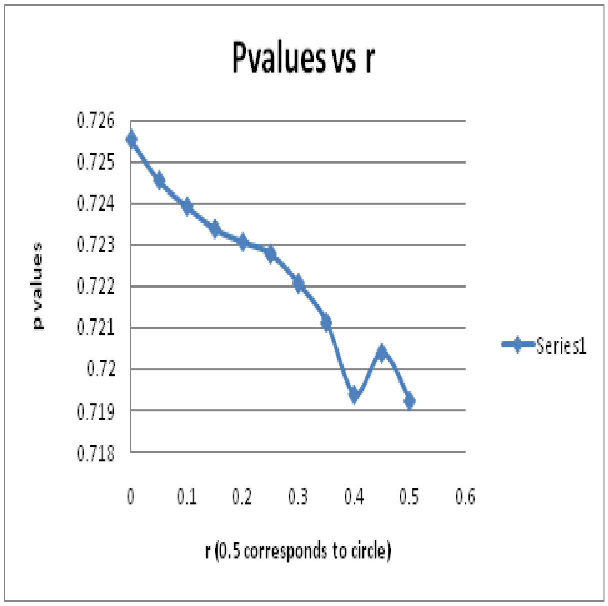


Figure 3-9: Probability values of simulations done in squares with rounded corners (from square to circle)

decreases as the figure from which random points are sampled becomes more and more circular.

3.8 Circle to Ellipse Experiment

In this section a simulation based experiment is conducted with ellipses to further study the effect of shape on the probability of forming an obtuse triangle. It was observed in Section 3.5 that the probability increases when the rectangle from which points are sampled becomes thinner (if the rectangle is of dimension 1: L, probability increases when L increases and tends quickly to 1 when L is more than 20). Hence asymptotically it can be expected that when a rectangle approaches a line the probability is 1. The motivation for this approach was to think of a suitable transition from a circle to a line. This can be modeled by ellipses as is shown in Figure 3-10. The circle can be shrunk to ellipses and it will finally approach a line. The ellipse is measured by its eccentricity (the eccentricity of the circle is 1 and that of a line is 0). Hence the experiment is started with an ellipse of eccentricity 1 (i.e. a circle) and continued with ellipses of decreasing eccentricity until one reaches a line (eccentricity 0).

In each case we sample random points (10,000 random triangles for each simulation

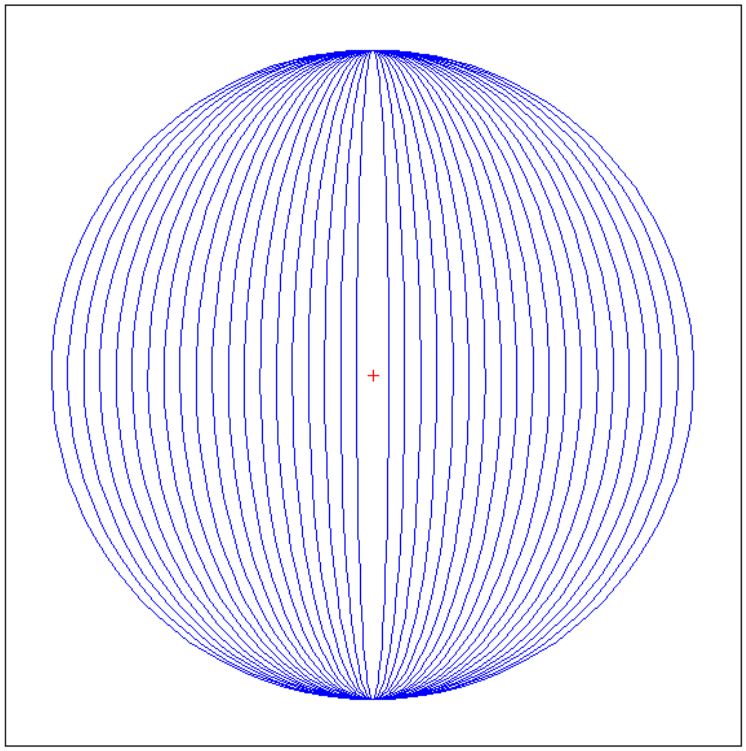


Figure 3-10: Shrinking Circle Experiment

and a total of 10 simulations are run) from the figures and estimate the probability by finding out the proportion of the triangles which are obtuse. The proposition is that if the obtuse fraction is minimum for a circle and increases continuously for all the intermediate figures and asymptotically reaches 1 for a line, we can conclude that the probability is minimum for a circle and increases as the shape becomes less and less circular. The intermediate figures obtained by the simulation are shown in Figure 3-11. The mean simulation values obtained for each of the figures are listed in Table 3.6. The graph showing the mean values plotted for each of the intermediate figure is shown in Figure 3-12. It shows a constant increase in the probability with a minimum for the circle and asymptotically reaching 1 as straight line is reached.

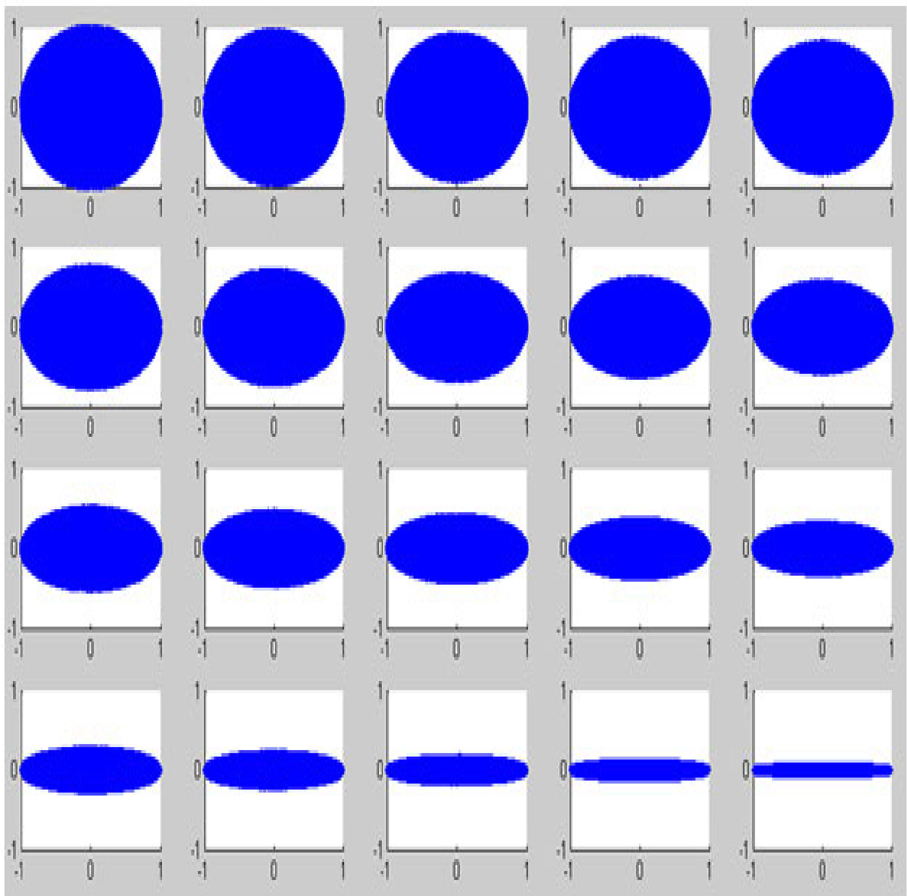


Figure 3-11: The circle to ellipse transition

Table 3.6: Simulation results of the various intermediate elliptical figures

Index of Figure	Value of b(semi major axis)	p value	Standard Deviation
1	1 (Circle)	0.7195	0.0018
2	0.95	0.7199	8.33E-04
3	0.9	0.7221	0.0011
4	0.85	0.7244	0.001
5	0.8	0.7286	0.0013
6	0.75	0.735	0.0011
7	0.7	0.7425	0.0017
8	0.65	0.7516	0.0018
9	0.6	0.7632	9.98E-04
10	0.55	0.7776	8.44E-04
11	0.5	0.7932	0.0013
12	0.45	0.8117	0.0012
13	0.4	0.8318	0.0012
14	0.35	0.8552	0.0011
15	0.3	0.8789	6.59E-04
16	0.25	0.9038	8.91E-04
17	0.2	0.929	9.65E-04
18	0.15	0.9534	6.42E-04
19	0.1	0.975	5.89E-04
20	0.05	0.9919	2.58E-04

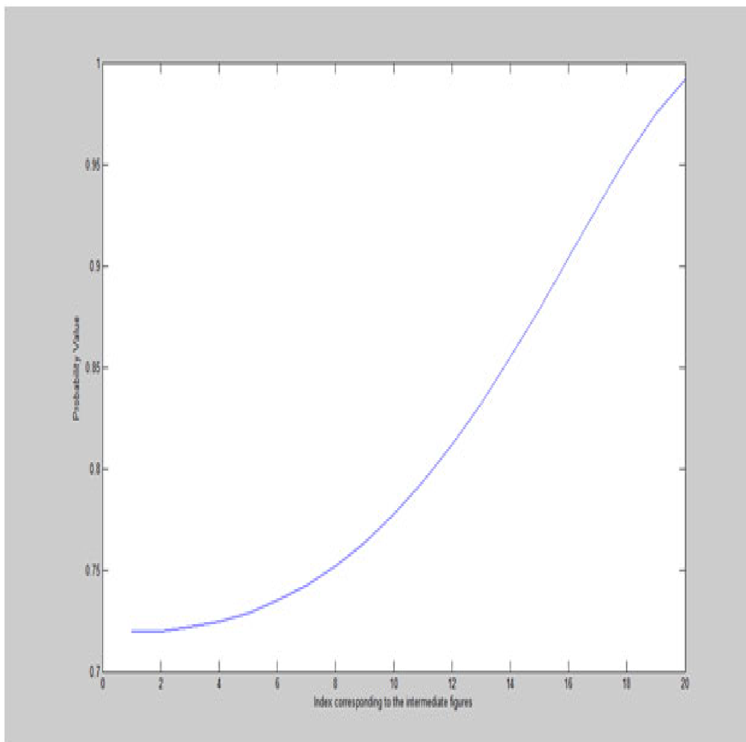


Figure 3-12: Graph showing probabilities in the circle to ellipse experiment. The probability reaches 1 as eccentricity of the intermediate ellipse approached 0

3.9 Simulation in Similar Triangles

Intuitively the probability of forming a random obtuse triangle should only depend on shape and not on the size of the figure from which points are sampled. To prove this by means of numerical simulation we can hypothesize that this probability will be same for all triangles (from which random points are taken) which are similar to each other. We generate similar triangles by the following steps:

- We draw a random triangle by choosing 6 numbers at random and assigning them as the coordinates of the three vertices.
- Let ABC be the triangle and AM be a median. Let E be the centroid of the triangle. A point A' is choosen which lies on AM such that AA'=$\frac{1}{10}$AE
- Similarly we choose B' and C' such that BB'=$\frac{1}{10}$BE and CC'=$\frac{1}{10}$CE.
- A'B'C' is similar to ABC. Steps are repeated to get a new triangle which is similar to A'B'C' and by the property of similar triangles is similar to ABC.
- Hence we generate 10 triangles all similar to each other.

All the triangles thus generated are shown in Figure 3-13 The next step is to generate

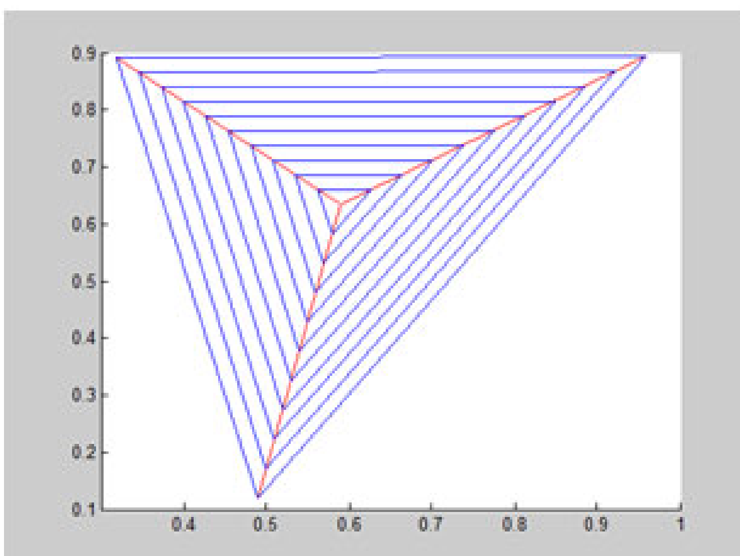
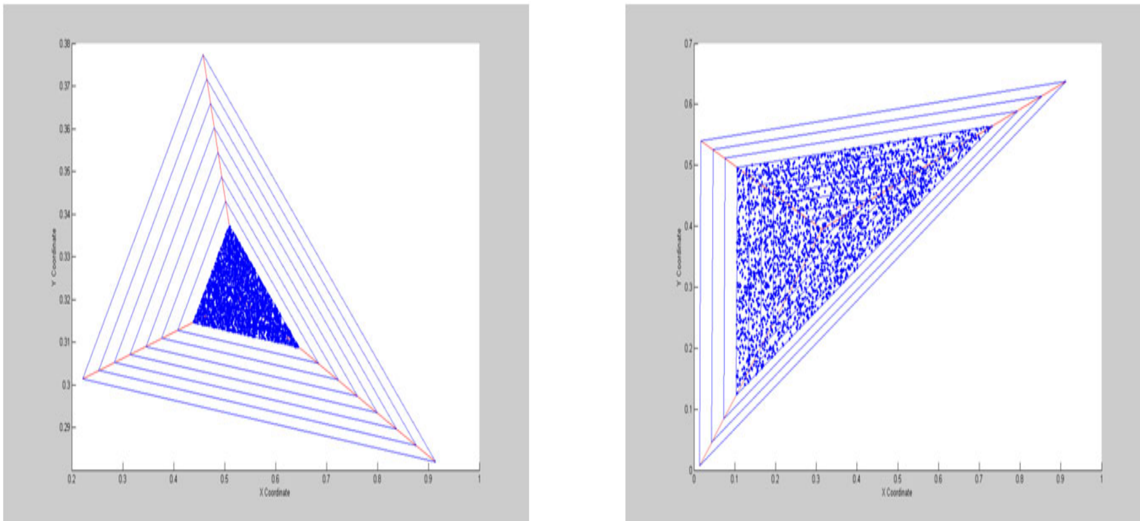


Figure 3-13: Generating similar triangles by the movement of vertices towards the centroid along the medians

random points in each of these triangles. We find out the proportion of triangles which are obtuse by sampling points from each of the triangles. The sampling of points in two of the triangles is shown in Figure 3-14. The mean values of simulation are



(a) Random Points sampled from the 4th triangle from the centroid

(b) Random Points sampled from the 8th triangle from the centroid

Figure 3-14: Random Points Sampling in Similar Triangles

plotted in Figure 3-15. In the Figure the triangle furthest away from the centroid is numbered 1 and the one closest to the centroid is numbered 10. As can be seen the probability is almost the same for all the triangles from which points are sampled. The only minor discrepancies are due to the intrinsic error of Monte-Carlo Simulation. This suggests that the probability is independent of the size and hence remains the same in similar triangles.

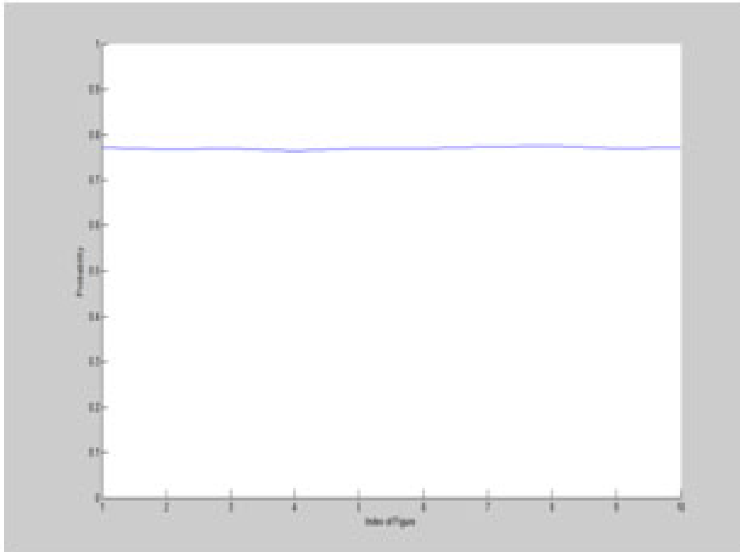


Figure 3-15: Mean probability values in similar triangles.

3.10 Random Walk of Triangles

This section deals with the effect of probability of a triangle being obtuse when we generate triangles by the random walk method.

3.10.1 Random Walk

A random walk is a trajectory that consists of taking successive random steps. The next state in such a model is defined by a transition from its original state by a transition function which is random in nature. Hence random walk is an example of Markov Process where the future behaviour is independent of the past history. Well known applications of such a model include Lévi Flight Problem [16] and the Drunkard's Walk [21].

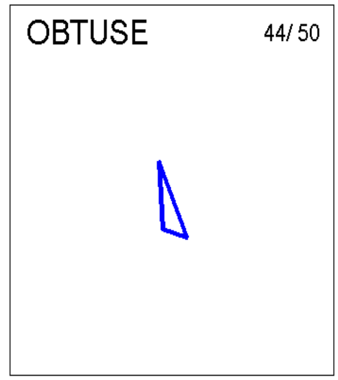
3.10.2 The Process

In this simulation we initially generate a triangle by generating the three vertices randomly. The random number generation is scaled to a particular value so that the sides of the triangle thus obtained is within a specified limit. Generation of successive triangles follows the MATLAB command:

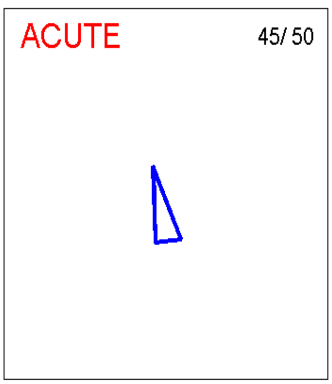
```
z = z + 0.03*(randn(3,1)+i*randn(3,1))
```

One way to comprehend this is to think that the triangle formed at time $t + 1$ is dependent on the triangle formed at time t and there is a random noise associated with it. Figure 3-16 shows snapshots of this random triangle generation. In this experiment 50 such triangles were generated using the Random Walk Algorithm. The Figure shows 5 of them namely the 43^{rd} to 47^{th} ones. As can be seen from the Figure, the 44^{th} triangle is generated by shifting the vertices of the 43^{rd} triangle by random amounts. This is continued for the other figures. For each of the triangles thus generated by simulation we check whether it acute or obtuse (mentioned at the top of every subfigure). We then calculate the proportion of obtuse triangles obtained by this simulation.

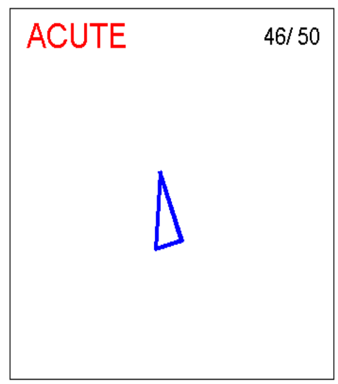
(a) 43



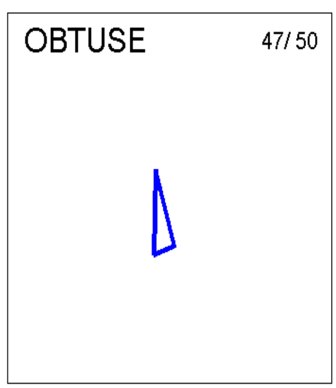
(b) 44



(c) 45



(d) 46



(e) 47

Figure 3-16: Random Walk in Triangles

3.10.3 Results

Monte Carlo simulation was performed using 10,000 triangles generated by Random Walk. We repeated the entire setup 10 times. The mean probabilities of forming an obtuse triangle (calculated by the proportion of resulting triangles that are obtuse) was plotted against the number of the iteration in Figure 3-17. The next aim was to

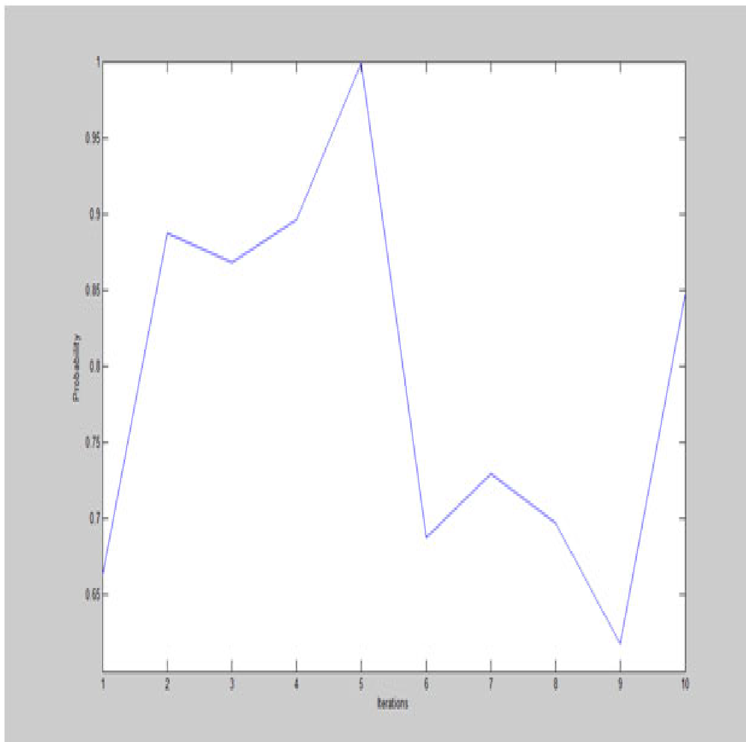


Figure 3-17: Proportion of obtuse triangles as obtained in the Random Walk Problem

check whether there was any pattern in these numbers. Hence formally the test can be defined as:

- Null Hypothesis (H_0): The probabilities obtained are random in nature.
- Alternate Hypothesis (H_1): The probabilities are not random and they follow a definite pattern.

We used the Wald-Wolfowitz Runs Test [20] to test for randomness. This is a test of the null hypothesis that the values (in this case the probabilities) come in random order, against the alternative that they do not. The p value for this test came to be 0.33. Hence we cannot reject the null hypothesis that the numbers are random and independent i.e. they do not follow a particular pattern.

3.11 Differential Equation in Triangle Space

3.11.1 The Process

In this section we describe an unique process which simulates the solving of a differential equation in triangle space. The differential equation is modeled by change in area. A typical experiment consists of the following steps:

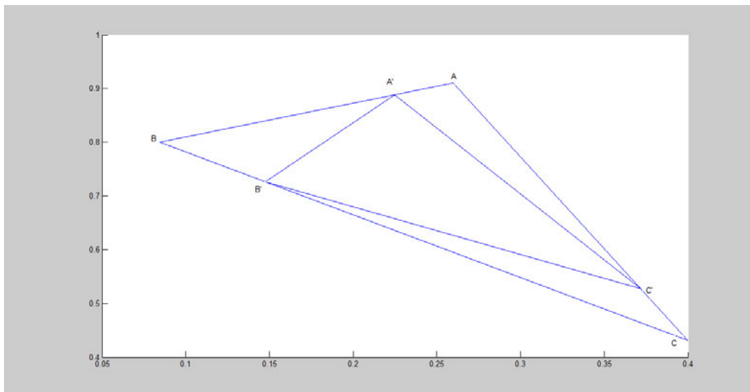
- Initially we choose a random triangle.
- We move the three vertices by a distance *deltah* towards their counterclockwise neighbour. Hence A moves towards B (along AB) to form the new vertex A' and so on. Hence we get triangle A'B'C'.
- We further obtain triangles by repeating the previous step in every iteration.

The entire procedure is shown in Figure 3-18. The loci of the three vertices in this experiment over 100 such iterations can be seen in part b of the figure.

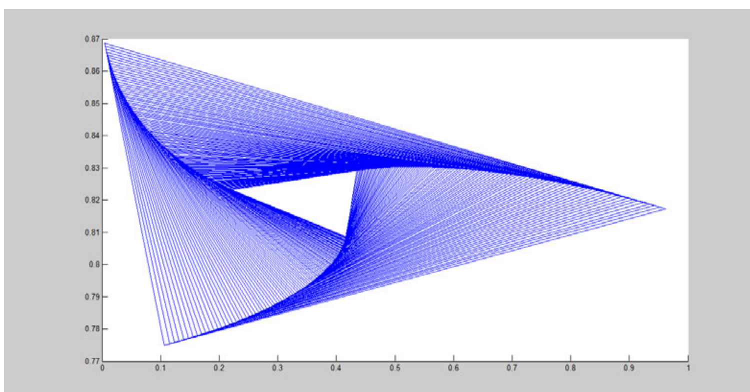
3.11.2 Characteristics of the Angles

We plot each of the angles over all such iterations to study the variation. In each case irrespective of the original random triangle chosen, we obtain the plot as a perfect wave as shown in Figure 3-19(a). We have considered 10,000 iterations to generate this Figure. Value of *deltah* used in each iteration is 0.001. This shows that the angle repeats itself after a number of iterations. We then plot all the three angles on the same graph against the number of iterations. This is shown in Figure 3-19(b). As can be seen from the Figure, all the angles follow exactly the same wave pattern with an initial phase difference. This observation is confirmed by repeating the experiment several times.

We repeated the entire experiment 100 times(each consisting of 100 such iterations). As was discussed we got a wave graph for each of the angles in each of the experiments. We then compared these waves over all the experiments performed. The

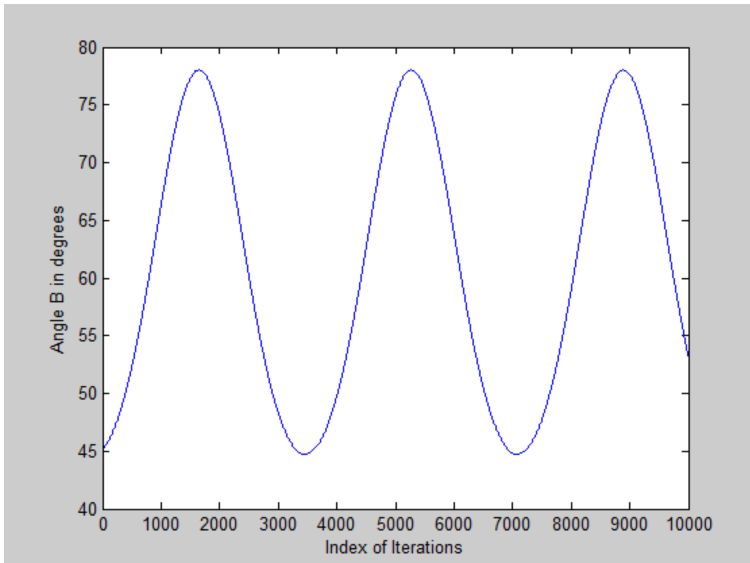


(a) The process by which a triangle at time t+δt is created

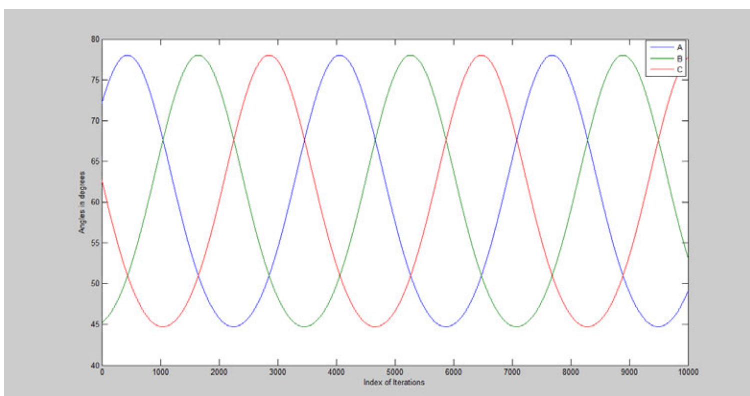


(b) The loci of the vertices is shown. The figure is created using 100 iterations.

Figure 3-18: Differential Equation in Triangle Space

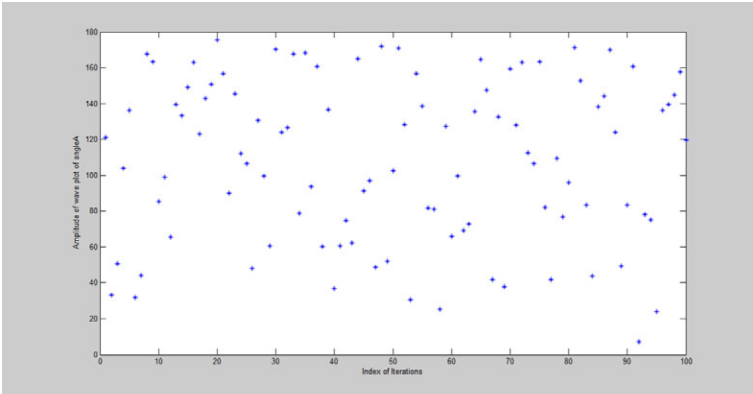


(a) The wavy nature of angle B. The angle repeats itself over several iterations

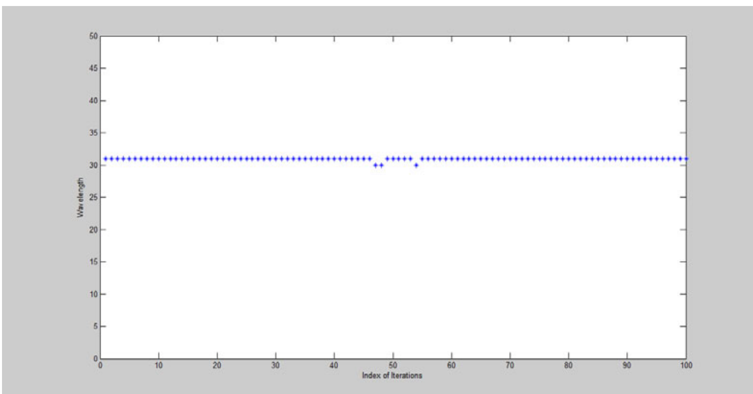


(b) All the angles are shown together. They follow the same wave exactly with an initial phase difference

Figure 3-19: Differential Equation in Triangle Space amplitude of the wave was different in each experiment (Figure 3-20(a)) showing that it depends on the initial random triangle chosen. However the wavelength remained the same throughout all the experiments as can be seen in Figure 3-20(b). The nature of the results remained unchanged when *deltah* was varied.



(a) Amplitude of the wave plot in several experiments. As can be seen visually they are random and hence can be assumed to depend on the initial triangle.



(b) Wavelength of the wave plot in several experiments. They are the same in all such experiments conducted.

Figure 3-20: Variation of Amplitude and Wavelength in different experiments

As the wavelength was the same for different experiments, we suspected that they will have a specific variation with *deltah*. Hence we carried out the experiment with different values of *deltah* varying from 0.001 to 0.1. It was noted that the number of iterations in one such experiment has to be adjusted according to the value of *deltah* chosen. We observed by trial and error that the number of iterations for the range considered can be calculated empirically as:

```
num = ceil(10/deltah)
```

where 'num' is the number of triangles which is equal to the number of iterations in an experiment. We then plotted the wavelengths against the deltah of the corresponding experiment. Figure 3-21 shows the plot. As we can see there is an inverse relationship between the wavelength and deltah.

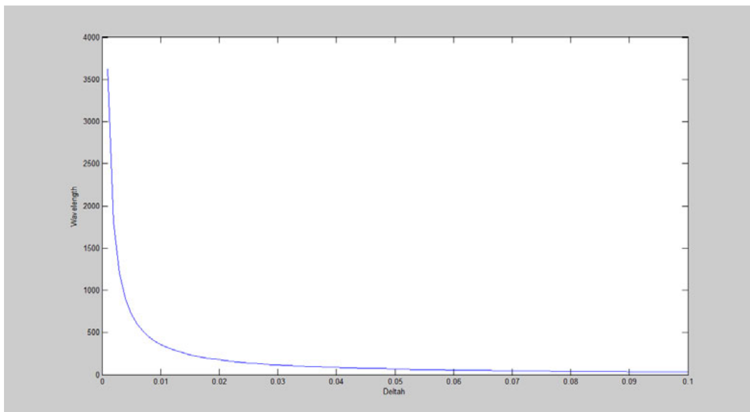


Figure 3-21: Graph showing wavelength as a function of deltah. An inverse relationship is observed.

It is to be noted that the wavelength is estimated by the difference in x-coordinates between two neighbouring optimal (maxima or minima) points on the graph. As the initial graph is a discrete set, the maximum or minimum points may not correspond to one of the discrete values. Hence approximation has been done to calculate the wavelength.

The nature of the plot suggests a negative exponential relationship between wavelength and deltah. We analyzed this using the Curve Fitting Tool in MATLAB. Two models and the corresponding goodness of fit measures are listed in Table 3.7. The first model fits the data almost perfectly. This is shown visually in Figure 3-22. Also this can be concluded by noting that both the R-square and the adjusted R-square measures of the fit is in the excess of 0.99. Hence we conclude that the relationship between wavelength(W) and deltah(δh) is best given by:

$$W = 6370e^{-757.6\delta h} + 618.8e^{-52.71\delta h} \tag{3.7}$$

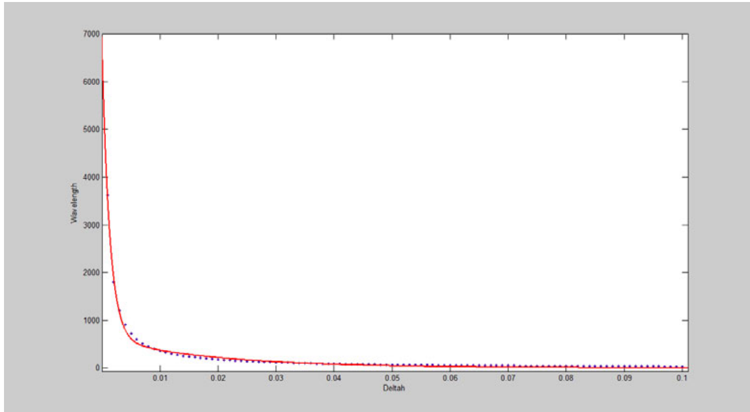


Figure 3-22: Exponential curve fitting on the plot of wavelength against deltah. The fitted curve corresponds to the general model of $f(x) = a * exp(b * x) + c * exp(d * x)$

Table 3.7: Two exponential models used to characterize the relationship between wavelength and deltah.

General model	**Coefficients (with 95% CI)**	**R-square**	**Adj. R-square**
f(x) = a*exp(b*x) + c*exp(d*x)	a = 6370 (6039, 6701)	0.9931	0.9929
	b = -757.6 (-804.6, -710.6)		
	c = 618.8 (557.6, 680)		
	d = -52.71 (-58.15, -47.28)		
f(x) = a*exp(b*x)	a = 4778 (4300, 5256)	0.9131	0.9122
	b = -384.1 (-428.5, -339.7)		

3.11.3 Obtuse/Acute Nature of Triangles Formed

The process explained earlier starts with a random triangle. Hence it is interesting to note whether the new triangles which are generated are acute or obtuse. Intuition might suggest that it will depend on the initial random triangle. A sample experiment is shown in Figure 3-23 where we determine whether the triangle is obtuse or acute everytime it is generated in an iteration. A value of 1 indicates the triangle is obtuse and that of 0 indicates that it is acute. The Figure shows that all the triangles generated are acute including the first one.

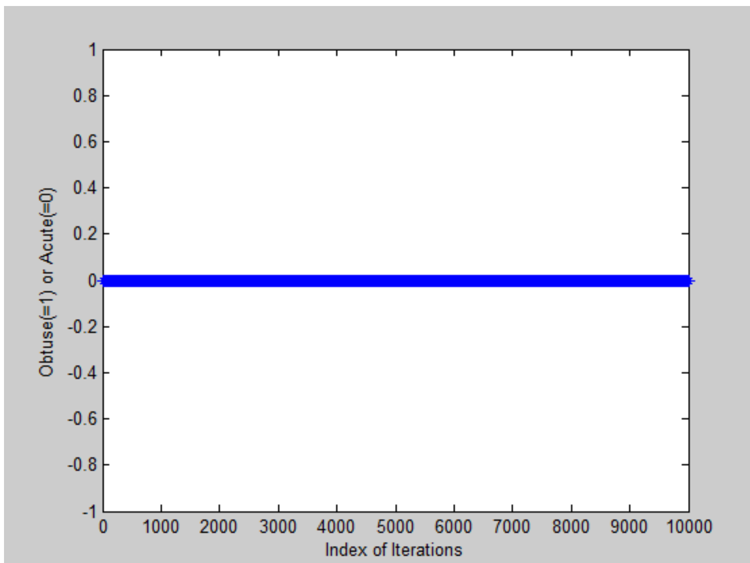


Figure 3-23: Obtuse/Acute triangle in several iterations. A value of 1 indicates the triangle is obtuse whereas a value of 0 indicates that it is acute. In this particular experiment all triangles obtained are acute in nature.

We repeated the entire set of experiments 100 times. In each case we obtain the proportion of the triangles which are obtuse. The results are plotted in Figure 3-24. The following can be concluded based on the results obtained.

- Most of the times we obtain a high proportion of obtuse triangles (almost 1 in many cases). This is expected as the probability that a random triangle is obtuse is almost 0.75 as was seen in this chapter.
- In some cases the proportion of obtuse triangles is 0 and hence that of acute triangles is 1.

- There are cases when the proportion is almost 0.5. This suggests that this proportion is independent of the nature of the initial random triangle.

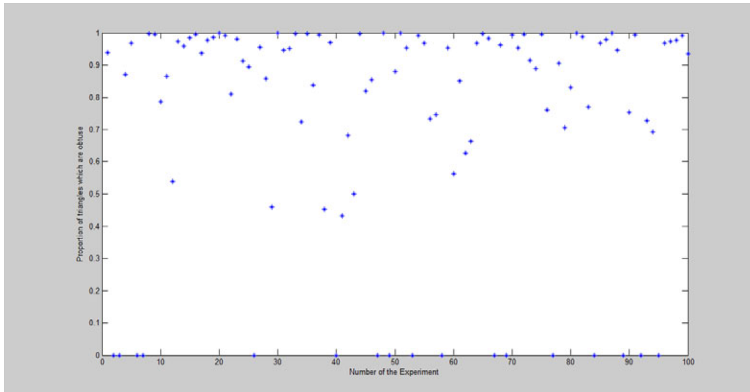


Figure 3-24: Proportion of obtuse triangles obtained in several experiments performed. As in some cases we see values near 0.5 we cannot conclude anything definitely about the pattern of 'obtuseness' in this simulation.

Chapter 4

Random Quadrilaterals

This chapter deals with the problem of finding the probability that a random quadrilateral will be convex. We will use the term 'non-concave' quadrilaterals instead of 'convex' as will be explained in Section 4.1. We shall consider various shapes to sample random points and also discuss the effect of these shapes on the probability.

4.1 Definition

There are three different types of quadrilaterals: convex,crossed and concave. Figure 4-1 shows samples of each one of them. There are several ways to define and distinguish between the three kinds of quadrilaterals. One of them is based on diagonals. The diagonals of a convex quadrilateral are both inside the quadrilateral, in case of crossed both the diagonals are outside whereas in case of a concave one only one diagonal is inside and the other one is outside the quadrilateral. This can also be verified from Figure 4-1. The method discussed here corresponds to one of the Euclidean postulates. Checking that ABCD is a convex quadrilateral entails checking that each of its four vertices is contained in the interior of its opposite angle; and each such containment reduces in turn to two statements - for example, to say that D is in the interior of angle(ABC) means that A and D are on the same side of BC; C and D are on the same side of BA. There are eight such statements in all. However, when the definitions are expanded out, it is seen that four of them are redundant,

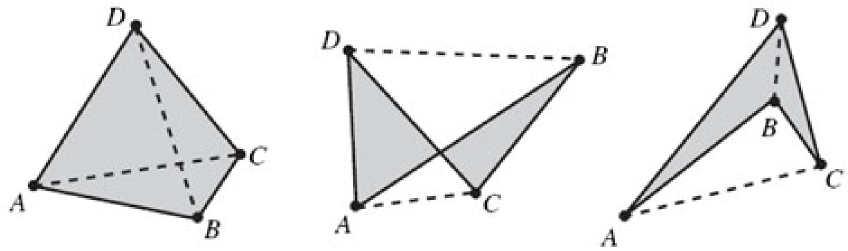


Figure 4-1: Convex,Crossed and Concave Quadrilaterals [19]

so in fact ABCD is a convex quadrilateral if and only if it satisfies the following four conditions:

- A and B are on the same side of CD,
- B and C are on the same side of AD,
- C and D are on the same side of AB
- D and A are on the same side of BC.

This can again be verified using Figure 4-1. Only Convex quadrilaterals obey the four conditions and both concave and crossed ones do not. It should also be noted that ABCD is taken to be a directional quadrilateral. This is important as it is possible to draw more than one quadrilateral using the same four points as shown in Figure 4-2. Considering the above definition, we then sample points from an unit square (corresponding to 100,000 random quadrilaterals) and determine the probability of forming a convex quadrilateral. This results in a mean probability value of 0.5853 for random numbers generated using 'rand'. We also sampled points using 'randn' and obtained a mean probability of 0.5783. However we shall broaden our definition in Section 4.2 and perform simulation in all regular shapes.

4.2 Sylvester Problem

The Sylvester Problem incorporates the crossed quadrilateral case as convex. The problem now is slightly changed. Instead of finding out the probability to form a convex quadrilateral, we are now interested in knowing the probability whether

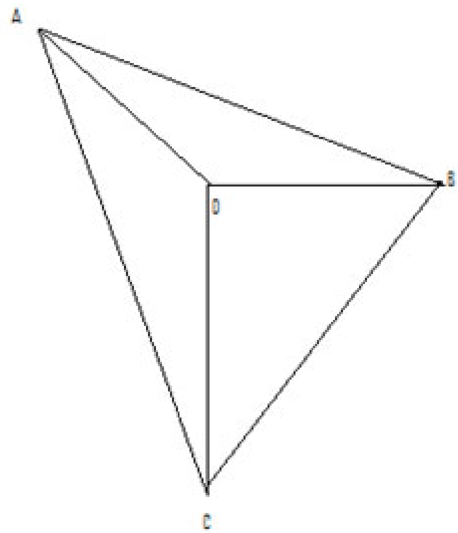


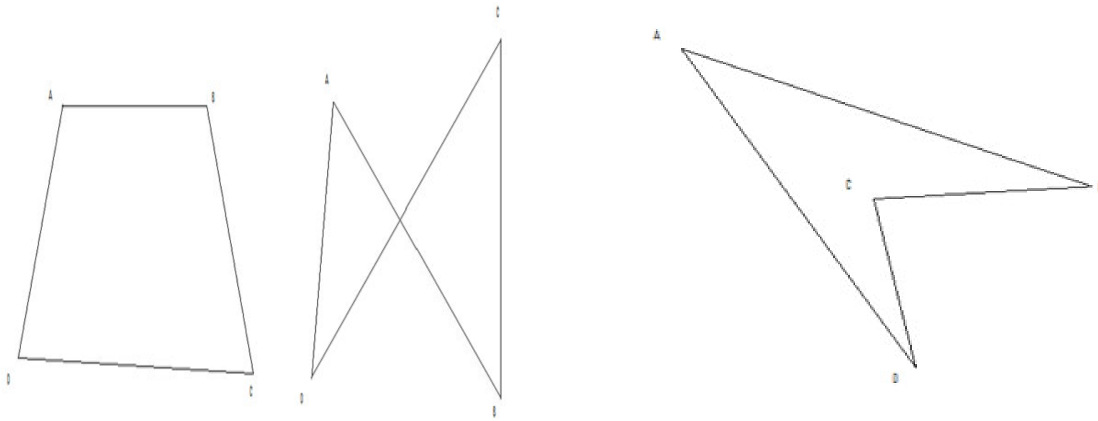
Figure 4-2: ADCB and ACDB are two different quadrilaterals that are possible with the same four points A, B, C and D.

ABCD forms a non-concave quadrilateral. In the Sylvester's Problem [8] a convex quadrilateral is defined as one formed by four points such that "none of the points is inside the triangle formed by the other three". It also restricts the four points to be taken inside a convex domain. As can be seen from Figure 4-3 both convex and crossed quadrilaterals obey this definition. Hence solving this problem is equivalent to finding out the probability of forming a (convex or crossed) quadrilateral when four points are taken randomly in some given space. The following steps describe the general algorithm used in this thesis for solving the Sylvester Problem.

1) Four points are randomly generated in a figure of a particular shape say square. Let the points be A, B, C and D.

2) ABCD is a convex quadrilateral if and only if it satisfies the following four conditions:

   - A and B are on the same side of CD,
   - B and C are on the same side of AD
   - C and D are on the same side of AB
   - D and A are on the same side of BC



(a) Non-Concave Quadrilaterals (b) Concave Quadrilateral

Figure 4-3: Non-Concave Vs Concave Quadrilaterals

3) If condition 2 is not satisfied the quadrilateral is either concave or crossed. Two of the four points are taken which doesn't satisfy the condition stated in 2. Hence we take A and B if they are on the opposite side of CD. The point of intersection of the line joining the points (A and B) with the line joining the other two points (C and D) is found out. Let this point be X.

4) Let length1 be the distance between A and B. Also let length2 be the distance between C and D.

5) ABCD is a crossed quadrilateral if it satisfies all the four conditions

   - Distance from X to A is less than distance1
   - Distance from X to B is less than distance1
   - Distance from X to C is less than distance2
   - Distance from X to D is less than distance2

   If any one of the above four conditions are not satisfied it is a concave quadrilateral.

6) Finally the required probability is obtained by:

$$Ratio = \frac{Convex + Crossed}{Convex + Crossed + Concave}$$

7) The simulations are iterated ten times (each simulation consists of 100,000 sets of points) and the mean of the ten ratios is taken.

This algorithm is applied to sample points from figures of different shapes in Section 4.3

4.3 Monte-Carlo Simulations of Quadrilaterals

4.3.1 Square/Rectangle

In this simulation random points are taken inside a square initially. The results for the ten simulations are shown in Table 4.1. The probability obtained by simulation (0.6943) is very similar to the analytical result [8] of 25/36 (= 0.6944). The experiment is repeated in rectangles of unit area but changing the ratio of the sides (corresponds to changing L in $1 : L$). The mean probability turns out to be exactly same in case of a rectangle irrespective of L.

Table 4.1: Simulation Results in a Square/Rectangle

Simulation	**Probability**	**Mean**	**Variance**
1	0.6934		
2	0.6932		
3	0.6954		
4	0.6945		
5	0.6933	0.6943	1.70E-06
6	0.6933		
7	0.6967		
8	0.694		
9	0.6929		
10	0.6958		

4.3.2 Triangle

The probability of forming a nonconcave quadrilateral when random points are taken inside a triangle is shown for various simulations in Table 4.2. The analytical value [8] obtained is 0.6667.

Table 4.2: Simulation Results in a Triangle

Simulation	Probability	Mean	Variance
1	0.6655		
2	0.6658		
3	0.6668		
4	0.6674		
5	0.6655	0.6664	5.23E-07
6	0.6666		
7	0.666		
8	0.666		
9	0.6675		
10	0.6665		

4.3.3 Regular Hexagon

The probability of forming a nonconcave quadrilateral when random points are taken inside a regular hexagon is shown for various simulations in Table 4.3. The analytical value [8] obtained is 0.7028.

Table 4.3: Simulation Results in a Regular Hexagon

Simulation	Probability	Mean	Variance
1	0.6998		
2	0.7043		
3	0.7018		
4	0.7035		
5	0.7016	0.7025	1.64E-06
6	0.7023		
7	0.7038		
8	0.7023		
9	0.7031		
10	0.7027		

4.3.4 Circle

The probability of forming a nonconcave quadrilateral when random points are taken inside a circle is shown for various simulations in Table 4.4. The analytical value does not exist in the literature. However according to Deltheil [8]

For all such figures the probability is not greater than the value for a circle (or ellipse). This has apparently not yet been proved.

Table 4.4: Simulation Results in a Circle

Simulation	Probability	Mean	Variance
1	0.7044		
2	0.7055		
3	0.7058		
4	0.705		
5	0.7044	0.7046	9.05E-07
6	0.7033		
7	0.7038		
8	0.7032		
9	0.7045		
10	0.7058		

4.3.5 Normally Generated PseudoRandom Numbers

Monte Carlo simulations were carried out using normally distributed random numbers using the command 'randn' in MATLAB. The probability to form a nonconcave quadrilateral was found out to be 0.6483. In Section 4.1 it was shown that the probability of forming a convex quadrilateral using randn is 0.5783. The increase in probability is due to the incorporation of the crossed case.

Table 4.5: Simulation Results using Normally Generated PseudoRandom Numbers

Simulation	Probability	Mean	Variance
1	0.6454		
2	0.6488		
3	0.6495		
4	0.6473		
5	0.6474	0.6483	1.88E-06
6	0.648		
7	0.6493		
8	0.6497		
9	0.648		
10	0.6496		

Hence it was found out that among the various standard shapes to sample the random points, the probability for forming a nonconcave quadrilateral is maximum for a circle and minimum for a triangle. Also the probability is least if normally distributed random numbers (randn) are used.

4.4 Simulation in Square with Rounded Corners

The simulation experiment done in Section 3.7 is repeated here for the case of convex quadrilaterals.The main aim is to show that the probability increases as the geometrical figure from which random points are sampled becomes more and more circular in nature. The shapes (squares with round edges) studied in this simulation were intermediate between a square and a circle. We performed Monte-Carlo Simulations (10 iterations of 100,000 quadrilaterals in each case) using random points sampled from each of these shapes. As is seen from Figure 4-4, the probability of forming a nonconcave quadrilateral constantly increases as the shape of the figure becomes more and more circular. This probability is least for a square and maximum for a circle. The mean probability values are also listed in Table 4.6 for all intermediate shapes.

Table 4.6: Probabilities obtained in the 'Square with Rounded Corners' experiment. The values are obtained by sampling random points from a square, a circle and all intermediate figures.

Figure	**Mean Probability**	**Variance**
Square	0.6952	0.0017
	0.6955	0.0011
	0.6965	0.0012
	0.6974	0.0013
	0.6993	0.0012
	0.701	0.0012
	0.7024	0.0015
	0.7032	0.0013
	0.7032	0.0011
	0.7035	0.001
Circle	0.7041	0.0015

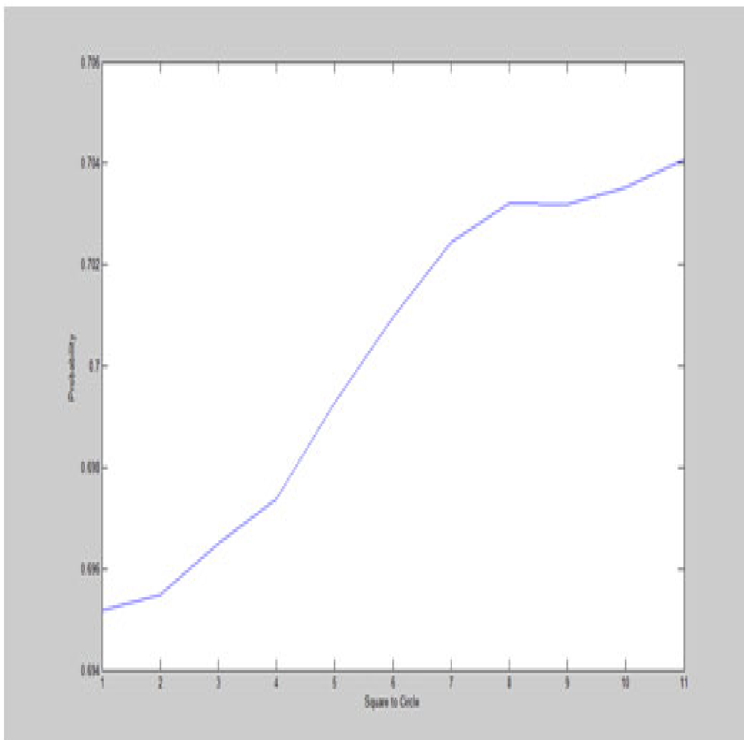


Figure 4-4: Probabilities are shown for figures ranging from square (indexed 1) to a circle (indexed 11).

4.5 Circle to Ellipse Experiment for Quadrilaterals

This section performs the same experiment as was done in Section 3.8. We expect that the simulation values will almost be same. The reason is the 'disk triangle picking' (Section 1.2.2) problem predicted the same values for the circle and the ellipse. Each simulation considered $100,000$ random quadrilaterals over each of the intermediate shapes (circles to ellipses of reducing eccentricity) and the mean probability of the formation of nonconcave quadrilaterals was found out over 10 iterations. The values obtained are shown in Table 4.7. As can be seen from the Table the values indicate

Table 4.7: Simulation results in ellipses of decreasing eccentricity. The figures vary from circle to narrow ellipses approaching a line. Values are same in all cases (till 2^{nd} order of decimal)

Index of Figure	Value of b(semi major axis)	p value
1	1 (Circle)	0.705
2	0.95	0.7044
3	0.9	0.7036
4	0.85	0.7046
5	0.8	0.7044
6	0.75	0.7044
7	0.7	0.7048
8	0.65	0.7047
9	0.6	0.7035
10	0.55	0.7042
11	0.5	0.7045
12	0.45	0.7037
13	0.4	0.7048
14	0.35	0.7044
15	0.3	0.7053
16	0.25	0.7045
17	0.2	0.704
18	0.15	0.7049
19	0.1	0.7051
20	0.05	0.7046

that the mean probabilities is constant over ellipses of all eccentricities to the second place after decimal. Small changes (third place after decimal) is due to intrinsic error involved in Monte-Carlo simulation.

Chapter 5

Conclusions

We performed various Monte-Carlo simulations in the thesis to find the answers to two problems. The problems involved generation of random triangles and quadrilaterals. The method we used to generate random numbers inside a given figure is to initially generate them in the rectangle of minimum area which covers the figure entirely. Then we check whether the random point thus generated lies within the area of interest and accept or reject it accordingly. Normal random numbers were generated using the Ziggurat Algorithm.

The first problem involved finding out the probability that a random triangle is obtuse. We generated angles randomly in the plane $x + y + z = 180$ and this method yielded a probability value of 0.75. We also tried to solve the problem by choosing random numbers corresponding to the sides of the triangle and then considering only those in the simulation which satisfied the triangle inequality. It was observed that 50 percent of the cases resulted in triangle formation. The probability obtained by this method was significantly lower than in the other simulations. The broken stick problem was modified to yield a probability of 0.68 of forming an obtuse triangle given that the pieces satisfy the triangle inequality in the first place. Monte-Carlo Simulations were performed by chosing random points inside various shapes. Circle gave the minimum value of the probability (0.720146) whereas very thin rectangles(approaching a line) gave the maximum value (almost equal to 1).We also showed by means of simulation that the probability continuously decreases when the figure from which random points are sampled becomes more and more circular in nature. In particular we show a continuous increase in probability when the figure changes from a circle to a square and also from a circle to ellipses of decreasing eccentricity. Our simulation results also matched with the analytical ones available in literature. We then carried out a simulation based experiment to show that the probability is independent of the area of the figure from which points are sampled and is hence same for triangles which are similar to each other. We introduced the concept of random walk of triangles. The probability that a triangle is obtuse in random walk is concluded to be itself a random number and no visible pattern was observed in the various simulations that were performed. Finally we also introduced the idea of differential equations in triangle space.

The second problem which we discussed involved finding out the probability that a random quadrilateral is non-concave. Because of the discrepancies over the definition of the problem we tried to solve the Sylvester Four Point Problem and hence included crossed quadrilaterals in the solution. Strangely in this case the circle and ellipse gave the same result. This probability value of 0.7046 corresponded to the maximum probability limit [2] that can be obtained in any shape. We obtained probabilities by sampling points from all regular shapes. We then carried out simulation in intermediate figures between a square and a circle (square with round corners). An increasing trend was obtained showing that the probability increases when the figure becomes more circular. The circle to ellipse experiment however gave the same results for all intermediate figures. The probability was also found to be exactly the same when random points were sampled from a square and a rectangle (irrespective of the ratio of the sides of the rectangle).

The normally generated random numbers resulted in extreme results in both the problems. In the quadrilateral problem we got a probability of 0.6483 when we used normally generated pseudorandom numbers for sampling random points. This result is of special interest as it is less than the minimum predicted value [2] of $\frac{2}{3}$ for all shapes of the Sylvester Four Point Problem. In case of the obtuse triangle problem we got the probability of 0.7497 when we used 'randn'to generate random numbers.

This result is greater than almost all (except too thin rectangles) shapes considered to sample random points.

As mentioned earlier the programs to generate the simulation results as well as the graphs are available online. The author hopes that the thesis helped the reader to have a better and in depth understanding of these two beautiful geometrical probability problems.

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