VII. NOISE IN ELECTRON DEVICES*

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A. QUANTUM NOISE IN THE LASER OSCILLATOR WITH FINITE MATERIAL BANDWIDTH

1. Introduction

Haus[1] has obtained results for the phase and amplitude fluctuations of the laser oscillator above threshold by means of a classical model in which the semiclassical Van der Pol equation contains noise sources that correctly represent the properties of the field below threshold. We have shown in a previous report[2] that this model is "equivalent" to the quantum analysis (apart from small "saturation" and "quantum" corrections), provided the bandwidth of the material is much wider than the cold-cavity bandwidth. In this report we describe an equivalent classical model for a laser in which the material bandwidth is arbitrary. "Sufficiently" below threshold it correctly predicts the moments of the normally ordered products of the creation and annihilation operators of the field. "Sufficiently" above threshold it predicts the field moments with an accuracy that is very good but not good enough to distinguish between normally and unnormally ordered products. The proof of the equivalence of this model is published elsewhere.[3]

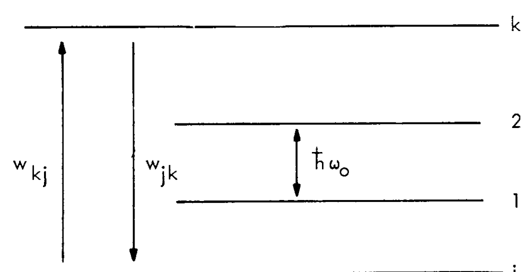

Fig. VII-1. Level pair (1,2) of the many-level system; transition probabilities w between two arbitrary levels.

2. Laser Model

The laser oscillates in one mode with resonance angular frequency ω_o. The material system consists of a large number of many-level systems. In each of these there is one level pair with resonance angular frequency ω_o (Fig. VII-1). This level pair is dipole coupled the field mode. The modulus κ of the coupling constant is assumed to be independent of

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position. The upper and lower levels of such a level pair will be denoted by subscripts 2 and 1, respectively. The field mode is also coupled to a loss system that represents radiation into the space outside; the material system is also coupled to a material reservoir that introduces pumping, nonradiative decay (or radiative decay into black-body modes) and randomization into the material system.

The notations used in this report includes the following: the decay constant of the field as caused by the loss system is denoted μ and the cold-cavity bandwidth is then 2μ; the amplification constant of the field as caused by the unsaturated material (if the material bandwidth were infinitely wide) is denoted by γ; the hot-cavity bandwidth is defined $\omega_h = 2(\gamma-\mu)$; the material bandwidth is denoted 2Γ; we shall also introduce the decay constant Γ_p of the inversion.

3. Loaded LC Circuit Driven by Noise

An LC circuit loaded by a conductance G and driven by a noise-source current i_n (Fig. VII-2) is described by the equations

$$-L\frac{dI}{dt} = V; \qquad C\frac{dV}{dt} + GV - I = i_n. \tag{1}$$

If we set $\omega_o = (LC)^{-1/2}$; $V = (\hbar\omega_o/2C)^{1/2}\, i(a-a^+)$ and $I = (\hbar\omega_o/2L)^{1/2}\,(a+a^+)$, so that the energy $\frac{1}{2}(LI^2+CV^2)$ in the LC circuit equals $\hbar\omega_o a^+a$, then Eqs. 1 can be transformed to

$$\begin{aligned} \frac{da}{dt} &= -i\omega_o a - (G/2C)(a-a^+) - i(2\hbar\omega_o C)^{-1/2}\, i_n \\ \frac{da^+}{dt} &= i\omega_o a^+ - (G/2C)(a^+-a) + i(2\hbar\omega_o C)^{-1/2}\, i_n. \end{aligned} \tag{2}$$

We introduce new slowly time-variant variables by setting $i_n = i_n^+(t)\exp(i\omega_o t) + i_n^-(t)\exp(-i\omega_o t)$; $a = a(t)\exp(-i\omega_o t)$; $a^+ = a^+(t)\exp(i\omega_o t)$. We assume that the LC circuit has a high Q and that $i_n^+(t)$ and $i_n^-(t)$ contain only slow time variations with respect to $\exp(\pm i\omega_o t)$. We can then neglect the double-frequency drives in Eqs. 2 and obtain

$$\frac{da(t)}{dt} + ga(t) = x^-(t); \qquad \frac{da^+(t)}{dt} + ga^+(t) = x^+(t), \tag{3}$$

where

$$g = (G/2C); \qquad x^{\pm}(t) = \pm i(2\hbar\omega_o C)^{-1/2}\, i_n^{\pm}(t). \tag{4}$$

The plus and minus signs in Eq. 4 correspond to each other. Eventually we shall use time-domain and frequency-domain notation. Spectra and correlation functions are related by

$$\left\langle u(t+\tau)\, v(t)\right\rangle = \int_{-\infty}^{+\infty} \left\langle u(\omega)\, v(-\omega)\right\rangle e^{i\omega\tau} \frac{d\omega}{2\pi}$$

$$\left\langle u(\omega)\, v(-\omega)\right\rangle = \int_{-\infty}^{+\infty} \left\langle u(t+\tau)\, v(t)\right\rangle e^{-i\omega\tau}\, d\tau. \tag{5}$$

If spectra are said to be "frequency independent," this means that they are frequency-independent over a range around $\omega = 0$ which is very large compared with the various

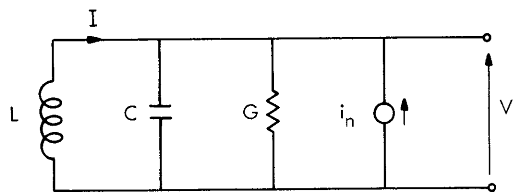

Fig. VII-2. Noisy loaded LC circuit.

relaxation constants, but very small compared with ω_o. We shall denote such spectra by $\langle uv\rangle$. The conversion from frequency domain to time domain is described by

$$u(t) \rightarrow u(\omega); \quad \frac{du(t)}{dt} \rightarrow i\omega u(\omega). \tag{6}$$

4. Steady State below Threshold

The equivalent classical model is obtained by assuming that the (normalized) conductance g consists of a loss conductance and a frequency-dependent gain conductance. With each of these are associated independent Gaussian noise sources with zero mean. In the frequency-domain we have

$$g(\omega) = \mu - \frac{\Gamma}{i\omega + \Gamma}\gamma$$

$$x^-(\omega) = x_L^-(\omega) + \frac{\Gamma}{i\omega + \Gamma} x_m^-(\omega); \quad \text{and complex conjugate,} \tag{7}$$

where 2μ is the cold-cavity bandwidth, 2Γ is the material bandwidth, and γ is the amplification constant of the field (at zero frequency or infinite material bandwidth) caused by the inverted material. Operation below threshold requires $\mu > \gamma$. The spectra of the noise sources $x_L^{\pm}$ and $x_m^{\pm}$ are frequency-independent and are given by

$$\left\langle x_L^+ x_L^-\right\rangle = \left\langle x_L^- x_L^+\right\rangle = 2\mu\beta_L; \quad \left\langle x_m^+ x_m^-\right\rangle = \left\langle x_m^- x_m^+\right\rangle = 2\gamma(1+\beta_m)$$

$$\left\langle x_L^+ x_L^+\right\rangle = \left\langle x_L^- x_L^-\right\rangle = 0; \quad \left\langle x_m^+ x_m^+\right\rangle = \left\langle x_m^- x_m^-\right\rangle = 0. \tag{8}$$

If T_L is the temperature of the loss system and P_2^o and P_1^o are the populations of the upper and lower level as established by the material reservoir alone, then

$$\beta_L = [\exp(\hbar\omega_o/kT_L)-1]^{-1}; \quad \beta_m = P_1^o/\left(P_2^o - P_1^o\right). \tag{9}$$

Note that the spectrum of the noise source associated with the frequency-dependent gain conductance is also frequency dependent:

$$\left\langle \frac{\Gamma}{i\omega + \Gamma} x_m^+(\omega) \frac{\Gamma}{-i\omega + \Gamma} x_m^-(-\omega) \right\rangle = \frac{\Gamma^2}{\omega^2 + \Gamma^2} 2\gamma(1+\beta_m). \tag{10}$$

Equations 3 can be written in the time-domain

$$\frac{d^2 a(t)}{dt^2} + (\Gamma+\mu)\frac{da(t)}{dt} + \Gamma(\mu-\gamma)\, a(t) = \left(\frac{d}{dt} + \Gamma\right) x_L^-(t) + \Gamma x_m^-(t); \quad \text{and complex conjugate.} \tag{11}$$

It is straightforward to obtain the spectrum $\langle a^+(\omega)\, a(-\omega)\rangle$ from Eq. 11.

$$\langle a^+(\omega)\, a(-\omega)\rangle = (A+B\omega^2)/D(\omega),$$

$$A = \Gamma^2[2\mu\beta_L + 2\gamma(1+\beta_m)]; \quad B = 2\mu\beta_L;$$

$$D(\omega) = [\Gamma(\mu-\gamma)-\omega^2]^2 + (\Gamma+\mu)^2\omega^2. \tag{12}$$

By means of Eq. 5, we can obtain the correlation function $\langle a^+(t+\tau)\, a(t)\rangle$. Higher order moments are obtained from the fact that $a^+(t)$ and $a(t')$ are Gaussian. A discussion of Eq. 12 has been given elsewhere.[3]

5. Steady-State above Threshold

The equivalent classical model is obtained by setting

$$g(\omega) = \mu - \frac{\Gamma}{i\omega + \Gamma}(\kappa^2/\Gamma)\, P(\omega); \quad x^-(\omega) = x_L^-(\omega) + \frac{\Gamma}{i\omega + \Gamma} x_m^-(\omega), \tag{13}$$

where $x_L^\pm$ and $x_m^\pm$ are independent Gaussian noise sources with zero mean, the frequency-independent spectra of which are given by

$$\langle x_L^+ x_L^-\rangle = 2\mu\left(\frac{1}{2} + \beta_L\right); \quad \langle x_m^+ x_m^-\rangle = 2\mu\left(\frac{1}{2} + \beta_m^s\right);$$

$$\beta_L = [\exp(\hbar\omega_o/kT_L)-1]^{-1}; \quad \beta_m^s = P_1^s/\left(P_2^s - P_1^s\right), \tag{14}$$

where T_L is the temperature of the loss system, and P_2^s and P_1^s are the steady-state populations of upper and lower levels, respectively, as established by the field and the material reservoir. Equations 3 can now be written in the time domain as

$$\frac{d^2a(t)}{dt^2} + (\Gamma+\mu)\frac{da(t)}{dt} + \Gamma[\mu-(\kappa^2/\Gamma)P(t)]\,a(t) = \left(\frac{d}{dt}+\Gamma\right)x_L^-(t) + \Gamma x_m^-(t); \quad \text{and complex conjugate.} \tag{15}$$

The net population difference $P(t) = P_2(t) - P_1(t)$ is determined by a set of coupled rate equations that express conservation of populations and photon number, that is,

$$\frac{dP_k(t)}{dt} + (\delta_{2k}-\delta_{1k})\left[\frac{da^+a(t)}{dt} + 2\mu a^+a(t)\right] + \Gamma_k P_k(t) - \sum_j w_{kj}P_j(t)$$
$$= X_k + (\delta_{2k}-\delta_{1k})\left(a^+x_L^- + x_L^+a\right), \tag{16}$$

where w_{kj} is the transition probability from level j into level k ($w_{kk} = 0$), and $\Gamma_k = \sum_j w_{jk}$ is the total transition probability out of level k. The variables on the right-hand side of Eq. 16 are Gaussian shot-noise sources with zero mean, associated with population transfer induced by the material reservoir, and with photon transfer between the field mode and the loss system. The second-order moments of these noise sources will be described below.

If one sets all noise sources equal to zero, one obtains the semiclassical steady-state solution (subscript s) with

$$\mu = (\kappa^2/\Gamma)P_s; \quad a_s = R_s e^{-i\theta_s}; \quad a_s^+ = R_s e^{i\theta_s};$$
$$(\delta_{2k}-\delta_{1k})\left(2\mu R_s^2\right) + \Gamma_k P_k^s - \sum_j w_{kj}P_j^s = 0. \tag{17}$$

Equations 15 and 16 are solved by setting

$$a(t) = [R_s+r(t)]\,e^{-i\theta(t)}; \quad a^+(t) = [R_s+r(t)]\,e^{i\theta(t)}$$
$$P_k(t) = P_k^s + p_k(t), \tag{18}$$

and linearizing in r, p_k, and $d\theta/dt$. We approximate da/dt and d^2a/dt^2 by

$$\frac{da}{dt} = \left[-iR_s\frac{d\theta}{dt} + \frac{dr}{dt}\right]e^{-i\theta}; \quad \text{and complex conjugate}$$
$$\frac{d^2a}{dt^2} = \left[-iR_s\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\right]e^{-i\theta}; \quad \text{and complex conjugate} \tag{19}$$

and obtain

$$R_s\left[\frac{d^2\theta}{dt^2}+(\Gamma+\mu)\frac{d\theta}{dt}\right]=\left(\frac{d}{dt}+\Gamma\right)C_L+\Gamma C_m \tag{20a}$$

$$\frac{d^2(2R_sr)}{dt^2}+(\Gamma+\mu)\frac{d(2R_sr)}{dt}-2\kappa^2R_s^2p(t)=\left(\frac{d}{dt}+\Gamma\right)(2R_sS_L)+\Gamma(2R_sS_m) \tag{20b}$$

$$\frac{dp_k}{dt}+(\delta_{2k}-\delta_{1k})\left[\frac{d}{dt}(2R_sr)+2\mu(2R_sr)\right]+\Gamma_kp_k-\sum_j w_{kj}p_j$$

$$=X_k+(\delta_{2k}-\delta_{1k})(2R_sS_L), \tag{20c}$$

where

$$C_L=\frac{1}{2i}\left[x_L^+e^{-i\theta}-x_L^-e^{i\theta}\right];\qquad C_m=\frac{1}{2i}\left[x_m^+e^{-i\theta}-x_m^-e^{i\theta}\right]$$

$$S_L=\frac{1}{2}\left[x_L^+e^{-i\theta}+x_L^-e^{i\theta}\right];\qquad S_m=\frac{1}{2}\left[x_m^+e^{-i\theta}+x_m^-e^{i\theta}\right]. \tag{21}$$

In order to evaluate the moments of the S and C noise sources, one treats θ as being independent of $x_L^{\pm}$ and $x_m^{\pm}$. One obtains

$$\langle C_LC_L\rangle=\langle S_LS_L\rangle=\mu\left(\frac{1}{2}+\beta_L\right);\qquad \langle C_mC_m\rangle=\langle S_mS_m\rangle=\mu\left(\frac{1}{2}+\beta_m^s\right) \tag{22a}$$

$$\langle X_kX_k\rangle=\Gamma_kP_k^s+\sum_j w_{kj}P_j^s=(1)^2\text{ (atomic rate in + out)} \tag{22b}$$

$$\langle X_kX_j\rangle=-\left(w_{kj}P_j^s+w_{jk}P_k^s\right)=-(1)^2\text{ (atomic transfer rate)} \tag{22c}$$

$$\langle X_kS_m\rangle=(\mu R_s/\Gamma)\left[\frac{1}{2}\Gamma_k(\delta_{2k}+\delta_{1k})-\frac{1}{2}(w_{k2}+w_{k1})\right]. \tag{22d}$$

The noise sources C_L and C_m are independent of the noise sources S_L, S_m, and X_k. The noise source S_L is also independent of X_k. The moments of the noise sources X_k, $2R_sS_L$, and $2R_sS_m$ describe, respectively, shot noise associated with population transfer induced by the material reservoir, photon transfer between the field mode and the loss reservoir (Fig. VII-3), and photon emission and absorption by the material system. The cross moment of the noise sources X_k and $2R_sS_m$ is hard to interpret; it is proportional to $\left(2\mu R_s^2/\Gamma\right)$, that is, to the number of photons emitted by the material in the mean time $(\Gamma)^{-1}$; this number is a measure of the expectation value of the off-diagonal (1, 2)-elements of the material density matrix in the steady state. The noise sources of Eq. 20a have been said to be independent of the noise sources of Eqs. 20b and 20c. In the quantum analysis the expectation value of the anticommutator of the corresponding operator noise sources is indeed zero, but the expectation value of the commutator is not zero; for example,

$$\left\langle [X_k, C_m] \right\rangle = \frac{\mu R_s}{i\Gamma} \left[(\delta_{2k}-\delta_{1k})\Gamma_k+(w_{k1}-w_{k2})\right]$$

$$\left\langle [S_m+S_L, C_m+C_L] \right\rangle = \frac{\mu}{i}\left(2\mu R_s^2/\Gamma P_s\right). \tag{23}$$

These commutators are neglected in our equivalent analysis, which is the cause of its lack of "quantum accuracy," that is, our analysis is not accurate enough to distinguish between the various orderings of the creation and annihilation operators of the field mode in the field moments. In particular, it does not allow us to calculate the field commutator or yield the difference between the second-order Glauber function and the correlation function of the photon number operator.

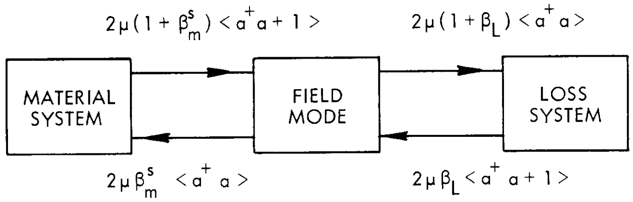

Fig. VII-3.

Transfer rates of photons between the field mode and the loss system and emission and absorption rates of photons in the material system.

Equation 20a can be solved immediately for phase fluctuations. We neglect $d^2\theta/dt^2$ and dC_L/dt and take into account the fact that the noise sources are Gaussian. Then we obtain

$$\left\langle \exp[i\theta(t+\tau)-i\theta(t)] \right\rangle = \exp\left(-\frac{1}{2}\omega_{ph}|\tau|\right) \tag{24a}$$

$$\omega_{ph} = \left(\frac{2\Gamma\mu}{\Gamma+\mu}\right)^2 \frac{\hbar\omega_o}{2P_{tr.}}\left(1+\beta_L+\beta_m^s\right), \tag{24b}$$

where $P_{tr.} = 2\mu\hbar\omega_o R_s^2$ is the average power in the laser beam. For $\Gamma \gg \mu$, this result reduces to the result derived in a previous report.[2]

The solution of the coupled equations (20b and 20c) for the moments of $2R_s r(t)$ is straightforward but can become very involved for an arbitrary material system. We give here only the solution for the case in which the material system consists of N strictly two-level systems. We introduce the following notation:

$$\Gamma_p = w_{21} + w_{12}; \quad P_o = N(w_{21}-w_{12})/\Gamma_p; \quad \gamma = (\kappa^2/\Gamma)P_o. \tag{25a}$$

From the semiclassical equations (Eqs. 17), it follows that Γ_p is the decay constant of the inversion, P_o is the inversion as established by the material reservoir alone, γ is the amplification of the field (at zero frequency) as caused by this inversion, and the field amplitude is given by

$$4\mu R_s^2 = \Gamma_p(P_o-P_s); \quad \mu = (\kappa^2/\Gamma)P_s. \tag{25b}$$

We also introduce

$$e = (\gamma-\mu)/\mu = aR_s^2; \quad a = 4\kappa^2(\Gamma\Gamma_p)^{-1}; \quad \omega_h = 2(\gamma-\mu)$$

$$\epsilon = \Gamma/(\Gamma+\mu+\Gamma e) \leqslant 1; \quad \delta = \Gamma_p/2\Gamma \leqslant 1. \tag{25c}$$

In previous reports[1,2] ω_h has been called the "hot-cavity bandwidth" (the bandwidth of the intensity fluctuations if the material has infinite bandwidth); e is the ratio of the hot-to-cold cavity bandwidth and is directly proportional to power output and normally very small close to threshold; e^{-1} has been called the "enhancement factor"; a is the nonlinear constant of the material; ϵ and δ are two new parameters; δ is smaller or equal to one because $\Gamma = \frac{1}{2}\Gamma_p + \Gamma_{ph}$.[3] From the equivalent model (Eqs. 20-22), we obtain

$$\left\langle r(\omega)\, r(-\omega)\right\rangle = \frac{A + B\omega^2 + C\omega^4}{D(\omega)} \tag{26a}$$

$$A = \mu\epsilon^2\left[\left(\frac{1}{2}+\beta_L\right)(1+e)^2 + \left(\frac{1}{2}+\beta_m^s\right)(1+e) - \frac{1}{2}\frac{P_o}{N}(1+2\delta)\, e\right] \tag{26b}$$

$$B\omega^2 = \mu\epsilon^2\left[\left(\frac{1}{2}+\beta_L\right)(1+4\delta^2-4e\delta) + \left(\frac{1}{2}+\beta_m^s\right)\right]\left(\omega^2/\Gamma_p^2\right) \tag{26c}$$

$$C\omega^4 = \mu\epsilon^2\left(\frac{1}{2}+\beta_L\right)\left(\omega^4/\Gamma_p^4\right) 4\delta^2 \tag{26d}$$

$$D(\omega) = \left[\epsilon\omega_h - \left(\omega^2/\Gamma_p\right)(1-\epsilon e+2\epsilon\delta)\right]^2 + \omega^2\left[1 - (\omega/\Gamma_p)^2 2\epsilon\delta\right]^2 \tag{26e}$$

$$D(\omega) = (\epsilon\omega_h)^2 + \omega^2[1-2(\epsilon\omega_h/\Gamma_p)(1-\epsilon e+2\epsilon\delta)]$$
$$+ \omega^2(\omega/\Gamma_p)^2\,[(1-\epsilon e)^2+4\epsilon^2\delta(\delta-e)] + \omega^2(\omega/\Gamma_p)^4\, 4\epsilon^2\delta^2. \tag{26f}$$

By means of Eqs. 5 we can calculate $\langle r(t+\tau)\, r(t)\rangle$. From Eqs. 18 and 24 and the fact that θ is independent of r, we can then derive moments of $a^+(t)$ and $a(t')$.

The spectrum of Eq. 26 is of third order in ω^2. It is hard to discuss it in general without plugging in specific numbers for various lasers. We can, however, clarify some aspects of these results through the following remarks.

(i) Neglecting the shot-noise sources that drive the rate equations (16) results in setting $e = 0$ in Eqs. 26b, 26c, and 26d. Since on the one hand the region close to threshold is the most important one for intensity fluctuations (verify that for $\omega = 0$), and on the other hand e is much smaller than 1 in that region, we conclude that these noise sources are not very important and that the important noise sources are those that drive Eq. 15 and that are described in Eqs. 13 and 14.

(ii) Neglecting d^2a/dt^2 and dx_L^-/dt in Eq. 15 results in setting $\delta = 0$ in Eqs. 26. A careful investigation of Eqs. 26 shows that such an approximation is justified if δ and $\epsilon\delta$ are much smaller than one (say, one order of magnitude smaller than one). Equations 15 and 16 can then be transformed to "modified" rate equations. These have been discussed elsewhere.[3] The spectrum of r becomes then of second order in ω^2. It can have a resonance peak for ω somewhere between $\epsilon\omega_h$ and Γ_p. If one also sets $\epsilon = (1+e)^{-1}$, the modified rate equations reduce to the rate equations discussed by McCumber[4]; the conditions $\delta \ll 1$, $\epsilon = (1+e)^{-1}$ are fulfilled if the material bandwidth 2Γ is much larger than the cold-cavity bandwidth 2μ and the decay constant Γ_p of the inversion.

(iii) If $\Gamma_p \gg \epsilon\omega_h$, then one may approximate Eq. 26a by

$$\langle r(\omega)\, r(-\omega)\rangle = \frac{A}{\omega^2 + (\epsilon\omega_h)^2}. \tag{27}$$

Although one may lose an experimental verifiable resonance peak for $\omega \neq 0$ in this approximation, this resonance peak will be much smaller than the spectrum near $\omega = 0$. The result (27) was derived previously for $\epsilon = 1$.[1,2] We thus see that the bandwidth of the intensity fluctuations is narrowed by the factor $\epsilon = [1+(\mu/\Gamma)+e]^{-1}$, that is, mainly by the effect of a finite material bandwidth (e is present in this reduction factor ϵ because in this analysis we did not approximate the full nonlinear term $\left(1 + aR_s^2\right)^{-1}$ by $1 - aR_s^2$, as one does in the Van der Pol equation).

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