COMMUNICATION SCIENCES

AND

ENGINEERING

XI. STATISTICAL COMMUNICATION THEORY*

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A. WORK COMPLETED

1. CLASS-D AMPLIFICATION OF RADIOFREQUENCY AMPLITUDE-MODULATED SINE WAVES

This study has been completed by D. A. Feldman. It was submitted as a thesis in partial fulfillment of the requirements for the Degree of Master of Science, Department of Electrical Engineering, M. I. T., August, 1966.

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2. SPECTRUM ANALYSIS OF THE LOGARITHM OF A FUNCTION

This study has been completed by J. F. Kososki. It was submitted as a thesis in partial fulfillment of the requirements for the Degree of Master of Science, Department of Electrical Engineering, M. I. T., August, 1966.

A. V. Oppenheim

3. AN INVESTIGATION OF TIME JITTER IN NEON BULB THRESHOLD-CROSSING DETECTORS

This study has been completed by J. C. Stafford. It was submitted as a thesis in partial fulfillment of the requirements for the Degree of Master of Science, Department of Electrical Engineering, M. I. T., August, 1966.

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B. OPTIMUM QUANTIZATION OF A TWO-LEVEL SIGNAL AFTER CONTAMINATION BY NOISE

One of the important problems of statistical decision theory is the binary detection problem; that is, the problem of deciding after reception which of two possible signals has been transmitted over a noisy channel. In this paper we shall approach this problem from a point of view different from that traditionally taken.

We begin by considering the message signal s(t) which takes on values s_1 and s_2 with probabilities q and (1-q), respectively. For convenience we assume throughout the paper that $s_1 < s_2$. s(t) is contaminated by an additive, independent noise signal n(t). This results in a received signal x(t),

$$x(t) = s(t) + n(t). \tag{1}$$

The amplitude probability density of the noise signal n(t) is $p_n(\rho)$. Our problem is to determine the two-level quantizer which, when operating on x(t), yields a signal y(t),

$$y(t) = Q[x(t)], \tag{2}$$

that minimizes an appropriate mean value of the quantizer error signal, subject to the constraint that the quantizer output signal can assume only the values s_1 and s_2. That is, an appropriate mean value of the quantizer error signal will be minimized subject to the constraint that the quantizer's representation values are constrained to be s_1 and s_2.

Mathematically, this problem can be formulated in the following manner. We desire that the quantized signal Q[x(t)] be an instantaneous replica of the message portion s(t) of the quantizer-input signal. Generally, we demand more than the quantizer can accomplish. There will be an error,

$$e(t) = s(t) - Q[x(t)]. \tag{3}$$

We shall take an appropriate mean value of e(t) as a measure of how well the quantizer performs with respect to the demands. This measure of the error is given by

$$\mathscr{E} = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \{g[\eta - Q(\xi)]\, p_{x,s}(\xi,\eta)\}. \tag{4}$$

$p_{x,s}(\xi,\eta)$ is the joint amplitude probability density of the quantizer-input signal x and the message signal s (which is also the desired output signal). From our statement of the problem $p_{x,s}(\xi,\eta)$ is

$$p_{x,s}(\xi,\eta) = [q \cdot p_n(\xi - s_1)]\, u_o(\eta - s_1) + [(1-q) \cdot p_n(\xi - s_2)]\, u_o(\eta - s_2). \tag{5}$$

$u_o(t)$ is the unit-impulse function. $g[\eta-Q(\xi)]$ is a function of the error that we call the error-weighting function. $Q(\xi)$ is the quantizer input-output characteristic and is defined in the following way:

$$Q(\xi) = \begin{cases} s_1 & \xi \leq x_1 \\ s_2 & \xi > x_1. \end{cases} \tag{6}$$

Substituting Eqs. (5) and (6) into (4) we have for the measure of the error

$$\mathscr{E} = \int_{-\infty}^{x_1} d\xi \int_{-\infty}^{\infty} d\eta \, \{g(\eta-s_1)[q\cdot p_n(\xi-s_1)\, u_o(\eta-s_1) + (1-q)\cdot p_n(\xi-s_2)\, u_o(\eta-s_2)]\}$$
$$+ \int_{x_1}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \{g(\eta-s_2)[q\cdot p_n(\xi-s_1)\, u_o(\eta-s_1) + (1-q)\cdot p_n(\xi-s_2)\, u_o(\eta-s_2)]\} \tag{7}$$

or, upon integration with respect to η,

$$\mathscr{E} = q \cdot g(0) \int_{-\infty}^{x_1} p_n(\xi-s_1)\, d\xi + (1-q) \cdot g(s_2-s_1) \int_{-\infty}^{x_1} p_n(\xi-s_2)\, d\xi$$
$$+ q \cdot g(s_1-s_2) \int_{-\infty}^{x_1} p_n(\xi-s_1)\, d\xi + (1-q) \cdot g(0) \int_{x_1}^{\infty} p_n(\xi-s_2)\, d\xi. \tag{8}$$

Referring to Eq. (8) we observe that the error $\mathscr{E}$ is a function of only one variable, x_1. We find the value of s_1 which minimizes $\mathscr{E}$ by taking the first partial derivative of $\mathscr{E}$ with respect to x_1 and equating it to zero. Doing this we have

$$0 = q \cdot g(0)\, p_n(x_1-s_1) + (1-q) \cdot g(s_2-s_1)\, p_n(x_1-s_2) - q$$
$$\cdot\, g(s_1-s_2)\, p_n(x_1-s_1) - (1-q) \cdot g(0)\, p_n(x_1-s_2) \tag{9}$$

or, after algebraic manipulation,

$$\frac{q}{1-q}\, \frac{g(s_1-s_2) - g(0)}{g(s_2-s_1) - g(0)} = \frac{p_n(x_1-s_2)}{p_n(x_1-s_1)}. \tag{10}$$

The value of x_1 which satisfies (10) minimizes the error $\mathscr{E}$. It should be observed that in most cases of interest the error-weighting function will satisfy the condition

$$g(e) = g(-e). \tag{11}$$

Applying this condition to (10) we have

$$\frac{q}{1-q} = \frac{p_n(x_1-s_2)}{p_n(x_1-s_1)} \tag{12}$$

which is independent of the error-weighting function.

We now want to compare this result to that obtained using one of the traditional detection theory approaches. The approach which we select for comparison is that of the ideal observer. This point of view maximizes the probability of a correct decision. The x_1 which is optimum from the ideal observer point of view satisfies the equation (see, for example, Wainstein and Zubakov[1])

$$\left.\frac{p_x(\xi|s_2)}{p_x(\xi|s_1)}\right|_{\xi=x_1} = \frac{\Pr(s_2)}{\Pr(s_1)}. \tag{13}$$

For this particular problem (13) becomes

$$\frac{p_n(x_1-s_2)}{p_n(x_1-s_2)} = \frac{q}{1-q} \tag{14}$$

which is identical to (12). Therefore, in this case the detection theory point of view yields a result which is identical to the result obtained by applying optimum quantization techniques.

Consider the following example which indicates how (12) might be solved for x_1. We assume that the noise signal is Gaussian with mean $\bar{n}$ and mean-square value ψ_n. Then,

$$p_n(\rho) = \frac{1}{(2\pi\psi_n)^{1/2}} \exp\left[-\frac{(\rho-\bar{n})^2}{2\psi_n}\right] \tag{15}$$

Upon substitution of (15) in (12) we have

$$\frac{q}{(1-q)} = \frac{\dfrac{1}{(2\pi\psi_n)^{1/2}} \exp\left\{-\dfrac{[(x_1-s_2)-\bar{n}]^2}{2\psi_n}\right\}}{\dfrac{1}{(2\pi\psi_n)^{1/2}} \exp\left\{-\dfrac{[(x_1-s_1)-\bar{n}]^2}{2\psi_n}\right\}} \tag{16}$$

After algebraic manipulation this becomes

$$x_1 = \frac{\psi_n}{s_2 - s_1} \ln \left[\frac{q}{1-q}\right] + \frac{s_1 + s_2}{2} + \bar{n}. \tag{17}$$

which is the desired result.

J. D. Bruce

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C. NOISE IN MAGNETIC RECORDING SYSTEMS CAUSED BY THE RECORDING TAPE

1. Introduction

This report summarizes the results of a thesis[1] in which we studied the cause of noise in magnetic recording systems that is due to the recording tape.

Previous investigations have indicated that there are two types of noise which are due to the tape: background noise which limits the smallest signal that can be recorded, and noise that amplitude-modulates the recorded signal and thus broadens its spectrum. This modulation noise, as it is called, is also evidenced by an increase in the noise from the recording system when the tape is uniformly magnetized. It has been felt that these two noise terms are both due to randomness in the magnetic properties of the tape, but no generally accepted and verified theory exists for relating the noise to the physical properties of the tape.

2. Statement of the Problem

The purpose of this study was to relate the observed noise to the physical properties of the recording tape in order to determine its cause.

The procedure was to derive a power density spectrum for the noise flux in the reproduce head in terms of a model incorporating all possible random properties of the tape, which would account for the experimentally observed increase in the noise with uniform tape magnetization. This power density spectrum was then checked through further experiments. These experiments verified the model and made it possible to determine the cause of the noise.

3. Power Density Spectrum of the Flux through the Reproduce Head Because of the Magnetization of the Tape

The relation between the flux through the reproduce head and the magnetization of the tape[1,2] is given by

$$\phi = \int_V \overline{H} \cdot \overline{M}\, dv, \tag{1}$$

where $\overline{M}$ is the magnetic moment density of the tape, $\overline{H}$ is the field set up by the reproduce head for a unit current in the head coil, and v is the volume of the tape.

The magnetic coating of the tape is not continuous, however, but is an assembly of small magnetic particles dispersed in a binder that is coated on a plastic backing.[3]

The flux relation can thus be written

$$\phi(x) = W \sum_i \overline{M}_i \cdot \overline{H}(x-x_i, y_i), \tag{2}$$

under the assumptions that the field function can be adequately represented in just two dimensions,[2] and the variation of the field over the volume of a particle is negligible. W is the width of the tape.

The autocorrelation function of the head flux in terms of the random properties of the magnetic particles[1] is

$$R_\phi(\tau) = W^2 \int_0^\infty k^2 |F(k)|^2 e^{-2k(d+T/2)} \frac{\sinh^2\left(k\frac{T}{2}\right)}{\left(k\frac{T}{2}\right)^2} \left\{ \overline{N} E[m^2]\left(k\frac{T}{4}\right) \frac{\sinh\left(k\frac{T}{2}\right)}{\cosh\left(k\frac{T}{2}\right) - 1} + 2\overline{N} E^2[\cos\theta]\, E^2[|m|] \sum_{n=1}^{\infty} E[s_i s_{i+n}]\, \mathrm{Re}\, M_{D_{i,i+n}}(k) \right\} \cos(kv\tau), \tag{3}$$

where F(k) is the two-dimensional transform of the magnetic scalar potential of the head, d is the spacing between the head and tape, T is the thickness of the magnetic coating of the tape, $\overline{N}$ is the average number of particles per unit length of the tape, m is the particle magnetic dipole moment, θ is the angle from the x axis of the projection of the particle on the x-y plane, s is the sign of the particle, $M_{D_{i,i+n}}(k)$ is the characteristic function of the random distance between the i^{th} and the $(i+n)^{th}$ particles, and v is the tape velocity.

The power density spectrum of the flux is

$$S_\phi(\omega) = W^2\left(\frac{\omega}{v}\right)^2 \left|F\left(\frac{\omega}{v}\right)\right|^2 e^{-2\frac{\omega}{v}(d+T/2)} \frac{\sinh^2\left(\frac{\omega}{v}\frac{T}{2}\right)}{\left(\frac{\omega}{v}\frac{T}{2}\right)^2} \left\{ \overline{N} E[m^2]\left(\frac{\omega}{v}\frac{T}{4}\right) \frac{\sinh\left(\frac{\omega}{v}\frac{T}{2}\right)}{\cosh\left(\frac{\omega}{v}\frac{T}{2}\right) - 1} + 2\overline{N} E^2[\cos\theta]\, E^2[|m|] \sum_{n=1}^{\infty} E[s_i s_{i+n}]\, \mathrm{Re}\, M_{D_{i,i+n}}\left(\frac{\omega}{v}\right) \right\}. \tag{4}$$

Correcting this equation for the transfer characteristic of the head gives the power density spectrum of the magnetization:

$$S_M(\omega) = \overline{N}E[m^2]\left\{\left(\frac{\omega}{v}\frac{T}{4}\right)\frac{\sinh\left(\frac{\omega}{v}\frac{T}{2}\right)}{\cosh\left(\frac{\omega}{v}\frac{T}{2}\right)-1} + 2E^2[\cos\theta]\frac{E^2[|m|]}{E[m^2]}\right.$$

$$\left.\sum_{n=1}^{\infty} E[s_is_{i+n}]\,\text{Re}\,M_{D_{i,i+n}}\left(\frac{\omega}{v}\right)\right\}. \tag{5}$$

This power density spectrum can be rewritten

$$S_M(\omega) = \overline{N}E[m^2]\left\{\left(\frac{\omega}{v}\frac{T}{4}\right)\frac{\sinh\left(\frac{\omega}{v}\frac{T}{2}\right)}{\cosh\left(\frac{\omega}{v}\frac{T}{2}\right)-1} + 2E^2[\cos\theta]\frac{E^2[|m|]}{E[m^2]}(P_+-P_-)^2\sum_{n=1}^{\infty}\text{Re}\,M_{D_{i,i+n}}\left(\frac{\omega}{v}\right)\right.$$

$$\left.+\,8E^2[\cos\theta]\frac{E^2[|m|]}{E[m^2]}\sum_{n=1}^{\infty}\left(P_+P_- - P_{+-}^{(n)}\right)\text{Re}\,M_{D_{i,i+n}}\left(\frac{\omega}{\theta}\right)\right\}, \tag{6}$$

where $P_+ = p(s_i=+1)$, $P_- = p(s_i=-1)$, and $P_{+-}^{(n)} = p(s_i=+1, s_{i+n}=-1)$. Note also that $P_+ - P_-$ is proportional to the magnetization of the tape.

Examining Eq. 6, we see that the first term is a noise term independent of the magnetization, the second term contains the signal and can also contain a noise term that is dependent on the magnetization, and the third term is also a noise term dependent on magnetization. These two noise terms, which change with magnetization, are caused in two different ways: the first is due to the distribution of the magnetic particles, and the second to interaction between the particles.

4. Experimental Results

We shall now present the experimental results that allow us to determine the cause of the modulation noise. In Fig. XI-1 we show the power density spectra of the flux for various levels of magnetization from the erased state to the saturated state. These curves show the manner in which the power density increases with magnetization.

To determine the cause of the noise that increases with magnetization, we correct these spectra for the transfer function of the head and compare them with the theoretical spectrum from Eq. 6. We note that if the modulation noise is caused by the particle distribution alone, the shape of the spectrum in the erased state is

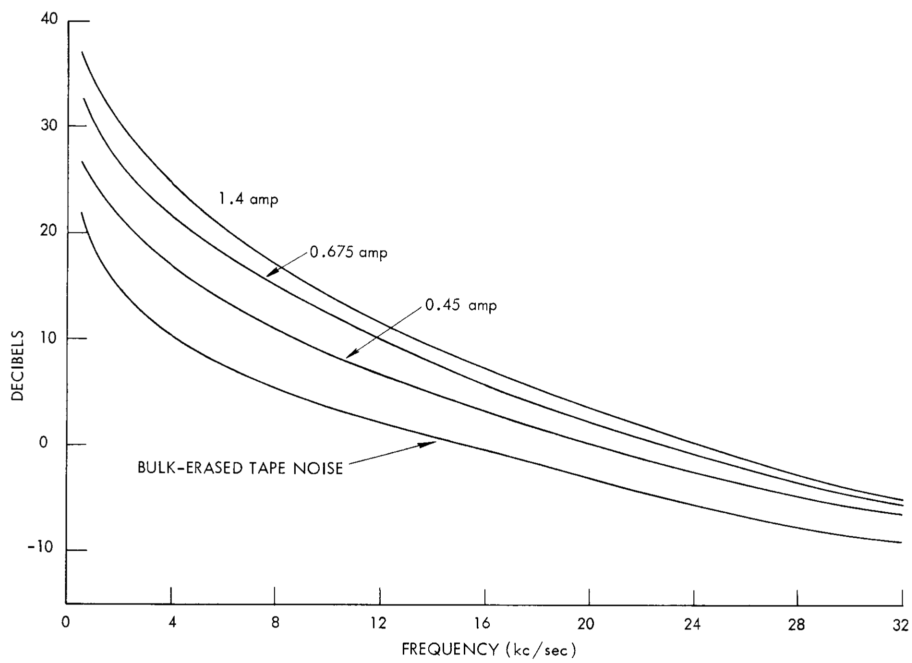

Fig. XI-1. Power density spectrum of noise through the reproduce head as a function of recording current. Tape speed: 7. 5 ips.

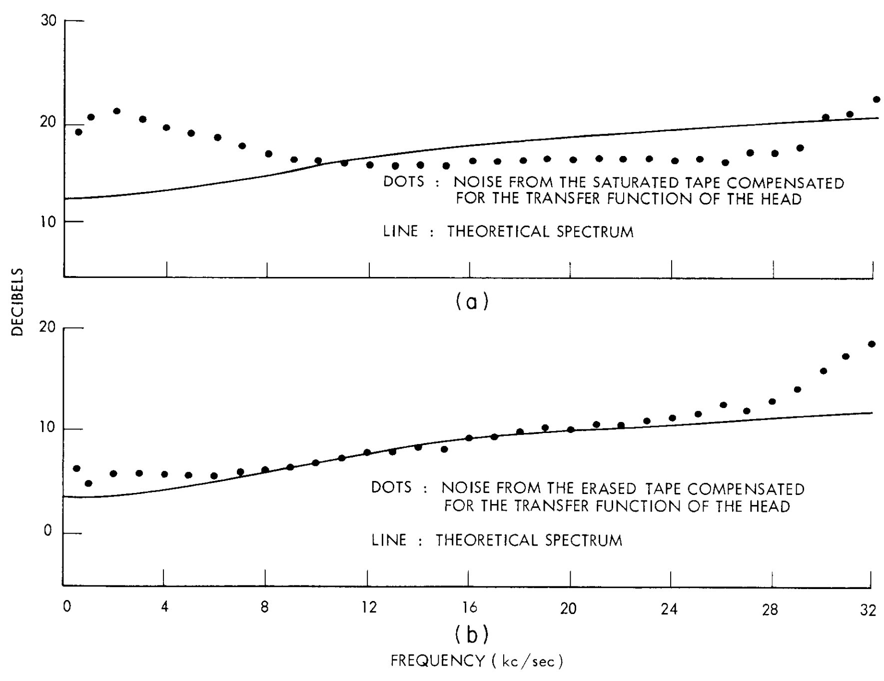

Fig. XI-2. Comparison of measured power density spectra in the (a) saturated and (b) erased states with $f(\omega)$.

$$f(\omega) = \left(\frac{\omega}{v}\frac{T}{4}\right) \frac{\sinh\left(\frac{\omega}{v}\frac{T}{2}\right)}{\cosh\left(\frac{\omega}{v}\frac{T}{2}\right) - 1}, \tag{7}$$

while if the modulation noise is caused by particle interaction alone the shape of the spectrum in the saturated state will be $f(\omega)$.

In Fig. XI-2 we compare the measured power density spectra in the erased and saturated states with $f(\omega)$. This shows that the measured noise spectrum from the erased tape most closely matches $f(\omega)$. This leads us to the conclusion that the modulation noise is caused by the manner in which the particles are distributed on the tape.

The spectral shape of the noise, which increases with magnetization, is determined by subtracting the noise that is due to the erased tape from the other spectra. The resulting power density spectra for various values of magnetization, corrected for the

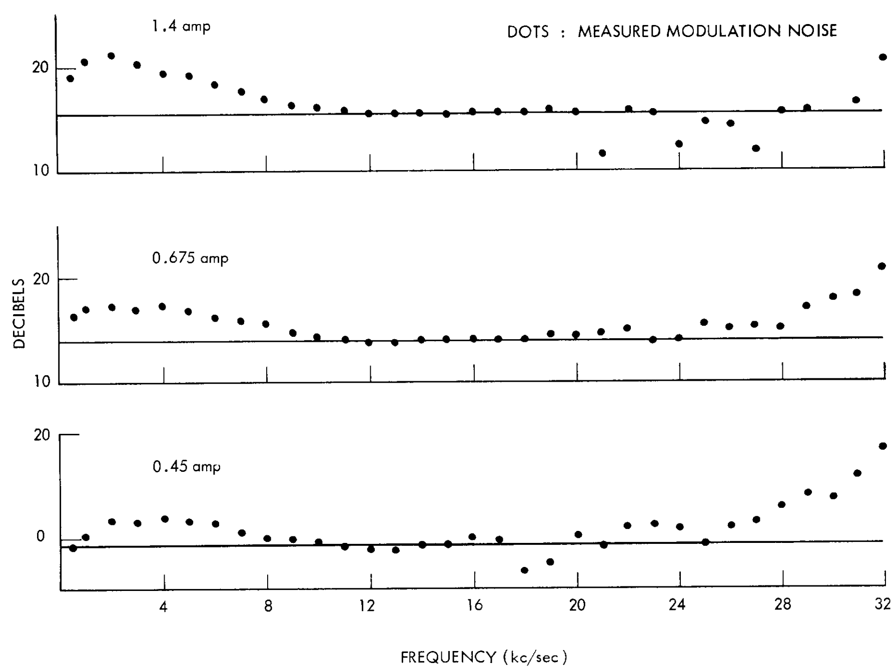

Fig. XI-3. Power density spectra for various values of magnetization corrected for the transfer function of the head.

transfer function of the head, are shown in Fig. XI-3. These plots indicate that the modulation-noise term has a relatively flat spectrum in the frequency range 0-30 kc.

5. Conclusions

We have derived the power density spectrum for the flux through the reproduce head of a magnetic recorder in terms of the random properties of the magnetic particles in the recording tape. This power density spectrum has two noise terms that are a function of tape magnetization and one noise term independent of the magnetic state of the tape.

Comparison of the theoretical spectrum with measured spectra shows that the noise which increases with magnetization is caused by the distribution of particles, not by interaction between the particles. This comparison also indicates that the spectrum of the noise which increases with magnetization is flat in the frequency range 0-30 kc.

R. F. Bauer

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