XVII. PROCESSING AND TRANSMISSION OF INFORMATION*

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A. ANGLE-OF-ARRIVAL DISPERSION OF A PLANE WAVE TRAVERSING A TWO-DIMENSIONAL CLOUD

An optical-frequency plane wave traversing a cloud emerges below it as a superposition of a large number of multiply scattered waves, distributed over a range of angles of arrival. As an initial step in analyzing this phenomenon, let us consider an idealized two-dimensional cloud of identical, round, lossless scattering particles of diameter a. The cloud has infinite parallel boundaries τ meters apart, as shown in Fig. XVII-1, and

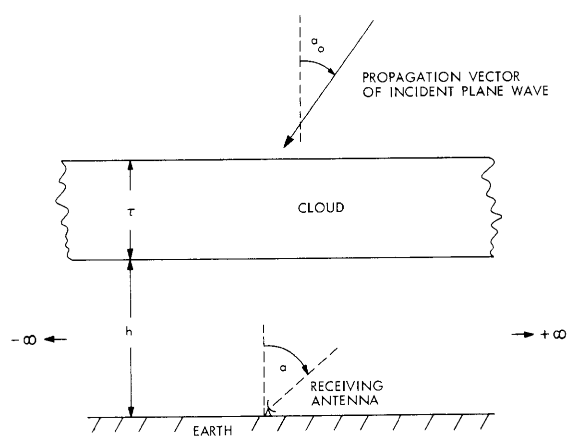

Fig. XVII-1. Two-dimensional cloud configuration.

has particles Poisson-distributed over it with average density d_a per square meter.

Water droplets in clouds, having diameters in the range[1] from approximately 5 μ to 20 μ, are many times larger than a visible-light wavelength. For such particles, scattered radiation is essentially confined to an angular spread[1] of

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$$\pm\theta_o = \pm\frac{\lambda}{2a} \text{ rad} \tag{1}$$

about the direction of the incident wave. For our idealized two-dimensional particles, we shall assume that the scattered intensity at radial distance r from a single drop illuminated by a unit-intensity plane wave is

$$I_{sc}(\gamma) = \frac{a}{2\theta_o r} \sqcap_{2\theta_o}(\gamma), \tag{2}$$

where the angle γ is measured from the propagation vector of the incident wave. The "rectangle-function" notation is

$$\sqcap_{w}(x) \triangleq \begin{cases} 1 & \text{if} \quad |x| < \frac{w}{2}; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The coefficient $\frac{a}{2\theta_o}$ in (2) follows from an energy conservation argument.

We define an angular intensity distribution function $I(\alpha)$, such that $I(\alpha)\, d\alpha$ is the total (incoherently added) intensity of all plane waves at a point in space having angles of arrival between α and $\alpha + d\alpha$. In this notation a unit-intensity plane wave arriving from direction α_o has the representation

$$I(\alpha) = u_o(\alpha-\alpha_o), \tag{4}$$

where $u_o(\ )$ is the unit impulse function.

Let the cloud be divided into N very thin parallel layers, of thickness

$$\ell_o = \frac{T}{N}, \tag{5}$$

and let all of the particles in each layer be collapsed onto a single horizontal line along the center of the layer. The particles will be Poisson-distributed on the line, with average density

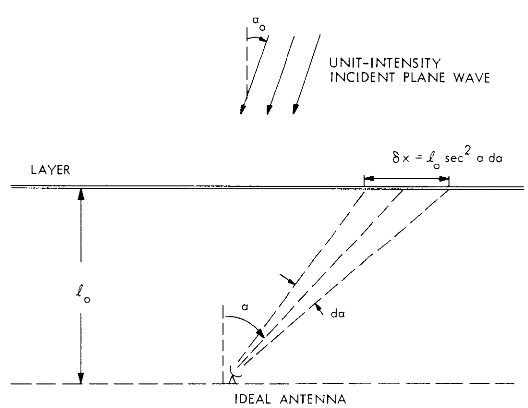

Fig. XVII-2. Geometry for single-layer impulse response.

$$\rho = d_a \ell_o \text{ m}^{-1}. \tag{6}$$

We begin by finding the average impulse response $h_1(\alpha, \alpha_o)$ of a single layer. This is the average angular intensity function along a line ℓ_o meters below the layer, when it is illuminated from above by the unit impulse (4). The quantity $h_1(\alpha, \alpha_o)\, d\alpha$ is the average intensity measured by the antenna illustrated in Fig. XVII-2. If a particle is present in the indicated interval δx on the layer, the antenna sees a scattered intensity of value

$$\frac{a}{2\theta_o \ell_o \sec \alpha} \lrcorner\overline{2\theta_o}\llcorner \, (\alpha-\alpha_o). \tag{7}$$

Since a particle is in δx with probability

$$\rho \delta x = \rho \ell_o \sec^2 \alpha \, d\alpha, \tag{8}$$

the average scattered intensity measured by the antenna is

$$\frac{\rho a}{2\theta_o} \sec \alpha \lrcorner\overline{2\theta_o}\llcorner \, (\alpha-\alpha_o) d\alpha. \tag{9}$$

The average intensity of the portion of the plane wave passing unscattered through the layer is represented as

$$(1-\rho a \sec \alpha) \, u_o(\alpha-\alpha_o). \tag{10}$$

Another energy conservation argument prompts us to replace $\sec \alpha$ in (9) and (10) by

$$\widehat{\sec}\, \alpha \triangleq \begin{cases} \sec \alpha & \text{if} \quad |\alpha| < \sec^{-1} \dfrac{1}{\rho a}; \\ \dfrac{1}{\rho a} & \text{otherwise.} \end{cases} \tag{11}$$

Equations 9 and 10 imply that

$$h_1(\alpha, \alpha_o) = (1-\rho a \, \widehat{\sec}\, \alpha_o) \, u_o(\alpha-\alpha_o) + \frac{\rho a}{2\theta_o} \widehat{\sec}\, \alpha \lrcorner\overline{2\theta_o}\llcorner \, (\alpha-\alpha_o). \tag{12}$$

Since the scattering process is linear, it follows that the average impulse response $h_N(\alpha_N, \alpha_o)$ of the entire array of N layers ℓ_o meters apart is given by the (N-1)-fold superposition integral

$$h_N(\alpha_N, \alpha_o) = \iint_{-\pi/2}^{\pi/2} \cdots \int d\alpha_{N-1} d\alpha_{N-2} \cdots d\alpha_1 \, h_1(\alpha_N, \alpha_{N-1}) \cdots h_1(\alpha_1, \alpha_o). \tag{13}$$

The limits $\pm \frac{\pi}{2}$ on the integrals express the assumption that we ignore all radiation that has been scattered so many times that it is traveling horizontally.

For many cases of interest, we shall find that essentially all of the area under the function $h_N(\ )$ is concentrated well within the interval

$$|\alpha_N| < \frac{\pi}{2},$$

as long as α_o is reasonably close to zero. We can then replace the integration limits in (13) by $\pm\infty$. If we make the substitution

$$\widehat{\sec}\ \alpha \cong 1,$$

which is accurate within about 10% for

$$|\alpha| \leq 0.45 \text{ rad},$$

then the approximate impulse response

$$h_1(\alpha, \alpha_o) \cong (1-\rho a)\, u_o(\alpha-\alpha_o) + \frac{\rho a}{2\theta_o} \lrcorner 2\theta_o \llcorner\, (\alpha-\alpha_o) \tag{14}$$

is a function only of the difference $(\alpha-\alpha_o)$. We are now able to apply the Central Limit theorem to Eq. 13 to obtain the result

$$h_N(\alpha_N, \alpha_o) \cong \frac{1}{\sigma_h\sqrt{2N\pi}} \exp\left[-\frac{(\alpha_N-\alpha_o)^2}{2N\sigma_h^2}\right] \tag{15}$$

in the limit as N goes to infinity, where

$$\sigma_h = \theta_o\sqrt{\frac{\rho a}{3}}.$$

Now, when N becomes infinite, the layer thickness

$$\ell_o = \frac{\tau}{N}$$

goes to zero. This causes the layer model to become exactly equivalent to the actual two-dimensional cloud. We note that the quantity

$$\begin{aligned} \sigma_h\sqrt{N} &= \theta_o\sqrt{\frac{N\rho a}{3}} \\ &= \theta_o\sqrt{\frac{N\ell_o d_a a}{3}} \\ &= \theta_o\sqrt{\frac{\tau d_a a}{3}} \end{aligned} \tag{16}$$

in (15) is independent of both N and ℓ_o. Now, the parameter

$$d_e = (d_a a)^{-1}$$

is the two-dimensional equivalent of the "extinction distance" D_e which enters into the familiar equation[1]

$$I(x) = I_o\, e^{-x/D_e}$$

for the attenuation of a plane wave traversing a distance x in a cloud of scattering particles. We can therefore make the substitution

$$\tau d_a a = \frac{\tau}{d_e} \triangleq N_e$$

in (16). The quantity N_e is the so-called "optical thickness" of the cloud. The average impulse response of the entire cloud is thus given by

$$h_{\text{cloud}}(\alpha, \alpha_o) = \frac{1}{\sigma_\alpha \sqrt{2\pi}} \exp\left[-\frac{(\alpha-\alpha_o)^2}{2\sigma_\alpha^2}\right], \tag{17a}$$

in which

$$\sigma_\alpha = \theta_o \sqrt{\frac{N_e}{3}}. \tag{17b}$$

Let us check the reasonableness of the assumption that $h_{\text{cloud}}(\ )$ is concentrated well within the range

$$|\alpha| < \frac{\pi}{2},$$

when α_o is quite close to zero. We know that 95% of the area under a Gaussian curve lies within the range $\pm 2\sigma$; hence, we require that

$$\sigma_\alpha < \frac{\pi}{4}. \tag{18}$$

Now, typical parameters for fair-weather cumulus clouds,[2] for example, are 5-6 μ for the particle size a and 10^8-10^{10} m^{-3} for the volume density d_v. From Eq. 1, we have

$$\theta_o \cong 0.05 \text{ rad}$$

for 0.5-μ light. Using the approximate relation[1]

$$D_e = \left(\frac{\pi a^2}{2} d_v\right)^{-1}$$

for three-dimensional clouds, we find that N_e might be as large as 500 for clouds up to a few thousand meters thick. But (17b) implies that (18) is satisfied for

$N_e < 740.$

We can now use (17a) to compute the total power $P_A(\alpha', \alpha_o)$ received by a two-dimensional antenna placed as in Fig. XVII-1 and aimed in direction α'. Suppose that the effective width of the antenna is W meters, and that its power gain pattern is $g(\theta)$. Let the intensity of the incident plane wave be I_p. We then have

$$P_A(\alpha', \alpha_o) = \frac{I_p W}{\sigma_\alpha \sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \exp\left[-\frac{(\alpha-\alpha_o)^2}{2\sigma_\alpha^2}\right] g(\alpha-\alpha') \, d\alpha. \tag{19}$$

H. M. Heggestad

References

1. H. C. van de Hulst, Light Scattering by Small Particles (John Wiley and Sons, Inc., New York, 1957).

2. H. J. aufm Kampe and H. K. Weickmann, Physics of Clouds, Meteorological Monographs, Vol. 3, July 1957, pp. 182-225.

B. ON THE FOURIER TRANSFORMS OF BOOLEAN FUNCTIONS

The classification and study of the structure of Boolean functions and their realizations have received considerable attention in the past ten years.[1-8] Both have benefited from what is variously known as the coordinate representation,[3,7] the discrete Fourier transform,[1,3,5] and the Rademacher-Walsh expansion of the functions.[1,4,8] Despite the amount of work done in this area, some interesting questions concerning discrete Fourier transforms have remained unanswered. Two of these questions will be considered here; one question will be answered, and some partial results will be discussed for the other.

1. Correspondence between Fourier Coefficient Sets and Equivalence Classes

Muller,[6] in an early work, found by exhaustive computation that there were only 8 distinct unordered sets of absolute values of Fourier coefficients among the 65,536 four-variable Boolean functions. Ninomiya[7] and Lechner[5] showed that each such distinct set corresponds to a single equivalence class under what Lechner has named the restricted affine group (n/1), RAG(n/1). (Note that n is the number of variables in the function.) Unlike most transformation groups that have been extensively studied, RAG(n/1) is not a direct product of a group operating on the domain, D, and a group operating on the range, R. It is the subset or restriction of affine transformations on $D \times R$ such that it is the largest group operating on $D \times R$ which does not involve either feedback from R to D or nonlinear operations on the domain and range coordinates. Here, linearity is used in the wider sense to include affine transformations.

Ninomiya[7] conjectured that 1:1 correspondence between distinct sets (in the unordered magnitude sense) of Fourier coefficients and equivalence classes under RAG(n/1) would not hold for functions of more than four variables. Lechner[5] provided part of the information necessary to establish this conjecture by calculating that there are 48 equivalence classes under RAG(5/1). While this provides an upper bound on the number of distinct sets of Fourier coefficients, it provides no information about how to calculate these sets.

Ninomiya's conjecture has been established by the author using a semiexhaustive tabulation of the unordered sets of Fourier coefficients of five-variable functions. This tabulation produced 40 such sets, 8 less than the known number of equivalence classes under RAG(5/1). Thus there must be different equivalence classes having the same set of unordered Fourier coefficients.

The forty sets are listed in Table XVII-1 and the Fourier transforms for a function from one of the equivalence classes represented by each set are listed in Table XVII-2. Note that 8 equivalence classes under RAG(5/1) are not represented in Table XVII-2. The problem of determining which sets of Fourier coefficients correspond to more than one equivalence class is now an unsolved problem.

The procedure used to generate the Fourier coefficients was an extension of a well-known method employed by Dertouzos.[1] The discussion here will be restricted to five-variable Boolean functions. Let the 16 Fourier coefficients for a four-variable function $R(x_1, x_2, x_3, x_4)$ be

$$b_1, b_2, b_3, b_4, b_o; \; b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}; \; b_{123}, b_{124}, b_{134}, b_{234}; \; b_{1234}. \tag{1}$$

Then if, $G(x_1, x_2, x_3, x_4, x_5)$ is any five-variable function, it is always possible to expand it as follows:

$$G(x_1, x_2, x_3, x_4, x_5) = x_5 R_1(x_1, x_2, x_3, x_4) + \overline{x}_5 R_2(x_1, x_2, x_3, x_4). \tag{2}$$

If R_1 has the 16 Fourier coefficients b_i^1 and R_2 has coefficients b_i^2, where i takes on the 16 values in (1), then G has the 32 Fourier coefficients given by

$$\begin{aligned} b_i &= b_i^1 + b_i^2 \\ b_{i5} &= b_i^1 - b_i^2. \end{aligned} \tag{3}$$

Table XVII-1. Unordered sets of Fourier Coefficients of five-variable functions.

	2	6	10	14	18	22	26	30
1	31	0	0	0	0	0	0	1
2	24	7	0	0	0	0	1	0
3	24	4	3	0	0	1	0	0
4	20	10	1	0	0	1	0	0
5	28	0	0	3	1	0	0	0
6	22	6	2	1	1	0	0	0
7	18	12	0	1	1	0	0	0
8	19	9	3	0	1	0	0	0
9	15	15	1	0	1	0	0	0
10	21	7	1	3	0	0	0	0
11	22	4	4	2	0	0	0	0
12	18	10	2	2	0	0	0	0
13	19	7	5	1	0	0	0	0
14	15	13	3	1	0	0	0	0
15	16	10	6	0	0	0	0	0
16	12	16	4	0	0	0	0	0

	0	4	8	12	16	20	24	28	32
17	31	0	0	0	0	0	0	0	1
18	16	15	0	0	0	0	0	1	0
19	24	0	7	0	0	0	1	0	0
20	12	16	3	0	0	0	1	0	0
21	16	12	0	3	0	1	0	0	0
22	12	14	4	1	0	1	0	0	0
23	0	30	0	1	0	1	0	0	0
24	10	15	6	0	0	1	0	0	0
25	28	0	0	0	4	0	0	0	0
26	14	14	0	2	2	0	0	0	0
27	22	0	8	0	2	0	0	0	0
28	10	16	4	0	2	0	0	0	0
29	11	14	4	2	1	0	0	0	0
30	9	15	6	1	1	0	0	0	0
31	19	0	12	0	1	0	0	0	0
32	7	16	8	0	1	0	0	0	0
33	16	10	0	6	0	0	0	0	0
34	12	12	4	4	0	0	0	0	0
35	0	28	0	4	0	0	0	0	0
36	10	13	6	3	0	0	0	0	0
37	8	14	8	2	0	0	0	0	0
38	6	15	10	1	0	0	0	0	0
39	16	0	16	0	0	0	0	0	0
40	4	16	12	0	0	0	0	0	0

Note: **Headings are magnitudes of Fourier coefficients. Each entry gives the number of coefficients in a particular set having that magnitude.**

Table XVII-2 (Part 1). Fourier transforms.

	1	2	3	4	5	0	12	13	14	15	23	24	25	34	35	45	123	124	125	134	135	145	234	235	245	345	1234	1235	1245	1345	2345	12345
1)	30	2	2	2	-2	2	-2	-2	-2	2	2	2	-2	2	-2	-2	-2	-2	2	-2	2	2	2	-2	-2	-2	-2	2	2	2	-2	2
2)	26	6	6	2	-2	2	-2	-2	2	6	2	-2	-6	-2	-6	-2	-6	-2	2	-2	2	-2	2	-2	2	2	2	6	2	2	-2	-2
3)	22	10	10	2	-2	6	-6	-6	2	-2	6	-2	2	-2	2	2	-10	-2	2	-2	2	2	2	-2	-2	-2	2	-2	-2	-2	2	2
4)	22	6	10	6	-2	6	-2	-6	-2	-2	2	-2	6	2	2	-2	-6	-2	-2	-6	2	6	2	2	-2	-6	2	-6	-2	2	2	2
5)	18	14	14	2	2	2	-2	-2	2	2	2	-2	-2	-2	-2	2	-14	-2	-2	-2	-2	2	2	2	-2	-2	2	2	-2	-2	2	2
6)	18	14	6	2	-6	10	-10	-2	2	2	2	-2	-2	-2	6	2	-6	-2	6	-2	-2	2	2	2	-2	-2	2	-6	-2	-2	2	2
7)	14	6	18	6	-2	6	-2	-6	6	6	2	-2	6	-6	-6	-2	-6	-2	-2	-6	2	-2	2	2	-2	2	2	-6	-2	2	2	2
8)	18	6	6	10	-6	10	-2	-2	-6	2	2	-2	6	-2	6	-6	-6	-2	-2	-2	-2	10	2	2	-2	-2	2	-6	-2	-2	2	2
9)	6	6	6	6	-6	18	-6	-6	-6	6	-6	-6	6	-6	6	6	-2	-2	2	-2	2	2	-2	2	2	2	2	-2	-2	-2	-2	10
10)	14	14	6	6	-2	14	-10	-2	-2	-2	-2	-2	-2	-2	6	6	-2	-2	6	-2	-2	-2	-2	-2	-2	-2	6	-2	-2	-2	-2	6
11)	10	14	14	2	-6	10	-2	-2	2	10	-6	-2	-2	-2	-2	2	-6	-2	-2	-2	-2	2	2	10	-2	-2	2	-6	-2	-2	2	2
12)	14	6	6	6	-2	14	-2	-2	-2	-2	-10	-2	6	-2	6	6	6	-2	-2	-2	-2	-2	-2	6	-2	-2	6	-10	-2	-2	-2	6
13)	10	6	14	10	-6	10	-2	-2	2	10	2	-2	6	-10	-2	-6	-6	-2	-2	-2	-2	2	2	2	-2	6	2	-6	-2	-2	2	2
14)	6	6	6	6	-2	14	10	-10	-2	6	-10	-2	-6	-2	6	6	-2	-2	6	-2	6	-2	-2	6	-2	-2	6	-2	-2	-2	-2	6
15)	10	6	6	10	-6	10	-10	-2	2	10	2	-2	6	-2	6	-6	-6	6	6	-2	-2	2	2	2	-2	-2	2	-6	-10	-2	2	2
16)	6	6	6	6	6	6	-10	-10	-2	6	-2	-10	6	6	6	6	-2	-2	6	-2	6	-2	-2	-2	6	-10	6	-2	-2	-2	-2	6

Table XVII-2. (Part 2).

	1	2	3	4	5	0	12	13	14	15	23	24	25	34	35	45	123	124	125	134	135	145	234	235	245	345	1234	1235	1245	1345	2345	12345
17)	32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18)	28	4	4	4	0	0	0	0	0	4	0	0	-4	0	-4	-4	-4	-4	0	-4	0	0	4	0	0	0	0	4	4	4	-4	0
19)	24	8	8	8	0	0	0	0	0	0	0	0	0	0	0	0	-8	-8	0	-8	0	0	8	0	0	0	0	0	0	0	0	0
20)	24	8	8	4	-4	4	-4	-4	0	0	4	0	0	0	0	4	-8	-4	4	-4	4	0	4	-4	0	0	0	0	-4	-4	4	0
21)	20	12	12	4	0	0	0	0	0	4	0	0	-4	0	-4	4	-12	-4	0	-4	0	0	4	0	0	0	0	4	-4	-4	4	0
22)	20	12	4	4	-8	8	-8	0	0	4	0	0	-4	0	4	4	-4	-4	8	-4	0	0	4	0	0	0	0	-4	-4	-4	4	0
23)	20	4	4	4	-4	4	-4	-4	-4	12	-4	-4	-4	4	-4	-4	-4	-4	4	-4	4	4	-4	4	4	4	4	4	4	4	4	-4
24)	8	20	8	8	0	8	-4	0	0	8	-4	-4	-4	0	0	0	-4	-4	-4	0	0	0	-4	4	4	-8	4	-4	-4	0	4	4
25)	16	16	16	0	0	0	0	0	0	0	0	0	0	0	0	0	-16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26)	16	16	4	4	-4	12	-12	0	0	0	0	0	0	-4	4	4	-4	-4	4	0	0	0	0	0	0	-4	4	-4	-4	0	0	4
27)	16	16	8	0	-8	8	-8	0	0	0	0	0	0	0	8	0	-8	0	8	0	0	0	0	0	0	0	0	-8	0	0	0	0
28)	16	16	4	4	4	4	-4	-8	0	0	8	0	0	-4	4	4	-4	-4	-4	0	8	0	0	-8	0	-4	4	-4	-4	0	0	4
29)	16	8	4	12	-4	12	-4	0	-8	0	0	0	8	-4	4	-4	-4	-4	-4	0	0	8	0	0	0	-4	4	-4	-4	0	0	4
30)	12	8	8	8	-8	16	-4	-4	-4	4	0	0	8	-8	0	0	-4	-4	-4	4	4	4	16	0	0	0	4	-4	-4	-4	0	4
31)	16	8	8	8	-8	8	0	0	-8	0	0	0	8	0	8	-8	-8	0	0	0	0	8	0	0	0	0	0	-8	0	0	0	0
32)	8	16	4	4	4	4	-4	0	0	8	-8	-8	0	-4	4	4	-4	-4	-4	0	0	0	-8	8	8	4	4	-4	-4	0	8	4
33)	12	12	12	12	0	0	0	0	0	12	0	0	-4	0	-4	-4	-4	-4	0	-4	0	0	-4	0	0	0	0	-4	4	-4	12	0
34)	12	4	12	12	-8	8	0	0	0	12	0	0	4	-8	-4	-4	-4	-4	0	-4	0	0	4	0	0	8	0	-4	-4	-4	4	0
35)	12	4	4	4	-12	12	-4	-4	-4	12	-4	-4	4	-4	4	4	-4	-4	4	-4	4	4	4	4	4	4	-4	-4	-4	-4	4	4

Table XVII-2 (Part 2 continued).

	1	2	3	4	5	0	12	13	14	15	23	24	25	34	35	45	123	124	125	134	135	145	234	235	245	345	1234	1235	1245	1345	2345	12345
36)	8	12	8	8	0	8	-4	0	0	8	-12	4	4	0	0	0	-4	-4	-4	0	0	0	4	12	-4	-8	4	-4	-4	0	-4	4
37)	8	8	12	4	-4	12	-4	0	-8	8	0	0	8	-4	-4	4	-4	-4	-4	8	0	8	0	0	0	-4	4	-4	-4	-8	0	4
38)	12	8	8	8	-4	8	-4	-4	-4	8	8	-8	4	0	4	-4	-4	-4	0	4	0	8	0	-4	4	-4	4	-8	0	-8	4	0
39)	8	8	8	8	-8	8	-8	0	0	8	0	0	8	0	8	-8	-8	8	8	0	0	0	0	0	0	0	0	-8	-8	0	0	0
40)	8	8	4	4	4	4	-4	-8	-8	8	8	-8	8	-4	4	4	-4	-4	-4	0	8	8	0	-8	8	-4	4	-4	-4	0	0	4

Note: The subscripts of the Rademacher-Walsh functions are listed at the top of the table. The magnitude and sign of the corresponding coefficient is listed below each subscript. The numbering of the transforms corresponds to the numbering in Table XVII-1.

Conceptually, we want to let R_1 and R_2 in (2) vary independently through every four-variable function. This would guarantee that every five-variable function was generated. Using (3) and the known Fourier coefficients of all four-variable functions, we could calculate the Fourier coefficients of all five-variable functions. These could then be classified according to the unordered magnitudes of coefficients. This would be a prohibitively long calculation. Therefore, a number of transformations are utilized which greatly reduce the number of four-variable functions that must be considered in (2).

Dertouzos[1] has shown that complementation of a function corresponds to negating each of its Fourier coefficients; that is, $b_i^* = -b_i$ for every i. Also, complementation of a single variable x_j of a function corresponds to negation of every Fourier coefficient containing j in its subscript.

Thus

1. Complementing the functions R_1 and R_2 causes a complementation of the function G

$$b_i^* = -b_i^1 - b_i^2 = -\left(b_i^1+b_i^2\right) = -b_i$$

$$b_{i5}^* = b_i^1 - \left(-b_i^2\right) = -\left(b_i^1-b_i^2\right) = -b_{i5}$$

2. Complementing the function R_1 causes a reordering and negation of the Fourier coefficients of G

$$b_i^* = -b_i^1 + b_i^2 = -\left(b_i^1-b_i^2\right) = -b_{i5}$$

$$b_{i5}^* = -b_i^1 - b_i^2 = -\left(b_i^1+b_i^2\right) = -b_i$$

3. Complementing the function R_2 causes a reordering of the Fourier coefficients of G

$$b_i^* = b_i^1 - b_i^2 = b_{i5}$$

$$b_{i5}^* = b_i^1 + b_i^2 = b_i$$

4. Complementing a variable, x_j, of the function R_1 causes a reordering and negation of half of the Fourier coefficients of G

$$\left.\begin{aligned} b_i^* &= b_i^1 + b_i^2 = b_i \\ b_{i5}^* &= b_i^1 - b_i^2 = b_{i5} \end{aligned}\right\} \text{ i not containing j}$$

$$\left.\begin{aligned} b_i^* &= -b_i^1 + b_i^2 = -\left(b_i^1 - b_i^2\right) = -b_{i5} \\ b_{i5}^* &= -b_i^1 - b_i^2 = -\left(b_i^1 + b_i^2\right) = -b_i \end{aligned}\right\} \quad i \text{ containing } j$$

5. Complementing a variable, x_j, of the function R_2 causes a reordering of half of the Fourier coefficients of G.

Accordingly, the set of unordered magnitudes of Fourier coefficients is invariant under negation of the functions R_1 and/or R_2 and/or the negation of variables of R_1 and/or R_2. Therefore, in order to generate all possible sets of Fourier coefficients (unordered magnitudes), it is sufficient for R_1 and R_2 to vary independently through the set composed of a single representative function from each NN class.

One further transformation may be conveniently utilized. Consider an arbitrary pair of functions chosen as above for R_1 and R_2. The function R_1 must be in one of the 8 equivalence classes under RAG(4/1), and therefore can be transformed into some canonic form, R_1^*. If the <u>same</u> transformation is applied to R_2, yielding R_2^*, the relative ordering of Fourier coefficients between the two functions will be preserved. This transformation can only reorder and/or negate coefficients. Since invariance under negations has been established and the relative ordering of the Fourier coefficients has been preserved, the function G^* formed from R_1^* and R_2^* according to (2) must have the same set of unordered coefficient magnitudes as G.

Thus it is only necessary to let R_1 be chosen from the set of representative functions of equivalence classes under RAG(4/1) and R_2 be chosen from the set of representative functions of equivalence classes under NN.

The Fourier coefficients for classes under RAG(4/1) have been obtained from Ninomiya.[7] The Fourier coefficients for the 222 classes under NPN were also found by Ninomiya. These were used to generate the coefficient for the NN classes.

2. Generation Problem

The most extensive (and probably earliest) study of the structure of Boolean functions through coordinate representation or Fourier transform techniques was done by Ninomiya.[7] One problem that he investigated was that of generating the Fourier transform of Boolean functions directly in the transform or spectral domain. While there is no lack of necessary and sufficient conditions for a set of numbers being the Fourier transform of a Boolean function, none of the conditions provide any reasonable method for <u>generating</u> the set of numbers in the transform domain.

Some necessary conditions have been derived by Ninomiya, and probably others that do allow the generation of all sets of unordered magnitudes that can be derived from the transforms of Boolean functions. These conditions happen to be sufficient for functions of four or fewer variables, the only cases considered by Ninomiya. His work in this area, then, can be separated into two parts: (i) generation of sets of unordered magnitudes of numbers; and (ii) ordering and signing each set so that it is the transform of a Boolean function while proving that all functions derivable from that set are equivalent under RAG(n/1).

An extension of Ninomiya's work to the five-variable case was attempted; this met with limited success because of some essentially different features of the five-variable problem.

First, the generation of sets of unordered magnitudes of coefficients was accomplished by using the following well-known necessary conditions (see, for example, Ninomiya[7]).

1. The coordinates of a Boolean function are integers, the sums of whose squares is 2^{2n}, and, for $n \geq 2$, all come from either the set $(0, 4, 8, 12, \ldots)$ or the set $(2, 6, 10, 14, \ldots)$.

2. For any function, the sum of the absolute values of any two coordinates never exceeds 2^n.

This operation generated 191 sets of coefficients – many more than the maximum of 48 sets established by Lechner or the 40 sets found earlier. Many of these can be eliminated, however, by using the following condition of Ninomiya:

3. For any function whose coordinates are in the set $(0, 4, 8, 12, \ldots)$ either all or exactly half the coordinates are equal to 0, modulo 8.

This reduces the number of sets to 84, fifty-three of which are of the $(2, 6, 10, 14, \ldots)$ type and 31 of the $(0, 4, 8, 12 \ldots)$ type. In the four-variable case, application of these three conditions reduces the number of sets to exactly 8 – which in fact is the correct number of sets. Thus the conditions that are sufficient at $n = 4$ are no longer so at $n = 5$.

Additional conditions are needed to eliminate further sets from consideration. Some of these conditions, based on work by Hatfield,[4] have been derived.

There are 2^n terms in the Fourier transform of a Boolean function of n variables. Choose any $k \leq n$ independent Rademacher-Walsh functions (or "frequencies") from these 2^n. Taking all possible products of these functions generates a subset of the 2^k R-W functions.

EXAMPLE 1. Let $n = 5$, $k = 3$ and choose r_1, r_{235}, r_{12}.

This generates the subset $r_o, r_1, r_2, r_{12}, r_{35}, r_{135}, r_{235}, r_{1235}$.

Now if $k < n$, consider any R-W function not in the subset just formed. Multiply each function in the subset by this new function. This generates a second (disjoint) subset. Repeat this process, using as the multiplying function any function not already included in any subset. This produces 2^{n-k} subsets.

EXAMPLE 2. The subset produced in Example 1 is

$$(r_o, r_1, r_2, r_{12}, r_{35}, r_{135}, r_{235}, r_{1235}).$$

Choose r_{23} as the multiplying function, thereby generating
$(r_{23}, r_{123}, r_3, r_{13}, r_{25}, r_{125}, r_5, r_{15})$.
Choose r_4, thereby generating
$(r_4, r_{14}, r_{24}, r_{124}, r_{345}, r_{1345}, r_{235}, r_{12345})$.
Choose r_{1234}
$(r_{1234}, r_{234}, r_{134}, r_{34}, r_{1245}, r_{245}, r_{145}, r_{45})$.
Note that these are $2^{5-3} = 4$ subsets.

Now consider just that portion of the R-W expansion (Fourier transform) of a function F involving R-W functions belonging to a subset formed as above. This can be written in the form

$$\underline{A} = b_a r_a + b_{ab} r_b r_a + r_c (b_{ac} r_a + b_{abc} r_b r_a)$$
$$+ r_d (b_{ad} r_a + b_{abd} r_b r_a) + r_c (b_{acd} r_a + b_{abcd} r_b r_a) + \dots,$$

where $r_b, r_c, r_d \dots$ are the k independent R-W functions, and r_a is r_o (if the subset contains r_o) or the multiplying function for that subset.

$\underline{A}$ can now be evaluated at a vertex by setting the r_i to ± 1, depending on the vertex.

As shown in Hatfield, the evaluation of $\underline{A}$ at a vertex can be written as

$$\underline{A} = \left\langle \pm r_a (1 \pm r_b)(1 \pm r_c)(1 \pm r_d) \dots F \right\rangle, \tag{4}$$

where the signs to be used are determined by which vertex A is being evaluated at, and $\langle\ \rangle$ denotes summation over all of the vertices of the function F.

It is also shown that the product $(1 \pm r_b)(1 \pm r_c)(1 \pm r_d) \dots$ is nonzero at exactly 2^{n-k} vertices of the function. Thus the summation over the vertices of F indicated in (4) reduces to a summation over 2^{n-k} vertices. Note also that $\pm r_a (1 \pm r_b)(1 \pm r_c)(1 \pm r_d) \dots$ is always equal to $\pm 2^k$ or 0.

Now consider various values of k. If $k = n-1$, the evaluation of $\underline{A}$ at a particular vertex reduces to the summation over 2 vertices. For convenience, consider all evaluations to be carried out at the vertex $x_1 = x_2 = \dots = 1$. This sets all the signs +. (The result at <u>any</u> vertex is sufficient for our purposes.) Thus (4) reduces to 2^{n-1} times the sum of $r_a F$ over two vertices (the particular ones being determined by $r_b, r_c, r_d \dots$). The function and r_a may be ± 1 at each vertex and thus the sum is ± 2 or 0, yielding $\pm 2^n$ or 0 as the value of $\underline{A}$ at that vertex.

Now since $k = n - 1$, there are $2^{n-(n-1)}$ subsets of this size. The other subset is specified by the same $r_b, r_c, r_d, \dots$ but a different r_a. Because of the orthogonality condition on the r_i (see Hatfield[4]), this new r_a must have the same sign as the old at half the nonzero vertices of the product $(1+r_b)(1+r_c)(1+r_d)$ and must differ at half. Thus if the first subset summed to $\pm 2^n (0)$, the second must sum to $0\ (\pm 2^n)$.

If $k = n - 2$, the summation of $r_a F$ is over 4 vertices. This sum can be ± 4, ± 2, or 0.

Table XVII-3. Summary of results for $k \geqslant n - 4$.

k	Magnitude of the sum of 2^k R-W coefficients in a subset	Magnitudes of possible sums of the other subsets
$n-1$	0	2^n
	2^n	0
$n-2$	0	$0,\ 2^n$
	2^{n-1}	2^{n-1}
	2^n	0
$n-3$	0	$0,\ 2^{n-1},\ 2^n$
	2^{n-2}	$2^{n-1},\ 3\text{x}2^{n-2}$
	2^{n-1}	$0,\ 2^{n-1}$
	$3\text{x}2^{n-2}$	2^{n-2}
	2^n	0
$n-4$	0	$0,\ 2^{n-2},\ 2^{n-1},\ 3\text{x}2^{n-2},\ 2^n$
	2^{n-3}	$2^{n-3},\ 3\text{x}2^{n-3},\ 5\text{x}2^{n-3},\ 7\text{x}2^{n-3}$
	2^{n-2}	$0,\ 2^{n-2},\ 2^{n-1},\ 3\text{x}2^{n-2}$
	$3\text{x}2^{n-3}$	$2^{n-3},\ 3\text{x}2^{n-3},\ 5\text{x}2^{n-3}$
	2^{n-1}	$0,\ 2^{n-2},\ 2^{n-1}$
	$5\text{x}2^{n-3}$	$2^{n-3},\ 3\text{x}2^{n-3}$
	$3\text{x}2^{n-2}$	$0,\ 2^{n-2}$
	$7\text{x}2^{n-3}$	2^{n-3}
	2^n	0

Table XVII-4. Eliminated coefficient sets.

	2	6	10	14	18	22	26	30
1	26	1	4	0	0	1	0	0
2	22	7	2	0	0	1	0	0
3	27	0	2	2	1	0	0	0
4	25	3	1	2	1	0	0	0
5	23	6	0	2	1	0	0	0
6	26	0	4	1	1	0	0	0
7	24	3	3	1	1	0	0	0
8	20	9	1	1	1	0	0	0
9	25	0	6	0	1	0	0	0
10	23	3	5	0	1	0	0	0
11	26	1	1	4	0	0	0	0
12	24	4	0	4	0	0	0	0
13	25	1	3	3	0	0	0	0
14	19	10	0	3	0	0	0	0
15	24	1	5	2	0	0	0	0
16	14	16	0	2	0	0	0	0

	0	4	8	12	16	20	24	28	32
17	14	13	2	2	0	1	0	0	0
18	25	0	4	0	3	0	0	0	0
19	13	16	0	0	3	0	0	0	0
20	12	15	2	1	2	0	0	0	0
21	15	12	0	4	1	0	0	0	0

Note: Headings are magnitudes of Fourier coefficients. Each entry gives the number of coefficients in a particular set having that magnitude.

Table XVII-5. Sets not corresponding to Boolean functions.

	2	6	10	14	18	22	26	30
1	18	13	0	0	0	1	0	0
2	21	6	4	0	1	0	0	0
3	17	12	2	0	1	0	0	0
4	13	18	0	0	1	0	0	0
5	23	4	2	3	0	0	0	0
6	20	7	3	2	0	0	0	0
7	16	13	1	2	0	0	0	0
8	23	1	7	1	0	0	0	0
9	21	4	6	1	0	0	0	0
10	17	10	4	1	0	0	0	0
11	13	16	2	1	0	0	0	0
12	11	19	1	1	0	0	0	0
13	9	22	0	1	0	0	0	0
14	22	1	9	0	0	0	0	0
15	20	4	8	0	0	0	0	0
16	18	7	7	0	0	0	0	0
17	14	13	5	0	0	0	0	0
18	10	19	3	0	0	0	0	0
19	8	22	2	0	0	0	0	0
20	6	25	1	0	0	0	0	0
21	4	28	0	0	0	0	0	0

	0	4	8	12	16	20	24	28	32
22	13	13	2	3	1	0	0	0	0
23	14	11	2	5	0	0	0	0	0

Note: Headings are magnitudes of Fourier coefficients. Each entry gives the number of coefficients in a particular set having that magnitude.

With $k = n - 2$, there is a total of 4 subsets; thus there are 3 other subsets to be considered. If the sum for the first subset is ±4, the other 3 must sum to zero, again because the new r_a agrees in sign for exactly half the nonzero vertices. If the first subset sums to ±2, the others must also and if the first sums to 0, the others sum to 0 or ±4. The value of $\underline{A}$ is then 2^{n-2} times this sum.

Similar results hold for smaller values of k. These results are summarized for $k \geq n - 4$ in Table XVII-3.

If we let $n = 5$ and $k = 5$ we find that each subset consists of exactly two R-W functions. Thus if the sum of corresponding R-W coefficients of a subset is ±32, Table XVII-3 requires that the other 15 subsets of coefficients sum to 0. Similarly, if two coefficients sum to ±28, the other 15 pairs must sum to ±4. The coefficient sets that can be eliminated on the basis of these requirements are listed in Table XVII-4.

This reduced the possible sets to 63. No test has been found for eliminating more sets. Included in these 63 sets are the 40 sets actually corresponding to Boolean functions given in Table XVII-1, and 23 sets not corresponding to Boolean functions given in Table XVII-5.

L. Hatfield

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