XVI. SIGNAL PROCESSING*

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A. FIRST-PASSAGE TIME STATISTICS OF A FIRST-ORDER MARKOV PROCESS WITH A LINEAR RAMP

1. Statement of the Problem

Some problems concerning time jitter in regenerative switches can be reduced to the calculation of the statistical characteristics of the first zero passage time (FPT) of the sum y(t) of a linear ramp $\overline{x}(t) = \epsilon t$ and a first-order Markov process x(t) defined by the differential equation

$$\frac{dx}{dt} + x = n(t). \tag{1}$$

We assume that n(t) is stationary white Gaussian noise with the autocorrelation function

$$R_n(\tau) = \overline{n(t)\, n(t+\tau)} = 2\delta(\tau), \tag{2}$$

where $\delta(\tau)$ is the unit impulse function. If the coefficients of Eq. 1 and the noise spectral density were arbitrary rather than as specified above, the problem could easily be reduced to that described with suitable scaling of x and t.

The problem is illustrated in Fig. XVI-1. Solution for the first zero of $y(t) = \epsilon t + x(t)$ is equivalent to determining the first time instant that $x(t) = -\epsilon t$. As the slope, ϵ, decreases, we would expect the mean FPT to become more negative, and the standard deviation of the FPT to increase. In this report, we have obtained asymptotic expressions for the probability distribution $w(\theta)$, the mean value $\overline{\theta}$, and the standard deviation $\sigma_\theta = \sqrt{\overline{\theta^2} - \overline{\theta}^2}$ of the FPT. The results are valid when $\epsilon \ll 1$, which turns out to be the region where computer simulation of the process is most difficult.

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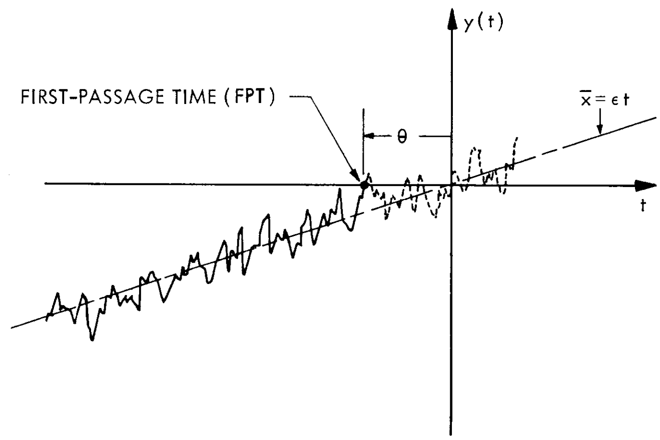

Fig. XVI-1. First-passage time for the sum y(t) of the linear ramp $\bar{x} = \epsilon t$ and noise x(t).

2. Method of Solution

The results were obtained by a Fokker-Planck equation approach. We utilized the series solution of the problem for $\bar{x}(t) = \text{const.}$ (obtained by Stumpers[1]) and proceeded to the case of slowly varying $\bar{x}(t)$. Asymptotic formulas for the parabolic cylinder functions[2] were used to present the results in a very simple form.

3. Statistical Characteristics of the FPT

The asymptotic distribution, w(θ), of the FPT is

$$w(\theta) = p(\epsilon\theta) = \begin{cases} \dfrac{1}{\epsilon\sqrt{2\pi}} (\epsilon\theta)\, e^{-(\epsilon\theta)^2/2} \exp\left[-\dfrac{1}{\epsilon\sqrt{2\pi}} e^{-(\epsilon\theta)^2/2}\right] & \theta \leq 0 \\ 0 & \theta > 0 \end{cases} \tag{3}$$

In terms of the integrated distribution function, $W(\theta) = \int_{-\infty}^{\theta} w(r)\, dr$ a simpler form

$$1 - W(\theta) = \begin{cases} \exp\left[-\dfrac{1}{\epsilon\sqrt{2\pi}} e^{-(\epsilon\theta)^2/2}\right] & \theta \leq 0 \\ 0 & \theta > 0 \end{cases} \tag{4}$$

is obtained. Examples of the distribution in Eq. 3 are shown in Fig. XVI-2.

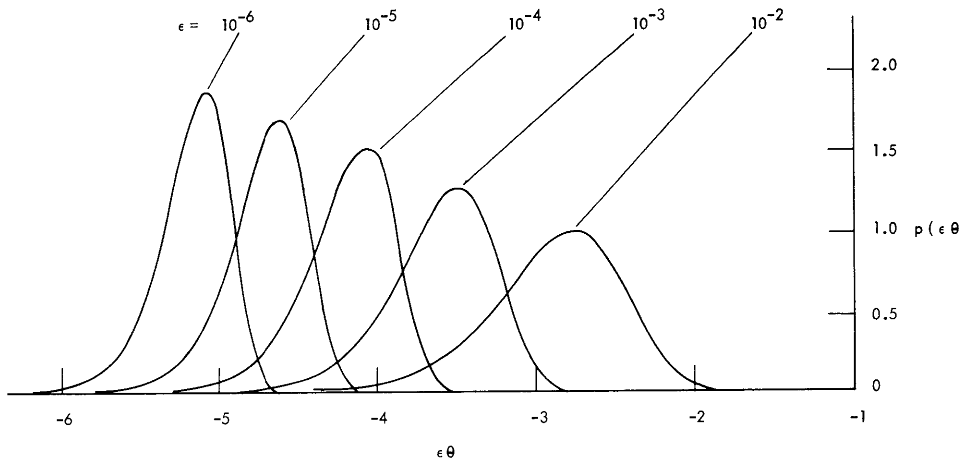

Fig. XVI-2. First-passage time probability distributions $p(\epsilon\theta)$ for several values of the ramp slope ϵ.

a. Average Value $\bar{\theta}$ of the FPT

We were unable to get an analytical expression for $\bar{\theta}$ from Eq. 3. A numerical evaluation of $\bar{\theta}$ was made and is presented in Fig. XVI-3. Fortunately, for $\epsilon \ll 1$, $\bar{\theta}$ is asymptotically close to the value $\hat{\theta}$ of the maximum of the distribution $w(\theta)$, which can easily be approximated. An expression for the mean $\epsilon\bar{\theta}(\epsilon)$ is thus found to be

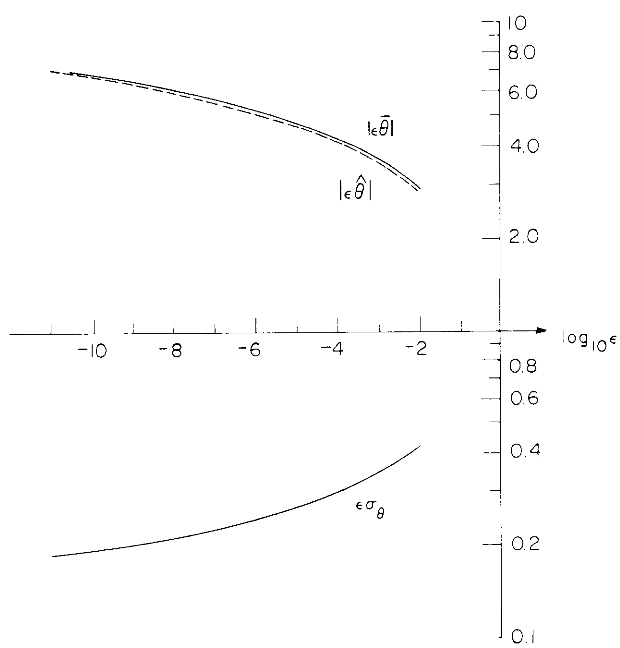

Fig. XVI-3. Plots of the mean value $\bar{\theta}$, mode $\hat{\theta}$, and standard deviation σ_θ of the FPT distribution as a function of ramp slope ϵ. The parameters are multiplied by ϵ to emphasize deviation of the laws from $1/\epsilon$.

$$\epsilon\bar{\theta} \approx \epsilon\hat{\theta} \approx -\sqrt{-2\ \ln\left[\epsilon\sqrt{2\pi}\left(1 - \frac{1}{-2\ \ln\left(\epsilon\sqrt{2\pi}\right)}\right)\right]} \tag{5}$$

which is also shown in Fig. XVI-3.

b. Standard Deviation σ_θ

The variation of the FPT standard deviation σ_θ with slope ϵ was calculated numerically and is also shown in Fig. XVI-3. An approximation for σ_θ can be obtained by using analytic results for the height of $w(\theta)$ at the maximum, $\hat{\theta}$, and approximating the shape of $w(\theta)$ by a unit-area Gaussian distribution. We obtain the approximate formula,

$$\epsilon\sigma_\theta \approx \frac{-e}{\sqrt{2\pi}\ \epsilon\hat{\theta}} = -\frac{1.083}{\epsilon\hat{\theta}}. \tag{6}$$

c. Range of Validity of the Asymptotic Results

The formulas above are certainly valid for $\epsilon \ll 0.005$ when the assumption $w(\theta) = 0$ for $\theta > 0$ gives negligible error. Under this condition, most of the first passages occur far back in time where the threshold, $-\epsilon t$, is still several standard deviations (of $x(t)$) high.

4. Discussion of Results

There are two points in these results that deserve to be mentioned.

The first is connected with applications. If we apply this FPT model to a calculation of jitter statistics in a regenerative switch where noise is caused by wideband shot and thermal noise, the region where $\epsilon < 10^{-2}$ is the range of practical interest.

The second point is concerned with the generalization of the results. Replacing $\bar{x}(t) = \epsilon t$ with any function with $\left|\frac{d\bar{x}}{dt}\right| < 0.01$ and using the same approach, we can get approximate analytical expressions for $w(\theta)$. It is also possible to adapt this analysis to the case in which the spectral density of $n(t)$ is multiplied by a slowly changing coefficient.

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References

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B. DISCRETE REPRESENTATIONS OF ANALOG SIGNALS

1. Introduction

In many applications we are concerned with implementing or simulating analog signal processing on a digital computer. In carrying out such an implementation, an analog (continuous) signal is represented by a sequence that is processed digitally, and the output sequence is then converted to an analog signal. The most common example of a discrete representation of continuous signals is periodic sampling. More generally, Steiglitz[1] has discussed the equivalence of digital and analog signal processing through an isomorphic mapping between the s-plane and the z-plane, and he suggests the bilinear transformation as a specific mapping that permits linear, time-invariant continuous filters to be represented by linear, shift-invariant discrete filters. Masry, Steiglitz, and Liu[2] discuss other isomorphisms between continuous and discrete signals. Piovoso and Bolgiano[3] have proposed the Poisson transform as a means for associating sequences with continuous functions to implement linear continuous time-invariant filters digitally.

A linear representation of a continuous function f(t) with a sequence f_n is of the form

$$f(t) = \sum_{n=-\infty}^{+\infty} f_n \phi_n(t), \tag{1}$$

where the functions $\phi_n(t)$ should, for convenience, be linearly independent so that the representation is unique, but need not be orthogonal. The requirement that a linear, continuous time-invariant filter be represented by a linear discrete shift-invariant filter corresponds to requiring that if f_n, h_n, and g_n are the discrete representations of f(t), h(t), and g(t), respectively, and if g(t) is the continuous convolution of f(t) and h(t), then g_n will be the discrete convolution of the sequences f_n and h_n. It is straightforward to show that this requires the Fourier transform of $\phi_n(t)$, denoted $\Phi_n(j\omega)$, to have the property

$$\Phi_n(j\omega)\, \Phi_m(j\omega) = \Phi_{n+m}(j\omega). \tag{2}$$

When f_n is derived by periodic sampling of f(t) so that $f_n = f(nT)$, the functions $\phi_n(t)$ are given by

$$\phi_n(t) = \frac{\sin \pi(t-nT)}{\pi(t-nT)}, \tag{3}$$

with

$$\Phi_n(j\omega) = \begin{cases} e^{-j\omega nT} & 0 \leq |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

so that (2) is satisfied.

In general, any mapping from the s-plane to the z-plane corresponding to expressing z as a function of s will lead to a representation of a time function f(t) in the form of Eq. 1 with the $\Phi_n(j\omega)$ satisfying Eq. 2. This can be seen in several ways. For example, from (1), the Laplace transform F(s) of f(t) can be expressed as

$$F(s) = \sum_{n=-\infty}^{+\infty} f_n \Phi_n(s). \tag{4}$$

The z-transform of the sequence f_n, denoted $\hat{F}(z)$, is given by

$$\hat{F}(z) = \sum_{n=-\infty}^{+\infty} f_n z^{-n},$$

and since the s-plane is mapped to the z-plane, we have z = G(s), and consequently

$$F(s) = \hat{F}[G(s)] = \sum_{n=-\infty}^{+\infty} f_n [G(s)]^{-n}. \tag{5}$$

Comparing Eqs. 4 and 5 yields

$$\Phi_n(s) = [G(s)]^{-n}, \tag{6}$$

so that Eq. 2 is clearly satisfied. In general there is no guarantee that the set of functions $\phi_n(t)$ corresponding to (6) is complete. The conditions under which they are have been discussed by Masry et al.[2]

Any mapping from the s-plane to the z-plane of a form corresponding to Eq. 6 will preserve convolution whether or not it maps the jω axis in the s-plane to the unit circle in the z-plane. For the case of the bilinear transformation, $z = G(s) = \left[\frac{a+s}{a-s}\right]$ so that $\Phi_n(s) = \left[\frac{a-s}{a+s}\right]^n$. With $z = e^{j\Omega}$ corresponding to the unit circle and $s = j\omega$,

$$j\omega = a\frac{e^{j\Omega}+1}{e^{j\Omega}-1} = ja \tan\frac{\Omega}{2}$$

or

$$\Omega(\omega) = 2 \tan^{-1} \left(\frac{\omega}{a}\right). \tag{7}$$

In particular, then, the bilinear transformation maps the $j\omega$ axis in the s-plane to the unit circle in the z-plane. The functions $\phi_n(t)$ corresponding to this transformation are

$$\phi_n(t) = \begin{cases} 2a(-1)^{n-1} e^{-at} L_{n-1}^{(1)}(2at)\, u_{-1}(t) + (-1)^n u_o(t), & n > 0 \\ u_o(t) & n = 0 \\ 2a(-1)^{n-1} e^{at} L_{-n-1}^{(1)}(-2at)\, u_{-1}(-t) + (-1)^n u_o(t), & n < 0 \end{cases} \tag{8}$$

where $L_{n-1}^{(1)}(x)$ is the first-order Laguerre polynomial of degree $(n-1)$ defined as

$$L_{(n-1)}^{(1)}(x) = -\frac{d}{dx}\left[L_n(x)\right]$$

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right). \tag{9}$$

In addition to simulation, it is often desirable to choose a discrete representation of analog signals which permits a determination of the analog spectrum digitally. This is motivated in part by the fact that an efficient algorithm exists for computing samples of the z transform on the unit circle. In order to utilize this algorithm for measuring an analog spectrum, we would want to choose a discrete representation that maps the $j\omega$ axis to the unit circle. A common procedure is periodic sampling corresponding to a representation in terms of Eqs. 1 and 3. If the sampling rate is sufficiently high so that aliasing is avoided, then a spectral analysis of the discrete sequence resulting in samples at equally spaced angles on the unit circle will correspond to a measurement of the analog spectrum at equally spaced frequencies on the $j\omega$ axis. Furthermore, since the computation of the spectrum of the discrete sequence must be carried out on a finite-length sequence, a finite-duration window is put on the sequence. This then corresponds to a "smearing" of the spectrum with a spectral window that is the same at all frequencies. Consequently, measurement of the analog spectrum by generating a sequence through periodic sampling, applying a window, and then computing the discrete Fourier transform is similar to measurement of the analog spectrum with equal-bandwidth filters. More generally, if we choose a discrete representation of the form of Eq. 1 with $\Phi_n(s)$ given by (6), then the $j\omega$ axis in the s-plane will be mapped onto the unit circle in the z-plane if $|G(j\omega)| = 1$, so that $\Phi_n(j\omega) = e^{-jn\Omega(\omega)}$.

The function $\Omega(\omega)$ then represents a warping between analog frequencies ω and digital frequencies Ω. In many instances, it is desirable to measure an analog spectrum with unequal resolution across the band, as obtained for example, with a constant-Q measurement. Such a spectral measurement can be obtained digitally by using a discrete representation corresponding to the bilinear transformation between the s- and z-planes. For Eq. 7 we see that frequencies equally spaced in Ω correspond to frequencies unequally spaced in ω with spectral samples closer at low frequencies than at high frequencies. Just as in periodic sampling, the discrete sequence must be truncated before computing the transform. If f_n corresponds to the discrete representation of f(t), and w_n is a finite-duration window, then the sequence to be transformed, g_n, would be given by

$$g_n = w_n f_n.$$

If $\hat{F}(\Omega)$, $W(\Omega)$, and $G(\Omega)$ represent the transform of the sequences f_n, w_n, and g_n, respectively, then

$$G(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\Omega-\phi)\, \hat{F}(\phi)\, d\phi. \tag{10}$$

Or if $F(j\omega)$ corresponds to the analog spectrum of f(t), then from (7)

$$G(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\Omega-\phi)\, F\left(aj\tan\frac{\phi}{2}\right) d\phi. \tag{11}$$

From Eq. 11 we see that the digital spectrum, $G(\Omega)$, is obtained by smearing the analog spectrum in such a way that spectral resolution decreases as the analog frequency increases. Samples of $G(\Omega)$ equally spaced in Ω, would then correspond to a measurement of the analog spectrum with a filter bank for which the filter bandwidth was proportional to the tangent of half the center frequency. Over a range of frequencies, this can be made to look similar to a constant-Q filter bank, with the specific shaping of the frequency resolution governed by the parameter a.

One of the major advantages of the discrete representation of analog signals through periodic sampling is the ease with which samples can be obtained. To obtain the coefficients in an expansion in terms of the set of functions given in Eq. 8, there are several alternatives. One possibility is to utilize the fact that for these functions,

$$\int_{-\infty}^{\infty} t\phi_n(t)\, \phi_m(t)\, dt = \begin{cases} 0 & n \neq m \\ |n| & n = m \end{cases} \tag{12}$$

so that

$$f_n = \frac{1}{|n|} \int_{-\infty}^{\infty} tf(t)\, \phi_n(t)\, dt. \tag{13}$$

If we consider, for convenience, $f(t)$ to be zero for $t < 0$, then f_n for $n > 0$ can be obtained by exciting the linear time-invariant network shown in Fig. XVI-4 with

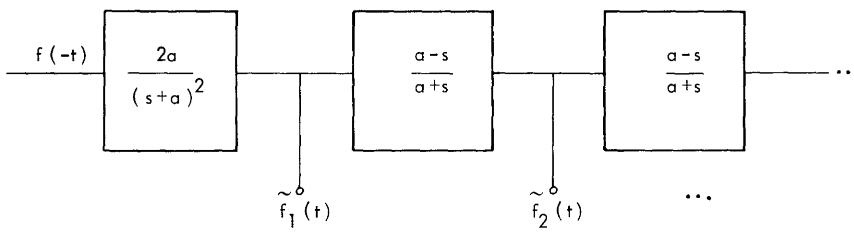

Fig. XVI-4. Analog network for obtaining the discrete representation corresponding to the bilinear transformation.

$f(-t)$. The coefficients f_n are then equal to the outputs at each of the taps at $t = 0$, that is,

$$f_n = \tilde{f}_n(0).$$

An alternative procedure is periodic sampling of $f(t)$ at a sufficiently high sampling rate to avoid aliasing, and then converting these samples to samples corresponding to a representation in terms of the functions in Eq. 8. If $f(nT)$ correspond to the periodic samples, and f_n correspond to the coefficients for an expansion in terms of the functions in Eq. 8, then

$$f_n = \sum_{k=0}^{\infty} f(kT)\, C_{nk} \tag{14}$$

$$C_{nk} = \frac{1}{n} \int_0^{\infty} t\phi_n(t)\, \frac{\sin \frac{\pi}{T}(t-kT)}{\frac{\pi}{T}(t-kT)}\, dt$$

corresponding to a matrix multiplication.

The transformation from periodic sampling to the samples f_n in Eq. 14 can be viewed in the z-plane as a transformation from the unit circle to the unit circle that preserves convolution and also provides a frequency warping. In future work, other means for accomplishing this frequency warping will be investigated.

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C. NONLINEAR PROCESSING OF SIGNAL ENVELOPES

In recent months our group has been considering some problems relating to the question of style in music. Some of these ideas were suggested by M. V. Cerrillo, in 1957.[1,2] At that time, he pointed out that the gross essence of style may be characterized

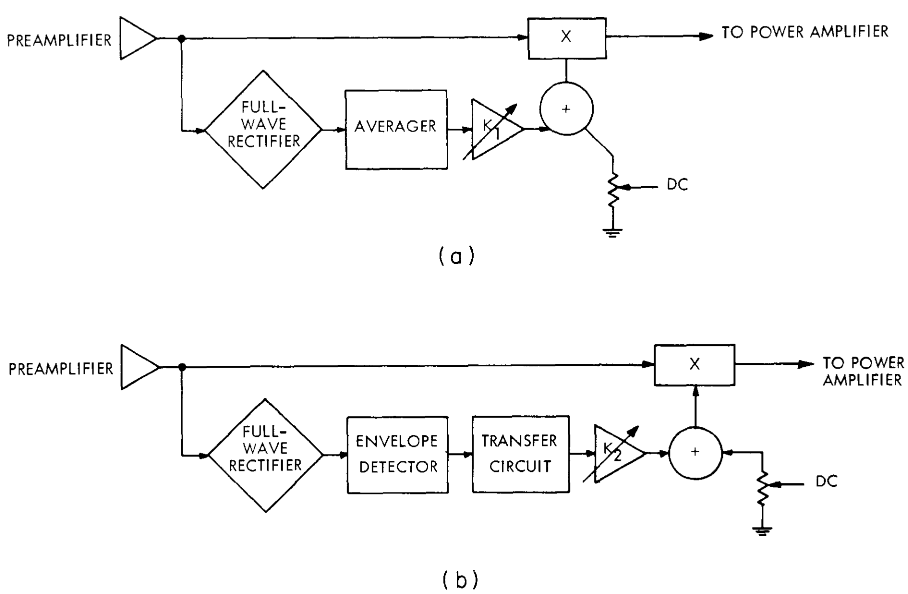

Fig. XVI-5. (a) Circuit for altering level contrast.
(b) Circuit for altering note-to-note contrast.

by a small number of parameters, such as contrast, attack, tempo, and so forth. These in turn may be related to more physical quantities, such as the time between two accented notes for tempo, or the magnitude of note envelopes for contrast. While tempo cannot be altered in real time, contrast or attack can be.

Thus far, we have concentrated mainly on contrast, which, qualitatively at least, is the difference between the loud and the soft passages of music. We have defined two different kinds of contrast. The first is level contrast, that is, the difference in level between two successive passages of music, while the second is note-to-note contrast, or the difference between the level of a note and the background. Save for the noticeable lag when a fast transition between two passages of different levels occurs, level contrast is easily accomplished (see Fig. XVI-5a). On the other hand, note-to-note contrast is somewhat more complicated (see Fig. XVI-5b). In order to avoid unnecessary delay, the envelope detector must be mainly a peak detector, rather than an averaging detector. The problem is to set the time constant. If it is too short, the detector will pick up carriers, as well as envelopes, while if it is too long, it will tend to skip notes, particularly any preceded by a much louder note or a very short one. Furthermore, while the music is playing, the peak detector seldom gets a chance to decay completely before the next note, so that a superfluous DC level is injected into the signal.

While there is no optimum solution to the question of a time constant, it is possible to find a working compromise so that only very few notes are skipped. Eliminating the excess DC, however, requires some kind of nonlinear transfer curve, and several such curves have been found that are effective to one degree or another. Both kinds of contrast can now be altered in a manner that sounds quite natural.

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