

GENERAL PHYSICS

I. MOLECULAR BEAMS*

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A. CALCULATION OF THE CROSS SECTION FOR He³-ROTON SCATTERING IN SUPERFLUID HELIUM

1. Introduction

We have found from our measurements of diffusion in dilute He³-He⁴ solutions that the cross section for the He³-roton interaction is energy-dependent, and increases with temperature over the temperature range of our experiment.¹ The magnitude of the cross section can be approximated from our measurements and we find that it ranges from $\sim 1.6 \times 10^{-14}$ cm² to $\sim 2.3-2.4 \times 10^{-14}$ cm² as the temperature increases from 1.27° to 1.69° K. To understand these results, we have analyzed a possible model for the interaction.

2. Model for the Interaction

In the picture developed by Feynmann and Cohen² a solute He³ atom in liquid He II can be thought of as a sphere moving in an inviscid background fluid, and the Feynmann-Cohen wave function for the He³ quasi particle includes the velocity potential of the dipole flow pattern of a sphere. The energy spectrum³ of a solute He³ quasi particle is consistent with this hydrodynamic picture:

$$E_3 = -\epsilon_0 + \hbar^2 k_3^2 / 2m_3^* \quad (1)$$

where ϵ_0 represents a binding energy to the liquid, and m_3^* is an effective mass which takes into account the backflow pattern around the sphere. Specific heat measurements⁴ in dilute solutions verify this picture, and measurements of the normal fluid fraction⁴ in solutions indicate that $m_3^* \approx 2.7 m_3$.

The roton excitation involves many atoms of the fluid. In the Feynmann-Cohen

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picture, it arises from the backflow pattern of each atom in the liquid in the presence of the hydrodynamic continuum formed by all of the other identical He^4 atoms. At a given atom, the exact velocity will depend on the pair correlation function, and correlation effects involving many atoms. At distances greater than an interatomic distance, however,⁵ the dipole form should give a good approximation

$$\vec{V}(\vec{R}) = (-A/m_4) \nabla(k_r/R^2),$$

where k_r determines the strength of the dipole and is nearly independent of energy, and the quantity m_4/A is approximately $4\pi\rho$, where ρ is the liquid density.

Thus, the situation is similar to that of two spheres moving in an inviscid background fluid where the dipole flow pattern of each moving sphere produces an added pressure on the other. To lowest order in $1/R$ (neglecting terms of $1/R^6$ and higher) this interaction is

$$U = 4\pi\rho[\vec{\mu}_1 \cdot \vec{\mu}_2 - 3(\vec{\mu}_1 \cdot \hat{R})(\vec{\mu}_2 \cdot \hat{R})]/R^3, \quad (2)$$

where $\mu = 1/2 a^3 u$ is the dipole moment, with a the radius, and u the velocity of the sphere. For the case of the He^3 -roton interaction, we use the same form for U , with

$$\mu_3 = \frac{3}{4\pi\rho} \left(\frac{m_3^* - m_3}{m_3^*} \right) \hbar k_3, \quad \mu_r = \frac{A}{m_4} \hbar k_r \approx \frac{\hbar k_r}{4\pi\rho}. \quad (3)$$

At very small distances, the higher order terms are important, and in addition, the dipole approximation for the roton will probably not be valid. To take account of these effects in a crude way, the wave function will be cut off at a small distance a , and U will be taken as ∞ for $R < a$.

3. Method of Solution

The problem is to calculate the cross section of the velocity-dependent interaction (2) for two particles, one having the classical energy spectrum (1), the other having the spectrum $E_r = \Delta + (\hbar^2/2\mu)(k_r - k_0)^2$. Putting the problem into the center-of-momentum system does not reduce the Hamiltonian function to a one-body problem as is usually the case. A partial wave expansion, then, does not seem appropriate. Instead, we use first-order perturbation theory, with two-particle wave functions. A perturbation theory calculation of the He^3 -roton interaction has been previously derived in the well-known work of Khalatnikov and Zharkov,⁶ who took $U(\vec{R}) = V_0 \delta(\vec{R})$ and found that σ was independent of energy. In the calculation presented here, the dipole-dipole interaction (2) will be used with the cutoff described above. The initial roton velocity will also be taken

into account. (The roton was assumed to be initially at rest by Khalatnikov and Zharkov.⁶) It should be noted that the dipole-dipole interaction (2) has been recently used in connection with the roton-roton interaction.^{7, 8}

The He³ diffusion cross section is the transition rate divided by the input He³ current, which for a single particle is $V_{\text{relative}}/\bar{V}$. The expression for the cross section is

$$d\sigma = \frac{2\pi}{\hbar} |\langle \psi_3 \psi_r | U(\vec{R}) | \psi'_3 \psi'_r \rangle|^2 \delta(E_3 + E_r - E'_3 - E'_r) f \\ \times \frac{1}{2} \frac{(1 - \cos \psi)}{V_{\text{rel}}/\bar{V}} \frac{d^3 k'_3 d^3 k'_r}{\left(\frac{(2\pi)^3}{\bar{V}}\right)^2}, \quad (4)$$

where f is a factor involving the change in variable to be made in the delta function from E to k' , and ψ is the scattering angle of the He³ quasi particle.

The solution will involve determination of the matrix element and integration of expression (4) over the allowed phase space. Since two-particle wave functions are used, momentum conservation is not implicit as with the usual one-body case derived in the CM frame. The three equations from momentum conservation, plus the energy delta function, impose 4 constraints on the integration so that the phase space of allowed states will be two-dimensional. The problem is simplified by changing variables for the final states to $d^3 \Delta k d^3 \Delta k'$, where $\Delta \vec{k} = \vec{k}'_3 - \vec{k}_3$, $\Delta \vec{k}' = \vec{k}'_r - \vec{k}_r$, and, by conservation of momentum, $\Delta \vec{k} = -\Delta \vec{k}'_O$. After adding the appropriate delta functions to take this into account, and integrating over $d^3 \Delta k'$, we are left with a single integral over $d^3 \Delta k$. The relation between $|\Delta \vec{k}|$ and (θ, ϕ') , the angles defining $\Delta \vec{k}$, determines a surface in phase space over which the integration proceeds.

4. Kinetics – Conservation of Energy and Momentum in Collisions

To first order, k_r is constant, equal to k_O , and an elementary calculation shows that $\Delta k = 2k_r \cos \phi$, where ϕ is the angle between \vec{k}_r and $\Delta \vec{k}$ (see Fig. I-1a). To second order, let $(\Delta k - 2k_r \cos \phi)$ be a small quantity. With this approximation, conservation of energy leads to

$$|\Delta \vec{k}| = \frac{2(V_r \cos \phi - V_3 \cos \theta)}{V_r + k_r/m_3^*} k_r, \quad (5)$$

where $V_r \equiv (k_r - k_O)/\mu$, $V_3 \equiv k_3/m_3^*$, and $\cos \theta \equiv \hat{k}_3 \cdot \Delta \hat{k}$, $\cos \phi \equiv \hat{k}_r \cdot \Delta \hat{k}$. With $\alpha \equiv$ angle of incidence between \vec{k}_3 and \vec{k}_r , and ϕ' the azimuthal angle between $\Delta \vec{k}$ and

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the plane formed by \vec{k}_3 and \vec{k}_r , we have

$$\cos \phi = \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos \phi'. \quad (6)$$

(See Fig. I-1b.)

To perform the integration, the argument of the delta function expressing energy

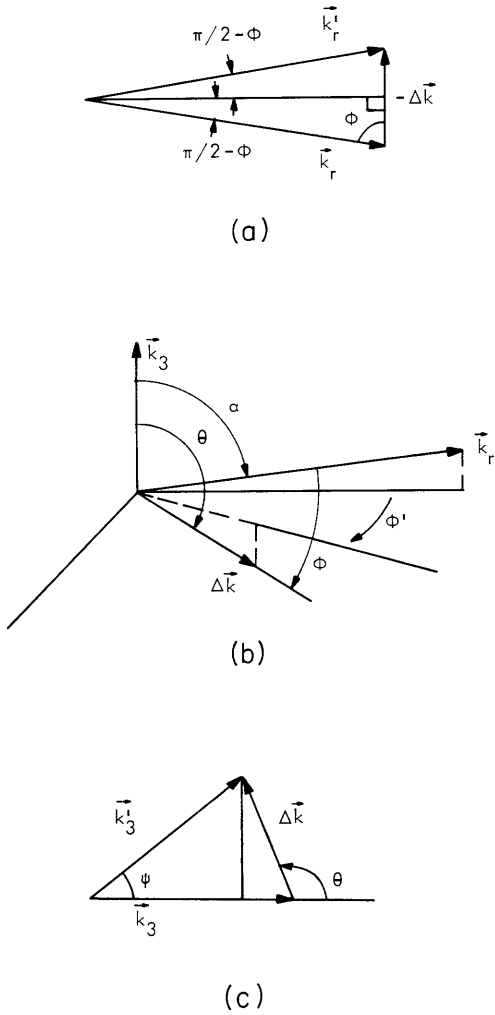


Fig. I-1.

- (a) Lowest order approximation, $\Delta k = 2k_r \cos \phi$.
 (b), (c) Angles defining the orientation of the various momentum vectors.

conservation must be written in terms of Δk with an added factor $1 / \frac{d(\arg)}{d\Delta k}$, since the energy delta function is of the form $\delta(f(\Delta k))$. This turns out to be

$$1 / \frac{d(\arg)}{d(\Delta k)} = 1 / \hbar^2 \left[(k_r - k_0) \cos \phi / \mu - k_3 \cos \theta / m_3^* \right] \quad (7)$$

or just $2kr / (\text{numerator in the } |\Delta k| \text{ expression})$ (Eq. 5).

5. Evaluation of the Matrix Element

The initial and final wave functions are plane waves

$$\psi_i = \frac{1}{\sqrt{V}} e^{i\vec{k}_3 \cdot \vec{r}_3} e^{i\vec{k}_r \cdot \vec{r}_r} \quad \psi_f = \frac{1}{\sqrt{V}} e^{i\vec{k}'_3 \cdot \vec{r}_3} e^{i\vec{k}'_r \cdot \vec{r}_r}.$$

To evaluate $\langle \psi_i | U | \psi_f \rangle$ (where $U = \infty$, for $r < a$, and U is the expression in Eq. 2 for $r > a$), it is simple to show that what is needed is

$$V_{12} \equiv \frac{1}{\sqrt{V}} \int_{\Delta ka}^{\infty} e^{-i\Delta\vec{k} \cdot \vec{R}} U(\vec{R}), \quad (8)$$

with $\vec{R} = \vec{r}_3 - \vec{r}_r$. The integral can be shown to be

$$\frac{\hbar^2}{\pi\rho} \left(\frac{m_3^* - m_3}{m_3^*} \right) [3(\vec{k}_3 \cdot \hat{\Delta\vec{k}})(\vec{k}_r \cdot \Delta\vec{k}) - \vec{k}_3 \cdot \vec{k}_r] \left(\frac{3 \int_1^{\Delta ka} (\Delta ka)}{\Delta ka} \right). \quad (9)$$

If we put $|\Delta\vec{k}|$ as a function of (θ, ϕ') in (9), we get a clumsy complicated expression. For $\Delta ka < 1$, however, $3 \int_1^{\Delta ka} (\Delta ka) / (\Delta ka) \approx 1$, and the expression is greatly simplified. The assumption made in this simplification is that the "hard core" radius multiplied by the momentum change in the collision is small. (For Δk of the order of k_3 , and a of the order of a fraction of an angstrom, the approximation is seen to be valid.)

6. Diffusion Cross-Section Factor $1/2 (1 - \cos \psi)$

The main purpose of this factor is to eliminate the effect of small-angle scattering and emphasize large-angle scattering in the integral. The problem is being solved by using θ , the angle between Δk and k_3 , rather than ψ , the angle between \hat{k}_3' and k_3 , so an approximation will be used. From Fig. I-1 it is seen that

$$\cos \psi = (k_3 + \Delta k \cos \theta) / k_3'. \quad (10)$$

A simple calculation shows that the average thermal energy of a roton is $1/2 kT$, while that of a He^3 atom is $3/2 kT$, so that on the average the energy transferred to a He^3 atom in a collision will not be large, and $|\vec{k}_3|$ will not be too different from $|\vec{k}_3'|$. Then

$$\cos \psi \approx 1 + (\Delta k / k_3) \cos \theta \quad \text{or} \quad \frac{1}{2} (1 - \cos \psi) = -\frac{\Delta k}{k_3} \cos \theta. \quad (11)$$

Note that the $\cos \theta$ factor eliminates small-angle scattering and increases the amplitude for large-angle scattering, as desired.

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7. Evaluation of the Phase Space Integral

We have previously derived a relation between $|\Delta\vec{k}|$ and (θ, ϕ') . The differential area of the surface defined by this relation is

$$dA = (\Delta k)^2 \sin \theta d\theta d\phi' / \cos \gamma, \quad (12)$$

where γ is the angle between the normal to the surface and $\Delta\vec{k}$. If the surface defined by (5) is written as $g(\Delta k, \theta, \phi') = 0$, then $\cos \gamma = \nabla g \cdot \hat{\Delta k} / |\nabla g|$ and,

$$dA = \Delta k^2 \sin \theta d\theta d\phi' \left[1 + \frac{1}{\Delta k^2} \left(\frac{\partial \Delta k}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta \Delta k^2} \left(\frac{\partial \Delta k}{\partial \phi} \right)^2 \right]^{1/2}. \quad (13)$$

By using (5) and evaluating (13), we find that

$$dA = \frac{2k_r}{(v_r + k_r/m_3^*)} |\vec{v}_r - \vec{v}_3| \Delta k \sin \theta d\theta d\phi'. \quad (14)$$

Note that the factor of the relative velocity in the input current is canceled by a similar factor in the differential area of the allowed phase space.

The cross section can now be evaluated by substituting the various factors in (4) from (7), (8), (9), (11), and (14), using (5) and (6), and the approximation of our evaluation of the matrix element. The result is

$$\sigma = \left\{ (2/\pi^3) \frac{(m_3^* - m_3)^2}{\rho^2} k_r^5 k_3^2 / (k_r/m_3^* + v_r)^3 \right\} I,$$

and I is a somewhat involved angular integral. After evaluating the integral, the cross section is determined for a given \vec{k}_3 and \vec{k}_r . By integrating the result over all angles of incidence α , and substituting the average values of k_3 and k_r at a given temperature, we determine $\sigma(T)$. We now use

$$k_3 = 0.71 \times 10^8 T^{1/2} \text{ cm}^{-1}$$

$$(k_r - k_0)/k_0 = 0.060 T^{1/2}$$

$$m_3^* = 2.7 m_3$$

$$\mu = 0.16 m_4 \text{ (curvature parameter in the roton spectrum)}$$

$$k_o = 1.92 \times 10^8 \text{ cm}^{-1}$$

$$\rho = 0.141 \text{ gm/cm}^3 \text{ (He}^4 \text{ density).}$$

The result is

$$\sigma = 2.7 \times 10^{-14} T \frac{(1+.06 T^{1/2})^5}{(1+.82 T^{1/2})^3} \text{ cm}^2.$$

To compare this result with the values for σ derived from our diffusion measurements, we evaluate σ at the end points of our experimental temperature range:

$$T = 1.27^\circ \quad \sigma = 0.67 \times 10^{-14} \text{ cm}^2$$

$$T = 1.69^\circ \quad \sigma = 0.76 \times 10^{-14} \text{ cm}^2.$$

Thus σ increases with energy and is of order 10^{-14} cm^2 . The experimental and calculated values for σ agree within a factor of two and a half to three in magnitude and the calculated values increase less rapidly with temperature. Note that the short-range part of the interaction has been dealt with very crudely, and only first-order perturbation theory has been used. Nevertheless, a plausible model has been presented which leads to an expression for the cross section which increases with energy, and is of the correct order of magnitude, without containing adjustable parameters.

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