TIDAL EVOLUTION OF THE URANIAN SATELLITES

by

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ABSTRACT

The major satellites of Uranus are not currently involved in any low-order mean-motion commensurabilities, but they may have encountered a number of resonances as the orbits evolved due to tidal friction. There are significant chaotic zones associated with these resonances. Due to the presence of these chaotic zones, the standard theory describing passage through orbital resonances is not applicable. In particular, there are significant changes in the mechanism and probability of capture into resonance. Large changes in the orbital elements of some of the satellites may have resulted from evolution through these commensurabilities.

Ariel and Umbriel have passed through the 5:3 mean-motion commensurability if the specific dissipation function (Q) of Uranus is less than about 100,000. The orbits may have been chaotic for a significant interval of time during passage through this resonance. Tidal evolution within the chaotic zone may have driven orbital eccentricities to values two or three times higher than the values approaching the resonance, with some probability of escape from resonance remaining. Eccentricities high enough to have a significant effect on the thermal history of Ariel have not been found.

Miranda and Umbriel have passed through the 3:1 mean-motion commensurability if the Q of Uranus is less than about 39,000. There are three second-order inclination resonances and three second-order eccentricity resonances associated with this commensurability. Temporary capture into either of the resonances involving the orbital inclination of Miranda can account for the anomalously high ($\approx 4^\circ$) current inclination of Miranda. As the satellites approach the commensurability at low orbital inclinations, the coupling between the resonances is very weak, and capture into either of the resonances involving the orbital inclination of Miranda is likely. The evolution of the system after capture into one of these resonances is initially described well by the standard theory of evolution through isolated mean-motion resonances. However, as the orbital inclination of Miranda increases, and the coupling between the resonances becomes stronger, the separatrices associated with the resonances broaden into chaotic zones and eventually merge, creating a sizable chaotic region. Escape from resonance occurs via a qualitatively new dynamical mechanism. The trajectory encounters low-order commensurabilities between the libration frequency of the resonant argument and other fundamental frequencies in the system. If the trajectory is captured into any of these secondary resonances, it is dragged into the chaotic region, whereupon the system can escape the mean-motion commensurability. Miranda retains a high orbital inclination comparable to the current value. Since the anomalously large inclination of Miranda arises naturally upon passage through the 3:1 commensurability, it is likely that the satellites have encountered this resonance, and the Q of Uranus may be constrained to be less than 39,000.

The orbits of Miranda and Umbriel also become chaotic during evolution through the eccentricity resonances associated with the 3:1 commensurability. During evolution through the large chaotic zone, the orbital eccentricity of Miranda can be driven to a value comparable to or larger than the equilibrium eccentricity. At the maximum eccentricities observed during the numerical simulations of passage through this resonance, tidal friction may heat the interior of Miranda at a rate about an order of magnitude higher than the current radionuclide heating rate. If materials such as ammonia hydrate were present in quantity, significant partial melting of the interior may have occurred.

Miranda and Ariel passed through the 5:3 mean-motion commensurability if the Q of Uranus is less than about 12,000. During evolution through this commensurability, the semimajor axis ratio decreases, and there is no known mechanism of capture into this resonance. As the orbits enter a large chaotic zone associated with this commensurability, both the orbital eccentricity and inclination of Miranda jump to values three to four times higher than the values approaching the resonance. Upon escaping from the resonance, the orbit of Miranda may have retained the high eccentricity and inclination, or e_M and i_M may have decreased back to values comparable to those approaching the resonance. Tidal heating of Miranda was probably not significant during passage through this commensurability.

If the Q of Uranus was smaller than about 11,000, then Ariel and Umbriel would have encountered the 2:1 mean-motion commensurability. Capture into this resonance is very likely if the eccentricities of the orbits approaching the resonance were comparable to the current values: the probability of escape would not have been significant unless the initial eccentricities were of order 0.04. It is therefore unlikely that these satellites encountered this resonance, and the Q of Uranus may be constrained to be greater than 11,000.

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Table of Contents

Abstract	2
Acknowledgements	4
Table of Contents	5
 Chapter 1: General Introduction	 8
1.1 The Uranian Satellites	13
1.2 Orbital Evolution	15
1.2.1 Motion of Satellites about a Planet	15
1.2.2 Tides	18
1.2.3 Mean-Motion Resonances	20
- The Single Resonance Theory	21
1.3 Mean-Motion Commensurabilities in the Uranian Satellite System	26
1.4 Chaotic Behaviour	29
1.4.1 What is Chaos?	29
1.4.2 Origin of Chaotic Behaviour in the Uranian Satellite System	32
1.4.3 Ergodic Adiabatic Invariants	34
1.5 Tidal Heating of Planetary Satellites	36
 Chapter 2: The Ariel-Umbriel 5:3 Commensurability	 38
2.1 Introduction	38
2.2 The Planar-Eccentric Problem	40
2.2.1 Resonance Coordinates	43
2.2.2 Resonance Dynamics	46
- Surfaces of Section	46
- Onset of Chaos	57
2.2.3 Evolution through the Resonance	65
- Tidal Dissipation	65
- Numerical Runs	71
- Statistics of Escape	91
2.3 Secular Perturbations	98
2.4 The Eccentric-Inclined Problem	101
2.5 Discussion	106
 Chapter 3: The Miranda-Umbriel 3:1 Commensurability	 112
3.1 Introduction	112
3.2 The Circular-Inclined Problem	115
3.2.1 Resonance Coordinates	117
3.2.2 Evolution through the Resonance	119
- Rate of Evolution	119
- Numerical Results	125

- Secondary Resonances	146
- Extent of the Chaotic Zone	149
3.2.3 Secular Perturbations	156
3.3 The Planar-Eccentric Problem	159
3.3.1 Resonance Coordinates	161
3.3.2 Evolution through the Resonance	162
- Rate of Evolution	162
- Numerical Results	166
3.3.3 Secular Perturbations	204
3.4 The Eccentric-Inclined Problem	207
3.5 Discussion	211
3.5.1 Inclination Resonances	211
3.5.2 Eccentricity Resonances	212
3.5.3 Tidal Heating of Miranda	212
3.5.4 Q of Uranus	223
3.5.5 Conclusions	224
 Chapter 4: The Miranda-Ariel 5:3 Commensurability	225
4.1 Introduction	225
4.2 The Planar-Eccentric Problem	228
4.2.1 Resonance Coordinates	229
4.2.2 Evolution through the Resonance	230
- Rate of Evolution	230
- Numerical Results	234
4.3 Secular Perturbations	259
4.4 The Eccentric-Inclined Problem	262
4.5 Discussion	269
 Chapter 5: The Ariel-Umbriel 2:1 Commensurability	271
5.1 Introduction	271
5.2 The Planar-Eccentric Problem	273
5.2.1 Resonance Coordinates	274
5.2.2 Evolution through the Resonance	275
- Rate of Evolution	275
- Numerical Results	279
5.3 Secular Perturbations	310
5.4 The Eccentric-Inclined Problem	314
5.5 Discussion	318
 Chapter 6: Summary	321
 Chapter 7: Appendix I: The Averaged Resonant Hamiltonian	324
7.1 Planar-Eccentric or Circular-Inclined Models	324
7.1.1 Ariel-Umbriel 5:3 Planar-Eccentric Problem	327
7.1.2 Miranda-Umbriel 3:1 Circular-Inclined Problem	332

7.1.3 Miranda-Umbriel 3:1 Planar-Eccentric Problem	334
7.1.4 Miranda-Ariel 5:3 Planar-Eccentric Problem	338
7.1.5 Ariel-Umbriel 2:1 Planar-Eccentric Problem	341
7.2 Secular Perturbations	345
7.2.1 Ariel-Umbriel 5:3 Planar-Eccentric Problem	347
7.2.2 Miranda-Umbriel 3:1 Circular-Inclined Problem	349
7.2.3 Miranda-Umbriel 3:1 Planar-Eccentric Problem	351
7.2.4 Miranda-Ariel 5:3 Planar-Eccentric Problem	353
7.2.5 Ariel-Umbriel 2:1 Planar-Eccentric Problem	355
7.3 Eccentric-Inclined Models	357
7.3.1 Ariel-Umbriel 5:3 Resonance	359
7.3.2 Miranda-Umbriel 3:1 Resonance	362
7.3.3 Miranda-Ariel 5:3 Resonance	364
7.3.4 Ariel-Umbriel 2:1 Resonance	367
 Chapter 8: Appendix II: Mapping for Two Degree-of-Freedom Models	 370
 Chapter 9: Appendix III: Standard Theory of Evolution through Resonances	 377
9.1 Adiabatic Invariance	377
9.2 Second-Order Resonances	379
9.2.1 Capture Probabilities	383
9.2.2 Numerical Results	386
9.3 First-Order Resonances	390
 Biographical Note	 395
 Bibliography	 398

Chapter 1

General Introduction

Tidal friction is an important process in the evolution of planetary satellite systems. As first shown by Goldreich (1965), many of the existing orbital resonances among planetary satellites may be the result of evolution of the orbits due to tidal friction. The major Uranian satellites are not currently involved in any low-order mean-motion commensurabilities. However, some of the dynamical properties of the Uranian satellite system, such as the anomalously high orbital inclination of Miranda and the anomalously high orbital eccentricities of the inner large satellites (Squyres *et al.* 1985), cannot be explained by the current interactions between the satellites (Laskar 1986, Dermott and Nicholson 1986), and therefore suggest origins in past dynamical interactions. In addition, the relatively young surfaces of Miranda and Ariel (Smith *et al.* 1986) suggest that tidal heating may have been important in the thermal histories of these satellites. Passage through mean-motion resonances as the orbits tidally evolved may account for many of these observations.

In cases where the motion near a resonance is dominated by the perturbations of a single resonant argument, for instance where the resonances near a mean-motion commensurability are well separated by a large planetary oblateness, the dynamics can be adequately described by the standard integrable theory of evolution through resonances (Goldreich and Peale 1966; Counselman and Shapiro 1970; Yoder 1979; Henrard 1982; Henrard and Lemaître 1983; Borderies and Goldreich 1984; Lemaître

1984). However, in cases where there are many essential contributions to the motion near a resonance, for example where there is a significant coupling between resonances and/or a strong secular interaction, the assumptions made in deriving the single-resonance formulae may not apply. Specifically, the motion in a relatively large region of phase space near the separatrix at a resonance may be chaotic and therefore non-integrable. Because of the irregular nature of chaotic motion, the evolution of the satellites through the resonances may be significantly different from the evolution predicted by the single-resonance theory.

In the Uranian satellite system, the satellite-to-planet mass ratios are relatively high and the planetary oblateness is small. This results in significant coupling between the resonances at mean-motion commensurabilities, and there are strong secular interactions between the satellites. The single-resonance theory does not apply to this system.

We have been carrying out a thorough, systematic study of mean-motion resonances which may have been encountered by the Uranian satellites as the orbits evolved due to tidal friction. We have found significant chaotic zones at low-order mean-motion commensurabilities between the Uranian satellites (Titemore and Wisdom, 1988a,b,c). Due to the presence of these chaotic zones, there are significant changes in the mechanisms and outcomes of resonance passage.

In this study, the most important interactions involving the orbital eccentricities or inclinations of satellites near a resonance are approximated by a two degree of freedom Hamiltonian system with a parameter which slowly evolves due to tidal friction. A two degree of freedom Hamiltonian model is especially useful because it is both complex enough to preserve the essential features of the motion, including chaotic behavior, and simple enough that the dynamics can be studied in detail using surfaces of section (Hénon and Heiles 1964). Many trajectories are numerically integrated through the resonance, in order to determine the natures and proba-

bilities of various outcomes. The phase space available to trajectories at various points during the evolution is studied using surfaces of section, in order to obtain a global view of the qualitative dynamics as the system evolves. The picture of resonance passage thus developed is then tested by including interactions between the eccentricity resonances and inclination resonances, and by including nonresonant perturbations, such as the perturbations due to the other satellites in the system.

Once the dynamical processes and outcomes are determined, it is then possible to proceed with an analytic approach to the problem. Although chaotic behavior is non-integrable, it is possible in some cases to construct a semianalytic theory of the evolution through a mean-motion commensurability in which chaotic behavior is important. By understanding the fundamental frequencies of the Hamiltonian and the structures of the individual unperturbed resonances, one can apply analytic estimates of the widths of chaotic zones and the resonance overlap criterion for onset of large-scale chaotic behavior (Chirikov 1979) to the problem in order to understand the origin and extent of the chaotic motion. When this information is combined with numerically estimated probabilities of the various outcomes of resonance passage, it is possible to produce a global qualitative picture of the evolution paths available to a trajectory, and the probability that a trajectory will follow a given path. By comparing this picture to the present dynamical configurations of the satellites, taking into consideration such effects as the tidal damping of eccentricities since resonance passage, it is then possible to determine the most likely scenario for the evolution of the real satellite system.

This thesis outlines the problems, procedures, and results to date of this study. Some of the work in this thesis is a synthesis of the two papers Titemore and Wisdom 1988a and 1988b. The rest has not yet been published.

We begin with a general introduction. The purpose of Chapter 1 is to familiarize the reader with the Uranian system and the dynamical processes which are impor-

tant in the long-term evolution of the satellites. In the following Chapters, we then explore the dynamics of various mean-motion resonances, proceeding backwards in time from the present.

Chapter 2 explores the dynamics of the Ariel-Umbriel 5:3 mean-motion commensurability. During evolution through this resonance, the orbits of Ariel and Umbriel may have been chaotic for significant periods of time. The orbital eccentricities may have increased by factors of two or three before the satellites escaped from the resonant interaction, but eccentricities high enough to significantly affect the thermal evolution of Ariel have not been found so far in this study. Final orbital eccentricities of the satellites are consistent with the current values.

Chapter 3 explores the evolution of Miranda and Umbriel through the 3:1 mean-motion commensurability. Both the orbital eccentricity and inclination of Miranda may have increased dramatically during passage through this resonance, and the high values would have been retained after the satellites escaped from the resonant interaction. In the time since the satellites encountered this commensurability, the inclination of Miranda's orbit would have remained high, explaining the current anomalously high value, while the orbital eccentricity would have damped to the current value. The rate of tidal heating of Miranda during resonance passage may have been about an order of magnitude higher than the current radiogenic heating rate, but it probably would not have been high enough to result in the melting of large quantities of water ice. However, if materials such as ammonia hydrate were present, some partial melting of the satellite interior may have occurred.

The 5:3 mean-motion commensurability involving Miranda and Ariel is discussed in Chapter 4. This interaction differs from the others considered here in the sense that the semimajor axis ratio decreases, and there is no known mechanism of capture into this resonance. The orbital eccentricity and inclination of Miranda may have increased modestly during evolution through this commensurability, but tidal

heating of this satellite was probably not significant.

Finally, the Ariel-Umbriel 2:1 mean-motion commensurability is discussed in Chapter 5. The interaction at this resonance involves terms linearly proportional to the eccentricities as well as second-order terms. The probability of capture was very high if the satellites approached this resonance with orbital eccentricities comparable to the current values; eccentricities of order 0.04 would have been required for the probability of escape to be significant. Once captured into the resonance, the satellites would have evolved to the equilibrium eccentricity of Ariel without the resonance becoming unstable. Therefore, it is unlikely that the satellites ever encountered this commensurability.

There are significant chaotic zones associated with all of the resonances considered in this investigation. The dynamics are exciting, affecting the mechanisms and outcomes of resonance passage in ways not predictable by integrable theories of resonance passage. There are some similarities between different resonances, but each has unique properties, resulting in a rich variety of dynamical behaviour in the Uranian satellite system. The dynamics of the individual resonances may be used to constrain the evolution of the system as a whole. Since passage through the 3:1 resonance with Umbriel can explain the current inclination of Miranda's orbit, it is likely that the satellites have tidally evolved at least enough to encounter this commensurability. This may be used to set an upper limit of 39,000 on the specific dissipation function (Q) of Uranus. Requiring that Ariel and Umbriel did not encounter the 2:1 commensurability may be used to constrain Q to be greater than 11,000. This rather narrow range may be useful in modelling the interior of Uranus.

1.1 The Uranian Satellites

There are 5 major satellites, 10 much smaller satellites, 10 narrow rings, and a sheet of diffuse material known to be orbiting the planet Uranus. Much of what we know about the major Uranian satellites has been obtained recently during the encounter of the Voyager 2 spacecraft with the Uranian system. The nominal orbital and physical characteristics of these satellites are summarized in Table 1.

Table 1.1: Uranian Satellites

Satellite	$R(km)^a$	m/M_U^a	a/R_U^a	$T(days)^a$	e^b	$i(radians)^c$
Miranda	242 ± 5	$8.63 \pm 2.6 \times 10^{-7}$	4.96	1.41	0.0014 ± 0.0002	0.0737 ± 0.0028
Ariel	580 ± 5	$1.55 \pm .28 \times 10^{-5}$	7.29	2.52	0.0017 ± 0.0002	0.0054 ± 0.0019
Umbriel	595 ± 10	$1.47 \pm .28 \times 10^{-5}$	10.15	4.15	0.0043 ± 0.0002	0.0063 ± 0.0014
Titania	805 ± 5	$4.00 \pm .21 \times 10^{-5}$	16.65	8.70	0.0025 ± 0.0001	0.0025 ± 0.0005
Oberon	775 ± 10	$3.37 \pm .19 \times 10^{-5}$	22.27	13.46	0.0003 ± 0.0001	0.0018 ± 0.0004

^aStone and Miner 1986. $M_U = 8.69 \times 10^{25}$ kg, and $R_U = 26,200$ km (French *et al.* 1985).

^bPeale 1988

^cViellet 1983

Compared with other planetary satellites, these are intermediate in size. The densities indicate that they are probably composed of mixtures of icy and rocky materials. The near-infrared spectra of these objects (see *e.g.* Cruikshank and Brown 1981) are consistent with the existence of water ice on the surfaces. From Voyager observations (Smith *et al.* 1986), Umbriel and Oberon appear to have primitive, cratered surfaces, but the other satellites have surprisingly interesting geological histories, including evidence of resurfacing. Both Ariel and Titania display considerable fracturing and faulting of the surface, and Ariel in particular shows evidence of flow features on the surface. Miranda, the smallest of the major satellites, was expected to be a primitive, cratered body, but instead was found to have a bizarre potpourri of geological terrains. In addition to cratered regions, there are areas of grooved and banded terrain, enormous faults and scarps, and flow features. The existence of resurfaced material on these small, icy low-temperature satellites indi-

cates that some mechanism significantly heated the interiors after heavy cratering of the surfaces ended.

The orbital configuration of this satellite system is quite regular, but there are dynamical oddities. For example, examination of the orbital periods indicates that there are no low-order mean-motion commensurabilities among the satellites at the present time, in contrast to the satellite systems of Saturn and Jupiter. The orbital eccentricities of the inner large satellites, while quite small, are nonetheless anomalously large when the timescale of damping of orbital eccentricities is taken into consideration over the age of the solar system (Squyres *et al.* 1985). The orbital inclinations (in radians) are comparable to the eccentricities, except in the case of Miranda, which has an orbital inclination more than an order of magnitude higher than any of the others. Voyager results indicate that all of the major satellites are in synchronous rotation states.

1.2 Orbital Evolution

In this Section, we discuss properties of orbital evolution relevant to the Uranian satellite system. In addition to the zero-order Keplerian motion, we must take into consideration the effects of planetary oblateness, interactions between satellites, and the effects of tidal torques in order to adequately describe the long-term evolution of the satellite system.

1.2.1 Motion of Satellites about a Planet

The major Uranian satellites are relatively close to the planet, and their masses are much less than the mass of the planet; hence, the short-term motion of a satellite is well approximated as Keplerian two-body motion. However, there are perturbations of the Keplerian motion which can be important over timescales much longer than the orbital periods. The most important of these perturbations for the Uranian satellites are due to the nonspherical shape of the planet and the mutual interactions between the major satellites.

In Keplerian satellite motion, the orbit is described as a closed ellipse with a semimajor axis a and an eccentricity e . The plane of the orbit has an inclination (i) to the planet equator. The orientation of the orbit relative to a planet-centered set of coordinates is given by the longitude of the ascending node (Ω) and the argument of pericenter (ω), with the longitude of pericenter $\varpi = \Omega + \omega$ (see Fig. 1.1). The orbital period (T) depends only on the semimajor axis. The position of the satellite in the orbit relative to the argument of pericenter is given by the true longitude w . Alternatively, the position of the satellite can be specified in terms of the mean longitude $\lambda = l + \varpi$, where the mean anomaly l is defined as the product of the mean motion $n = 2\pi/T$ and the time since pericenter passage.

For a non-spherical planet symmetric about its rotation axis, the gravitational

Figure 1.1: The two-body orbit in space.

potential can be expanded in terms of the spherical harmonics J_l :

$$U_i = -\frac{GMm_i}{r_i} \left[1 + \sum_{l=2}^{\infty} J_l \left(\frac{R}{r_i} \right)^l P_l(\sin \varphi_i) \right] \quad (1.1)$$

where G is the gravitational constant, M is the mass of Uranus, m_i is the mass of the satellite, r_i is the planet-to-satellite distance, R is the radius of Uranus, φ_i is the satellite latitude, and the P_l are Legendre polynomials. The gravitational force no longer depends on the inverse square of the distance between the planet and satellite, and the orbits are in general no longer periodic. The dominant effect of this perturbation on the long-term evolution of satellite orbits with low eccentricities and inclinations is to induce a relatively slow precession of the pericenter

$$\dot{\omega}_i \simeq \frac{3}{2} n_i J_2 \left(\frac{R}{a_i} \right)^2 \quad (1.2)$$

and a regression of the node

$$\dot{\Omega}_i \simeq -\frac{3}{2} n_i J_2 \left(\frac{R}{a_i} \right)^2. \quad (1.3)$$

The gravitational interaction between two satellites depends on the intersatellite distance, which can be expanded in terms of Keplerian elements in the disturbing function $\mathcal{R}$:

$$\mathcal{R} = \sum A e_1^{q_5} e_2^{q_4} i_1^{q_6} i_2^{q_5} \cos(p_1 \lambda_1 + p_2 \lambda_2 + p_3 \varpi_1 + p_4 \varpi_2 + p_5 \Omega_1 + p_6 \Omega_2) \quad (1.4)$$

with the constraints $q_i - |p_i| \geq 0$ and even, and $q_5 + q_6$ and $p_5 + p_6$ must be even. Also, because of rotational invariance, $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 0$. The interaction strength is of order m/M weaker than the planet-satellite interaction. There are terms in $\mathcal{R}$ which are independent of angles, and behave as the J_2 terms, inducing precession of the pericenters and regression of the nodes. Other terms are explicitly angle-dependent, and result in more complicated behavior.

To first order in the satellite masses, then, the total Hamiltonian for the system may be written:

$$\mathcal{H} = -\sum_{i=1}^{n-1} \frac{GMm_i}{2a_i} \left[1 + \sum_{l=2}^{\infty} J_l \left(\frac{R}{a_i} \right)^l P_l(\sin \varphi_i) \right] - \sum_{i < j < n} \frac{Gm_i m_j}{a_j} \mathcal{R}_{ij} \quad (1.5)$$

For the Uranian satellites, the perturbations due to planetary oblateness and the satellite interactions are of comparable strengths. These satellites are not currently involved in any low-order mean-motion resonances, but the eccentricities and inclinations vary significantly over timescales of a few decades due to secular interactions between the satellites, which can be modelled by terms in the disturbing function which are independent of the mean longitudes.

1.2.2 Tides

Due to the dependence of the gravitational force on distance, the material within the volume of an object is not all attracted equally by a nearby mass. Hence, the planet and satellite tidally deform each other. Friction induced in the interiors of the bodies by time-dependent tidal potentials can affect the satellite orbits.

Uranus spins on its axis more rapidly than the major satellites revolve about their orbits. Therefore, the planet rotates with respect to the tidal potential of a satellite. Due to friction, the response of the planet to the tidal potential is delayed: the tidal bulge raised by a satellite is pulled slightly ahead of the orbital position of the satellite by the rotation of the planet. Due to this asymmetric orientation of the tidal deformation with respect to the satellite, there is a torque between the two. The rate of rotation of the planet is slowed down, and the semimajor axis of the satellite increases at a rate (Darwin 1880)

$$\frac{\dot{a}_i}{a_i} = 3k_2 n_i \frac{m_i}{M} \left(\frac{R}{a_i} \right)^5 \frac{1}{Q} \quad (1.6)$$

where k_2 is the potential Love number of the planet and Q is the specific dissipation function. $1/Q$ is the amount of energy dissipated in one tidal cycle. A similar torque causes the orbital inclination to decrease at a rate (Darwin 1880)

$$\frac{\dot{i}_i}{i_i} \approx -\frac{1}{4} \frac{\dot{a}_i}{a_i} \quad (1.7)$$

because the tidal bulge lags slightly behind a satellite in an inclined orbit as it moves above and below the equator of the planet during an orbit period. The eccentricity

of the satellite orbit increases at a rate (Kaula 1964):

$$\frac{\dot{e}_i}{e_i} = \frac{19}{8} \frac{\dot{a}_i}{a_i} \quad (1.8)$$

The tidal bulge raised on a synchronously rotating satellite by the planet varies with time if the orbit is eccentric. The magnitude of the tidal deformation is sensitive to the distance from the planet, and the orientation of the deformation on a synchronously rotating satellite changes as the satellite librates. As the satellite figure adjusts to the changing tidal potential, energy is dissipated in the interior. This causes the eccentricity of the orbit to damp at a rate (Goldreich 1963)

$$\frac{\dot{e}_i}{e_i} = -\frac{21}{2} k_{2i} n_i \frac{M}{m_i} \left(\frac{R_i}{a_i} \right)^5 \frac{1}{Q_i} \quad (1.9)$$

where k_{2i} and Q_i are, respectively, the potential Love number and specific dissipation function of the satellite. The ratio of the contributions due to dissipation in the planet and dissipation in the satellite is given by

$$\frac{\dot{e}_{is}}{\dot{e}_{ip}} = -\frac{28}{19} D_i \quad (1.10)$$

where

$$D_i = \frac{k_{2i}}{k_2} \left(\frac{M}{m_i} \right)^2 \left(\frac{R_i}{R} \right)^5 \frac{Q}{Q_i} \quad (1.11)$$

For the Uranian satellites, the rates of eccentricity damping due to tidal friction in the satellite are about two orders of magnitude greater than the rates of increase in eccentricity due to tidal friction in the planet.

Tidal friction in a satellite also damps the orbital inclination, at a rate which depends on the equilibrium obliquity of the Cassini rotation state (see Section 3.2). For the Uranian satellites, this mechanism is of comparable or less importance than tidal friction in the planet.

In the absence of external torques, the total angular momentum of the planet-satellite pair is conserved. Since tidal dissipation in the satellite does not directly affect the rotation rate of the planet, the orbital angular momentum $H_i \approx$

Table 1.2: Tidal Parameters

Satellite	ρ [g/cm ³] ^a	μ [dynes/cm ²] ^b	k_2 ^c	Q
Miranda	1.26 ± 0.39	4×10^{10}	1.4×10^{-3}	100
Ariel	1.65 ± 0.30	10^{11}	4.3×10^{-3}	100
Umbriel	1.44 ± 0.28	10^{11}	3.3×10^{-3}	100

^aStone and Miner, 1986

^bshear modulus, $\approx 4 \times 10^{10}$ dynes/cm² (water ice), $\approx 6.5 \times 10^{11}$ dynes/cm² (rock)

^c $k_{2i} = \frac{3/2}{1 + \frac{19\mu_i}{2\rho_i g_i R_i}} \approx \frac{3\rho_i g_i R_i}{19\mu_i}$, $k_2 = 0.104$ (Gavrilov and Zharkov 1977)

$m_i \sqrt{GM a_i (1 - e_i^2)} \cos i_i$ is approximately invariant. The energy dissipated by tidal friction in a synchronously rotating satellite must come from the orbit, so the semi-major axis decreases at a rate

$$\frac{\dot{a}_i}{a_i} \approx 2e_i \dot{e}_i + 2i_i \dot{i}_i \quad (1.12)$$

due to tidal friction in the satellites.

Tidal parameters used for the satellites considered in this Thesis are given in Table 1.2.

1.2.3 Mean-Motion Resonances

The orbits of satellites generally expand at different rates. As they do so, the mean motions decrease, and the magnitudes of the perturbations on the Keplerian motion change. As the orbits differentially expand, they may encounter low-order mean-motion commensurabilities, where a combination of integer multiples of the mean motions, $in_2 - jn_1$, is near zero. The combination of mean longitudes $i\lambda_2 - j\lambda_1$ is therefore slowly varying. In general, terms involving slowly varying angles are more important to the long-term evolution of the orbit than are the short period terms, since the effects of terms in the disturbing function involving rapidly circulating angles tend to average out over long timescales.

Goldreich (1965) first showed that the interactions between satellites involved in a mean-motion commensurability may be stable under further tidal evolution of the orbits, and that tidal evolution could therefore account for the statistically significant abundance of resonances in planetary satellite systems. Why does the Uranian satellite system not presently have any stable low-order mean-motion commensurabilities between satellites?

At a mean-motion commensurability, there may be several important resonant terms involving the resonant combination of mean longitudes. For example, at the 3:1 mean-motion commensurability, a second-order resonance, the most important resonant terms involve the following combinations of Keplerian elements:

$$\begin{aligned}
& e_1^2 \cos(3\lambda_2 - \lambda_1 - 2\varpi_1) \\
& e_1 e_2 \cos(3\lambda_2 - \lambda_1 - \varpi_1 - \varpi_2) \\
& e_2^2 \cos(3\lambda_2 - \lambda_1 - 2\varpi_2) \\
& i_1^2 \cos(3\lambda_2 - \lambda_1 - 2\Omega_1) \\
& i_1 i_2 \cos(3\lambda_2 - \lambda_1 - \Omega_1 - \Omega_2) \\
& i_2^2 \cos(3\lambda_2 - \lambda_1 - 2\Omega_2)
\end{aligned} \tag{1.13}$$

A pair of satellites is involved in a mean-motion resonance if one or more of the resonant arguments is librating.

The Single-Resonance Theory

If the planetary oblateness is large and the satellites are not very massive, the J_2 terms in the Hamiltonian will dominate the terms in the disturbing function. Because of the large difference between the forced precession of ϖ_1 and ϖ_2 and between the forced regression of Ω_1 and Ω_2 , the above resonance terms will be well separated; when one combination of angles is slowly varying, the others will circulate much more rapidly. Therefore, as a pair of satellites evolves through a

mean-motion resonance, the effects of each term will be encountered separately. In this case, the standard theory of evolution through resonances (Goldreich and Peale 1966; Counselman and Shapiro 1970; Yoder 1979a; Henrard 1982; Henrard and Lemaître 1983; Borderies and Goldreich 1984; Lemaître 1984) can be used to describe the effects on the orbits. In the single resonance model, a particular resonant interaction is reduced to consider only perturbations of a single resonant argument. If the orbital eccentricities and inclinations are small enough, only the lowest order term in the disturbing function involving the resonant argument need be considered. The Hamiltonian for the resonant interaction then takes a standard form which depends on the order $i - j$ of the resonance, and the formulae of the single-resonance theory may be used to describe the evolution of the system through the resonance.

In such cases, the Hamiltonian near the resonance can be approximated by the standard generalized pendulum form

$$\mathcal{H} = \frac{1}{2}\alpha p^2 - \frac{\omega_0^2}{\alpha} \cos q \quad (1.14)$$

where q is the slowly varying angle, p is the conjugate momentum, and ω_0 is the frequency of small-amplitude librations. The pendulum Hamiltonian is the prototype form for all integrable resonance problems, and shows all of the essential features of resonant motion. There is only a single *degree of freedom* in this model, corresponding to the q, p pair of generalized coordinates. If the numerical value of $\mathcal{H}$ is time-independent, it is an *integral of the motion*. A one degree of freedom problem with an integral of the motion is integrable: the evolution of the equations of motion can be expressed in terms of standard analytic functions, *e.g.* elliptic integrals for the pendulum. Given a value of $\mathcal{H}$ and one coordinate, one can solve for the other coordinate explicitly using Eq. 1.13. In order to qualitatively understand the motion, one can plot curves of constant $\mathcal{H}$ in the *phase space* defined by p and q . The phase space of the generalized pendulum is shown in Fig. 1.2.

A trajectory started on a level curve of $\mathcal{H}$ remains on the curve if $\mathcal{H}$ is constant. There are three types of motion possible in the pendulum problem. The region at the top of the figure corresponds to positive circulation of the coordinate angle q , the central region corresponds to libration (oscillation) of q and the lower region corresponds to negative circulation of q . The curve dividing these regions is called the separatrix, and corresponds to infinite-period libration/circulation. In terms of a pendulum model, the fixed point on the separatrix at $q = \pm\pi$ corresponds to the unstable equilibrium at the top of a pendulum swing. The fixed point at the center of the libration region corresponds to the stable equilibrium at the bottom of the pendulum swing. The phase space representation of a Hamiltonian system is very useful, as the full range of dynamical behavior can be displayed in one easily interpreted plot.

In the orbital resonance model, tidal evolution of the orbits causes changes in the coefficients of the Hamiltonian, and the value of $\mathcal{H}$ is no longer constant. However, if the evolution is slow enough, there is another integral of the motion: the adiabatic invariant (Lenard 1959, Kruskal 1962, Arnold 1963). The action of a trajectory J ,

$$J = \oint pdq \quad (1.15)$$

or the area enclosed by a trajectory in phase space, is an adiabatic invariant to first order in the rate of evolution of $\mathcal{H}$. The fate of a tidally evolving trajectory may be viewed qualitatively as a slow evolution of the level curves of $\mathcal{H}$, maintaining constant area on the phase plane, except at the point of transition across the separatrix. The adiabatic invariant theory requires that the dynamical timescales of the Hamiltonian be much shorter than the tidal evolution timescale, and therefore breaks down at the point of transition across the separatrix, which has an infinite period. However, the fate of a given trajectory can be estimated using a probabilistic approach, by evaluating certain integrals along the (assumed) regular separatrix and assuming that the phases before transition are uniformly distributed. The ac-

Figure 1.2: Phase space of the pendulum.

tual models used for these calculations for the orbital resonances are a bit more complicated than the generalized pendulum equation (1.14), but are very similar to it (see Appendix III).

As a trajectory tidally evolves through the resonance, a particular resonant argument may either be captured into libration or reverse its direction of circulation. Because of the requirement that the action be conserved after crossing the separatrix, capture into libration can only occur if the area of the libration regions on the phase plane increase with further tidal evolution. For mean-motion resonances, this occurs only if the semimajor axis ratio is increasing. If the particular resonant argument of the system is captured into libration, one or both eccentricities or inclinations increase with further evolution, depending on which combination of angles is involved (see *e.g.* Eqs. 1.13). If the system is not captured into the resonance, then one or both eccentricities or inclinations decrease suddenly as the separatrix is crossed. If the semimajor axis ratio is decreasing, then there is a sudden increase in one or both eccentricities or inclinations as the separatrix is crossed.

This theory can be applied to the resonant interactions between the inner satellites of Saturn, where the planetary oblateness is large and the satellites are small. However, as we will show, the single resonance theory cannot be applied to the mean-motion resonances in the Uranian satellite system. The single resonance theory is explained in more detail in Appendix III.

1.3 Mean-Motion Commensurabilities in the Uranian Satellite System

The major low-order mean-motion commensurabilities which may have been encountered as the satellite orbits evolved due to tidal friction are shown in Fig. 1.3. This figure plots the mean motion ratios of pairs of satellites as a function of time for the minimum specific dissipation function of Uranus $Q = 6600$. Since $\dot{a}_i \propto a_i^{-11/2}$ from Eq. 1.6, or equivalently, $\dot{n}_i \propto n_i^{16/3}$, it can be shown that

$$n_i^{-13/3} - n_{i0}^{-13/3} = Nm_i(t - t_0) \quad (1.16)$$

where

$$N \equiv -\frac{13}{3} \left(-\frac{9}{2} k_2 \frac{1}{M} \frac{R^5}{(GM)^{5/3}} \frac{1}{Q} \right) \approx 2 \times 10^{-18} [\text{yr}^{10/3} \text{M}^{-1}] \quad (1.17)$$

is a tidal factor common to all satellites in the system (*e.g.* Peale 1988). If $Q = 6600$, Miranda and Ariel were at the same distance from the planet 4.6 billion years ago, using the nominal masses given in Table 1.1. Since this is likely to be a very unstable configuration, allowing close approaches and catastrophic consequences, it is unlikely that the satellites would have passed through this state. Therefore, we may consider $Q = 6600$ to be an upper limit on the rate of dissipation in Uranus (Peale 1988). This upper limit allows us to display in Fig. 1.3 all of the important orbital commensurabilities the satellites may have encountered over the age of the solar system. The real Q of Uranus, of course, determines the actual amount of tidal evolution which has taken place in the satellite system. The timescale on Fig. 1.3 depends linearly on the value of Q , so the amount of tidal evolution the satellites have undergone can be scaled accordingly on this figure.

The solid line in Fig. 1.3 represents the mean-motion ratio of Miranda and Ariel, the dashed line represents the mean-motion ratio of Ariel and Umbriel, and the dot-dashed line represents the mean-motion ratio of Miranda and Umbriel, all as functions of time from the present. Note that in the cases of Miranda-Umbriel

Figure 1.3: Mean-motion commensurabilities in the Uranian satellite system. The scale of time, here shown for $Q = 6600$, depends linearly on the value of Q .

and Ariel-Umbriel, the mean-motion ratio is decreasing (semimajor axis ratio is increasing), while in the case of Miranda-Ariel, the mean-motion ratio is increasing (semimajor axis ratio is decreasing). The mean-motion commensurabilities discussed in this Thesis, denoted by the diamonds in Fig. 1.3, include the Ariel-Umbriel 5:3, which would have been encountered if $Q < 100,000$, the Miranda-Umbriel 3:1 ($Q < 39,000$), the Miranda-Ariel 5:3 ($Q < 12,000$), and the Ariel-Umbriel 2:1 ($Q < 11,000$). As we will see, the dynamics of passage through these resonances further constrain Q to lie within the values 11,000 and 39,000, by requiring that the system avoided the Ariel-Umbriel 2:1 commensurability but evolved through the Miranda-Umbriel 3:1 commensurability.

1.4 Chaotic Behaviour

1.4.1 What is Chaos?

In 1963, Edward Lorenz of M.I.T. discovered that a low-dimensional dynamical model of the Earth's weather patterns behaved in a fundamentally unpredictable manner, despite the fact that the equations of motion were fully deterministic (Lorenz 1963). Slightly different initial conditions would rapidly evolve to completely different states, *i.e.* the evolution of the system was extremely sensitive to initial conditions. Therefore, the length of time over which the behaviour could be reliably predicted depended exponentially on how precisely the initial state of the system could be determined. About this same time, Hénon and Heiles (1964) found similar behaviour in a conservative system developed as a considerably simplified model of the motions of stars in our galaxy. From certain initial conditions, the system would evolve in a regular, predictable manner, while from other initial conditions the evolution would proceed in an irregular manner, characterized by exponential divergence of nearby trajectories with time: again, extreme sensitivity to initial conditions. Since then, the study of chaotic dynamical systems has become very widespread, with applications in many fields. The dynamical models we study here are slowly evolving Hamiltonians, so we concentrate on this type of system.

One feature of chaotic behaviour is that when represented as, say, a time series, it *looks* irregular: one doesn't see nice regular patterns. Another characteristic of chaotic behaviour is the existence of a broadband component in the frequency spectrum: quasiperiodic trajectories possess a finite number of discrete frequencies of the motion.

In order to objectively determine the presence of chaotic behaviour, the common practice is to numerically integrate the equations of motion for the system and evaluate the *Lyapunov Characteristic Exponent* (LCE), which may be used to estimate the rate of divergence of nearby trajectories in a dynamical system, and is defined

as

$$\lambda = \lim_{t \rightarrow \infty} \frac{\ln \left(\frac{d(t)}{d(t_0)} \right)}{t - t_0} \quad (1.18)$$

where $d(t)$ is the Euclidean distance between nearby trajectories in the phase space. Numerical methods for calculating λ are discussed in Benettin *et al.* 1976, 1980a, b. If the trajectory is quasiperiodic (regular), then trajectories will on average diverge linearly with time, and λ is zero. If the trajectory is chaotic, nearby trajectories diverge exponentially with time, and λ is positive.

The extent of chaotic behaviour in a dynamical system may be determined via graphical methods. Hénon and Heiles (1964) used the Poincare map, or surface of section, to study their Hamiltonian model. The surface of section may be used to study the phase space of Hamiltonian systems with two degrees of freedom. It is a generalization of the phase diagram described in the last Section for one-degree of freedom systems. A two degree of freedom system evolves in a four-dimensional phase space defined by two coordinates (*e.g.* q_1, q_2) and their conjugate momenta (*e.g.* p_1, p_2). The fact that the energy of the Hamiltonian is conserved may be used to eliminate one of the coordinates or momenta in terms of the others. The surface of section is usually a plane defined by setting one of the three remaining coordinates or momenta to have a value of zero. Successive intersections of this plane by a trajectory define points on the surface of section (Fig. 1.4), which reveal the structure of the phase space in a readily interpreted two-dimensional form.

As discovered by Hénon and Heiles, regular and chaotic trajectories show quite different characteristics on these plots. Quasiperiodic trajectories generate successive points on the surface of section which lie on curves; chaotic trajectories generate points which fill areas on the phase plane in an irregular manner. This allows one to very quickly determine the qualitative character of a trajectory by eye. In general, Hamiltonian systems with two degrees of freedom display a divided phase space: part of it is filled with quasiperiodic curves, and part of it is filled with chaotic

Figure 1.4: Surface of Section, or Poincaré Map.

points. As we will see, the ability to display the essential features of the phase space is very useful in determining the dynamical mechanisms which are important in the evolution of the Uranian satellites through mean-motion resonances.

In Hamiltonian systems with more than one degree of freedom, there are chaotic zones found in every region of the phase space: chaotic separatrices exist wherever there are resonances between the fundamental frequencies. These chaotic zones may occupy a majority of the phase space, or they may be so narrow that they have no discernable effect on the dynamics. It is important to note, however, that they are not *everywhere*: the existence of stable quasiperiodic motion has been mathematically proven in the Kolmogorov-Arnold-Moser (KAM) theorem (*e.g.* Arnold 1963).

A property of two degree-of-freedom systems is that quasiperiodic regions may separate chaotic zones which therefore do not communicate: a trajectory in an isolated chaotic zone will remain there. For systems with more than two degrees of freedom, though, the chaotic zones are all connected, forming the *Arnold Web* everywhere dense in the phase space (see *e.g.* Chirikov 1979, Hénon 1983). Thus, a trajectory started in the Arnold Web will come arbitrarily close to every point in the phase space, through a mechanism known as *Arnold Diffusion*. This Arnold Diffusion proceeds at a much slower rate than the diffusion of a trajectory within the primary chaotic zone associated with a resonance between fundamental frequencies.

1.4.2 Origin of Chaotic Behaviour in the Uranian Satellite System

In the case of the Uranian satellites, the oblateness of the planet is small ($J_2 = 0.0033$) and the satellites are relatively massive, so the resonance terms are not well separated. In addition, the secular interactions, which are proportional to the second power in eccentricities and inclinations, can significantly influence the orbital interactions near a resonance. Therefore, the assumptions made in deriving the

single resonance formulae do not apply. Numerous effects can combine to destroy the simple picture of evolution in the single resonance theory.

The proximity of the libration regions of many resonance terms in phase space may severely deform the shapes of the regions and the surrounding separatrices. The separatrix motion assumed in the evaluation of certain integrals in the single-resonance picture may become quite different, and may require re-evaluation.

A more serious problem concerns the nature of motion on the separatrix. The single-resonance theory assumes that the separatrix is regular. When the coupling between resonances and the secular interactions are considered, the problem can no longer be modelled by a single degree of freedom Hamiltonian, making it no longer integrable. As in most resonant systems, chaotic behavior is a likely result. In its mildest manifestation, chaotic behavior appears when high frequency perturbations broaden the separatrix into a chaotic zone (*e.g.* Chirikov 1979). The half-width of the chaotic zone may be expressed in terms of the chaotic variation of the energy (numerical value of the Hamiltonian) of the unperturbed pendulum problem (Eq. 1.14) at the separatrix:

$$\frac{\Delta \mathcal{E}}{\mathcal{E}_{sz}} = 4\pi\epsilon\lambda^3 e^{-\frac{\pi\lambda}{2}} \quad (1.19)$$

where ϵ is the relative strength of the perturbing term and λ is the ratio of the perturbing frequency to the frequency of small-amplitude librations. The chaotic zone may be exceedingly narrow for large λ , for example if the difference between frequencies of neighbouring resonance terms is large due to planetary oblateness. In such cases, the motion near the chaotic separatrix may be very close to that assumed by the single resonance theory, and the trajectory may cross this region in a time too short for the chaotic nature of the motion to manifest itself. However, the width of the chaotic separatrix depends exponentially on λ . If chaotic motion is important during transition of a mean-motion resonance, then there is no reason to believe that the single resonance formulae may be applied to the problem.

Large-scale chaos results when the chaotic separatrices surrounding different libration centers merge. The motion can be extremely irregular, with different resonance arguments alternately circulating and librating. The onset of large-scale chaos can be predicted using the resonance overlap criterion (Chirikov 1979), by estimating the point at which the unperturbed libration zones would overlap in the phase space.

Because of the presence of strong secular terms in some of the resonant interactions between Uranian satellites, the presence of significant chaotic behaviour may be predicted in another way. Wisdom (1985a) has demonstrated how in the 3:1 Kirkwood gap, secular terms drive a trajectory across the separatrices associated with the mean-motion resonances and cause large-scale chaotic behavior.

Chaotic behavior is interesting because of its extreme irregularity: the orbital parameters can change by large amounts during passage through a resonance. This is important in terms of explaining the large inclination of Miranda and the large eccentricities of the inner satellites. In addition, large chaotic variations of the eccentricities of satellites may have significantly affected their thermal histories. Interestingly enough, although the chaotic zones significantly affect the orbits of the satellites, the semimajor axes change by only a few tens of kilometers during evolution through the resonances.

1.4.3 Ergodic Adiabatic Invariants

The concept of an adiabatic invariant in a one degree of freedom Hamiltonian with slow time dependence can be generalized to higher dimensions. For a trajectory which is ergodic on the energy surface, it has been shown (Ott 1979) that the volume enclosed by the trajectory in phase space is an adiabatic invariant to order $\epsilon^{1/2}$, where ϵ is the slow rate at which the Hamiltonian energy changes. Numerical experiments (Brown *et al.* 1987a, b) have verified this result for ergodic billiard

motion. These authors have also shown that for ergodic trajectories which may temporarily mimic regular behaviour, for example due to the presence of neutrally stable equilibria, the error in the ergodic adiabatic invariant scales as $\sqrt{\epsilon \ln(\epsilon^{-1})}$. Brown *et al.* also considered a Hamiltonian system with small quasiperiodic islands in the phase space, and found that the ergodic adiabatic invariant scaled as $\epsilon^{\xi/2}$, $0 < \xi < 1$.

These results have important implications for the work presented here: they imply that for sufficiently slow tidal evolution rates, the dynamics in the chaotic zone should essentially be independent of the rate. In principle, the evolution of the Hamiltonian energy may be predictable if there is an ergodic adiabatic invariant. We have found evidence for ergodic invariants during evolution through the Uranian satellite mean-motion resonances: for sufficiently slow evolution rates, the evolution of the energy of the Hamiltonian appears to be very regular, even when the trajectory is chaotic.

1.5 Tidal Heating of Planetary Satellites

The importance of resonantly forced tidal heating to the thermal histories of satellites was demonstrated by Peale *et al.* (1979), in their dramatic prediction of extensive melting of the interior of Io and its possible consequences just prior to the discovery of active volcanism on that Jovian satellite by Voyager 1 (Morabito *et al.* 1979). Miranda and Ariel show evidence of significant resurfacing of material, but are not currently involved in resonant interactions which can account for this activity. This suggests that some of these satellites may have been tidally heated in the past. Chaotic evolution through mean-motion commensurabilities may have provided a mechanism for the generation of internal heat in satellites, without leading to permanent capture into a resonant interaction.

In the case of Io, the relatively high orbital eccentricity forced by the resonant interaction with Europa and Ganymede causes the tides raised on the satellite by Jupiter to vary during an orbit period. As the figure of Io adjusts to the changing tidal potential, friction heats the interior of the satellite. The orbital eccentricity is maintained near an equilibrium value by the opposing actions of two processes: the dissipation of this tidally generated heat, which tends to damp the eccentricity, and the tidal evolution of the orbit within the resonance, which tends to increase the eccentricity. Because the resonant interaction is regular, the rate of tidal heating in Io can be analytically estimated using an equilibrium model (*e.g.* Peale and Cassen 1978):

$$\frac{dE_i}{dt} = -\frac{21}{2} \frac{k_{2i} M n_i^3 R_i^5}{a_i^3 Q_i} e_i^2 \quad (1.20)$$

During passage through a resonance in which the orbital parameters vary chaotically, the mechanism of tidal dissipation in the satellites is more complicated: the tidal potential varies chaotically. The rate of tidal heating may vary considerably during passage through the resonance, so an equilibrium eccentricity is not neces-

sarily achieved. If the eccentricities increased to large values for significant periods of time during passage through a resonance, though, tidal heating may have had significant effects on the thermal histories of some of the satellites. We have found this to be possibly the case at the Miranda-Umbriel 3:1 commensurability (Chapter 3).

If the rotation axis of a satellite is not normal to the orbit plane, the tidal bulge raised by the planet is pulled above and below the equator as the satellite moves about its orbit. This also causes frictional heating in the satellite at a rate (Peale and Cassen 1978):

$$\frac{dE_i}{dt} = -\frac{3}{2} \frac{k_{2i} M n_i^3 R_i^5}{a_i^3 Q_i} \theta_i^2 \quad (1.21)$$

where θ_i is the obliquity of the spin axis of the satellite. This is probably not an important source of internal heat for the Uranian satellites (see Chapter 3).

Chapter 2

The Ariel-Umbriel 5:3 Commensurability

2.1 Introduction

In this chapter, we consider the Ariel-Umbriel 5:3 mean-motion commensurability. The 5:3 commensurability between Ariel and Umbriel would have been the most recently encountered of the first and second order commensurabilities between the Uranian satellites. For this commensurability there are three important mean-motion resonances involving the eccentricities and three involving the inclinations, as well as a strong secular coupling. The 5:3 resonance therefore promises to be quite interesting.

In Section 2.2, the planar-eccentric problem is considered. Restricting the resonant interactions to a planar problem allows us to reduce the system to two degrees of freedom, and to explore the phase space of the problem in detail using surfaces of section. It is found that there is a large chaotic zone associated with this commensurability, in which the eccentricities of the satellites may increase by a factor of two or three over the initial values. The chaotic zone serves as a bridge between the initial quasiperiodic motion and a quasiperiodic region of escape. Capture into the resonance occurs, even for chaotic trajectories, when the boundary of the surface of section defined by conservation of the instantaneous energy integral divides the

phase space into independent regions which do not allow circulation of one or both resonant arguments.

Section 2.3 considers the secular perturbations on the planar-eccentric problem due to Titania. It is found that the evolution is somewhat more complicated in this model than in the planar-eccentric two-satellite model, but that the essential qualitative features of the motion are preserved.

In Section 2.4, the full three-dimensional two satellite resonant problem is considered. Both the eccentricities and the inclinations vary chaotically. The evolution of the eccentricities is qualitatively similar to that in the planar-eccentric model. The average inclination after escape from the resonance may differ from the average value before the resonance was encountered by a factor of two or so.

Section 2.5 discusses physical applications of these results. Final eccentricities of escaping trajectories are consistent with the present values. Eccentricities high enough to have had a significant effect on the thermal history of Ariel have not been found so far in this study: maximum tidal heating rates are comparable to the current rate of heating due to the decay of radioactive elements.

2.2 The Planar-Eccentric Problem

We consider the Ariel-Umbriel 5:3 resonant interaction in the planar approximation. The restriction to the planar problem has been made in an effort to get some understanding of the dynamical mechanisms involved in the passage through such a complicated resonance. The planar system has the advantage of being reducible to two degrees of freedom, which allows the detailed study of the phase space using the Poincaré surface of section technique. This technique has been found to be invaluable for understanding the qualitative features of motion of asteroids in the 3:1 Kirkwood gap (Wisdom 1985a).

In Section 2.2.1, the resonant coordinates of the system are discussed. Since the masses of the satellites are comparable, a treatment of the full resonant three-body problem is required. The evolution of the system is studied with the aid of an algebraic mapping with the same resonant structure as this three-body problem, analogous to that developed by Wisdom (1982, 1983) for the restricted problem to study resonant asteroid motion. The Hamiltonian and its derivation are given in Appendix I, and the map and its derivation are given in Appendix II.

In Section 2.2.2 some aspects of the dynamics of the 5:3 Ariel-Umbriel mean-motion commensurability are exhibited. Surfaces of section for this system reveal the presence of significant chaotic zones, even for relatively low eccentricities. A pair of islands on the surfaces of section may be associated with the normal modes of the linear secular theory. The development of large-scale chaotic zones is associated with the fixed point associated with one of these modes becoming unstable.

Section 2.2.3 explores the evolution of the system through the resonance with small tidal dissipation included. Results are compiled for five numerical runs with distinct families of initial conditions. In each case large numbers of trajectories have been followed to assess the probabilities of various outcomes. It is found that the satellites can be driven to relatively high eccentricities in the chaotic zone,

and still escape from the resonance. The maximum eccentricities during passage through resonance are always found to be higher than the initial values, and occur during the chaotic phase of the resonance encounter. The mechanism of capture is markedly different from that of the isolated resonance theory. In the single resonance picture capture or escape occurs depending on the phase of the trajectory as the regular separatrix is encountered. For the 5:3 commensurability the “decision” of capture or escape occurs while the motion is chaotic. The trajectory approaches the resonance from a quasiperiodic region of phase space in which the resonant arguments circulate. There is another quasiperiodic circulation region on the other side of the resonance, into which trajectories may escape. In the resonance region the phase space is dominated by a large chaotic zone. The chaotic zone acts as a bridge between these two quasiperiodic regions. If, during the period of chaotic evolution, the trajectory becomes trapped in the second region of quasiperiodic circulation, the trajectory escapes from the resonance. If, on the other hand, the trajectory remains in the chaotic zone, then the energy surface eventually divides the phase space into two regions in which libration of one or more resonant arguments must occur. At this point capture into resonance has occurred, *even though the trajectory may still be chaotic*. Upon further evolution those trajectories which were chaotic at the point of capture eventually become quasiperiodic librators. The probability of escape depends not only on the eccentricities of the satellites far from the resonance, which vary considerably due to the strong secular interaction, but also on the initial distribution of energy between the two secular modes. The coupled nature of the problem makes comparison with the standard single resonance model of evolution through resonances impossible. Even qualitative features of passage through resonance differ from the picture developed for passage through isolated resonances. For example, the average eccentricity of Umbriel after escape from the resonance can be higher than the average value before the resonance is encountered.

In the standard theory, the eccentricity after escape is always smaller.

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2.2.1 Resonance Coordinates

In the planar-eccentric problem, all terms in the disturbing function involving inclinations are set equal to zero. Since the inclinations are less than one degree, this is a reasonable first approximation. Recall, though, that taking into account the inclinations in the 3:1 asteroid-Jupiter-Sun problem significantly enhanced the variations in the eccentricities (Wisdom 1983, 1987).

We are interested in studying changes in the osculating eccentricities and longitudes of pericenter of the orbits caused by passage through the resonance. These changes occur over timescales which are generally much longer than the orbit periods. The major contributions to these changes in the elements are resonant interactions between Ariel and Umbriel involving the slowly-varying combination of mean longitudes $5\lambda_U - 3\lambda_A$, and the secular interactions between the satellites. Here and in the following development, the subscript A refers to Ariel, and the subscript U to Umbriel.

The secular interactions of the Uranian satellites are relatively strong (Dermott and Nicholson 1986, Laskar 1986). However, the largest contributions to the secular variations of Ariel and Umbriel arise from their mutual interaction. Initially, we have ignored the secular perturbations due to the other satellites. With this approximation the problem remains a two degree of freedom problem, allowing a detailed study of the complicated dynamics with the surface of section technique.

Near the resonance, the evolution of e_i and ϖ_i is dominated by the low frequency perturbations, with frequencies associated with changes in the resonant combination of longitudes $5\lambda_U - 3\lambda_A$ and the longitudes of pericenter ϖ_i . The high frequency contributions associated with the motions of λ_A and λ_U , and with other non-resonant combinations of the mean longitudes, are removed in first order by averaging.

The planar-eccentric model is developed in Appendix I. As resonance coordinates

we have chosen

$$\begin{aligned}\sigma_A &= \frac{1}{2}(5\lambda_U - 3\lambda_A - 2\varpi_A) \\ \sigma_U &= \frac{1}{2}(5\lambda_U - 3\lambda_A - 2\varpi_U)\end{aligned}\tag{2.1}$$

which together with λ_A and λ_U form a complete set of generalized coordinates. The momenta conjugate to σ_A and σ_U are, in terms of the Delaunay momenta $L_i \approx m_i \sqrt{GM a_i}$ and $G_i = L_i \sqrt{1 - e_i^2}$ (see Plummer 1960):

$$\begin{aligned}\Sigma_A &= L_A - G_A \approx \frac{L_A}{2} e_A^2 \\ \Sigma_U &= L_U - G_U \approx \frac{L_U}{2} e_U^2\end{aligned}\tag{2.2}$$

This choice of variables results in two integrals of motion, the momenta conjugate to $\gamma_A = \lambda_A$ and $\gamma_U = \lambda_U$, since in the new variables the resonant Hamiltonian is cyclic in these variables. The resonance integrals are:

$$\begin{aligned}\Gamma_A &= L_A + \frac{3}{2}(\Sigma_A + \Sigma_U) \\ \Gamma_U &= L_U - \frac{5}{2}(\Sigma_A + \Sigma_U)\end{aligned}\tag{2.3}$$

For low eccentricities, the coefficient in the Hamiltonian (Eq. 7.5) $A \approx \frac{1}{4}(5n_U - 3n_A)$. Near the 5:3 mean-motion resonance, this quantity is near zero. Tidal dissipation in the planet causes the orbits of the satellites to expand differentially (Goldreich 1965). Because of this differential expansion, the mean-motion ratio decreases (see section 1.3). Therefore, the quantity $(5n_U - 3n_A)$ increases as energy is dissipated in the planet, and changes sign when the mean motions are exactly commensurate. This provides a convenient measure of distance from the resonance. We define the parameter $\delta = 4A + 2(C + D)$ to be the resonance parameter in this problem (see Eq. 7.9). At small eccentricities it is proportional to the non-resonant contribution to $5n_U - 3n_A - \dot{\varpi}_A - \dot{\varpi}_U$, and it changes sign in the middle of the resonance region. The other coefficients are proportional to the semimajor axis ratio as

it appears in the Leverrier coefficients (see Appendix I). The change in semimajor axis ratio as the resonance is traversed is of order one part in a thousand, while the fractional change in δ is of order unity. The changes in the numerical values of the coefficients are therefore small compared to the changes in δ and consequently will be neglected.

The expressions for the coefficients and their values are given in Appendix I (Section 7.1.1).

2.2.2 Resonance Dynamics

Surfaces of Section

In problems such as that considered by Hénon and Heiles the presence of large-scale chaos is difficult to predict with the resonance overlap criterion or other methods. The Hamiltonian (7.9) is quite similar to the Hénon-Heiles Hamiltonian in this respect; it is a bit too complicated to rely on a simple criterion such as the resonance overlap criterion. In order to securely determine the extent of chaotic behavior, it is necessary to resort to numerical methods. For a system with two degrees of freedom, it is possible to study the structure of phase space using the Poincaré surface of section technique (see Hénon and Heiles, 1964). The basic idea of the surface of section is to study the intersections of a trajectory with a two-dimensional plane through the phase space, rather than the full four-dimensional phase space. These intersections reveal the qualitative character of the trajectories, i.e. whether they are chaotic or quasiperiodic. For quasiperiodic trajectories successive intersections will fall on smooth curves; for chaotic trajectories successive intersections appear to fill an area in an irregular manner.

Two sections have been chosen for study in this paper: plotting y_A vs. x_A when $x_U = 0$, which we designate section I, and plotting y_U vs. x_U when $x_A = 0$, which we designate section II. On these plots, the radial distance from the origin is proportional to eccentricity, and the resonance variables σ_i are polar angles. The choice of these section conditions simplifies the determination of initial conditions on the section.

A point on a section defined by a condition such as $x_i = 0$ does not necessarily correspond to a unique trajectory. It is useful to add further conditions on the section so that this will be the case. For instance, in the Hénon-Heiles (1964) problem, the usual surface of section is to plot p_y versus y whenever $x = 0$. Since the Hénon-Heiles Hamiltonian is quadratic in the momenta, each point on the section

corresponds to two possible values of p_x . Points on the section will correspond to unique trajectories if an additional condition, say that p_x be non-negative, is added. Our problem is more complicated. The Hamiltonian is quartic in the x_i and y_i . Due to the quartic nature of the Hamiltonian, for given values of δ , the energy, and the coordinates on the subspace defining the section, there can be two or four values of y_i conjugate to that x_i for which the section condition $x_i = 0$ has been chosen. It is desirable therefore, to further constrain the surface of section as belonging to one of four root “families.” When there are four roots these families are labelled a, b, c, d in order of decreasing numerical value. When there are only two roots they are labelled a and d in order of decreasing numerical value. As indicated by the nomenclature these two families join continuously the corresponding families in the four root case. Root families a and d will be referred to as the “outer” pair and root families b and c will be referred to as the “inner” pair. The sections for these particular root families will be referred to as Ia, Ib, Ic, Id , and IIa, IIb, IIc , and IId .

Inspection of the Hamiltonian (7.9) allows us to determine regions in the δ -energy parameter space in which these root families may be found. For energies less than zero, only two real roots of the quartic equation exist. This is the outer pair. For energies greater than zero and $\delta > -2(C - D) + 4F$, the inner pair of root families exists for points on both sections near the origin.

For energies greater than

$$E_1 = \begin{cases} 0, & \delta < 2(C - D) + 4H \\ -(\delta - 2(C - D) - 4H)^2 / 64B, & \delta \geq 2(C - D) + 4H \end{cases} \quad (2.4)$$

there are no real roots for points at the origin of sections I. The boundary of the energy surface on the section divides it into two separate regions, and σ_A must librate on sections I. In this case libration on the section corresponds to an actual libration of σ_A in the full phase space.

Similarly, for energies greater than

$$E_2 = \begin{cases} 0, & \delta < -2(C - D) + 4F \\ -(\delta + 2(C - D) - 4F)^2 / 64B, & \delta \geq -2(C - D) + 4F \end{cases} \quad (2.5)$$

σ_U must librate on sections II and in the full phase space (see Fig. 2.6). Note that $E_1 > E_2$.

The most interesting dynamical behavior occurs for energies near E_1 and E_2 . To more easily display the results of our calculations we introduce the new parameter $\Delta E = \mathcal{E} - E_2$, where $\mathcal{E}$ is the numerical value of the Hamiltonian. ΔE naturally reflects the types of motion possible on sections. For $\Delta E \geq 0$, σ_U must librate; for $\Delta E \geq E_1 - E_2$, both σ_A and σ_U must librate. For $0 > \Delta E \geq -E_2$, the resonance variables may circulate or librate, and there are four root families. For $\Delta E < -E_2$ (equivalently $\mathcal{E} < 0$), there are only two root families, and the resonance variables circulate. This is summarized in Fig. 2.1. The shaded regions in the upper left corners of each plot are regions of $\delta, \Delta E$ in which there are no real roots for either subspace, and are therefore forbidden regions.

Far from the resonance, the resonant combination of mean longitudes $5\lambda_U - 3\lambda_A$ circulates rapidly, and the interaction is dominated by the secular interaction. The terms in (Eq. 7.5) containing the angles $2\sigma_A$, $2\sigma_U$, and $\sigma_A + \sigma_U$ oscillate rapidly; the angle $\sigma_A - \sigma_U = -(\varpi_A - \varpi_U)$ varies slowly. If a canonical transformation to new variables which are the sum and difference of σ_A and σ_U is made, the combination $\Sigma_A + \Sigma_U$ is conjugate to the rapidly circulating sum and is therefore approximately conserved. After removing the high-frequency resonant terms by averaging, the new Hamiltonian becomes

$$\mathcal{H}' = \frac{1}{4}(\delta + 2(C - D))(x_A^2 + y_A^2) + \frac{1}{4}(\delta - 2(C - D))(x_U^2 + y_U^2) + E(x_A x_U + y_A y_U) \quad (2.6)$$

which is the linear secular Hamiltonian for these two satellites in the resonance coordinates. For a given value of the Hamiltonian, a point on a surface of section defined by $x_i = 0$ can have either of two values of the conjugate coordinate y_i , which

Figure 2.1: $\delta, \Delta E$ parameter space at two scales. Crosses indicate parameters of the surfaces of section shown in this paper. Various types of possible motion: QC – quasiperiodic circulation, QL – quasiperiodic libration, CC – chaotic circulation, CL – chaotic libration. S_1 is the energy of the separatrix associated with the Ariel (e_A^2) resonance, and S_2 is the energy of the separatrix associated with the Umbriel (e_U^2) resonance. For energies greater than E_2 , σ_U must librate. For energies greater than E_1 , both σ_U and σ_A must librate.

may be found by solving a quadratic equation. The points on the section are made to correspond to unique trajectories by specifying which of the two root families is being plotted. These two root families correspond to the outer pair of roots in the more complete resonance problem.

This Hamiltonian can be diagonalized and normal modes found, yielding the standard Lagrange solution of the secular problem (see e.g. Brouwer and Clemence 1961). The solution is of the form:

$$\begin{aligned}x_i &= A_{i1} \cos(\alpha_1 t + \beta_1) + A_{i2} \cos(\alpha_2 t + \beta_2) \\y_i &= A_{i1} \sin(\alpha_1 t + \beta_1) + A_{i2} \sin(\alpha_2 t + \beta_2)\end{aligned}\tag{2.7}$$

where α 's are the eigenfrequencies.

Periodic orbits are found wherever the amplitude of one of the modes is zero, and the eccentricities are constant. In these cases, the solutions are of the form:

$$\begin{aligned}x_A &= A_{Aj} \cos(\alpha_j t + \beta_j) \\y_A &= A_{Aj} \sin(\alpha_j t + \beta_j) \\x_U &= A_{Uj} \cos(\alpha_j t + \beta_j) \\y_U &= A_{Uj} \sin(\alpha_j t + \beta_j)\end{aligned}\tag{2.8}$$

If the section condition is, for instance, $x_U = 0$, then $\alpha_j t + \beta_j = \pi/2$ or $3\pi/2$, and therefore $x_A = 0$ also. There are two fixed points on the section, with $x_A = 0$ and $y_A = \pm A_{Aj}$. The “roots,” the values of the coordinate conjugate to the section variable $x_U (= 0)$ for the fixed points on the section, are $y_U = \pm A_{Uj}$, one root in family *a* and one root in family *d*. There are two such fixed points associated with the other eigenmode. We therefore expect, that for a given section condition, there are four fixed points on the *y*-axis, two of which belong to root family *a* and two to root family *d*. Thus sections Ia, Id, IIa, and IId each have two fixed points in the linear secular problem.

Representative surfaces of section for the full 5:3 resonance problem before the resonance is encountered are shown in Figs. 2.2 and 2.3. Figs. 2a,d display sections Ia, d respectively and Figs. 2.3a,d display sections IIa, d respectively. The same trajectories are plotted on all figures. As expected, on each section there are two fixed points corresponding to each pure mode, each surrounded by concentric invariant curves. Note that the regions dominated by each mode are not periodic islands associated with a resonance phenomenon, and they are not divided by an infinite period separatrix with an unstable equilibrium. The position of a curve on the section depends on the relative strengths of the two eigenmodes. For a particular trajectory points alternately appear on the sections corresponding to the two root families. For instance the points forming the small loop at the top of Fig. 2.3a and the small loop at the bottom of Fig. 2.3d were generated by the same trajectory. Similarly, the big loops on Figs. 2.3a and 2.3d were generated by the same trajectory. Each section is dominated by one of the modes. The other mode appears on the sections near the boundary of the energy surface. Note that even though the resonance angle σ_i may not encircle the origin on the section for a particular trajectory, the resonance variable is not librating. The apparent libration on the section is simply a matter of one resonance variable being strobed by the other. We designate the mode dominating sections I as Mode I, and that dominating sections II as Mode II.

The surfaces of section in Figs. 2.4 and 2.5 show more interesting behavior. The same trajectories on both sections Ia – d (Fig. 2.4) and sections IIa – d (Fig. 2.5) are shown. For these parameters all four root families exist. All sections for all root families continue to show the bimodal structure, but now the modes are separated by a large chaotic region. The fixed point at the center of the large quasiperiodic island on sections Ia and Id is part of a continuous family of periodic orbits which becomes Mode I far inside the resonance region ($\delta \ll 0$). Similarly, the fixed

Figure 2.2: Surfaces of section showing quasiperiodic secular normal modes, at $\delta = 1.4263$, $\Delta E = -2.1626 \times 10^{-5}$. (a): section Ia (largest quartic root family). (d): section Id (smallest quartic root family). The Mode I fixed point occupies the central region of both figures. The Mode II fixed point occupies the small regions close to the energy surface boundary, near the bottom of section Ia and near the top of section Id.

Figure 2.3: Surfaces of section showing quasiperiodic secular normal modes, at $\delta = 1.4263$, $\Delta E = -2.1626 \times 10^{-5}$. (a): section IIa (largest quartic root family). (d): section IIId (smallest quartic root family). The curves shown in these figures are generated by the same trajectories as generated the curves in Fig. 2.2. The Mode II fixed point occupies the central region of both figures. The Mode I fixed point occupies the small regions close to the energy surface boundary, near the top of section Ia and near the bottom of section Id.

point at the center of the large quasiperiodic island on sections *Ib* and *Ic* is part of a continuous family of periodic orbits which becomes Mode I far outside the resonance region ($\delta \gg 0$). The quasiperiodic zone surrounding these latter islands is the region of phase space into which trajectories escape from the resonance. We continue then to refer to the islands surrounding these fixed points as Mode I. Trajectories in Mode I on sections *Ia* and *Id* alternately visit each of these sections. In a similar manner, Mode I trajectories on sections *Ib* and *Ic* alternately visit each of the sections. No Mode I trajectory generates points on all the sections corresponding to all four root families. For chaotic trajectories successive points appear on the sections corresponding to all 4 root families in a typically chaotic manner. The small islands in the lower parts of Figs. 2.4a and 2.4b (sections *Ia* and *Ib*) are alternately visited by the trajectories which generate them. The periodic orbits at the center of these islands are associated with Mode II, but in a more complicated way than was the case for Mode I (see next Section). A similar pair of small islands exists on sections *Ic* and *Id*, but trajectories belonging to these islands were not computed on the figure. Trajectories on these small islands librate in σ_A and may librate in σ_U . Mode I trajectories do not librate. The surfaces of section in Fig. 2.5 show the same features as in Fig. 2.4, but in Umbriel variables (sections II). The large features in Fig. 2.5 were generated by the same trajectories as generated the large features in Fig. 2.4. The small islands in both Figs. 2.4 and 2.5 were generated by the same trajectories.

The surfaces of section in Fig. 2.6 show the same trajectories as in Fig. 2.5, but this time computed using the averaged Hamiltonian (Eq. 7.9) with a relative precision of 10^{-9} per time step. The correspondence between trajectories is excellent, indicating that the mapping represents the Hamiltonian (Eq. 7.9) well. The slight difference in appearance of the chaotic zone is expected, since using a different integrator is equivalent to beginning with a slightly different initial set of coordinates.

Figure 2.4: Sections I in the region of parameter of space in which large-scale chaotic behavior is possible, at $\delta = 2.258$, $\Delta E = -8.78 \times 10^{-6}$. Quartic root families are ordered (a) - (d) according to numerical value. Much of the region surrounding the Mode II fixed point on sections Ia and Id in Fig. 2.2 has become chaotic. Escape from the resonance occurs in the large quasiperiodic zone in roots (b) and (c). The fixed point of this large quasiperiodic region becomes Mode I far from the resonance. The small quasiperiodic curve near the bottom of section Ia and the small quasiperiodic curve near the bottom of section Ib are generated by a trajectory for which σ_A librates.

Figure 2.5: Sections II in the region of parameter space in which large-scale chaotic behavior is possible, at $\delta = 2.258$, $\Delta E = -8.78 \times 10^{-6}$. Quartic root families are ordered (a) - (d) according to numerical value. The curves shown in these figures are generated by the same trajectories as generated the curves in Fig. 2.4. Escape from the resonance occurs in the large quasiperiodic zone in roots (b) and (c). The small quasiperiodic curves in the midst of the chaotic zone on sections IIc and II d are generated by the same trajectory that generates the small quasiperiodic curves in Figs. 2.4a and 2.4b, for which σ_A librates.

In a chaotic region, two trajectories with slightly different initial coordinates diverge exponentially with time. While the detailed evolution within the chaotic zone will differ, the qualitative behavior is the same. In the long run the same region of phase space will be filled.

The surface of section in Fig. 2.7 is at $\Delta E = 0$, computed with the mapping. The energy surface has split the section into two independent regions, and σ_U librates for trajectories in each region, independently of the quasiperiodic or chaotic character of the trajectory. The libration is forced by the topology of the energy surface. Successive appearances of points on the section are confined to one region or the other. At still higher ΔE , the two regions move apart, and the large chaotic zone disappears. The forced libration of chaotic trajectories is a new feature of this problem. More usually, the chaotic zone occurs at the boundary between libration and circulation, and thus a chaotic trajectory alternately librates and circulates. This new feature provides a new mechanism for capture into resonance. *Trajectories may be captured into resonance while they are still chaotic.*

Fig. 2.8 shows results of a computation of the maximum Lyapunov characteristic exponent λ (see Chapter 1). The Lyapunov characteristic exponent provides a measure of the exponential divergence of trajectories in chaotic zones. Trajectories from two different regions of parameter space have been represented: one in the large chaotic zone shown in Figs. 2.4, 2.5 and 2.6 and one in the large chaotic zone in the librating region shown in the lower half of Fig. 2.7. In both cases, λ approaches a value $\approx 0.03 \text{ yr}^{-1}$. This corresponds to an e-folding time for the distance function of about 30 years, which is comparable to the secular evolution timescale.

Onset of Chaos

The lightly shaded regions in the lower right of Fig. 2.1 show the parameter

Figure 2.6: Sections II in the region of parameter space in which large-scale chaotic behavior is possible, at $\delta = 2.258$, $\Delta E = -8.78 \times 10^{-6}$, computed using *differential equations* for the averaged planar Hamiltonian. Quartic root families are ordered (a) - (d) according to decreasing numerical value. The curves shown in these figures have the same initial coordinates as the curves in Fig. 2.5.

Figure 2.7: Sections II at $\delta = 3.5$, $\Delta E = 0$, on the boundary in parameter space above which capture into the resonance occurs. (a): section IIa (largest quartic root family). (d) section II d (smallest quartic root family). σ_U librates in the full phase space for all trajectories on these plots. All trajectories shown, even chaotic trajectories, have been captured into the resonance.

Figure 2.8: Maximum Lyapunov characteristic exponents for trajectories in the chaotic zones in the previous figures. The unit of time is one year. The upper curve at low t was computed for a trajectory in the chaotic zone in the lower libration region shown in Fig. 2.7, and the lower curve at low t was computed for a trajectory in the chaotic zone in Figs. 2.4 and 2.5.

regions where “macroscopic” chaotic zones are accessible some place on the surface of section. The extent of the region was determined by evaluating the rate of growth of distance between two nearby trajectories for 20 initial conditions evenly spaced along the y-axis between the energy boundary limits on a sections II for a given δ and ΔE — if one of these was found to indicate chaos, the section is considered to have a “macroscopic” chaotic zone. Note that the two single resonance separatrix energies, denoted S_1 and S_2 , run through the middle of the chaotic zone (see Appendix III).

The onset of large-scale chaos is associated with an instability of the Mode II fixed point. A tidally evolving system with increasing δ will encounter the region in δ - ΔE parameter space in which large-scale chaos is present on the surfaces of section from the lower left of Fig. 2.1. Near the boundary of this region of large-scale chaos the fixed point of Mode II goes through two period-3 bifurcations (see e.g. Meyer 1970, Hénon 1970). These bifurcations are associated with a 3:1 resonance between the degrees of freedom in the problem. After these bifurcations the nature of the fixed points has changed, with the result that σ_A librates in the small quasiperiodic island surrounding the Mode II fixed point. Before the bifurcations σ_A circulated in Mode II.

Fig. 2.9 illustrates these bifurcations on section IIb. The changes in δ and ΔE in the series 2.9a - 2.9e simulate possible variations during tidal evolution (see next Section). These surfaces of section therefore illustrate the changes in the structure of the phase space available to a trajectory as it crosses the boundary. Successive appearances of points on the section near the Mode II stable fixed point before the bifurcation (Fig. 2.9a) alternate between sections IIb and IIc, and σ_A and σ_V circulate. As the parameters evolve towards larger δ and ΔE , the region of stability near the Mode II fixed point shrinks, the extent of the chaotic zone in phase space increases, and eventually the Mode II fixed point becomes unstable (Fig. 2.9b). After the first bifurcation, a stable island appears around the fixed point

(Fig. 2.9c), but now the points alternate between sections IIa and IIb. σ_A librates for trajectories within these islands. A second period-3 bifurcation occurs (Figs. 2.9d-e). Note that the orientation of the roughly triangular trajectory surrounding the stable fixed point has inverted, a characteristic feature of this type of bifurcation (see e.g. Hénon 1966, 1969, 1970). Again, the points alternate between sections IIa and IIb, and σ_A librates. The trajectories generating these islands librate in σ_A and, as tidal evolution continues, eventually in σ_U around $\pi/2$ or $3\pi/2$. (A similar pair of islands appears on sections IIc and IId.) Note that throughout the sequence of Figs. 2.8a-e the region surrounding the Mode I fixed point remains stable.

Another qualitative change in phase space concerns the accessibility of the root families. Before the bifurcation, trajectories generate points alternately on the sections corresponding to the “outer” pair of root families (*a* and *d*) or the “inner” pair (*b* and *c*). As the first bifurcation occurs, the large chaotic zone appears around the fixed point on the section corresponding to the periodic orbit associated with Mode II, which has become unstable. Successive points generated by a chaotic trajectory appear on the sections corresponding to all four root families. Therefore, the appearance of the chaotic zone supplies a dynamical mechanism by which trajectories which generate points on sections corresponding to one pair of root families gain access to the sections corresponding to the other pair of root families. This is important in the mechanism of escape from the resonance, as seen in the next Section.

For very low eccentricities, the coupling between resonances is weakened, and one might expect to observe behavior characteristic of a perturbed single resonance problem. Fig. 2.10 illustrates this for values of δ and ΔE near those where the Umbriel resonance is first encountered in the single resonance approximation (see Appendix III). There is a relatively narrow chaotic zone with the characteristic “figure 8” shape of the separatrix in the single resonance case.

Figure 2.9: Period-3 bifurcation on section Πb . (a): $\delta = 2.066$, $\Delta E = -9.34 \times 10^{-6}$, (b): $\delta = 2.09$, $\Delta E = -9.1 \times 10^{-6}$, (c): $\delta = 2.11$, $\Delta E = -8.9 \times 10^{-6}$, (d): $\delta = 2.13$, $\Delta E = -8.7 \times 10^{-6}$, (e): $\delta = 2.15$, $\Delta E = -8.5 \times 10^{-6}$. This corresponds to a 3:1 resonance between the degrees of freedom.

Figure 2.10: Sections II for low eccentricities, $\delta = -0.32$, $\Delta E = -1.0 \times 10^{-9}$. (a): section IIa (largest quartic root family). (d) section IIId (smallest quartic root family). The chaotic zone is characteristic of a perturbed separatrix.

2.2.3 Evolution through the Resonance

Tidal Dissipation

The results of the previous Section illustrate the behavior of the Hamiltonian (7.9) for fixed values of the parameters δ and ΔE . This Section discusses the tidal evolution through the resonance. Tidal friction enters in two distinct ways. The principal modification of the model is that the coefficients of the Hamiltonian become time-dependent. This is primarily due to the differential tidal expansion of the orbits. There is also the direct action of tidal friction in the satellites on the degrees of freedom. Thus tidal friction gives rise to time-dependent coefficients as well as explicit friction terms in the equations of motion.

Of the various coefficients in the resonance Hamiltonian, only δ depends strongly on the resonant combination of mean-motions; during passage through the resonance, the fractional change in the parameter δ is large compared to the fractional changes in the other coefficients. The other coefficients will therefore be taken to be constant. In addition, the value of δ is influenced by the damping of eccentricities due to tidal dissipation in the satellites. The time rate of change of δ is

$$\dot{\delta} \simeq 4\dot{A} \quad (2.9)$$

From the expansion of A to first order in Σ_i :

$$\begin{aligned} \dot{\delta} &\simeq \frac{d}{dt} [(5n_U - 3n_A) - 16B(\Sigma_A + \Sigma_U)] \\ &= (5\dot{n}_U - 3\dot{n}_A) - 16B(\dot{\Sigma}_A + \dot{\Sigma}_U) \end{aligned} \quad (2.10)$$

The most significant terms in the above expression are (see Chapter 5 for a similar, more complete analysis):

$$\dot{\delta} = -3\dot{n}_A - 16B\dot{\Sigma}_A \quad (2.11)$$

and therefore

$$\dot{\delta} = -3\dot{n}_{Ap} \left\{ 1 - \frac{35}{2} D_A e_A^2 \left[1 + \frac{5}{3} \frac{m_A}{m_U} \left(\frac{a_A}{a_U} \right)^2 \right] \right\} \quad (2.12)$$

where $D_A = 60.4$ for $Q = 6600$ and $Q_A = 100$ (see Eq. 1.11).

δ will be stationary at the equilibrium eccentricity

$$e_A = \left\{ \frac{1}{\frac{35}{2} D_A \left[1 + \frac{5}{3} \frac{m_A}{m_U} \left(\frac{a_A}{a_U} \right)^2 \right]} \right\}^{1/2} = \sqrt{\frac{1}{33.1 D_A}} = 0.022 \quad (2.13)$$

For the current eccentricity of Ariel the $\dot{n}_A$ term is dominant in the expression for $\dot{\delta}$. The $\dot{\Sigma}_A$ term becomes comparable to the $\dot{n}_A$ term only for eccentricities approaching the equilibrium value, although it is not clear whether an equilibrium state can be reached if the orbits are chaotic. Since we are studying the evolution for systems with initial eccentricities much less than this, we have only retained the $\dot{n}_A$ term in the expression for $\dot{\delta}$. However, if large increases in the eccentricity result from resonant interactions between the satellites, this approximation will require some reconsideration. In our approximation, then,

$$\dot{\delta} \approx \frac{9G^2 M^2 m_A^3 \dot{a}_A}{2\Gamma_A^3 a_A} \quad (2.14)$$

Tidal friction also introduces explicit frictional terms into the equations of motion. The most important frictional terms arise from the damping of the orbital eccentricities from dissipation within the satellites. From Eq. 1.9, with rigidities $\mu_A = \mu_U \approx 10^{11} \text{ dynes/cm}^2$ (see Table 1.2), the eccentricity damping timescale for Ariel is about $1.5 \times 10^6 Q_A$ years, and that for Umbriel is about $1.4 \times 10^7 Q_U$ years. For $Q_A \approx Q_U \approx 100$, these timescales are on the order of 10^8 years and 10^9 years, respectively. Whether $\dot{e}_i$ is important in our study depends on the timescale of resonance passage compared to the eccentricity damping timescale. The timescale of resonance passage depends in turn on the rate of orbital evolution, which is determined by the specific dissipation function Q of Uranus, as discussed below.

The value of Q can be constrained by considering reasonable scenarios for the orbital evolution of the satellites of Uranus (see Section 1.3). The rate at which the system evolves is inversely proportional to Q . By extrapolating the evolution

backwards in time, and eliminating unreasonable configurations of the satellites from the present to the time of formation of the solar system, it is possible to find upper and lower limits to the possible values Q can take. A lower limit $Q \approx 6600$ places Miranda and Ariel at the same distance from Uranus at the time of formation of the solar system. This study is only relevant if the Q of Uranus is less than about 100,000, since this Q places Ariel and Umbriel at the 5:3 resonance at the time of formation of the solar system. Higher values of Q are possible, but then Ariel and Umbriel would never have encountered the 5:3 resonance.

Whether or not $\dot{e}_i$ is important depends on Q . The maximum time of resonance passage would occur for the maximum $Q = 100,000$. We have assumed that during the passage through resonance δ varies linearly with time due to the decrease of n_A . The change in δ during resonance passage is of order $1 - 3 \text{ yr}^{-1}$. This indicates that the maximum time of resonance passage is less than approximately 3.1×10^8 years. As discussed above, the tidal damping timescales of the satellites are approximately 10^8 years for Ariel and 10^9 years for Umbriel. These timescales are comparable to the maximum time of resonance passage. For the maximum value of Q , then, direct tidal damping of eccentricities will be important. The rate of evolution is inversely proportional to the value of Q assumed for Uranus. Therefore, for values of Q near the minimum allowed Q , the timescale of resonance passage will be much shorter (less than $\approx 2 \times 10^7$ years), and damping of eccentricities should have a much smaller effect on the dynamics of the system. The direct tidal damping of e_i could be important, but we neglect it in this initial study.

The numerical study of this system is constrained by finite computer resources. Unfortunately, even the lower limit of Q discussed above requires enormous expenditures of computer time to make a satisfactory study of the evolution of trajectories, even with the algebraic mapping. It is necessary, therefore, to artificially increase the rate of tidal evolution of the system above the maximum limit imposed by

the dynamics of the Uranian satellite system. In the single, integrable resonance model of passage through resonance the outcome of the resonance encounter does not depend on the rate of passage through the resonance as long as the passage is sufficiently slow. The single resonance formulae only require that the fractional change in the libration frequency over a libration period be much less than unity. For the problem under consideration here, where the dynamics and mechanism of passage through resonance are different than those of the single resonance theory, it is no longer clear that the outcome of resonance passage will become independent of the rate of passage for sufficiently slow passage (However, recent work on ergodic adiabatic invariants (see Section 1.4.3) indicate that this may be possible for chaotic systems). It is however plausible that this should be the case, whenever the timescale for resonance passage becomes much greater than all dynamical timescales. In any case, in numerical studies of passage through complicated resonances it is not satisfactory to simply choose an arbitrary rate of passage through resonance and hope that it is adequately slow. The burden of showing that the results of numerical experiments of passage through resonance reflect the true dynamics rests with the experimenter. As we illustrate below, the rate of passage of through the 5:3 Ariel-Umbriel resonance must be extremely slow in order to avoid artifacts.

We parameterize the rate of evolution in our numerical model by the effective specific dissipation function of Uranus. We designate this effective specific dissipation function by the script $\mathcal{Q}$, in order to emphasize that it is the parameter in our numerical model and not the physical Q of Uranus. With limited computer resources we must increase the rate of evolution determined by $\mathcal{Q}$ as much as possible without affecting the outcome of the resonance passage. In order to draw conclusions about the physical system we must show that the dynamics of passage through resonance become independent of $\mathcal{Q}$ for sufficiently large $\mathcal{Q}$.

To illustrate the effect of $\mathcal{Q}$ on the dynamics of the full resonant problem, Fig. 2.11 displays the probability of escape from the resonance against dissipation rate. The probability of escape was determined by integrating sets of 100 trajectories through the resonance with different dissipation rates (see next subsection for detail of initial conditions). The rate of dissipation increases to the right in the figure, and the value of $\mathcal{Q}$ decreases. The minimum physically reasonable value of $Q = 6600$ corresponds to the minimum dissipation rate shown. The choice of $\mathcal{Q}$ apparently can have a significant effect on the results: there is a dependence of escape probability on rate, especially for high dissipation rates (small $\mathcal{Q}$). Most of the runs described in this paper have used $\mathcal{Q} = 110$, or 60 times the minimum physical dissipation rate. This is the minimum dissipation rate for which a data point is shown in Fig. 2.11. This value of $\mathcal{Q}$ corresponds to $\dot{a}_A/a_A \simeq 1.4 \times 10^{-11}$ per orbit period during passage through the resonance. For somewhat larger dissipation rates, the measured escape probability is similar to that for $\mathcal{Q} = 110$, indicating that changes in $\mathcal{Q}$ near this value do not greatly affect the dynamics. It is plausible that the numerical experiments at this $\mathcal{Q}$ reflect the dynamics at the physical Q . Limitations on available computer time have not allowed us to further verify this numerically. However, trajectories evolved through the resonance with $\mathcal{Q} = 3300$ do show features similar to these with $\mathcal{Q} = 110$ (see next Section). As a further check of the mapping, we have computed runs with a higher mapping frequency $\Omega = 80 \text{ yr}^{-1}$ (diamond in Fig. 2.11) and with differential equations of the averaged Hamiltonian with a relative precision of 10^{-9} per time step (square) at $\mathcal{Q} = 3.3$. These have similar escape probabilities to those runs with $\Omega = 20 \text{ yr}^{-1}$ (circles in Fig. 2.11).

Fig. 2.11 emphasizes that for much higher dissipation rates, the dynamics are strongly affected by the choice of $\mathcal{Q}$. It is apparent that a run with $\mathcal{Q} = 3.3$, or $\dot{a}_A/a_A \simeq 4.8 \times 10^{-10}$ per orbit period, will produce results significantly different

Figure 2.11: Measured probability of escape versus tidal dissipation rate for the full resonance problem, in terms of the effective specific dissipation function of Uranus Q . Circles: $\Omega = 20 \text{ yr}^{-1}$, diamond: $\Omega = 80 \text{ yr}^{-1}$, square: differential equations for analytically averaged Hamiltonian.

from the physical system, i.e there are significant artifacts. This is even more evident when the energy evolution of the trajectories is compared: the trajectories in Fig. 2.12 have the same initial conditions as those in Fig. 2.15 (see next subsection), but the dissipation rate is 1000 times higher ($\dot{a}_A/a_A \simeq 1.4 \times 10^{-8}$ per orbit period). The faster evolution gives a more exciting picture of the evolution through resonance, with large increases in eccentricity, but *it is wrong*.

Numerical Runs

Figures 2.13 through 2.17 show results of numerical runs in which, for each of 5 initial values of δ and ΔE , 100 trajectories were evolved through the resonance using the mapping (see Appendix II) with slow tidal evolution ($\mathcal{Q} = 110$). The initial values of δ and ΔE , and the physical parameters corresponding to these values, are given in Table 2.1. The 100 initial points for Run 3 were on an invariant curve of sections II. For the other runs each set of 100 initial points was obtained by mapping the trajectory forward in time 7 (Runs 1, 2, and 4) or 10 (Run 5) mapping iterations from the initial point with no tidal dissipation, in order to thoroughly mix the phases σ_A and σ_U , but keep the “actions” the same. Since the trajectories are initially quasiperiodic there is, in principle, a transformation to action-angle variables. Under slow variation of parameters these actions are to first order conserved; the actions are first order adiabatic invariants. For slightly different initial conditions the dynamical frequencies are slightly different, and the initial angles will rapidly be spread over all values. (Motion on the n -torus is ergodic.) Thus the essential parameters which determine the probabilities of various outcomes are the actions; the angles are physically unknowable and considered to be uniformly distributed in the calculation of probabilities. In the single resonance theory the phase corresponding to these angles is also assumed to be uniformly distributed in the calculation of probabilities.

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.12: Trajectories evolved through the resonance with a high tidal dissipation rate ($Q = 0.11$). Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. The types of behavior possible for each orbit may be determined by comparing this figure with Fig. 2.1a. Evolution of these trajectories is strongly affected by the rate of tidal dissipation: the extremely wide range of outcomes is an artifact of evolving the system too rapidly through resonance. These trajectories have the same initial conditions as the trajectories in Run 3 (Fig. 2.15), but the dissipation rate is 1000 times higher. This picture is more exciting than that presented in Fig. 2.15, but *this picture is wrong*.

Table 2.1: Initial Parameters

Run	ΔE	δ	a_A a_U	e_A e_U	σ_A σ_U	x_A x_U	y_A y_U
1	-9.4511×10^{-7}	-0.1781	7.1633 10.0703	0.0017 0.0043	270.0 270.0	0.0 0.0	-1.4649×10^{-3} -3.9289×10^{-3}
2	-5.6281×10^{-6}	-0.1869	7.1625 10.0703	0.005 0.005	90.0 90.0	0.0 0.0	4.3084×10^{-3} 4.5685×10^{-3}
3	-1.9623×10^{-5}	-0.0283	7.1611 10.0703	0.0102 0.0031	90.0 45.0	0.0 0.002	8.7907×10^{-3} 0.002
4	-2.1626×10^{-5}	1.4263	7.1610 10.0703	0.01 0.01	0.0 0.0	8.6168×10^{-3} 9.1371×10^{-3}	0.0 0.0
5	-6.0706×10^{-5}	2.0000	7.1588 10.0703	0.0125 0.0125	0.0 0.0	1.0771×10^{-2} 1.1421×10^{-2}	0.0 0.0

The trajectories in δ , ΔE parameter space are represented by solid lines when the evolution is quasiperiodic and by dashed lines when the evolution is chaotic, as determined by evaluation of the rate of growth of the distance between nearby trajectories (Lyapunov exponent) at discrete intervals in δ along the trajectories.

The trajectories appear to have nearly the same evolution in ΔE until they first become chaotic. This can be understood in terms of the first order adiabatic invariance of the actions of the trajectories while they are quasiperiodic. The paths begin to diverge in the chaotic region.

Sample trajectories from each of Runs 2-5 have been plotted in Fig. 2.18. Figs. 2.19 - 2.22 show the eccentricity variations of Ariel and Umbriel for these trajectories. The maximum and minimum eccentricities in small intervals of δ (Fig. 2.19: $\Delta\delta \approx 0.0007$, Fig. 2.20: $\Delta\delta \approx 0.001$, Fig. 2.21: $\Delta\delta \approx 0.003$, and Fig. 2.22: $\Delta\delta \approx 0.002$) are plotted versus δ . These “envelopes” are smooth when the trajectory is quasiperiodic, but the maxima and minima vary in an irregular fashion in the chaotic zone. In each case, the maximum eccentricity in the chaotic zone is higher than the initial eccentricity. Note that in some cases, the average eccentricity of Umbriel after escape from the resonance is higher than the average eccentricity

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.13: Trajectories in $\delta, \Delta E$ parameter space for Run 1 with $Q = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. The types of behavior possible for each orbit may be determined by comparing this figure with Fig. 2.1b. None of the trajectories in this set escaped from the resonance.

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.14: Trajectories in $\delta, \Delta E$ parameter space for Run 2 with $Q = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. The types of behavior possible for each orbit may be determined by comparing this figure with Fig. 2.1a. Only one of these trajectories escaped from the resonance (see Table 2.2), closely following the curve of $E = 0.0$.

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.15: Trajectories in $\delta, \Delta E$ parameter space for Run 3 with $Q = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. The types of behavior possible for each orbit may be determined by comparing this figure with Fig. 2.1a. Of the 100 trajectories, 29 escaped from the resonance.

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.16: Trajectories in $\delta, \Delta E$ parameter space for Run 4 with $Q = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. The types of behavior possible for each orbit may be determined by comparing this figure with Fig. 2.1a. All of the trajectories which did not escape by the end of the run displayed were eventually captured into the resonance. Of the 100 trajectories, 28 escaped from the resonance. The trajectories which appear linear and quasiperiodic between $\delta \approx 2.5$ and $\delta \approx 4.5$ are temporarily captured into a quasiperiodic region of phase space in which σ_A librates.

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.17: Trajectories in $\delta, \Delta E$ parameter space for Run 5 with $Q = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. The types of behavior possible for each orbit may be determined by comparing this figure with Fig. 2.1a. All of the trajectories which did not escape by the end of the run displayed were eventually captured into the resonance. Of the 100 trajectories, 39 escaped from the resonance.

before the resonance is encountered (e.g. Fig. 2.20b), a phenomenon not possible in the standard theory of evolution through isolated mean-motion resonances.

Fig. 2.23 shows the eccentricity variations of a trajectory from Run 5 integrated with $Q = 3300$, which is close to the maximum dynamically allowed tidal dissipation rate in Uranus. The plot was produced in the same manner as Figs. 2.19-2.22, showing the maximum and minimum eccentricities in intervals of $\Delta\delta \approx 0.003$. The decrease of effective dissipation rate by a factor of 30 does not dramatically affect the dynamics – features which are seen in this figure are similar to those seen in the previous figures. This trajectory was captured into the resonance.

Fig. 2.24 shows the chaotic variations in eccentricity of part of the trajectory shown in Fig. 2.22, with the scale of δ greatly enlarged. The eccentricities of the satellites vary in an irregular manner. This behavior is similar to that observed for asteroid motion in the chaotic zone associated with the 3:1 Kirkwood gap (Wisdom 1982, 1983). Note that for $6.008 < \delta < 6.017$ the mean eccentricity of Ariel is relatively high while the mean eccentricity of Umbriel is relatively low. There are relatively sudden changes in the eccentricities at the boundaries of this range in δ . Outside of this range of δ , the mean eccentricity of Ariel is relatively low, while the mean eccentricity of Umbriel is relatively high. This behavior is typical for trajectories in the chaotic zone: there are time intervals during which the mean eccentricity of Ariel is relatively high while that of Umbriel is relatively low, during other time intervals the mean eccentricity of Umbriel is high while that of Ariel is low, with relatively sudden transitions between these intervals. This behavior results in the erratic alternations between high and low maximum eccentricity seen in Figs. 2.19-2.23.

Since the tidal dissipation is small, the change in energy over a short time interval is also small, and the system can be considered approximately Hamiltonian. It is possible to study the qualitative behavior of a trajectory by “freezing” the value of

[illegible]

80

Figure 2.19: Eccentricity variations of the trajectory from Run 2 which escaped from the resonance. Shown are the maximum and minimum eccentricities of Ariel (a) and Umbriel (b) in intervals of $\Delta\delta \approx 0.0007$.

Figure 2.20: Eccentricity variations of a trajectory from Run 3 which escapes from the resonance. Shown are the maximum and minimum eccentricities of Ariel (a) and Umbriel (b) in intervals of $\Delta\delta \approx 0.001$. In this example, the average eccentricity of Umbriel after escape from the resonance is higher than the average value before the resonance is encountered.

Figure 2.21: Eccentricity variations of a trajectory from Run 4 which is temporarily captured into a resonant island. Shown are the maximum and minimum eccentricities of Ariel (a) and Umbriel (b) in intervals of $\Delta\delta \approx 0.003$. Between $\delta \approx 2.5$ and $\delta \approx 4.5$, the eccentricity of Ariel increases during quasiperiodic libration of σ_A . After entering the chaotic region again, the trajectory is ultimately captured into the resonance.

Figure 2.22: Eccentricity variations of a trajectory from Run 5. Shown are the maximum and minimum eccentricities of Ariel (a) and Umbriel (b) in intervals of $\Delta\delta \approx 0.002$. For this trajectory, the average eccentricity of Umbriel after escape from the resonance is also higher than the average value before the resonance is encountered.

(a)

(b)

Figure 2.23: Eccentricity variations of a trajectory with the same initial conditions as the trajectory shown in Fig. 2.22, but with the tidal dissipation rate 30 times smaller. Shown are the maximum and minimum eccentricities of Ariel (a) and Umbriel (b) in intervals of $\Delta\delta \approx 0.003$. Between $\delta \approx 4$ and $\delta \approx 4.4$, σ_A was captured into temporary quasiperiodic libration. This trajectory is ultimately captured into the resonance.

Figure 2.24: The eccentricity variations of Ariel (a) and Umbriel (b) in the chaotic zone for the trajectory shown in Fig. 2.22 are shown at a greatly expanded scale. The time resolution of this plot is one mapping period ($\Delta t = 2\pi/20$ yr). The eccentricities of both satellites vary in an irregular manner. At $\delta \approx 6.008$, the average eccentricity of Ariel increases suddenly, while the average eccentricity of Umbriel decreases. At $\delta \approx 6.017$, the average eccentricity of Ariel decreases suddenly, while the average eccentricity of Umbriel increases.

δ and computing surfaces of section. This allows us to determine which regions of phase space are accessible to a trajectory at particular values of δ and ΔE , and the types of behavior that are possible in each region.

As discussed in Section 2.2.2, for chaotic trajectories points appear in an irregular manner on the sections corresponding to all four root families. The trajectories are observed to spend some time with low values of both e_U and e_A , during which time points appear on sections Ib, Ic, IIb, and IIc. Trajectories may jump back and forth between regions near to and far from the origin, during which time points appear on all sections corresponding to all four root families. They may also spend time with high values of both e_U and e_A , during which time points appear on sections Ia, Id, IIa, and IIc.

During the periods of time when points appear on sections corresponding only to either the outer root families or the inner root families, there is a correlation between the eccentricities of Ariel and Umbriel. At such times, a relatively high eccentricity for Ariel requires a relatively low eccentricity for Umbriel, and vice versa. This results in the interesting behavior of the eccentricities displayed in Fig. 2.24: over relatively short timescales, the mean eccentricity of one satellite is high while that of the other satellite is low. The eccentricities alternate between high and low mean values in an irregular manner. Eccentricities tend to be higher when points are appearing on the outer root sections.

While the trajectory is chaotic, ΔE evolves in a surprisingly regular manner to higher values with increasing δ , provided that the rate of tidal evolution is slow enough. This regularity may be associated with the conservation of an ergodic adiabatic invariant while the evolution is chaotic (see Section 1.4.3). Qualitative changes in the evolution of ΔE have been associated with temporary capture onto the librating island surrounding the Mode II fixed point, or with the trajectory “sticking” to a quasiperiodic island for a period of time.

Fig. 2.21 shows some of this interesting behavior: after becoming chaotic, the trajectory apparently becomes quasiperiodic again, during which time the eccentricity of Ariel increases considerably. It then becomes chaotic again, exhibiting eccentricity behavior similar to that of the other trajectories plotted. By plotting surfaces of section along this “quasiperiodic interlude”, it was found that the trajectory was on one of the islands in the chaotic zone and on which σ_A librates in the full phase space. The trajectory became unstable again at a later time, possibly due to a change in the size of the island. It may also have been a case of “sticking”, where the trajectory spent time near an island, and exhibited behavior which only appeared to be quasiperiodic (e.g. Wisdom 1983). The fact that two of the trajectories in Fig. 2.16 which exhibit this behavior become unstable at similar values of δ indicates that the phenomenon is not extremely sensitive to initial conditions, suggesting that the trajectory actually was captured onto the quasiperiodic island and not stuck to it. Temporary quasiperiodic libration has served as a conduit to higher eccentricity for Ariel, resulting in maximum eccentricities more like those of the system shown in Fig. 2.22, which has higher initial eccentricities. The energy evolution of these two trajectories in the chaotic zone further indicates the similarity. This particular trajectory is eventually captured, but examination of the trajectory in Fig. 2.18 indicates that escape is possible for similar trajectories. The trajectory from Run 5 in Fig. 2.18, which escapes to large negative ΔE , overlaps the trajectory from Run 4, indicating a period in which evolution through the chaotic zone occurred in the same region of phase space. This mechanism of temporary quasiperiodic libration allows for the possibility that trajectories approaching the resonance with relatively low eccentricities could be driven to much higher eccentricities than they would with the evolution in the chaotic zone alone, and the satellites might escape from the resonance with higher eccentricities than they had approaching the resonance. The trajectory shown in Fig. 2.23 also shows this phenomenon of temporary capture.

Trajectories may escape from the resonance almost immediately, or may spend a considerable period of time in the chaotic zone before they escape, during which time the maximum eccentricities can be driven to relatively high values.

Eventually, the trajectories become quasiperiodic again. The final states of the trajectories fall into two major categories: trajectories which escape from the resonance and trajectories which are captured into resonance. For trajectories which are captured ΔE increases and ends up in the upper half of the parameter space; for trajectories which escape ΔE decreases to large negative values.

Those trajectories which evolve to large negative ΔE escape into the large quasiperiodic region on the sections corresponding to root families b and c , in which both resonant arguments circulate. This requires that both eccentricities have low mean values in the chaotic zone (with points appearing only on sections Ib, c and IIb, c) for a period of time long enough to allow evolution into this quasiperiodic zone. Then, as δ evolves to higher values, the system moves away from the resonance region, and the dynamics of the system are dominated by the secular interactions between the satellites, as they were before the system encountered the commensurability.

Examination of section II for $\Delta E \geq 0$ (see Fig. 2.7) indicates that capture into resonance occurs when the two regions of the section defined by the energy surface boundary separate. Trajectories appear to be confined to one region of phase space or the other, and σ_V must librate. We therefore have capture into resonance through evolution of the energy surface. There is a relatively small region of parameter space for ΔE just larger than zero where the trajectory can still be chaotic (see Fig. 2.1). A trajectory can therefore be captured into resonance, yet still be chaotic. This phenomenon is a novel feature of our problem. As the system continues to evolve, the two “bubbles” move apart on sections II, and the eccentricity of Umbriel increases. In addition, the large chaotic zone disappears, and trajectories

become quasiperiodic, indicating permanent capture into the resonance involving the longitude of pericenter of Umbriel's orbit. If further evolution of the trajectory results in $\Delta E > E_1 - E_2$, the energy surface also divides sections I into two regions in which σ_A must librate (see Section 2.2.2). As the system continues to evolve, these two "bubbles" also move apart on sections I, and the eccentricity of Ariel increases. The orbits of both satellites are trapped into quasiperiodic libration.

The island surrounding the Mode II fixed point could provide another mechanism of capture, although the ultimate stability of the island is unknown.

The observed spread in outcomes contrasts with the behavior of trajectories in the single resonance approximation (see e.g. Borderies and Goldreich 1984), in which the outcomes are probabilistic due to an unknown phase, but are otherwise quite predictable. The presence of a chaotic zone in this problem indicates that the outcome is inherently unpredictable (though still deterministic), due to the extreme sensitivity of the motion to initial conditions. Two trajectories entering the chaotic zone with even very similar phases will diverge exponentially, making it impossible to predict their comparative evolutions over the long term. The width of the chaotic zone when a trajectory enters it may be important. The probability of escape may be much larger for a trajectory if the chaotic zone is narrow and the quasiperiodic region of escape on sections corresponding to root families b and c occupies a majority of the phase space. This is inferred from observations that most escapes occur very shortly after the trajectories become chaotic, when the chaotic zone occupies a relatively small region of phase space.

The trajectories in each of the runs described so far in this paper originally all have the same actions. The actions depend on the positions of the invariant curves on a section. There appears to be a dependence of the probability of escape on the initial distribution of energy between the normal modes, and therefore on the actions. Fig. 2.25 shows the energy evolution of trajectories with the same

δ 's and ΔE 's as the five runs in Figs. 2.13-2.17, but the trajectory on which the initial points are computed generates invariant curves on sections I and II which are now very close to the fixed point associated with Mode I. 10 trajectories were integrated for each run. For these trajectories, the evolution of energy is markedly different from the evolution of the corresponding trajectories in Figs. 2.13-2.17: the trajectories all evolve to much higher values of ΔE before becoming chaotic. All of the trajectories were captured into resonance. The details of the secular interactions between the satellites before the resonance is encountered have a significant effect on the outcome of the passage through the resonance.

Statistics of Escape

Overall escape statistics are given in Table 2.2. There is a trend towards higher probability of escape from the resonance as the mean eccentricities of the satellites before the resonance is encountered are increased.

For both satellites, the maximum eccentricities in the chaotic zone are higher than the initial maximum eccentricities. The mean of the time-averaged final eccentricities for each satellite after escape from the resonance (last column in Table 2.2) is comparable to or lower than the mean initial eccentricity (third column). Fig. 2.26 shows the distribution of time-averaged final eccentricities after escape from the resonance. Note that in some cases the average final eccentricity of Umbriel after escape from the resonance is higher than the value before the resonance was encountered. This may be contrasted with the single resonance model in which the eccentricity after escape is always lower than the initial eccentricity.

There is a tendency for the average final eccentricity of Ariel to be lower than that of Umbriel. This can be understood by examining the structure of the sections corresponding to root families *b* and *c* (see Figs. 2.4b,c and 2.5b,c). The trajectories

Ariel/Umbriel 5:3 Resonance Energies

Figure 2.25: Trajectories in δ , ΔE parameter space with initial coordinates near the Mode I fixed point. The initial values of δ and ΔE for each set of 10 trajectories are the same as the initial values of δ and ΔE for the corresponding sets of trajectories in Runs 1 - 5 (Figs. 2.13 - 2.17). Note that the trajectories become chaotic much closer to $\Delta E = 0$ than the corresponding trajectories in Figs. 2.13 - 2.17. None of the trajectories shown escaped from the resonance.

Table 2.2: Escape Statistics

Run	δ_0	$\langle e_{A0} \rangle^a$ $\langle e_{U0} \rangle$		$\langle \delta_{esc} \rangle$	$\langle e_{Amax} \rangle$ $\langle e_{Umax} \rangle$		P_{esc}^b	δ_f	$\langle e_{Af} \rangle$ $\langle e_{Uf} \rangle$
2	-0.187	0.00493	+0.00066 -0.00108	0.412	0.0078		0.01	3.000	0.00034
		0.00503	+0.00120 -0.00073		0.0074				0.00123
3	-0.028	0.00960	+0.00064 -0.00110	1.182 ± 0.301	0.01170 ± 0.00052		0.29 ± 0.046	10.000	0.00345 ± 0.00101
		0.00417	+0.00172 -0.00117		0.01071 ± 0.00143				0.00365 ± 0.00220
4	1.426	0.00837	+0.00168 -0.00288	2.841 ± 0.463	0.01546 ± 0.00139		0.28 ± 0.036	8.114	0.00378 ± 0.00073
		0.01116	+0.00218 -0.00122		0.01577 ± 0.00100				0.00937 ± 0.00161
5	2.000	0.01037	+0.00215 -0.00315	4.757 ± 0.812	0.01916 ± 0.00216		0.39 ± 0.048	10.000	0.00541 ± 0.00092
		0.01401	+0.00223 -0.00153		0.01953 ± 0.00136				0.01285 ± 0.00210

^aTime average and range of eccentricity on initial trajectory

^bError for P_{esc} is standard deviation of the mean. All other standard deviations for single value.

Figure 2.26: Average final escape eccentricities. (a) and (b) refer to Run 3, (c) and (d) to Run 4, and (e) and (f) to Run 5. The average final eccentricity of Umbriel tends to be higher than the average final eccentricity of Ariel. Compare with average initial eccentricities, given in Table 2.2. It is possible for the average final eccentricity of Umbriel to be higher than the average value before the resonance is encountered.

escape into the quasiperiodic regions surrounding the fixed point associated with Mode I, which is near the origin in Ariel variables and near the boundary in Umbriel variables determined by the energy integral.

Fig. 2.27 shows the distribution of times spent in the chaotic zone in terms of δ for the escaping trajectories. For $Q = 6600$, $\Delta\delta = 1.0$ corresponds to about 6.9×10^6 years. There is a trend, with increasing initial eccentricity, for the distribution to spread out and for escaping trajectories to spend more time in the chaotic zone.

Fig. 2.28 shows the distributions of maximum eccentricities in the chaotic zone for the escaping trajectories. There is a correlation between the length of time spent in the chaotic zone and the maximum eccentricity achieved — corresponding essentially to the upper “envelope” of eccentricity in the chaotic zones in Figs. 2.19 - 2.23. The trajectories travel up the chaotic zone to higher and higher eccentricities. The largest eccentricity reached in any of the escaping trajectories was the maximum of the trajectory in Fig. 2.22, approximately 0.023.

Note that some of the trajectories in runs 4 and 5 (see Figs. 2.16 and 2.17) are still chaotic at the end of the run. When further evolved, none of these trajectories escaped.

Figure 2.27: $\Delta\delta$ spent in the chaotic zone for escaping trajectories. (a) - (c) refer to Runs 3 - 5, respectively. $\Delta\delta = 1.0$ corresponds to 6.9×10^6 years of evolution for $Q = 6600$. Trajectories with higher initial eccentricities tend to spend more time in the chaotic zone.

Figure 2.28: Maximum eccentricities in chaotic zone. (a) and (b) refer to Run 3, (c) and (d) to Run 4, and (e) and (f) to Run 5. The maximum eccentricities of the two satellites in the chaotic zone tend to be comparable.

2.3 Secular Perturbations

In this section we consider the perturbations on the eccentricity-type resonances produced by the secular variations due to the other satellites. To get a qualitative idea of the effects of secular perturbations, we consider only the perturbations on the circular inclined problem due to Titania as a first approximation, since they are the most significant (see Dermott and Nicholson 1986, Laskar 1986). By including these perturbations, the number of degrees of freedom in the problem is increased, no longer allowing detailed study of the phase space using surfaces of section. However, the evolution of the eccentricities with time in this model can be directly compared to the evolution in the two-satellite problem.

The development of the Hamiltonian is given in Appendix I (Section 7.2.1), with $\phi = 3\lambda_A - 5\lambda_U$, $p_i \approx e_i\sqrt{L_i}\cos(-\varpi_i)$, and $q_i \approx e_i\sqrt{L_i}\sin(-\varpi_i)$. The momentum conjugate to ϕ is $\Phi = \frac{1}{3}(L_A - L_{AR})$, where the resonant value L_{AR} is defined at $\frac{\partial \mathcal{H}}{\partial \Phi} = 0$:

$$L_{AR} = \frac{3L_U + 5L_A}{5 + 3\left(\frac{5}{3}\right)^{1/3}\frac{m_U}{m_A}} \quad (2.15)$$

In this model, evolution due to tidal dissipation changes the value of Φ at a rate of $\dot{\Phi} = \frac{1}{3}(\dot{L}_A - \dot{L}_{AR})$. The value of Φ can be converted into a value for δ .

Fig. 2.29 shows the evolution of the eccentricities of Ariel, Umbriel, and Titania for a trajectory evolved through the 5:3 commensurability using this model. The initial orbital parameters are: $a_A = 7.2074$, $a_U = 10.1353$, $a_T = 16.6500$, $e_A = 0.0102$, $e_U = 0.0031$, $e_T = 0.0021$, $\sigma_A = 90^\circ$, $\sigma_U = 45^\circ$, and $\sigma_T = 0^\circ$. Titania has been given its nominal current orbital eccentricity, and the initial orbital eccentricities of Ariel and Umbriel correspond to those used in Run 3 of the planar-eccentric problem (see *e.g.* Fig. 2.20). This trajectory was evolved through the resonance by numerically integrating the averaged differential equations for the above model using the Busirsch-Stoer (1966) integrator with a relative precision of 10^{-9} per time

step of approximately 0.6 year.

The most immediate impression one gets from this figure is that the motion is considerably more complicated than in the planar-eccentric model. With the addition of another degree of freedom, more resonances between the frequencies associated with the degrees of freedom are possible. In addition, higher-dimensional chaos, or Arnold diffusion, is possible. At $\delta \approx 0.7$, even before the trajectory enters the large chaotic zone, it encounters one of these secondary resonances, at which time the evolution of the eccentricities suddenly changes. Evolution within the chaotic zone is more irregular, with the upper eccentricity limit less well defined than in the planar-eccentric case. Note that the eccentricities of all three satellite orbits are varying chaotically. Despite the presence of perturbations due to Titania, though, the maximum orbital eccentricity achieved in the chaotic zone is still only a factor of two to three higher than the average initial values. For this particular trajectory, the orbits of Ariel and Umbriel are eventually captured into the 5:3 resonance, although the evolution after capture is complicated due to the presence of the secondary commensurabilities between the degrees of freedom. After the trajectory leaves the large chaotic zone, the eccentricity of Titania's orbit resumes its quasiperiodic behaviour. Note that even though the rate of evolution through the resonance used for this trajectory is similar to the rates used in the planar case, the eccentricity "envelope" in the quasiperiodic regime is considerably less well defined, due to the more complicated motion. However, the qualitative behaviour of the trajectory is not significantly different from the planar-eccentric case: the increases in eccentricity during evolution through the chaotic zone are similar.

Figure 2.29: Variations in the orbital eccentricities of Ariel (a) and Umbriel (b) during evolution through the 5:3 commensurability including secular perturbations due to Titania (c). The evolution is more complicated than in the two-satellite problem, but the increases in the orbital eccentricities during evolution through the chaotic zone are similar. This trajectory is captured into resonance.

2.4 The Eccentric-Inclined Problem

The inclinations were neglected in the above models of resonance passage. Lowest order terms in the disturbing function for inclination involve terms of order $\sin^2 \frac{I}{2}$, where I is the mutual inclination of the two orbits. For current values of the inclinations, this quantity is of the same order as e^2 . Therefore, the inclination terms could have contributed significantly to the evolution of the system. It has been found (Wisdom 1983, 1987b) that in the case of resonant asteroid motion, including inclination terms can lead to much higher eccentricities than in the planar case. A similar qualitative change in the behavior of this system may result. As in the case where secular perturbations due to Titania were included, though, the number of degrees of freedom in this model is increased, preventing the study of the phase space using surfaces of section. However, the evolution of this model may be qualitatively compared to the planar-eccentric problem by looking at the evolution of the orbital eccentricities and inclinations during passage through the resonance.

In this section we consider the effect of the interactions between the eccentricity-type resonances and the inclination-type resonances associated with the Ariel-Umbriel 5:3 mean-motion commensurability on the evolution of the orbits. This study uses the model developed in Appendix I for second order resonances.

We define

$$\begin{aligned}
 \sigma_i &= \frac{1}{2}(5n_U - 3n_A - 2\varpi_i) \\
 \Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
 \psi_i &= \frac{1}{2}(5n_U - 3n_A - 2\Omega_i) \\
 \Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
 \Gamma_A &= L_A + \frac{3}{2}(\Sigma_A + \Sigma_U + \Psi_A + \Psi_U) \\
 \Gamma_U &= L_U - \frac{5}{2}(\Sigma_A + \Sigma_U + \Psi_A + \Psi_U)
 \end{aligned} \tag{2.16}$$

This is now a four degree of freedom Hamiltonian problem with slow time dependence. There are 3 eccentricity resonances, 3 inclination resonances, and secular interactions. The eccentricities are coupled to the inclinations through the nonlinear terms. δ is defined to be the nonresonant contributions to $5n_U - 3n_A - \dot{\omega}_A - \dot{\omega}_U - \dot{\Omega}_A - \dot{\Omega}_U$.

For low eccentricities and inclinations approaching the resonance, capture into resonance is highly probable. An ensemble of 10 trajectories was evolved through the resonance for the nominal current values of the eccentricities and inclinations (see Table 1.1) approaching the resonance, with $a_{A0} = 7.2070$ and $a_{U0} = 10.1353$. The 10 initial conditions for these trajectories were generated by mapping 66 iterations ($\Omega = 40$) forward without tidal dissipation from an initial point with $\sigma_A = 90^\circ$, $\sigma_U = 270^\circ$, $\psi_A = 270^\circ$, and $\psi_U = 90^\circ$. Each of these trajectories was evolved through the commensurability at a slow rate ($\dot{a}_A/a_A = 1.4 \times 10^{-11}$ per orbit period). Of these 10 trajectories, 9 were captured into the i_A^2 resonance. The remaining trajectory, after escaping from the inclination-type resonances, was captured into the eccentricity-type resonances.

A similar numerical experiment was carried out using somewhat higher initial eccentricities and inclinations, from a point with initial parameters $a_A = 7.2050$, $a_U = 10.1353$, $e_A = 0.005$, $e_U = 0.005$, $\sigma_A = 90^\circ$, $\sigma_U = 270^\circ$, $i_A = 0.005$ radians, $i_U = 0.005$ radians, $\psi_A = 270^\circ$, and $\psi_U = 90^\circ$. 10 initial conditions were generated as in the experiment described above, and integrated through the commensurability at the rate $\dot{a}_A/a_A = 1.4 \times 10^{-11}$ per orbit period. Of this set of trajectories, 6 were captured into the i_A^2 resonance, 2 were captured into the eccentricity-type resonances, and 2 escaped from the resonances altogether. One of the trajectories which escaped is shown in Fig. 2.30. Note that the initial eccentricities are similar to the values used in Run 2 in the planar-eccentric approximation (see *e.g.* Fig. 2.19). The evolution of the eccentricities is qualitatively similar to that in the

planar-eccentric problem, except that the evolution is somewhat more irregular in the chaotic zone, with the upper eccentricity “envelopes” less well defined. The inclinations vary chaotically as well, typically by a factor of two or so. Upon escaping from the resonance, the average inclination may be a factor of two or so larger or smaller than the average value before the resonance was encountered. Since the inclinations are small to begin with, this change is not significant.

For the trajectory shown in Fig. 2.31, the orbital parameters defining the initial conditions are the following: $a_A = 7.2050$, $a_U = 10.1353$, $e_A = 0.0075$, $e_U = 0.0075$, $\sigma_A = 90^\circ$, $\sigma_U = 270^\circ$, $i_A = 0.0075$ radians, $i_U = 0.0075$ radians, $\psi_A = 270^\circ$, and $\psi_U = 90^\circ$. Again, the evolution of the eccentricities is qualitatively similar to that in the planar-eccentric problem, if somewhat more irregular in the chaotic zone. As in the planar case, the average final inclination of Ariel tends to be somewhat lower than that of Umbriel, which may be higher than the average initial value. The inclinations do not change significantly.

For both Figs. 2.30 and 2.31, the increases in the eccentricities during evolution through the chaotic zone are similar to those in the planar-eccentric problem for similar initial eccentricities. The interactions between eccentricity-type resonances and inclination-type resonances do not significantly change the qualitative picture of resonance passage at this commensurability.

Figure 2.30: Eccentricity and inclination variations for a trajectory evolved through the 5:3 commensurability with $\frac{\dot{a}_A}{a_A} = 1.4 \times 10^{-11}$ per orbit period. The maximum and minimum eccentricities are plotted in intervals of $\Delta\delta = 0.0038$. The evolution of the eccentricities is qualitatively similar to that in the planar-eccentric approximation for comparable initial values. The inclinations do not change significantly.

Figure 2.31: Eccentricity and inclination variations for a trajectory evolved through the 5:3 commensurability with $\frac{\dot{a}_A}{a_A} = 1.4 \times 10^{-11}$ per orbit period. The maximum and minimum eccentricities are plotted in intervals of $\Delta\delta = 0.0035$. The evolution of the eccentricities is qualitatively similar to that in the planar-eccentric approximation for comparable initial values. The inclinations do not change significantly.

2.5 Discussion

Well before they encounter the 5:3 resonance, the orbital evolution of Ariel and Umbriel is dominated by their mutual secular perturbations. The eccentricities of the satellites oscillate between minimum and maximum values determined by the relative strengths of the two normal modes. The variations are quasiperiodic, with two fundamental frequencies associated with the modes, and with high-frequency perturbations superimposed. The semimajor axes slowly increase due to tidal dissipation in Uranus. The orbit of Ariel expands more rapidly than that of Umbriel, and as they approach the 5:3 mean-motion commensurability resonant perturbations become stronger. One of the secular modes eventually becomes unstable, and the system enters a large chaotic zone. Further tidal dissipation within Uranus may drive the eccentricities up the chaotic zone to relatively high values. They may then either escape or be captured into resonance.

When the system enters the chaotic zone, a new low-eccentricity region of phase space is made accessible to the system. While in the chaotic zone, the system spends time in both regions of phase space, and the eccentricities irregularly alternate between high and low values. If the system is caught in the circulating quasiperiodic region in the low eccentricity region of phase space, it escapes from the resonance, and the orbits continue to differentially expand due to dissipation in Uranus. The average eccentricities after escape from the resonance tend to be lower than the values before the resonance is encountered, but in some cases the average eccentricity of Umbriel increases, a phenomenon not predicted by the standard single resonance theory of evolution through resonances. It may also be possible for the system to be temporarily captured into a librating island during passage through the resonance (see Section 2.2.3), with the result that the average eccentricities after escape from the resonance could be significantly higher than the values before the resonance is encountered. After escaping from the resonance, the evolution of the orbits is

again dominated by their mutual secular perturbations, and the eccentricities slowly damp due to tidal dissipation within the satellites.

If the system does not escape, the energy surface divides into two separate regions. When this happens, the system is captured into resonance, even though the trajectory may still be chaotic. Initially, only σ_U librates, and tidal dissipation drives the orbit of Umbriel to higher eccentricity. The trajectory becomes quasiperiodic at some point. Eventually, σ_A may also be trapped into a librating state, and the eccentricity of Ariel is also driven to high values. Both orbits are then captured into the resonance.

The outcome of resonance passage does not depend simply on the mean eccentricities; the individual amplitudes of the secular modes appear to play a significant role. The probability of escape is higher for larger initial average eccentricities, but also depends on which secular mode is stronger. Since the satellites are not presently in the resonance, they must have escaped. It appears that in order for the probability of escape to have been significant, the average eccentricities before the resonance was encountered were probably higher than current values. The largest increases in eccentricity occur when the trajectory is temporarily captured into libration during evolution through the resonance.

The relatively high current eccentricities must be explained. For the highest dynamically allowed rate of tidal dissipation in Uranus ($Q = 6600$), the time since resonance passage is about 3×10^8 years. For the lowest rate of tidal dissipation in Uranus which allows Ariel and Umbriel to encounter this commensurability, the time since resonance passage is of order 4×10^9 years. As discussed in Section 4.1, the timescales of direct tidal damping of the satellite eccentricities are about 10^8 years for Ariel and about 10^9 years for Umbriel. After escape from the resonance, the eccentricity of Ariel tends to be smaller than that of Umbriel, which is also the case at the present time. If the rate of tidal dissipation is near the maximum

dynamically allowed value, the damping timescale for Umbriel is about a factor of 3 longer than the time since resonance passage, and that of Ariel is about a factor of 3 smaller. This indicates that the eccentricity of Umbriel has probably not changed significantly since resonance passage, and that the eccentricity of Ariel, which was probably lower than that of Umbriel immediately after escape from the resonance, has damped to still lower values, but not necessarily to negligible values, which is consistent with the current observations. The run with outcomes most closely resembling the current physical configuration is Run 3 (Fig. 2.15). The current relatively high eccentricities, then, are remnants of relatively high eccentricities after passage through the resonance. Of course, the average eccentricities of the satellites for the initial conditions of Run 3 ($e_A \simeq 0.01$ and $e_U \simeq 0.003$) are quite high, so the problem is not really solved. We still require some mechanism to provide the satellites with high eccentricities before the resonance is encountered, unless the temporary capture into a quasiperiodic librating state occurred. In this case a trajectory approaching the resonance with low eccentricities could escape from the resonance with significantly higher eccentricities. It is difficult to determine if this happened to Ariel and Umbriel, but it is a tantalizing possibility. On the basis of our numerical experiments, temporary quasiperiodic capture seems to occur with relatively low probability. We point out, though, that our single run with $Q = 3300$ showed this phenomenon.

If the rate of tidal dissipation in Uranus is near the minimum value which allows passage through this resonance ($Q \approx 100,000$), then the damping timescale for Umbriel is about a factor of 4 smaller than the time since resonance passage, and that of Ariel is about a factor of 40 smaller. In this case, the eccentricities would have damped significantly since passage through the resonance. Since the present eccentricities are high, this seems to argue for a Q of Uranus closer to the minimum allowed value.

Although Ariel and Umbriel have similar masses and radii (see Table 1.1), the surfaces of the two satellites are quite different. Ariel has had a geologically active past (Smith *et al.* 1984, Plescia 1987), with faults, fractures and flow features remaining on its surface. Umbriel presents a bland, heavily-cratered, primitive surface. The density of Ariel is $1.65 \pm 0.30 \text{ g cm}^{-3}$ (see Table 1.2). If the interior is differentiated, it will have a rocky core ($\rho \approx 3.0 \text{ g cm}^{-3}$) about $0.7 R_A$ in radius, surrounded by an icy mantle. If the mantle is solid, the rate of heat conduction through the mantle can be approximated by

$$\frac{dE}{dt} = -k4\pi R_A^2 \frac{\Delta T}{\Delta R} \quad (2.17)$$

where k is the thermal conductivity of ice ($k \approx 2.2 \times 10^5 \text{ erg cm}^{-1} \text{ sec}^{-1} \text{ deg}^{-1}$), R_A is the radius of Ariel (see Table 1.1), ΔT is the difference in temperature between the base of the mantle and the surface, and ΔR is the mantle thickness. If the base of the mantle is maintained just below the melting temperature of $273^\circ K$, conduction through a rigid ice mantle can remove approximately 10^{17} ergs/sec of thermal energy. This is approximately the thermal energy that would be released by radioactive decay of materials in the core with chondritic abundances, with an energy production of about $10^{-7} \text{ ergs g}^{-1} \text{ s}^{-1}$ (Kaula 1968). It is also equivalent to the tidal heating produced if the eccentricity of Ariel is about 0.03, using the expression for the energy dissipation in a synchronously rotating satellite (Eq. 1.20) with reasonable physical parameters for Ariel (see Tables 1.1 and 1.2). However, if the temperature at the base exceeds about $230 - 240^\circ K$, the mantle can become unstable to solid state convection (see Chapter 3 for a discussion of solid state convection). The steady state configuration for the mantle in this case is a convecting solid layer topped by a rigid surface layer. This configuration is capable of removing heat at a significantly higher rate than the conducting configuration, and can still remain mostly solid.

Peale (1988) estimates that the solid state convection process could remove up

to approximately 7×10^{17} ergs/sec of thermal energy from the interior of Ariel. The equivalent tidal dissipation rate in the satellite would result if the eccentricity of Ariel was about 0.09 at the 5:3 resonance, not taking into consideration possible enhancement due to radioactive decay in the interior. A higher eccentricity would result in large-scale melting of water ice in the interior of Ariel. The maximum eccentricity reached by Ariel in this study for an escaping trajectory is approximately 0.023. Considering our results in the planar approximation, it seems unlikely that passage through the 5:3 resonance alone could have melted much of the interior of Ariel. Furthermore, the rate of energy dissipation in Ariel equals the rate of energy input to the orbit due to dissipation in Uranus for an eccentricity of about 0.05 (for $Q = 6600$). This provides an upper limit on the eccentricity that can be maintained through tidal evolution of the orbit. Since this eccentricity is lower than that estimated for the melting of the interior of Ariel, the eccentricity of Ariel may never reach the value required for melting. Umbriel is clearly not thermally affected by passage through the resonance – it requires an eccentricity of about 4 times that of Ariel for an equivalent tidal thermal input.

Note, however, that if ammonia hydrate is present in quantity, significant partial melting of this material may have occurred at temperatures greater than about $175^\circ K$, even without the presence of tidal heating. The addition of tidally generated heat may have increased the volume throughout which partial melting may have occurred.

Perturbations on the planar-eccentric problem due to the secular interactions between the satellites and Titania and the coupling between the eccentricity-type resonances and inclination-type resonances do not appear to significantly change the qualitative features of passage through this resonance, although the motion is more complicated.

The Hamiltonian studied in this project neglects dissipation in the satellite dur-

ing the relatively brief period of resonance passage. The damping of the eccentricities can slow down the rate of evolution of δ , as well as directly affecting the x_i and y_i . A possible consequence is that the system could have remained in the chaotic zone for a longer period of time, and may have reached higher eccentricities before escaping.

The passage of Ariel and Umbriel through the 5:3 mean-motion commensurability is significantly affected by the presence of a large chaotic zone. The mechanism of capture is qualitatively different from the isolated resonance capture mechanism. Relatively large eccentricities are obtained while the trajectory is chaotic even if the trajectory ultimately escapes. Peak eccentricities for escaping trajectories are typically of order 2 to 3 times the initial eccentricities. Eccentricities increases large enough to have a significant affect on the thermal history of Ariel have not been found in this study.

Chapter 3

The Miranda-Umbriel 3:1 Commensurability

3.1 Introduction

The 3:1 mean-motion commensurability involving Miranda and Umbriel would have been encountered if $Q < 39,000$. This commensurability also involves three second-order eccentricity resonances and three second-order inclination resonances. This resonance differs somewhat from the Ariel-Umbriel 5:3 commensurability in that the masses are not nearly equal: Umbriel is approximately 16 times more massive than Miranda. This implies that the resonant interaction will affect the orbit of Miranda more strongly than that of Umbriel. In addition, the orbits are further apart, and the secular interaction is not as strong. However, because of the small J_2 of Uranus, the resonances at this commensurability are not well separated, and significant chaotic behaviour is expected. We have indeed found significant chaotic zones at the Miranda-Umbriel 3:1 mean-motion commensurability, and during passage through this resonance, both the orbital elements and the thermal history of Miranda may have been significantly affected by the presence of these chaotic zones.

The inclination-type resonances would have been encountered first during passage through this commensurability. Section 3.2 explores the evolution through these resonances in the circular-inclined approximation. This is comparable to

studying the planar-eccentric approximation in the last chapter: the circular-inclined problem can be reduced to two degrees of freedom, allowing detailed study of the phase space. It is found that temporary capture into one of the resonances involving the orbital node of Miranda can explain the current anomalously high orbital inclination.

In Section 3.3, we explore evolution through the eccentricity-type resonances in the planar-eccentric approximation. The orbital eccentricity of Miranda can reach a value comparable to the equilibrium eccentricity during evolution through a large chaotic zone associated with these resonances, and then escape from the resonant interaction. The high orbital eccentricity retained by Miranda can damp to the current value during the time since resonance passage.

Both of the mechanisms described above are robust to secular perturbations. In the next Section, we consider the eccentric-inclined problem. Evolution through the inclination-type resonances is qualitatively unaffected by the low initial eccentricities: the orbital inclination of Miranda jumps to a value comparable to its current value. During evolution through the large chaotic zone associated with the eccentricity-type resonances, the high inclination of Miranda affects the eccentricities somewhat through higher-dimensional chaos, causing larger variations of e_M within the chaotic zone and allowing the orbital eccentricity of Miranda to reach larger values than in the planar-eccentric approximation. Upon escaping from the resonance, Miranda is left with both a high orbital eccentricity and a high orbital inclination, while the orbital elements of Umbriel remain relatively unaffected.

Section 3.5 discusses the importance of these results for the Uranian satellite system. By providing a plausible explanation of the current anomalously high orbital inclination of Miranda, this resonance can be used to constrain the Q of Uranus. Possible effects of resonance passage on the thermal history of Miranda are discussed: it is found that although tidal heating during the high-eccentricity

phase of resonance passage probably did not melt large quantities of water ice, it is possible that partial melting of ammonia hydrate may have occurred over much of the volume of the satellite, possibly explaining the bizarre thermal history indicated by the unusual surface of this satellite.

3.2 The Circular-Inclined Problem

The Miranda-Umbriel semimajor axis ratio increases as a result of tidal evolution, and the resonant combination of mean longitudes $3\lambda_U - \lambda_M$ passes from negative circulation to positive circulation during evolution through the resonance. Because the orbital nodes regress due to the planetary oblateness and the orbital pericenters precess, resonant arguments involving the nodes are stationary earlier in the evolution than those involving the pericenters, and therefore the inclination-type resonances are encountered before the eccentricity-type resonances.

We begin by studying the circular-inclined approximation, in order that we may attempt to gain some understanding of the dynamics of the 3:1 mean-motion resonance which are relevant to the evolution of the inclinations of the orbits of these two satellites. In Section 3.2.1, the resonance coordinates are discussed; the development of the Hamiltonian is given in Appendix I (Section 7.1.2). The circular-inclined Hamiltonian can be reduced to two degrees of freedom, allowing us to study the phase space of the problem in detail, using the Poincaré surface of section technique. In order to speed the numerical calculations, we have employed the second-order resonant mapping developed in Appendix II.

Numerical integrations of the evolution through this commensurability (Section 3.2.2) reveal very interesting behaviour. As the satellites approach the commensurability at low inclinations, the system behaves as if the resonances are isolated, with a significant probability of capture into one of the second-order resonances involving the orbital node of Miranda. Following capture into one of these resonances, the evolution of the system is temporarily quasiperiodic. The trajectory occupies a region of phase space in which one of the resonant arguments involving the orbital node of Miranda librates, and the evolution can initially be described well by the single resonance theory. However, as the orbital inclination of Miranda increases, the separatrix associated with the resonance broadens into a sizable chaotic zone.

Large-scale chaos ensues after the separatrices of the individual second-order resonances merge. The onset of large-scale chaos is well predicted by the resonance overlap criteria. The presence of a large chaotic region does not by itself guarantee that the trajectory will become chaotic, though, since the chaotic zone does not fully engulf the libration regions. However, there is another dynamical mechanism at work in this system. As the system evolves within one of the resonances, the frequency of small-amplitude libration of the resonant argument increases. The system encounters low-order commensurabilities between the libration frequency and other fundamental frequencies. These form a secondary resonant structure in the pendulum-like phase space of this dynamical system. If the system is captured into one of these secondary resonances, the trajectory is eventually “dragged” away from the center of the libration region and into the large chaotic zone, and the system can escape from the primary resonance, with the orbit of Miranda retaining a relatively high inclination comparable to its current value. The results of our numerical experiments agree well with analytic predictions of the width of the chaotic separatrix, the onset of large-scale chaos accompanying the overlap of primary resonance regions, and the locations of the secondary resonances between degrees of freedom in this problem.

In Section 3.2.3, the investigation of this resonance is generalized to include the effects of the secular interactions between Ariel and the satellites involved in the inclination resonances. The interesting mechanisms found in the circular-inclined problem are still present—the orbital inclination of Miranda still jumps to a high value before the system escapes from the resonance.

3.2.1 Resonance Coordinates

We refer the reader to Appendix I for a development of the Hamiltonian of the circular-inclined problem. In this Chapter, the subscript M refers to Miranda, and the subscript U to Umbriel.

We choose as resonance coordinates

$$\begin{aligned}\sigma_M &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\Omega_M) \\ \sigma_U &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\Omega_U)\end{aligned}\tag{3.1}$$

which with the mean longitudes λ_M and λ_U form a complete set of canonical variables for our problem. In terms of the Delaunay momenta $L_i \approx m_i \sqrt{GM a_i}$ and $H_i = L_i \cos(i_i)$, the momenta conjugate to σ_M and σ_U are

$$\begin{aligned}\Sigma_M &= L_M - H_M \approx \frac{L_M}{2} i_M^2 \\ \Sigma_U &= L_U - H_U \approx \frac{L_U}{2} i_U^2\end{aligned}\tag{3.2}$$

The momenta conjugate to $\gamma_M = \lambda_M$ and $\gamma_U = \lambda_U$:

$$\begin{aligned}\Gamma_M &= L_M + \frac{1}{2}(\Sigma_M + \Sigma_U) \\ \Gamma_U &= L_U - \frac{3}{2}(\Sigma_M + \Sigma_U)\end{aligned}\tag{3.3}$$

are integrals of the motion, since we are averaging over motion on timescales of the orbit periods.

For low inclinations, the coefficient in the Hamiltonian (Eq. 7.5) $A \approx \frac{1}{4}(3n_U - n_M)$. Because of the differential expansion of the orbits due to tidal dissipation in Uranus, the mean-motion ratio decreases, and $A \approx 0$ where the mean motions are exactly commensurate. We define the parameter in the problem, δ , to be the nonresonant contributions to $3n_U - n_M - \dot{\Omega}_M - \dot{\Omega}_U$, or $4A + 2(C + D)$. It provides us with a measure of distance from the resonance, and changes sign in the middle of the resonance region. During resonance passage, the fractional change in δ , proportional

to the fractional change in A , is much larger than the fractional changes in the other coefficients, which are proportional to changes in the semimajor axis ratio as it appears in the Leverrier coefficients. We therefore ignore changes in the coefficients other than A .

3.2.2 Evolution through the Resonance

Rate of Evolution

Tidal dissipation within Uranus results in differential decreases in the satellite mean motions. During passage through the resonance, the fractional change in the parameter δ is large compared to the fractional changes in the other coefficients, which will therefore be taken to be constant. The time rate of change of δ is

$$\dot{\delta} = 4\dot{A} \approx (3\dot{n}_U - \dot{n}_M) - 16B(\dot{\Sigma}_M + \dot{\Sigma}_U) \quad (3.4)$$

The orbits of the satellites expand due to tidal friction (see Eq. 1.6):

$$\frac{\dot{a}_i}{a_i} = 3k_2 n_i \frac{m_i}{M} \left(\frac{R}{a_i} \right)^5 \frac{1}{Q} \quad (3.5)$$

and the inclinations damp at a rate (Eq. 1.7):

$$\frac{\dot{i}_i}{i_i} \approx -\frac{1}{4} \frac{\dot{a}_i}{a_i} \quad (3.6)$$

due to tidal friction in the planet, where k_2 and Q are, respectively, the potential Love number and specific dissipation function for Uranus. Using numerical values appropriate for the Uranian satellites (see Table 1.1):

$$\frac{3\dot{n}_U}{\dot{n}_M} = \frac{3m_U}{m_M} \left(\frac{a_M}{a_U} \right)^8 \approx 0.12 \quad (3.7)$$

so the Miranda:Umbriel semimajor axis ratio increases. The timescales of inclination damping due to tidal friction in the planet are very long, at minimum of order 10^{11} years for Miranda and 10^{12} years for Umbriel.

The inclinations also damp due to tidal dissipation in the satellites. Neglecting variations in eccentricity, Eq. 1.12 yields the relationship

$$\frac{\dot{i}_i}{i_i} \approx -\frac{1}{2i_i^2} \frac{\dot{E}}{E} \quad (3.8)$$

and from Eq. 1.21,

$$\frac{\dot{i}_i}{i_i} \approx -\frac{3}{2} k_2 n_i \frac{M}{m_i} \left(\frac{R}{a_i}\right)^5 \frac{1}{Q} \left(\frac{\theta_i}{i_i}\right)^2 \quad (3.9)$$

where θ_i is the obliquity of the satellite rotation axis.

Any free obliquity a satellite may have quickly damps to an equilibrium value on a timescale comparable to the despinning timescale, given by (*e.g.* Peale 1977):

$$\tau_{despin} \approx \frac{2}{15} \frac{\Delta\omega m_i a_i^6 Q_i}{k_{2i} G M^2 R_i^3} \quad (3.10)$$

where $\Delta\omega$ is the change in spin rate. For Miranda, τ_{despin} is less than about 10^4 years.

For a satellite in a fixed orbit, the equilibrium obliquity is zero, and no significant inclination damping occurs. However, if the orbit precesses, for example due to planetary oblateness, the satellite occupies a *Cassini state* (*e.g.* Peale 1969, 1977), which relates the direction of the satellite spin vector, the orbit normal, and the normal to the invariable plane, which for the Uranian satellite system may be taken to be the planet's equatorial plane. The equilibrium Cassini states are given by the following relation (see Peale 1969):

$$\left[\frac{3}{4} n_i \left(\frac{C-A}{C} \right)_i - \frac{3}{16} n_i \left(\frac{B-A}{C} \right)_i \right] \sin 2\theta_i + \frac{3}{16} n_i \left(\frac{B-A}{C} \right)_i \sin \theta_i = \dot{\Omega}_i \sin(i_i - \theta_i) \quad (3.11)$$

where A , B , and C are the principal moments of inertia of the satellite listed in increasing order of magnitude, and $\dot{\Omega}_i$ is the rate of nodal regression (Eq. 1.3). For a satellite relatively close to its primary, the most important stable Cassini state (Cassini state 1) is near the normal to the orbit plane. If we approximate the triaxial hydrostatic figure of a synchronously rotating, tidally deformed satellite as a prolate spheroid ($B = C$), then the relationship between i_i and θ_i is just

$$\frac{i_i}{\theta_i} \approx \left(\frac{a_i}{R} \right)^2 \frac{1}{J_2} \left(\frac{B-A}{C} \right)_i \quad (3.12)$$

where numerical factors have been left out. Since for a relatively small tidal deformation Δr_T due to the tidal potential U_T

$$\left(\frac{B-A}{C}\right)_i \approx \frac{\Delta r_T}{R_i} \approx \frac{h_f U_T}{R_i g_i} \approx h_f \frac{M}{m_i} \left(\frac{R_i}{a_i}\right)^3 \quad (3.13)$$

at the satellite equator, where $h_f = 5/2$ is the fluid Love number appropriate for long-timescale deformation, then

$$\frac{i_i}{\theta_i} \approx 2.5 \frac{M}{m_i} \left(\frac{a_i}{R}\right)^2 \left(\frac{R_i}{a_i}\right)^3 \frac{1}{J_2} \quad (3.14)$$

For Miranda, this expression yields $i_M/\theta_M \approx 150$, and the timescale of inclination damping due to tidal friction in the satellite is of order 10^{13} years.

Inclination damping is therefore not important during resonance passage, or even during the subsequent evolution on timescales comparable to the age of the solar system.

The most significant term in Eq. 3.4 is therefore:

$$\dot{\delta} = -\dot{n}_M \approx \frac{3G^2 M^2 m_M^3}{2\Gamma_M^3} \frac{\dot{a}_M}{a_M} \quad (3.15)$$

The rate of evolution through the resonance depends on the specific dissipation function Q for Uranus. Q is constrained by the dynamical history of the satellite system (see Chapter 1). The lower limit of 6600 places Miranda and Ariel at the same distance from Uranus at the time of formation of the solar system. Upper limits on Q can be established if the current orbital configuration can only result from passage through a particular resonance. The upper limit for Q is the value that would allow passage of that resonance near the time of formation of the solar system.

Numerical exploration of this system is constrained by availability of computer time. Lower dissipation rates (higher Q) require longer integration times. It may not be feasible to integrate a trajectory through a resonance with an effective dissipation rate which is within the constraints described above. We parameterize the

effective dissipation rate as script $\mathcal{Q}$, to distinguish the numerical parameter which determines the rate of evolution in our simulation from the physical parameter. We desire a value of $\mathcal{Q}$ low enough that we can adequately study the system, but high enough that artifacts in the dynamics do not appear. In Paper I, it was found that artifacts appeared in the dynamics of the 5:3 resonance even at very slow dissipation rates. We have carried out a similar study to determine the influence of the rate on the dynamics of the 3:1 resonance.

This study involved integrating sets of trajectories through the resonance at different dissipation rates. The initial coordinates for each trajectory in a set were determined as follows. For an initial trajectory, with physical parameters $a_M = 4.8642$, $a_U = 10.1179$, $i_M = 0.005$ radians, $i_U = 0.005$ radians, $\sigma_M = \pi/2$, and $\sigma_U = 3\pi/2$, the coordinates $x_M = 0.0$, $y_M = 0.0009212$, $x_U = 0.0$, and $y_U = -0.004574$, and the parameter $\delta = -0.364003$ were computed. These inclinations are comparable to the current inclinations of the Uranian satellites excluding Miranda. From this initial point, the coordinates of 19 additional points, spaced in time by 66 mapping periods ($T = \frac{2\pi}{40}$ yr) were computed using the mapping without tidal dissipation. These 20 points formed the initial coordinates of the trajectories, each starting with the same energy, δ , and action (area enclosed by a trajectory in phase space), but with different phases.

The results of this study are shown in Fig. 3.1. As discussed in the introduction, the trajectories are generally captured into one of the resonances involving the orbital node of Miranda, evolve to high inclination, and escape from the resonance. Plotted in Fig. 3.1 are the mean final orbital inclinations of Miranda and Umbriel for each set as a function of dissipation rate. The mean final orbital inclination of Miranda is quite high, of order a few degrees, for $\mathcal{Q} > 2$. For smaller values of $\mathcal{Q}$, the mean inclination drops sharply, and for much higher dissipation rates the final orbital inclinations of Miranda and Umbriel are the same as the initial values:

they have been dragged through the resonance without displaying any interesting behaviour.

The sudden change in mean final inclination with rate can be understood in the following way. At low inclinations, the resonance behaviour is described well by the theory of evolution through isolated mean-motion resonances. This theory assumes that the timescale of evolution of the parameter δ is much longer than the periods of the fundamental frequencies in the problem. These frequencies are the secular frequencies and the frequencies of low-amplitude libration in each of the resonances. The libration frequency of a given resonance increases as a captured trajectory evolves within the resonance, whereas the nonresonant frequencies remain nearly constant over the timescale of resonance passage. We can express the rate of tidal evolution as the change of libration frequency in one libration period divided by the libration frequency ($\Delta\omega_L/\omega_L$). If this dimensionless quantity is of order unity, it is expected that the assumption of adiabatic invariance of the action will no longer be valid, since the timescale of evolution is similar to the dynamical timescale.

The resonance involving only the orbital node of Miranda is the first resonance encountered and we shall refer to it as resonance 1. The Hamiltonian for this resonance considered independently includes only one of the angle-dependent terms in Eq. 7.5:

$$\mathcal{H}_1 = \frac{1}{2}(\delta + 2(C - D) + 16B\Sigma_U)\Sigma_M + 4B\Sigma_M^2 + 2F\Sigma_M \cos(2\sigma_M) \quad (3.16)$$

To find the libration frequency, we expand the Hamiltonian about the resonance libration center (pendulum approximation) and consider the motion near the stable fixed point. The frequency of small-amplitude librations for this resonance can thus be expressed:

$$\omega_L^2 = -4F(\delta - \delta_0) \quad (3.17)$$

where $\delta_0 = -2(C - D) + 4F - 16B\Sigma_U$ is the value at which the libration zones first appear in the phase space of resonance 1 (see Appendix III for a discussion

Figure 3.1: Mean escape orbital inclinations of Miranda and Umbriel for ensembles of 20 trajectories as a function of tidal dissipation rate, expressed in terms of the effective specific dissipation function of Uranus Q . For $Q < 3.3$ ($\dot{a}_M/a_M > 2 \times 10^{-10}$ per orbit period), the trajectories are dragged through the resonance without displaying any interesting behavior.

of isolated second-order resonances). The change in the libration frequency in one period is $\Delta\omega_L \simeq -4\pi F\dot{\delta}/\omega_L^2$. From Eq. 3.15, with $\dot{a}_M/a_M \approx 9.4 \times 10^{-14}$ per orbit period for $Q = 6600$, $\dot{\delta} \approx 4.1 \times 10^{-4}/Q$ in our units. For $i_M = i_U = 0.005$ radians well before the resonance is encountered, $\delta_0 = 0.531180$. At the point that the trajectory is captured into the resonance, $\delta = 0.563$, and $\Delta\omega_L/\omega_L \approx 3.6/Q$. Therefore, for $Q \approx 3.6$, artifacts should begin to appear in the dynamics. In the numerical experiments, significant artifacts appear for $Q < 3.3$ ($\dot{a}_M/a_M > 2 \times 10^{-10}$ per orbit period), in excellent agreement with the analytic prediction.

The slowest rate used to integrate many trajectories, $Q = 110$ ($\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period), with $\Delta\omega_L/\omega_L = 0.03$, is slow enough that the artifacts do not influence the dynamics. This has been verified by integrating individual trajectories through the resonance with lower dissipation rates ($\dot{a}_M/a_M \approx 9 \times 10^{-13}$ per orbit period).

Numerical Results

In Fig. 3.2, the distributions of time-averaged final inclinations for Miranda and Umbriel for $Q = 110$ are shown. A few of the trajectories escaped from the resonances altogether, but for most trajectories the orbital inclination of Miranda reached at least a few degrees. It appears from this figure that there is a significant probability that Miranda could have escaped from the 3:1 resonance with Umbriel with an orbital inclination close to the present value of $4^\circ.22$. Clearly, the orbit of Umbriel can be significantly affected by the resonant interaction as well. The restricted three-body problem would not be a good approximation for this system.

In Fig. 3.3, the energies of the trajectories are plotted vs. δ . We have parameterized the energy in a manner analogous to that used in Chapter 2: $\Delta E = \mathcal{E} - E_1$, where $\mathcal{E}$ is the value of the Hamiltonian, and

$$E_1 = \begin{cases} 0, & \delta < 2(C - D) + 4H \\ -(\delta - 2(C - D) - 4H)^2 / 64B, & \delta \geq 2(C - D) + 4H \end{cases} \quad (3.18)$$

Figure 3.2: Distributions of escape orbital inclinations of Miranda and Umbriel for an ensemble of 20 trajectories evolved through the resonance with $\mathcal{Q} = 110$, or $\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period. Both orbits are affected by the resonant interaction. An orbital inclination of about 4° for Miranda can clearly be attained during passage through the 3:1 commensurability with Umbriel.

The dynamical significance of the parameterization used in Chapter 2 was that positive ΔE indicated forced libration of one of the resonant arguments, due to the division of the energy surface into two regions. In this problem, the phase space is not as complicated (see below). The phase space more nearly resembles the pendulum-like phase space of the single-resonance theory. Libration is possible on quasiperiodic islands surrounded by a chaotic separatrix. As in the surfaces of section described in Chapter 2, the quartic nature of the Hamiltonian means that the section condition $x_U = 0$ can have 2 or 4 values of y_U conjugate to x_U , forming 4 root ‘families’. Positive ΔE indicates the appearance of an excluded region near the origin of the surface of section which plots y_M vs. x_M when $x_U = 0$. The appearance of the excluded region provides a connection in phase space between the ‘outer’ families of roots and the ‘inner’ root families via the chaotic separatrix. Therefore, with positive ΔE , a trajectory can evolve into the quasiperiodic zone on the inner root families, which allows it to escape from the resonance. As in Chapter 2, the heavily shaded region in the upper part of the plot is the region in $\delta, \Delta E$ space in which evolution is not allowed. The region where ‘macroscopic’ chaotic regions may be found, which is most of the accessible region with positive ΔE , is lightly shaded. There is a tiny region of parameter space near the boundary of the prohibited region where the energy surface divides the phase space into librating regions, but it does not appear to be significant (it is not even visible at this scale). The new mechanism for capture discovered in the program of research described in Chapter 2 does not appear to be important in the Miranda-Umbriel 3:1 resonant interaction.

In the parameter space, captured trajectories are linear, and escaping trajectories decrease quadratically to large negative ΔE . For the set of trajectories shown in Fig. 3.3, evolution occurs along three ‘branches’. Quasiperiodic evolution is shown as solid lines, and chaotic behaviour is shown as dotted lines. There are two linear

Miranda/Umbriel 3:1 Resonance Energy

Figure 3.3: Trajectories in δ , ΔE parameter space with $Q = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. Evolution occurs along one of three “branches”. The upper linear branch involves temporary capture into resonance 1 (i_M^2), and the lower linear branch involves temporary capture into resonance 2 ($i_M i_U$). Trajectories escape to large negative ΔE .

(capture) branches, and an escape branch. Trajectories on the escape branch escape from all three of the resonances, and evolve to large negative ΔE .

The upper linear branch involves capture into resonance 1. Half of the trajectories in this set were captured into this resonance. The single-resonance theory predicts a capture probability of 0.42 for $i_M = 0.005$ before the resonance is encountered.

The lower linear branch involves capture into the mixed resonance, which we will refer to as resonance 2. Seven of the ten trajectories not captured into resonance 1 were captured into resonance 2. For low values of the inclinations, when the resonance is first encountered, the inclination of Umbriel can be approximated as constant, since the mass of Umbriel is much greater (a factor of about 17) than the mass of Miranda. The mixed resonance then takes the form of a first-order resonance (see Peale, 1988), and the probability of capture can be estimated (see Appendix III). In this approximation, the single-resonance theory predicts a capture probability of ≈ 0.69 for $i_M \approx 0.003$ after escaping from the first resonance, and $i_U = 0.005$.

None of the trajectories in this set were captured into the resonance involving only the orbital node of Umbriel, or resonance 3. The single-resonance theory predicts a very low probability of capture into this resonance of only 0.017 for $i_U = 0.005$ before the resonance is encountered. The orbital inclination of Umbriel does not decrease significantly upon escape from the resonances involving the orbital node of Umbriel.

In both of the resonances involving the node of Miranda, captured trajectories evolve within the resonance for some period of time, become chaotic, then escape to large negative ΔE . In both cases, escape is only possible within certain ranges of δ , rather than occurring randomly. The dynamical explanation for this is given below.

The energy of the Hamiltonian evolves slowly as δ changes. Since the evolution of the energy is slow compared to the dynamical timescales, we can “freeze” the energy and δ at any point in the evolution and study the structure of the phase space by computing surfaces of section, in order to understand the qualitative behaviour of the trajectory. The surface of section chosen for study plots y_M vs. x_M when $x_U = 0$. As indicated earlier, the phase space of this problem is not as complicated as the phase space of the Ariel-Umbriel 5:3 resonance. The interesting behaviour takes place on the outer pair of root families, until the trajectory escapes into that part of phase space including the inner pair. The structures of the phase space on the two root families forming the outer pair are similar. Therefore, the evolution of the trajectory through the resonance can be adequately studied on the root family with the largest numerical value. Furthermore, we have chosen to plot Σ_M vs. σ_M rather than y_M vs. x_M , in order to better display the pendulum-like structure of the phase space.

Fig. 3.4 shows the evolution of the inclinations with δ for a trajectory captured in resonance 1. The maximum and minimum inclinations of the satellites within a short interval of δ ($\Delta\delta \approx 0.004$) are plotted vs. δ . Before the resonance is encountered, the secular interaction between the satellites is weak, and the inclinations are nearly constant. When the resonance is encountered at low inclination, the trajectory crosses a narrow separatrix into one of two libration zones. In Fig. 3.5, the trajectory generates the closed curve in the region of libration corresponding to positive σ_M , which it occupies until escaping from the resonance. The separatrix is also displayed in the figure. After capture into the resonance, the average orbital inclination of Miranda increases proportional to the square root of δ , as predicted by the single-resonance theory. During the quasiperiodic evolution within the resonance, the inclination oscillates smoothly about a gradually increasing mean value. Meanwhile, the average orbital inclination of Umbriel stays nearly constant, but

with gradually increasing oscillations about the mean value. As the inclination increases, perturbations due to the other resonances broaden the separatrix into a chaotic region (Fig. 3.6). The libration region of the mixed resonance has appeared in the phase space, also surrounded by a chaotic separatrix. In this figure, the trajectory generates the closed loop in the libration region corresponding to positive σ_M .

At $\delta \approx 2.4$ in Fig. 3.4, there is a small but noticeable change in the magnitude of the oscillations about the mean value of the orbital inclination of Miranda. Fig. 3.7 displays a surface of section showing the trajectory at this point in the evolution. The frequency of small-amplitude librations has been increasing as δ increases, and at this point there is a 1:4 commensurability between the libration frequency and the frequency of circulation of σ_U . Note that the chaotic regions have continued to increase in width, and the libration region of the two resonances are nearly overlapping. A secondary resonant structure with four islands has formed within the libration zone of resonance 1, due to the fact that there is a 1:4 commensurability between the frequency of libration of σ_M and the motion of σ_U . As δ evolves, the chain of islands moves away from the libration center, eventually encountering the trajectory. In Fig. 3.7, the trajectory traces out the separatrix associated with this secondary resonance. The trajectory must cross the separatrix, and may either become trapped in the island structure or escape to the region near the libration center. The trajectory in Fig. 3.4 passes through the secondary commensurability without becoming trapped on the islands, but as it does so, the area enclosed by the trajectory on the phase plane decreases from a value corresponding to the area enclosed by the outer boundary of the separatrix at the point of transition to a value corresponding to the area enclosed by the inner boundary of the separatrix at the point of transition. When the area enclosed by the trajectory decreases, so does the amplitude of oscillation of the inclination about the mean value in the libration

Figure 3.4: Variations in inclination for a trajectory captured into resonance 1 ($\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period). The maximum and minimum inclination of each satellite are plotted in intervals of $\Delta\delta \approx 0.004$. The evolution is regular and quasiperiodic at low inclination, but secondary resonances between the degrees of freedom pull the trajectory into the chaotic zone at high inclination. The trajectory escapes from the mean-motion resonance via the chaotic zone, leaving the orbit of Miranda with an inclination comparable to its present value.

Figure 3.5: Surface of section showing the phase space of the trajectory shown in Fig. 3.4 just after capture into resonance 1 ($\delta = 0.578$). The evolution at this point is well described by the single-resonance theory: there are two libration regions surrounded by a regular looking separatrix. The trajectory occupies only one of the libration regions.

Figure 3.6: Surface of section for the trajectory shown in Fig. 3.4 at $\delta = 1.206$. The libration zone of the mixed ($i_M i_U$) resonance has appeared on the section. The chaotic separatrix has become significantly wider. The width of the chaotic separatrix agrees well with analytic estimates.

region, as seen in Fig. 3.4 at $\delta \approx 2.4$.

At a somewhat higher value of δ ($\delta \approx 3.7$), the trajectory encounters a 1:3 commensurability between the same two frequencies (Fig. 3.8). This time, the trajectory is trapped in the island structure, and as δ increases, the islands are pulled away from the libration center (Fig. 3.9). As this takes place, the amplitude of the oscillations of the inclination of Miranda increases (see Fig. 3.4) as the trajectory visits each island on the surface of section sequentially. At the same time, the mean orbital inclination of Umbriel decreases slightly. At $\delta \approx 4.8$, the trajectory is dragged into the chaotic zone.

Fig. 3.10 shows the phase space just before the trajectory enters the chaotic zone. The three loops generated by the trajectory in the previous figure have themselves broken into chains of islands: the trajectory has been captured into a *tertiary* resonance. In Fig. 3.11, we see the neighborhood of one set of these islands at an expanded scale. The trajectory, which is still quasiperiodic at this point, generates the 7 star-shaped islands, which are now fully surrounded by the chaotic zone. The shapes of these islands are characteristic of a period-4 instability bifurcation (see *e.g.* Hénon 1969), which is a *quaternary* resonance in this system! As the unstable equilibria associated with this bifurcation move towards the centers of the tertiary islands, the region of stability on each island shrinks, until eventually the trajectory becomes unstable and enters the large chaotic zone. Note that there are 6 islands associated with another tertiary resonance visible in the region of stability associated with the secondary period-3 resonance. This cascade of higher and higher order resonant structure is characteristic of the development of instabilities in chaotic dynamical systems.

During evolution through the chaotic zone, the variations of inclination are irregular, but the average inclination of Miranda continues to increase. The chaotic zones surrounding the libration zones of the resonances now overlap, and the chaotic

Figure 3.7: Surface of section for the trajectory shown in Fig. 3.4 at $\delta = 2.358$. The chaotic zones have continued to increase in width. The trajectory has just encountered a secondary 1:4 resonance between the libration frequency in the i_M^2 resonance and the frequency of circulation of σ_U , and it therefore traces out the separatrix associated with this secondary resonance. This particular trajectory is not captured into the secondary resonance.

Figure 3.8: Surface of section for the trajectory shown in Fig. 3.4 at $\delta = 3.614$. The two libration regions overlap, in agreement with the predictions of the resonance overlap criterion. The trajectory has just encountered a 1:3 secondary resonance between the libration frequency in the i_M^2 resonance and the frequency of circulation of σ_U . The trajectory is captured into this secondary resonance.

Figure 3.9: Surface of section for the trajectory shown in Fig. 3.4 at $\delta = 4.242$. The trajectory, which occupies the three quasiperiodic islands in the libration region, is being dragged away from the libration center due to capture into the 1:3 secondary resonance.

Figure 3.10: Surface of section for the trajectory shown in Fig. 3.4 at $\delta = 4.766$. The trajectory generates the 21 tiny islands which are at this point fully surrounded by the chaotic zone. Three stable regions associated with the 1:3 secondary resonance are still visible. The trajectory has been dragged from these stable regions by a period-7 *tertiary* resonance.

Figure 3.11: This is a portion of the phase space shown in the previous figure, but at a larger scale and with more trajectories plotted. The still-quasiperiodic trajectory generates the seven star-shaped islands in the chaotic zone, which are themselves undergoing a period-4 *quaternary* bifurcation. This bifurcation is unstable, and the trajectory enters the large chaotic zone. Note the period-6 tertiary resonance visible in the large quasiperiodic region.

zone is a dominant feature of the phase space. As δ continues to increase, the extent of the chaotic region reaches a maximum, and decreases for much larger values of δ . It never completely engulfs the libration zones. Although escape from the orbital resonance is possible during this period of chaotic evolution, the trajectory shown in Fig. 3.4 enters the libration zone of resonance 1 again at $\delta \approx 5.0$ and becomes quasiperiodic, with a much larger amplitude of oscillation than it had before encountering the 1:3 secondary commensurability.

At $\delta \approx 8.0$ there is a short period of chaotic behaviour when the trajectory encounters the 1:2 commensurability between the same two frequencies (Fig. 3.12). The chaotic separatrix associated with this secondary commensurability has a visible width. As the trajectory crosses this narrow chaotic zone, it exhibits the brief period of chaotic behaviour seen in Fig. 3.4. It is then trapped into the island structure. The oscillation amplitude of Miranda's orbital inclination again increases with δ , while the mean orbital inclination of Umbriel decreases slightly. The trajectory is eventually dragged by the 1:2 islands into the large chaotic zone associated with the primary resonance at $\delta \approx 9.8$. Shortly thereafter the system escapes from the resonance, with orbital inclinations of about $4^\circ.6$ for Miranda and $0^\circ.26$ for Umbriel.

Fig. 3.13 shows the behavior of the inclinations for a trajectory captured into resonance 2. Again, well before the resonance is encountered, the mutual perturbations are weak, and the inclinations are nearly constant. This time, the trajectory escapes from the first resonance, and the mean orbital inclination of Miranda decreases suddenly as it crosses the separatrix. The trajectory is then captured into the mixed resonance, and both inclinations oscillate about mean values which increase with time (δ). At $\delta \approx 6$ and $\delta \approx 7.8$ there are sudden changes in the oscillation amplitudes similar to those described above in Fig. 3.4. Fig. 3.14 shows the surface of section for this trajectory at $\delta \approx 12.5$, just before it enters the chaotic region. Note that the trajectory generates 3 islands, which are being pulled away from the

Figure 3.12: Surface of section for the trajectory shown in Fig. 3.4 at $\delta = 8.011$. The trajectory, having returned to the libration region from the large chaotic zone, has just encountered the 1:2 secondary resonance between the libration frequency in the i_M^2 resonance and the frequency of circulation of σ_U . The chaotic separatrix associated with this secondary commensurability has a visible width. The trajectory becomes chaotic briefly while crossing this narrow chaotic zone, causing the feature at $\delta \approx 8$ in Fig. 3.4. The trajectory is captured into this secondary resonance and is again dragged into the large chaotic zone, from which it escapes the mean-motion resonance.

libration center as δ increases. These islands occupy a region of phase space in which the argument of the mixed resonance $(\sigma_M + \sigma_U)/2$ librates, although this is not obvious in the variables plotted. At $\delta \approx 12.7$, the trajectory is dragged into the chaotic zone, which also surrounds the neighbouring libration zone for resonance 1. It is possible for a trajectory, after entering the chaotic region, to temporarily enter the librating region of resonance 1. The trajectory in Fig. 3.13 escapes shortly after entering the chaotic zone, leaving Miranda with an average orbital inclination of about $3^\circ.7$, and Umbriel with an average orbital inclination of about $0^\circ.79$.

We have also carried out a numerical simulation of the evolution of trajectories through the 3:1 inclination resonances with higher initial $i_M \approx 1.0^\circ$ approaching the commensurability, in order to determine if the dynamical mechanisms described above are still important.

The initial coordinates for 10 trajectories in this ensemble were determined as follows. From an initial state, with physical parameters $a_M = 4.8612$, $a_U = 10.1179$, $i_M = 0.0175$ radians, $i_U = 0.005$ radians, $\sigma_M = \pi/2$, and $\sigma_U = 3\pi/2$, the coordinates of 9 additional states, spaced in time by 66 mapping periods ($T = \frac{2\pi}{40}$ yr) were computed using the mapping without tidal dissipation. As in the previously discussed ensemble, these 10 states formed the initial coordinates of the trajectories, each starting with the same energy, δ , and action (area enclosed by a trajectory in phase space), but with different phases. The trajectories were then tidally evolved through the resonances using the mapping, with $\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period.

Of these 10 trajectories, 2 were temporarily captured into the mixed $(i_M i_U)$ resonance, and the orbital inclination of Miranda evolved to a higher value. The other 8 trajectories escaped from the inclination resonances altogether. Fig. 3.15 shows the evolution of the orbital inclinations of the satellites for one of the captured trajectories. The evolution is very similar to that shown in Fig. 3.13: the orbital

Figure 3.13: Variations in inclination for a trajectory captured into resonance 2 ($\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period). The maximum and minimum inclination of each satellite are plotted in intervals of $\Delta\delta \approx 0.005$. The inclinations of both orbits increase during the temporary quasiperiodic phase of evolution within the resonance. Secondary resonances between the degrees of freedom pull the trajectory into the chaotic zone at high inclination. The trajectory can escape from the mean-motion resonance via the chaotic zone, leaving the orbit of Miranda with an inclination comparable to its present value.

Figure 3.14: Surface of section showing the phase space of the trajectory shown in Fig. 3.13 at $\delta = 12.512$, just before escaping from the resonance. The chaotic zone is noticeably wider than in the previous figures, but still does not engulf the libration regions. The trajectory occupies the three quasiperiodic islands in the libration region of the mixed resonance, and is being dragged away from the libration center. It is possible for a trajectory to jump from one libration region to another via the chaotic zone.

inclination of Miranda decreases suddenly upon passing through the i_M^2 resonance, then both inclinations increase after capture into the mixed resonance. After passing through a number of secondary resonances, the trajectory is dragged into the chaotic zone and soon thereafter escapes from the inclination resonances, leaving Miranda with an orbital inclination more than three times the initial value of about one degree. Again, the increase in i_U is relatively small.

The dynamical mechanisms leading to the spectacular increases in the orbital inclination of Miranda are therefore important even if the inclination is relatively high approaching the resonance.

Secondary Resonances

The mechanism of escape in all cases of temporary capture appears to require the presence of the secondary resonances between the libration frequency and other fundamental frequencies. The points in the evolution at which these secondary commensurabilities are encountered can be estimated by comparing the frequencies of small-amplitude libration for the resonances considered independently to the fundamental frequencies of the nonresonant part of the Hamiltonian.

The largest contributions to the nonresonant Hamiltonian are given by the expression:

$$\mathcal{H}_0 = \frac{1}{2}(\delta + 2(C - D))\Sigma_M + \frac{1}{2}(\delta - 2(C - D))\Sigma_U + 4B(\Sigma_M + \Sigma_U)^2 \quad (3.19)$$

Therefore, the “zeroth order” frequencies of the system are:

$$\begin{aligned} \dot{\sigma}_M &= \frac{\partial \mathcal{H}_0}{\partial \Sigma_M} = \frac{1}{2}(\delta + 2(C - D)) + 8B(\Sigma_M + \Sigma_U) \\ \dot{\sigma}_U &= \frac{\partial \mathcal{H}_0}{\partial \Sigma_U} = \frac{1}{2}(\delta - 2(C - D)) + 8B(\Sigma_M + \Sigma_U) \\ &= \dot{\sigma}_M - 2(C - D) \end{aligned} \quad (3.20)$$

For resonance 1, the average value of $\dot{\sigma}_M$ is zero. Therefore, $\dot{\sigma}_U \approx -2(C - D)$, which has the same magnitude as the difference in frequency between resonance 1

Figure 3.15: Variations in the orbital inclination of Miranda (a) and Umbriel (b) for a trajectory with initial $i_M \approx 1^\circ$ ($\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period). The maximum and minimum inclination of each satellite are plotted in intervals of $\Delta\delta \approx 0.0036$. Even with the large initial value of i_M , this trajectory is captured into the mixed ($i_M i_U$) resonance, and the orbital inclination of Miranda evolves to a relatively high value.

Table 3.1: Frequency Ratios: Resonance 1

$\omega_L : \dot{\sigma}_U$	δ	i_M
1:4	2.3	2.0°
1:3	3.6	2.7°
1:2	7.5	4.1°

and resonance 2. The commensurabilities between the degrees of freedom should occur where the libration frequency of resonance 1 (Eq. 3.17) and $\dot{\sigma}_U$ form integer ratios. The most important of these commensurabilities, and the values of δ and i_M at which they occur, are summarized in Table 3.1. There is a good correlation between the analytical predictions and the numerical results.

As indicated earlier, the mixed resonance can be approximated as a first order resonance when it is first encountered by the trajectory, and the inclination of Umbriel is nearly constant ($\Sigma_M \ll \Sigma_U$). After the trajectory is captured, the phase space of the mixed resonance is quite complicated (see also Sinclair 1974) as both inclinations increase. For large δ , the mixed resonance can be approximated as a second-order resonance in $(\sigma_M + \sigma_U)/2$, and $\Sigma_M \approx \Sigma_U$. In this approximation, the Hamiltonian for resonance 2 becomes

$$\mathcal{H}_2 = (\delta + 16B\Sigma_{U0})\Sigma_M + 16B\Sigma_M^2 + 2G\Sigma_M \cos(\sigma_M + \sigma_U) \quad (3.21)$$

The frequency of libration of the mixed resonance argument $(\sigma_M + \sigma_U)/2$ is therefore:

$$\omega_L^2 \approx 2G(\delta + 2G + 16B\Sigma_{U0}) \quad (3.22)$$

The average value of $\dot{\sigma}_M + \dot{\sigma}_U$ is zero in the mixed resonance, so the frequency of circulation of the other degree of freedom $\dot{\sigma}_M - \dot{\sigma}_U = 2(C - D)$. The most important commensurabilities between these two degrees of freedom are summarized in Table 3.2. Again, the correlation between these predicted values and numerical

Table 3.2: Frequency Ratios: Resonance 2

$\omega_L : (\dot{\sigma}_M - \dot{\sigma}_U)$	δ	i_M	i_U
1:5	5.8	2.4°	0.5°
1:4	8.6	3.0°	0.6°
1:3	14.6	4.0°	0.8°

experiments (*e.g.* Fig. 3.13) is very good. Note that the values of δ and i_M at which the trajectory enters the chaotic zone are larger than the values at which the secondary resonances are encountered, due to the time it takes for the islands to be dragged away from the libration center and into the chaotic zone.

The full Hamiltonian of this problem is coupled through both linear and non-linear terms. The strength of the coupling increases as the inclination(s) increase during temporary capture into one of the resonances. Therefore, the “zeroth order” frequencies described above are modified, and the motion becomes more complicated. Secondary resonances between the libration frequency and various frequency components of this more complicated motion are possible, but do not appear to significantly affect the evolution.

Extent of the Chaotic Zone

Because the chaotic zone does not completely engulf the libration regions for the trajectories discussed above, it is possible for a trajectory to be permanently captured into a resonance, if it passes through all of the secondary commensurabilities between the degrees of freedom.

The finite extent of the chaotic region can be understood by considering the theoretical estimate of the width of a perturbed separatrix (Chirikov 1979). The half-width of the separatrix is expressed in terms of the chaotic variation of the

energy integral for the perturbed resonance:

$$\frac{\Delta \mathcal{E}}{\mathcal{E}_{sz}} = 4\pi\epsilon\lambda^3 e^{\frac{-\pi\lambda}{2}} \quad (3.23)$$

where λ is the ratio of the perturbation frequency to the frequency of small-amplitude librations and the perturbation parameter ϵ is the ratio of the coefficients of the perturbing term to the coefficients of the perturbed term.

For perturbations of resonance 1 by resonance 2, the perturbed Hamiltonian can be written

$$\mathcal{H} = \mathcal{H}_1 + \epsilon\mathcal{V} \quad (3.24)$$

where $\mathcal{H}_1$ is given in Eq. 3.16 and

$$\epsilon\mathcal{V} = 2G\sqrt{\Sigma_M\Sigma_U} \cos(\sigma_M + \sigma_U) \quad (3.25)$$

Therefore

$$\lambda = \frac{\dot{\sigma}_M - \dot{\sigma}_U}{\omega_L} = \frac{2(C - D)}{\omega_L} \quad (3.26)$$

and

$$\epsilon = -\frac{G}{F}\sqrt{\frac{\Sigma_U}{\Sigma_{M_1R}}} \quad (3.27)$$

where

$$\Sigma_{M_1R} = \frac{(\delta - \delta_0)}{-16B} \quad (3.28)$$

is the libration center of resonance 1. Due to the secular interaction, ϵ and λ are time-dependent in the full problem. However, since the secular interaction is weak, the variations are small.

For low δ the separatrix width is exceedingly small, but it increases exponentially as the libration frequency increases. However, for higher δ , the exponential contribution approaches 1 asymptotically, while the other contributions decrease as $\frac{1}{\delta^2}$, so $\Delta\mathcal{E}/\mathcal{E}_{sz}$ decreases. The perturbation parameter ϵ is linear in i_U – hence, the chaotic zone in Fig. 3.14 is larger than the chaotic zone in Fig. 3.9, although the inclination of Miranda is similar for the two cases.

However, because of the overlap of the libration regions of resonance 1 and resonance 2 at large values of δ , the chaotic zone will be larger than predicted by Eq. 3.23. The resonance overlap criterion (Chirikov 1979) predicts large-scale chaos where the sum of the half-widths of the libration regions plus the half-widths of the chaotic separatrices exceeds the spacing between libration centers. The point at which the libration regions overlap can be estimated, using the formulae for the resonances considered independently. The extent of the libration region for resonance 1 is given by:

$$\Sigma_{M_1} = \Sigma_{M_1 R} \pm \Delta \Sigma_{M_1} \quad (3.29)$$

where

$$\Delta \Sigma_{M_1} = \begin{cases} \Sigma_{M_1 R}, & \delta_0 \leq \delta < -2(C - D) - 4F + 16B\Sigma_U \\ \frac{\sqrt{-F(-4B\Sigma_{M_1 R} + F)}}{-2B}, & \delta \geq -2(C - D) - 4F + 16B\Sigma_U \end{cases} \quad (3.30)$$

For large δ , the extent of the libration region for resonance 2 can be expressed (see Eq. 3.21):

$$\begin{aligned} \Sigma_{M_2} &= \Sigma_{M_2 R} \pm \Delta \Sigma_{M_2} \\ &\approx \Sigma_{M_2 R} \pm \frac{\sqrt{G(-16B\Sigma_{M_2 R} - G)}}{-8B} \end{aligned} \quad (3.31)$$

For a trajectory captured into resonance 1, σ_M is oscillating, while the argument of resonance 2, $(\sigma_M + \sigma_U)/2$, circulates with a frequency $-(C - D) \approx -0.165 \text{ yr}^{-1}$. Therefore, at the center of the resonance 2 island, $\dot{\sigma}_M = C - D \approx 0.165 \text{ yr}^{-1}$. Since $\dot{\sigma}_M = \partial \mathcal{H}_1 / \partial \Sigma_M$ for resonance 1, this frequency difference can be related to a difference in Σ_M between libration centers. The stable equilibrium on the surface of section corresponding to the largest quartic root ($\sigma_U = \pi/2$) is at $\sigma_M = -\pi/2$, so $\sigma_M + \sigma_U = 0$. Therefore, for a trajectory captured into the libration zone of resonance 1, the center of libration of resonance 2 is at

$$\Sigma_{M_2 R} = \frac{(\delta - (\delta_0 + 2(C - D)))}{-16B} \quad (3.32)$$

for $\delta \geq \delta_0 + 2(C - D)$. Σ_{M_2R} maintains a nearly constant separation from Σ_{M_1R} on the phase plane defined by σ_M and Σ_M until the chaotic separatrices merge.

For the case where resonance 1 is perturbed by resonance 2, the unperturbed libration widths, as well as $\Delta\mathcal{E}/\mathcal{E}_{sx}$ for resonance 1 ($i_U = 0.005$), are plotted in Fig. 3.16. The libration width plotted for resonance 2 is from the second-order resonance approximation described above, which is a good approximation only for the larger values of δ represented. The predicted value of δ at which the libration zones should overlap is about 4.3. For the trajectory shown in Fig. 3.4, the two libration zones overlap for $\delta \approx 3.6$. In the real system, of course, the shape of each libration zone is distorted by the presence of the other. The width of each chaotic separatrix tends to be larger on the side nearest the other libration region.

Fig. 3.17 illustrates the accuracy with which the Chirikov formula (Eq. 3.23) can be used to predict the width of the chaotic separatrix. The phase space in the neighborhood of one of the unstable fixed points predicted by the single resonance theory is displayed. The trajectory (not shown on the figure at this scale) has been captured into the resonance, but is at a point in the evolution well before the value of δ at which resonance overlap is predicted. The bar in the figure denotes the analytic estimate of the width $\Delta\sigma$ of the large chaotic zone near the unstable fixed point. There is good agreement between the predicted width and the $\Delta\sigma$ in which points appear in the large chaotic zone at its narrowest extent. Note, however, that the edge of the chaotic region is ill-defined: it breaks up into narrow connected bands surrounding chains of quasiperiodic islands, and forms a very complicated filamentary structure. Outside of the large chaotic zone, the narrow chaotic separatrices surrounding chains of islands are isolated by invariant curves. The predicted width is close to but within the $\Delta\sigma$ at which the invariant curves appear on the figure at the edge of the large chaotic zone. The accuracy of the analytic formula in predicting the width of the chaotic zone has been similarly verified over 5 orders

Figure 3.16: Analytic predictions of the positions and widths of the libration zones of the i_M^2 and $i_M i_U$ resonances relative to the libration center of the i_M^2 resonance (solid lines), and the width of the chaotic separatrix of the i_M^2 resonance (dashed lines). Large-scale chaos is present when the libration regions overlap.

of magnitude of $\Delta\mathcal{E}/\mathcal{E}_{sz}$.

The success of the single-resonance theory in describing this system at low inclinations can be understood by considering the width of the chaotic zone at the point of capture into the resonance. Eq. 3.23 predicts that the relative width of the separatrix at the point of capture is approximately $\Delta\mathcal{E}/\mathcal{E}_{sz} \approx 5.0 \times 10^{-15}$. For $Q = 6600$, the change in the Hamiltonian energy of the trajectory relative to the separatrix in one libration period is about *twelve orders of magnitude* larger than the separatrix width. The trajectory is very quickly pulled across this narrow separatrix by the tidal action. Therefore, chaotic behavior does not affect the process of initial capture into the resonance, if the inclinations of the satellites were comparable to the values we have chosen prior to encountering the resonance.

Miranda/Umbriel 3:1 Resonance

Figure 3.17: Surface of section showing the phase space of the trajectory shown in Fig. 3.4 shortly after capture into resonance 1 ($\delta = 0.788$), with $\Delta \mathcal{E} / \mathcal{E}_{zz} = 8.3 \times 10^{-4}$. The region near one of the unstable fixed points is shown. The bar denotes the analytic prediction of the width of the chaotic separatrix near the unstable fixed point. Although the boundary of the chaotic region is not well defined, the distances of the points furthest from the center of the chaotic zone at its narrowest extent agree well with the predicted width. The fact that the invariant curves just outside the large chaotic zone are slightly further from the center than the predicted width supports this result.

3.2.3 Secular Perturbations

In this section we consider the perturbations on the inclination-type resonances produced by the secular variations due to the other satellites. To get a qualitative idea of the effects of secular perturbations, we consider only the perturbations on the circular inclined problem due to Ariel as a first approximation, since they are the most significant (see Dermott and Nicholson 1986, Laskar 1986). The development of the Hamiltonian is given in Appendix I. The addition of these perturbations results in the problem no longer being reducible to two degrees of freedom, so it is not possible to study the phase space using surfaces of section.

We define $\phi = \lambda_M - 3\lambda_U$, $p_i \approx i_i\sqrt{L_i} \cos(-\Omega_i)$, and $q_i \approx i_i\sqrt{L_i} \sin(-\Omega_i)$. The momentum conjugate to ϕ is $\Phi = L_M - L_{MR}$. The resonant value L_{MR} is defined at $\frac{\partial \mathcal{H}}{\partial \Phi} = 0$:

$$L_{MR} = \frac{L_U + 3L_M}{3 + (3)^{1/3} \frac{m_U}{m_M}} \quad (3.33)$$

In this model, evolution due to tidal dissipation changes the value of Φ at a rate of $\dot{\Phi} = \dot{L}_M - \dot{L}_{MR}$. The value of Φ can be converted into a value for δ .

Fig. 3.18 shows the inclinations for a trajectory computed using this model. The initial parameters for the trajectory were: $a_M = 4.8617$, $a_A = 7.0846$, $a_U = 10.1179$, $i_M = 0.005$, $i_A = 0.00541$, $i_U = 0.005$, $\sigma_M = 90^\circ$, $\sigma_A = -263^\circ$, and $\sigma_U = 270^\circ$. Away from the resonance, both the inclination of Miranda and the inclination of Umbriel are strongly perturbed by Ariel, resulting in much larger variations of the inclinations about the mean values than were found in the circular-inclined case. This trajectory is captured into resonance 1. Again, there is a phase of quasiperiodic evolution within the resonance, during which various commensurabilities between the degrees of freedom are encountered. During this period, the amplitude of the variations in the orbital inclination of Ariel increases slightly, while the orbital inclination of Umbriel remains relatively unaffected. At $\delta \approx 4.2$, the trajectory encounters the 1:3 commensurability between the libration frequency and the frequency of circu-

lation of σ_U . The trajectory is pulled into the chaotic zone at $\delta \approx 5.4$, and the inclinations of all three orbits vary chaotically until $\delta \approx 6.8$. The trajectory escapes from the resonance, leaving Miranda with an average orbital inclination of just over 4 degrees, and leaving the orbital inclinations of Ariel and Umbriel virtually unchanged.

Other trajectories show similar behaviour. The presence of another degree of freedom provides a new set of secondary commensurabilities. The presence of Ariel complicates the structure of the resonance, and there appear to be resonant interactions involving the orbital node of Ariel. However, the mechanisms that allow evolution to high orbital inclination for Miranda in the circular-inclined two-satellite approximation are still present in this model.

Figure 3.18: Variations in the inclinations for a trajectory perturbed by the secular variations of Ariel. The variations in the orbital inclinations of Miranda and Umbriel are larger than those due only to their mutual perturbations. The orbital inclination of Miranda increases considerably during temporary capture into resonance 1.

3.3 The Planar-Eccentric Problem

Upon escaping from the inclination-type resonances at the 3:1 mean-motion commensurability, Miranda and Umbriel would then have encountered the eccentricity-type resonances. In this Section, we consider the interaction between the two satellites in the planar-eccentric approximation, for the moment ignoring the fact that it is likely that the orbital inclination of Miranda was comparable to the current value at the time of encountering the eccentricity resonances. We have chosen to explore the planar-eccentric problem because, as with the circular-inclined problem, it may be reduced to two degrees of freedom and studied in detail using surfaces of section. It is expected that, as in the space Ariel-Umbriel 5:3 resonant problem, the qualitative evolution of the eccentricities is similar to that in the planar-eccentric problem, because of the nature of the coupling between the eccentricity and inclination resonances: they are coupled only through nonlinear terms, so chaotic interaction between them involves higher-dimensional chaos, which in general is important only over much longer timescales than the primary chaotic zones associated with each type of resonance (see Chapter 1). This expectation is verified in Section 3.4, in which we explore the space Miranda-Umbriel 3:1 resonance problem.

First, in Section 3.3.1, the resonance coordinates are discussed. As discussed above, the planar-eccentric model can be reduced to two degrees of freedom, analogous to the circular-inclined problem. In Section 3.3.2, the evolution of trajectories through the eccentricity-type resonances is discussed. Numerical integrations using the second-order resonant mapping reveal that the orbital eccentricity of Miranda increases substantially during chaotic evolution through the resonances, with a high probability of escape from the commensurability. Temporary capture into one or more of the 3 eccentricity-type resonances may occur when they are first encountered, with capture into the mixed resonance likely. For low initial orbital eccentricities, this phase of evolution may be described by the single resonance theory. Again,

however, low-order secondary commensurabilities between the libration frequency of the mixed resonant argument and the difference between the circulation frequencies of the other two resonant arguments drag the trajectory into a large chaotic zone, similar to the behaviour in the circular-inclined problem. The chaotic zone is larger at the eccentricity-type resonances than at the inclination-type resonances, due in part to the presence of strongly overlapping secondary resonances between the circulation frequencies of the resonant arguments. Evolution through the resonance is dominated by this chaotic zone. The orbital eccentricity of Miranda may reach values comparable to the equilibrium eccentricity before the satellites escape from the resonance.

In Section 3.3.3, investigation of the eccentricity-type resonances is generalized to include the effects of the secular perturbations due to Ariel. The large chaotic zone is still present in this model: the orbital eccentricity of Miranda may still jump to a high value. The evolution is somewhat more complicated than in the unperturbed planar-eccentric model, but qualitatively similar.

3.3.1 Resonance Coordinates

The development of the Hamiltonian for the planar-eccentric case follows the standard development of second-order eccentricity-type resonances given in Appendix I (Section 7.1.3).

We choose as resonance coordinates

$$\begin{aligned}\sigma_M &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\varpi_M) \\ \sigma_U &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\varpi_U)\end{aligned}\tag{3.34}$$

which with the mean longitudes λ_M and λ_U form a complete set of canonical variables for our problem. In terms of the Delaunay momenta $L_i \approx m_i\sqrt{GMa_i}$ and $G_i = L_i \cos(e_i)$, the momenta conjugate to σ_M and σ_U are

$$\begin{aligned}\Sigma_M &= L_M - G_M \approx \frac{L_M}{2}e_M^2 \\ \Sigma_U &= L_U - G_U \approx \frac{L_U}{2}e_U^2\end{aligned}\tag{3.35}$$

The momenta conjugate to $\gamma_M = \lambda_M$ and $\gamma_U = \lambda_U$:

$$\begin{aligned}\Gamma_M &= L_M + \frac{1}{2}(\Sigma_M + \Sigma_U) \\ \Gamma_U &= L_U - \frac{3}{2}(\Sigma_M + \Sigma_U)\end{aligned}\tag{3.36}$$

are integrals of the motion, since we are averaging over motion on timescales of the orbit periods.

Analogous to the circular-inclined case, for low eccentricities, $A \approx \frac{1}{4}(3n_U - n_M)$, and we define the parameter in the problem, δ , to be the nonresonant contributions to $3n_U - n_M - \dot{\varpi}_M - \dot{\varpi}_U$, or $4A + 2(C + D)$.

3.3.2 Evolution through the Resonance

Rate of Evolution

The time rate of change of δ is

$$\dot{\delta} = 4\dot{A} \approx (3\dot{n}_U - \dot{n}_M) - 16B(\dot{\Sigma}_M + \dot{\Sigma}_U) \quad (3.37)$$

The most significant terms in the above expression are found to be (see Chapter 5 for a similar, more complete analysis):

$$\dot{\delta} = 4\dot{A} \approx -\dot{n}_M - 16B\dot{\Sigma}_M \quad (3.38)$$

yielding the following expression

$$\dot{\delta} = -\dot{n}_M \left(1 - \frac{21}{2} D_M e_M^2 \left[1 + 3 \frac{m_M}{m_U} \left(\frac{a_M}{a_U} \right)^2 \right] \right) \quad (3.39)$$

where $D_M = 80.8$ for $Q = 6600$ and $Q_M = 100$ (see Eq. 1.11). If an equilibrium between the rate of expansion of the orbits and tidal dissipation in Miranda is established, δ will be stationary at the equilibrium eccentricity

$$e_M = \sqrt{\frac{1}{\frac{21}{2} D_M \left(1 + 3 \frac{m_M}{m_U} \left(\frac{a_M}{a_U} \right)^2 \right)}} = \sqrt{\frac{1}{10.9 D_M}} = 0.034 \quad (3.40)$$

As we will see, the evolution through these resonances is strongly chaotic, and an equilibrium may not be established. In this initial investigation, we ignore the effects of dissipation in the satellites on the rate of evolution, and set the rate to be that used in the circular-inclined problem (Eq. 3.15). For low eccentricities comparable to the current value, this will be a good approximation.

In addition to influencing the rate of evolution through the resonance, tidal dissipation in the satellites also introduces explicit frictional terms into the equations of motion through the damping of the eccentricities. The importance of these frictional terms depends on the eccentricity damping timescale compared to the timescale of resonance passage. Using Eq. 1.9, an eccentricity damping timescale

of about 10^8 years is determined for Miranda and about 10^9 years for Umbriel ($Q_i = 100$). During evolution through these resonances, the parameter δ changes by about 10 yr^{-2} , indicating a resonance passage timescale of order 10^8 years. Therefore, direct tidal damping of e_M may be important, but have not been included in this initial study.

As in the circular-inclined problem, we have carried out a study to determine the influence of the rate on the dynamics of the 3:1 resonance. This study again involved integrating sets of trajectories through the resonance at different dissipation rates. The rate of tidal evolution is parameterized by the effective specific dissipation function of Uranus, which is designated by script $\mathcal{Q}$. The initial coordinates for each trajectory in a set were determined as follows. For an initial trajectory, with physical parameters $a_M = 4.8630$, $a_U = 10.117R$, $e_M = 0.005$, $e_U = 0.005$, $\sigma_M = \pi/2$, and $\sigma_U = 3\pi/2$, the coordinates $x_M = 0.0$, $y_M = 0.0009213$, $x_U = 0.0$, and $y_U = -0.004574$, and the parameter $\delta = -0.617450$ were computed. These eccentricities are comparable to the current eccentricities of the Uranian satellites. From this initial point, the coordinates of 19 additional points, spaced in time by 66 mapping periods ($T = \frac{2\pi}{40} \text{ yr}$) were computed using the mapping without tidal dissipation. These 20 points formed the initial coordinates of the trajectories, each starting with the same energy, δ , and action (area enclosed by a trajectory in phase space), but with different phases. These trajectories were then tidally evolved through the resonance using the mapping.

The results of this study are shown in Fig. 3.19. As discussed in the introduction, the trajectories are generally captured into the mixed resonance temporarily, enter a large chaotic zone, evolve to relatively high eccentricity of Miranda, and escape from the resonance. Plotted in Fig. 3.19 are the mean final orbital eccentricities of Miranda and Umbriel for each set as a function of dissipation rate. The mean final orbital eccentricity of Miranda is quite high, of order 0.02, for $\mathcal{Q} > 0.1$. For smaller

values of $\mathcal{Q}$, the mean eccentricity drops sharply, and for much higher dissipation rates the final orbital eccentricities of Miranda and Umbriel are the same as the initial values: they have been dragged through the resonance without displaying any interesting behaviour.

As in the circular-inclined problem, we can investigate the appearance of dynamical artifacts by comparing the rates at which they appear to the adiabatic rate. Again, we express the rate of tidal evolution as the change of libration frequency in one libration period divided by the libration frequency ($\Delta\omega_L/\omega_L$). If this dimensionless quantity is of order unity, it is expected that the assumption of adiabatic invariance of the action will no longer be valid, since the timescale of evolution is similar to the dynamical timescale.

Using Eqs. 3.21 and 3.22 with parameters appropriate for the eccentricity-type resonances, we find that the change in the libration frequency of the mixed ($e_M e_U$) resonance in one period is $\Delta\omega_L \simeq -2\pi G\dot{\delta}/\omega_L^2$. As in the circular-inclined problem, for $Q = 6600$, $\dot{\delta} \approx 4.1 \times 10^{-4}/\mathcal{Q}$ in our units. For $e_M = e_U = 0.005$ radians well before the resonance is encountered, $\delta_0 = -2G - 16B\Sigma_{U0} = 0.854$. The point that the trajectory is captured into the resonance is not well defined, due to the fact that the trajectory may diffuse through a complex network of chaotic zones before capture, as we will see later. However, from examination of surfaces of section, the value of δ at which capture occurs may be constrained to be greater than 0.869 and less than 0.975. Therefore, the adiabatic rate for these initial conditions should lie between $\mathcal{Q} = 0.3$ and $\mathcal{Q} = 6.9$. The lower limit is within a factor of 3 of the observed rate at which artifacts appear. However, because of the presence of significant chaotic zones at the point of capture (see *e.g.* Fig. 3.25), it is not expected that the adiabatic rate will be relevant, except for very low initial eccentricities.

The slowest rate used to integrate many trajectories, $\mathcal{Q} = 110$ ($\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period), with $\Delta\omega_L/\omega_L = 0.03$, is slow enough to represent

Figure 3.19: Mean escape orbital eccentricities of Miranda and Umbriel for ensembles of 20 trajectories as a function of tidal dissipation rate, expressed in terms of the effective specific dissipation function of Uranus $\mathcal{Q}$. For $\mathcal{Q} < 0.1$ ($\dot{a}_M/a_M > 6 \times 10^{-9}$ per orbit period), the trajectories are dragged through the resonance without displaying any interesting behavior.

the dynamics of resonance passage well. This has been verified by integrating individual trajectories through the resonance with lower dissipation rates, *e.g.* Fig. 3.39 ($\dot{a}_M/a_M \approx 9 \times 10^{-13}$ per orbit period) and Fig. 3.40 ($\dot{a}_M/a_M \approx 2 \times 10^{-13}$ per orbit period).

Numerical Results

In Fig. 3.20, the distributions of time-averaged final eccentricities for Miranda and Umbriel for $\mathcal{Q} = 110$ are shown. As in the circular-inclined problem, the orbit of Umbriel is also significantly affected by the resonant interaction, although the changes in eccentricity are small compared to those of Miranda. The high orbital eccentricities attained by Miranda can damp to the current value in the time since resonance passage.

In Fig. 3.21, the energies of the trajectories are plotted vs. δ . Again, we have parameterized the energy in a manner analogous to that used in Chapter 2: $\Delta E = \mathcal{E} - E_2$, where $\mathcal{E}$ is the value of the Hamiltonian, and

$$E_1 = \begin{cases} 0, & \delta < 2(C - D) + 4F \\ -(\delta - 2(C - D) - 4F)^2 / 64B, & \delta \geq 2(C - D) + 4F \end{cases} \quad (3.41)$$

Positive ΔE indicates the appearance of an excluded region near the origin of the surface of section which plots y_U vs. x_U when $x_M = 0$, in a manner analogous to the circular-inclined problem. Again, the appearance of the excluded region provides a connection in phase space between the ‘outer’ families of roots and the ‘inner’ root families via the chaotic separatrix. Therefore, with positive ΔE , a trajectory can evolve into the quasiperiodic zone on the inner root families, which allows it to escape from the resonance. The heavily shaded region in the upper part of the plot is the region in $\delta, \Delta E$ space in which evolution is not allowed. The region where ‘macroscopic’ chaotic regions may be found, which is most of the accessible region with positive ΔE , is lightly shaded. The extent of this region was determined by evaluating the rate of growth of the distance between nearby trajectories (Lyapunov

Figure 3.20: Distributions of escape orbital eccentricities of Miranda and Umbriel for an ensemble of 20 trajectories evolved through the resonance with $Q = 110$, or $\dot{a}_M/a_M \approx 6 \times 10^{-12}$ per orbit period. Both orbits are affected by the resonant interaction. The high orbital eccentricity of Miranda retained after escape from the resonance can damp to the current value in the time since resonance passage.

exponent) for 20 trajectories with initial conditions on the surface of section which plots y_M vs. x_M when $x_U = 0$, for different values of δ and ΔE . There is a tiny region of parameter space near the boundary of the prohibited region where the energy surface divides the phase space into librating regions. This region is significant only for trajectories with very small initial eccentricities.

For most trajectories, the period of temporary quasiperiodic capture is quite short compared to that in the circular-inclined problem. The trajectories are chaotic for most of the evolution through the resonance, as indicated by the dashed portions of the trajectories. Note that the evolution of the energy with increasing δ in the chaotic zone is very regular: although different trajectories take somewhat different paths in δ , ΔE space, the slopes of the chaotic portions of each trajectory are very nearly equal. This regularity of energy evolution is also observed in the Ariel-Umbriel 5:3 resonance, for sufficiently slow rates of evolution (see Section 2.2.3), and may be related to the conservation of an ergodic adiabatic invariant during the chaotic phase of the evolution (see Chapter 1). The two trajectories which spend a period of time near the boundary of the excluded region are temporarily captured into the e_U^2 resonance, but at some point, escape from this resonance, are temporarily captured into the mixed resonance, and eventually enter the large chaotic zone.

The diagonal lines plotted in the chaotic region of δ , ΔE space indicate the maximum Miranda eccentricity achieved by a trajectory *while it is evolving chaotically*. For each value of δ , a trajectory is confined to the 3-dimensional surface in phase space defined by the conservation of the instantaneous value of the Hamiltonian energy, or ΔE . As explained in previous Sections of this thesis, the projection of the energy surface onto a two-dimensional subspace defining a surface of section defines the limits of allowed motion on that surface of section. For the Poincaré section which plots x_M vs. y_M when $x_U = 0$, this energy surface boundary is ap-

Miranda/Umbriel 3:1 Resonance Energies

Figure 3.21: Trajectories in δ , ΔE parameter space with $Q = 110$. The initial eccentricities of both satellites were 0.005. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. Most trajectories are temporarily captured into quasiperiodic resonance, then evolve through a chaotic zone and escape with higher eccentricities to large negative ΔE . Diagonal lines indicate maximum e_M in the chaotic zone: a = 0.01, b = 0.02, c = 0.03, d = 0.04, e = 0.05, f = 0.06, g = 0.07, h = 0.08.

proximately circular (see below). The large chaotic zone occupies the region of phase space adjacent to this boundary, and therefore the boundary of the surface of section defines the maximum allowed eccentricity of Miranda's orbit possible in the chaotic zone. The radius of the energy boundary at $y_M = 0$ on the surface of section may be analytically determined, due to the quartic nature of the Hamiltonian (Eq. 7.9), therefore relating δ and ΔE to the maximum e_M . This allows the maximum eccentricity achieved by any of the trajectories during evolution in the chaotic zone to be read directly off of the δ , ΔE parameter space plot.

The distribution of maximum Miranda eccentricities in the chaotic zone for this set of trajectories is shown in Fig. 3.22. It is quite possible for the orbit of Miranda to attain orbital eccentricities comparable to the equilibrium value of about 0.034 during passage through this resonance, even in the planar approximation.

Again, for this problem we can “freeze” the energy and δ of a trajectory at any point in the evolution and study the structure of the phase space by computing surfaces of section, in order to understand the qualitative behaviour of the trajectory. The surfaces of section chosen for study plot y_M vs. x_M when $x_U = 0$, which we designate section I, and σ_U vs. Σ_U when $x_M = 0$ ($\sigma_M = \pi/2$), which we designate section II. As with the circular-inclined problem, the phase space is simple enough that we need only study one quartic root family for each surface of section: the second largest root for section I and the largest root for section II.

Fig. 3.23 shows the evolution of the orbital eccentricities of Miranda and Umbriel for one of the trajectories. As in previous Sections, the maximum and minimum orbital eccentricity are plotted in a small interval of δ . Well before the resonance is encountered, both orbital eccentricities are nearly constant: the *mutual* secular interactions are quite weak. As the resonance is approached, the amplitude of eccentricity variation increases slightly. At $\delta \approx 0.5$, the trajectory encounters the e_U^2 resonance. Fig. 3.24 illustrates the phase space at this point in the evolution on sec-

Figure 3.22: The distribution of maximum eccentricities of the orbit of Miranda attained during passage through the 3:1 resonance with Umbriel. e_M can easily attain a value comparable to the equilibrium eccentricity (see text).

tion II. The trajectory is evolving through the narrow chaotic separatrix associated with the e_U^2 resonance. Numerous other trajectories in the surrounding phase space have been displayed also to illustrate the structure of the phase space at this point. Although the eccentricities are quite low, the chaotic zone is has a significant width at the point of encountering this resonance. Constrast this with the narrow separatrix at the point of encountering the i_M^2 resonance *e.g.* Fig. 3.5. As the trajectory evolves briefly within this narrow chaotic zone, the orbital eccentricity of Umbriel increases slightly (Fig. 3.23b). The trajectory escapes from the e_U^2 resonance, and evolves into the portion of phase space displayed in the lower part of Fig. 3.24.

Soon, at $\delta \approx 0.869$, the libration of the mixed $(e_M e_U)$ resonance appears in the phase space (Fig. 3.25). Notice that in the relatively brief interval of δ since the last surface of section, the chaotic zone has expanded to fill most of the displayed phase space. We still see the quasiperiodic libration regions associated with the e_U^2 resonance, surrounded by a significantly wider chaotic separatrix. We also see the quasiperiodic zone of libration of the mixed resonance argument, also surrounded by a chaotic separatrix. Between these two mean-motion resonance, the phase space is filled with a complex, interconnected network of narrow chaotic zones surrounding islands of quasiperiodic motion. The periodic orbits at the centers of these islands are associated with secondary resonances between the circulation frequencies of the resonant arguments σ_U and σ_M . In the region of phase space containing the e_U^2 resonance, σ_U is librating (stationary on average) while σ_M is circulating. In the region of phase space containing the mixed resonance, the mixed resonance argument $(\sigma_M + \sigma_U)/2$ is librating, so σ_M and σ_U must be circulating at equal and opposite rates. Therefore, the ratio $\dot{\sigma}_U : \dot{\sigma}_M$ goes from zero at the e_U^2 resonance to minus one at the mixed resonance. Between these extremes, there are numerous low-order commensurabilities between these frequencies. The libration regions associated with the strongest of these, along with the chaotic separatrices, show up in

Figure 3.23: Evolution of the orbital eccentricities of Miranda (a) and Umbriel (b) during passage through the 3:1 commensurability in the planar-eccentric approximation with $\frac{\dot{a}_M}{a_M} = 6 \times 10^{-12}$ per orbit period. The maximum and minimum eccentricities are plotted in intervals $\Delta\delta = 0.0035$. After initially being captured into the mixed ($e_M e_U$) resonance, the trajectory enters a large chaotic zone and the orbital eccentricity of Miranda evolves to a high value before the trajectory escapes from the resonance.

Figure 3.24: Surface of section showing the phase space of the trajectory shown in Fig. 3.23 just after encountering the chaotic separatrix associated with the e_U^2 resonance, at $\delta = 0.550$. Other trajectories have been shown in this and subsequent figures in order to display the structure of the surrounding phase space.

the region of phase space between the two mean-motion resonances. Because these secondary commensurabilities are sufficiently strong, the chaotic zones overlap and merge, forming the observed complex interconnected chaotic region.

Fig. 3.26 shows the same region of phase space in section I. The quasiperiodic zone associated with the libration region of the mixed resonance occupies much of the central region of the section. The complex interconnected chaotic region, along with the quasiperiodic regions, fill the remaining phase space. The libration regions of the e_J^2 resonance are not visible on this plot. The trajectory diffuses through the complex chaotic zone for a while, producing the slightly irregular evolution of the orbital eccentricities at $\delta \approx 0.9$ in Fig. 3.23. The trajectory is then captured with high probability into the mixed resonance, and both eccentricities increase during a temporary phase of quasiperiodic evolution. Fig. 3.27 shows the phase space on section I just after capture into the mixed resonance. The trajectory generates the loop in the quasiperiodic zone in the lower part of the figure. The chaotic zone has become noticeably more uniform in structure, although the secondary resonances are still in evidence. A new quasiperiodic zone associated with escape from the eccentricity resonances has appeared near the center of the plot. With further evolution, this new quasiperiodic zone occupies a larger fraction of the phase space (Fig. 3.28). The libration zone of the mixed resonance has expanded and occupies a region of phase space adjacent to the energy surface boundary on section I. The trajectory generates the loop in the lower part of the figure. The chaotic zone is much more uniform, the secondary resonance quasiperiodic islands occupying only a tiny fraction of the phase space. Fig. 3.29 shows this same region of phase space on Section II. The trajectory generates the loop in the mixed resonance quasiperiodic region in the lower left of the figure.

As the trajectory continues to evolve within the mixed resonance, the libration zones associated with the e_M^2 resonance appear in the phase space of section I

Figure 3.25: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 0.869$. The trajectory has escaped from the e_U^2 resonance, and is evolving through a complicated network of connected chaotic zones associated with secondary resonances between the frequencies of circulation of the resonant arguments. The libration zone of the mixed ($e_M e_U$) resonance has appeared in the lower left of the figure.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.26: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 0.869$ in Miranda variables (section I). The mixed resonance libration region occupies the central area of the figure.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.27: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 0.975$. The trajectory has been captured into the mixed resonance, and generates the loop in the quasiperiodic region in the lower part of the figure. The quasiperiodic region into which trajectories may escape has appeared near the origin of this figure.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.28: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 1.294$. The quasiperiodic region of escape has increased in size. The secondary resonances are much less obvious, and the chaotic zone appears much more uniform. The trajectory still occupies the quasiperiodic region associated with the mixed resonance.

Figure 3.29: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 1.294$, this time in Umbriel variables (section II). The trajectory generates the loop in the mixed resonance libration region in the lower left of the figure. Compared with Fig. 3.24, the chaotic zone appears much more uniform.

(Fig. 3.30). Much of the large quasiperiodic zone near the origin in previous Figures has become unstable due to the appearance of the chaotic separatrix associated with this resonance. The trajectory still generates the loop in the lower part of the figure. With further evolution, the e_M^2 resonance libration zones move away from the origin, and the quasiperiodic region corresponding to escape from this resonance has appeared near the origin of section I (Fig. 3.31). Traces of the single resonance structure (see Appendix III) remain, but greatly modified by the presence of the chaotic zone. The zone of libration of the mixed resonance argument in the lower region of the figure has become unstable, due to the appearance of a period-3 secondary commensurability between the frequency of libration of the resonant argument and the frequency of circulation of σ_U , as can be clearly seen in Fig. 3.32. In this plot, only the large chaotic zone has been shown. The trajectory has entered this chaotic zone, and the orbital eccentricities are undergoing the relatively large irregular oscillations seen in Fig. 3.23.

With further evolution in the chaotic zone, the quasiperiodic zone of escape increases in extent, and the libration regions of the e_M^2 resonance become unstable (Fig. 3.33). Note that the nearly circular boundary of section I extends to quite high values of eccentricity, and that the chaotic zone occupies the region of phase space adjacent to the boundary. At any point in the evolution, the maximum e_M in the chaotic zone is constrained by the instantaneous energy boundary, resulting in the well-defined upper limit of Miranda's orbital eccentricity in the chaotic zone seen in Fig. 3.23a.

As the trajectory continues to evolve, the width of the chaotic zone in section I phase space remains nearly constant, with the result that the amplitude of libration of eccentricity in the chaotic zone seen in Fig. 3.23 does not vary much. An exception to this trend is seen at $\delta \approx 5.5$, when the lower limit to e_M in the chaotic zone suddenly decreases slightly. This is caused by a secondary resonance between

Figure 3.30: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 1.4$. The libration regions and chaotic separatrix associated with the e_M^2 resonance have appeared in the region of escape. Compare to Fig. 9.1b. The trajectory still occupies the mixed resonance region.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.31: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 1.506$. The e_M^2 resonance libration zones have moved away from the origin, and a quasiperiodic region of escape has appeared. The figure still bears some resemblance to the single resonance picture (*e.g.* Fig. 9.1c). The mixed resonance libration zone has become unstable, and the trajectory occupies the large chaotic zone.

Figure 3.32: Surface of section showing the phase space available to the chaotic trajectory in Fig. 3.23 at $\delta = 1.506$, in Umbriel variables (section II). Only the chaotic zone is shown. A period-3 bifurcation has occurred in the mixed resonance libration region, causing it to become unstable.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.33: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 2.037$. The region of escape occupies the central part of the figure. The libration regions of the e_M^2 resonance have become largely unstable. The trajectory still occupies the large chaotic zone.

the circulation frequencies on the escape side of the resonance. The sequence of Figs. 3.34 - 3.36 show the evolution of such a secondary resonance. The narrow separatrix surrounding the 3 islands associated with this secondary resonance is shown to be evolving away from the origin as the trajectory evolves in the chaotic zone between $\delta = 2.462$ (Fig. 3.34) and $\delta = 3.523$ (Fig. 3.35). At $\delta \approx 5.5$, this secondary resonance merges with the large chaotic zone. As it does so, the chaotic zone penetrates around the separatrix surrounding the 3 islands (see Fig. 3.36), so the trajectory can reach slightly lower eccentricities than previously. The remnants of the quasiperiodic islands associated with this secondary resonance are seen near the inner boundary of the chaotic zone. Fig. 3.36 also shows a period-2 secondary resonance (the curve in the quasiperiodic region, nearest the inner edge of the chaotic zone), but this resonance is much weaker. However, each time one of these secondary resonance structures merges with the large chaotic zone, the structure of the phase space near the inner boundary of the chaotic region becomes very complex. It is a generic feature of chaotic Hamiltonian dynamical systems that the merging of two large chaotic zones results in the formation of an extremely complex structure, in *e.g.* the Standard Map (Chirikov 1979), and in the overlap of mean-motion resonances in this problem (Fig. 3.25). It may be more likely at such times for the trajectory in the chaotic zone to more easily enter a quasiperiodic zone, say, associated with some high-order secondary resonance near the inner edge of the chaotic zone, and then evolve into the large quasiperiodic zone. Therefore, these secondary resonances may be vital to the mechanism of escape from the resonance. Many other high-order secondary resonances are present (*e.g.* Fig. 3.37), continually merging with the large chaotic zone as the system evolves.

Eventually, the trajectory leaves the large chaotic zone, with a final orbital eccentricity of Miranda of about 0.048. Fig. 3.38 shows the section I phase space just after escape. The trajectory in Fig. 3.23 generates the quasiperiodic loop just inside

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.34: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 2.462$. A period-3 secondary commensurability between the frequencies of circulation of the resonant arguments has appeared.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.35: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 3.523$. The period-3 secondary commensurability moves away from the origin as the system continues to evolve.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.36: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 5.753$. The period-3 secondary commensurability has merged with the large chaotic zone, slightly increasing its extent and causing the feature at $\delta \approx 5.5$ in Fig. 3.23.

Miranda/Umbriel 3:1 Resonance Surface of Section

Figure 3.37: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 7.346$. A secondary commensurability with $\dot{\sigma}_U : \dot{\sigma}_M = 7 : 3$ is shown just about to merge with the inner boundary of the chaotic zone.

the inner boundary of the chaotic zone. As the system evolves to large negative ΔE , the eccentricities become nearly constant, except for some modulation due to the weak secular interaction.

Figs. 3.39 and 3.40 show the eccentricity evolution of two trajectories evolved at rates 6 times and 30 times slower, respectively. Both display qualitatively similar behaviour, with minor differences during the initial, low δ stages of evolution, due to the complexity of phase space at these times. For example, the trajectory shown in Fig. 3.39, after escaping from the e_U^2 resonance, is captured briefly into the mixed resonance, then the e_M^2 resonance, before entering the large chaotic zone. Both trajectories evolve to eccentricities comparable to the equilibrium value. Due to the slower rates, the irregular variations of the eccentricities tend to average out more between plotted points, with the result that the eccentricity “envelopes” are better defined. Note that in both cases, there is a sudden change in the behaviour of the eccentricities shortly after escape from the chaotic zone. These “glitches” are caused by the trajectories passing through the separatrices associated with the period-3 commensurability between the circulation frequencies of the resonance arguments, *e.g.* Fig. 3.35.

Fig. 3.41 shows the phase space for one of the trajectories in Fig. 3.23 which was temporarily captured into the e_U^2 resonance. The trajectory generates the barely visible set of islands just inside the boundary between the large chaotic zone and the e_U^2 resonance libration zone centered at $\sigma_U = 0$. At this point in the evolution, the trajectory has been captured into a 2:9 secondary resonance between the libration frequency and the circulation frequency of σ_M , and is being dragged away from the libration center. Furthermore, it has been trapped into a period-11 tertiary resonance, thus generating the many tiny islands visible on the section. The trajectory is just about to enter the large chaotic zone. Also shown on this figure are the 1:4 secondary resonance, near the center of the libration zone, and the remnants of

Figure 3.38: Surface of section showing the phase space available to the trajectory in Fig. 3.23 at $\delta = 7.452$. The trajectory has become quasiperiodic, and generates the loop near the inner boundary of the chaotic zone.

Figure 3.39: Eccentricity variations of a trajectory evolved through the Miranda-Umbriel 3:1 resonance with $\frac{\dot{a}_M}{a_M} = 1 \times 10^{-12}$ per orbit period. The maximum and minimum eccentricities are plotted in intervals of $\Delta\delta = 0.0035$. Initial evolution through the resonances is quite complicated.

Figure 3.40: Eccentricity variations of a trajectory evolved through the Miranda-Umbriel 3:1 resonance with $\frac{\dot{a}_M}{a_M} = 2 \times 10^{-13}$ per orbit period. The maximum and minimum eccentricities are plotted in intervals of $\Delta\delta = 0.0035$. The evolution is qualitatively similar to that of trajectories with higher dissipation rates.

Table 3.3: Frequency Ratios: e_U^2 Resonance

$\omega_L : \dot{\sigma}_M$	δ	e_U
1:5	1.52	0.0073
2:9	1.94	0.0080
1:4	2.53	0.0090
1:3	4.74	0.0121

the 1:5 secondary resonance, still visible in the large chaotic zone surrounding the libration region.

The Hamiltonian for this resonance considered individually is given by

$$\mathcal{H} = \frac{1}{2}(\delta - \delta_0)\Sigma_U + 4B\Sigma_U^2 + 2H\Sigma_U \cos(\sigma_U) \quad (3.42)$$

where $\delta_0 = 2(C - D) - 16B\Sigma_M$. Using the expression for the frequency of low-amplitude libration at the e_U^2 resonance

$$\omega_L^2 = 4H(\delta - \delta_0) \quad (3.43)$$

and Eqs. 3.20 with parameters appropriate to the eccentricity resonances, the locations of the secondary resonances can be predicted. Note that $\delta_0 \approx -0.3$ for $e_M = 0.005$. The predicted locations of the secondary resonances are given in Table 3.3.

Fig. 3.41 was computed at $\delta = 2.568$, close to the predicted location of the 1:4 bifurcation. Indeed, we see a chain of 4 islands very close to the libration center, in excellent agreement with the calculation. It is clear from the figure that the trajectory encounters many potentially important secondary resonances at this point in its evolution, and that it is likely for the trajectory to be dragged into the chaotic zone. Both of the trajectories shown in Fig. 3.23 which were captured into the e_U^2 resonance eventually escaped.

Figure 3.41: Surface of section showing the phase space of a trajectory captured into the e_U^2 resonance. The trajectory generates the barely visible set of 9 islands just inside the boundary of the libration region at center. The trajectory has been captured onto a period-11 tertiary resonance, and is about to enter the large chaotic zone. Note the period-4 secondary resonance near the center of the libration region, and the remnants of a period-5 secondary resonance in the chaotic zone. The locations of these secondary resonances agree well with analytic predictions.

The width of the chaotic zone at this resonance can be estimated with the method used in Section 3.2. For the e_U^2 resonance perturbed by the mixed resonance, the parameters in Eq. 3.23 are given by:

$$\lambda = \frac{\dot{\sigma}_M - \dot{\sigma}_U}{\omega_L} = \frac{(C - D)}{\sqrt{H(\delta - \delta_0)}} \quad (3.44)$$

from Eq. 3.43, and

$$\epsilon = \frac{G}{H} \sqrt{\frac{\Sigma_M}{\Sigma_{UR}}} \quad (3.45)$$

where $\Sigma_{UR} = (\delta - \delta_0)/(-16B)$. By differentiating Eq. 3.23 with respect to δ , one finds that the maximum extent of the chaotic zone occurs at a value of $\delta - \delta_0 = \frac{\pi^2(C-D)^2}{64H} \approx 7.0$. For $e_M = 0.005$, the predicted $\Delta\mathcal{E}/\mathcal{E}_{sz} = 2.34$, indicating that the chaotic zone fully engulfs the libration zones. This may be seen by inspection of the pendulum Hamiltonian (Eq. 1.14): at the separatrix, $\mathcal{E}_{sz} = \frac{\omega_n^2}{\alpha}$, and at the stable equilibrium, $\mathcal{E}_{eq} = -\frac{\omega_n^2}{\alpha}$. Therefore, the relative width of the chaotic zone in terms of the separatrix energy $\frac{\Delta\mathcal{E}}{\mathcal{E}_{sz}} = 2$ for the chaotic zone to fully engulf the libration region. This, along with the presence of many important secondary resonances, indicates that for the initial conditions $e_M = e_U = 0.005$, the e_U^2 resonance is unstable. At the point in evolution shown in Fig. 3.41, Eq. 3.23 predicts a half width of the chaotic separatrix near the unstable point $\Delta\sigma_U \approx 1$ radian, but the actual width is affected by overlap of secondary resonances.

We have also explored the evolution of trajectories with lower initial eccentricities. Fig. 3.42 shows the evolution in $\delta, \Delta E$ parameter space of an ensemble of trajectories with initial eccentricities of 0.001 for both satellites. For this set, the initial coordinates for each trajectory in a set were determined in a similar fashion to the first set: For an initial trajectory, with physical parameters $a_M = 4.8640$, $a_U = 10.1179$, $e_M = 0.001$, $e_U = 0.001$, $\sigma_M = \pi/2$, and $\sigma_U = \pi/2$, the coordinates $x_M = 0.0$, $y_M = 0.0001843$, $x_U = 0.0$, and $y_U = 0.0009148$, and the parameter $\delta = -0.967002$. To determine the initial coordinates of the other 19 trajectories, the

trajectory was mapped forward in time 66 mapping periods ($T = 2\pi/40$) between points without tidal dissipation.

Of the 20 trajectories shown in Fig. 3.42, 7 were captured into the e_U^2 resonance. The single-resonance theory (Appendix III) predicts a capture probability of 0.32 for $e_U = 0.001$ approaching this resonance. All of the remaining trajectories were captured into the mixed ($e_M e_U$) resonance. As in the circular-inclined problem, the capture probability for this resonance can be computed by ignoring the variations in the orbital parameters of Umbriel and treating it as a first-order resonance (Peale 1988). The eccentricity of Miranda below which certain capture occurs for $e_U = 0.0007$ after escape from the first resonance is $e_M = 0.0036$. For an initial $e_M = 0.001$, then, capture into the mixed resonance is certain in the single-resonance approximation. The single-resonance approximation is a reasonable approximation during the initial stages of capture at these low initial eccentricities, since Eq. 3.23 with appropriate parameters for the eccentricity-type resonances and $e_M = e_U = 0.001$ at the point of encounter predict $\Delta\mathcal{E}/\mathcal{E}_{sz} \approx 10^{-40}$ for the e_U^2 resonance!!

The trajectories which are captured into the e_U^2 resonance quickly evolve into the tiny region of parameter next to the excluded region in Fig. 3.42. As discussed earlier, the energy surface boundary divides surfaces of section in this region of parameter space into two independent regions in which the resonant arguments librate. Therefore, the trajectories are captured into the resonances in a manner analogous to that in the Ariel-Umbriel 5:3 resonance (Chapter 2). With the phase space divided into independent librating regions which are not connected to regions of circulation via a chaotic separatrix, there appears to be no escape mechanism: the trajectories are permanently captured into resonance, at least in the planar approximation. Even though Eq. 3.43 predicts the existence of secondary resonances, capture into these secondary resonances do not lead to escape, since there is nowhere to escape to. (This is also found to be the case for low initial eccen-

tricies in the Ariel-Umbriel 2:1 commensurability. See Chapter 4). However, it requires extremely low eccentricities approaching the resonance ($e_U < 0.00037$) for capture into this resonance to be certain.

The trajectories which are captured into the mixed resonance all enter the large chaotic zone at approximately the same value of δ . None of these trajectories evolved to as high values of e_M as did in the other ensemble with higher initial eccentricities, but compared to the initial values, the jump in eccentricity is substantial. The highest orbital eccentricity of Miranda attained for this ensemble is comparable to the equilibrium value.

Fig. 3.43 shows the evolution of the orbital eccentricities of the two satellites for a trajectory with these initial conditions. The capture process appears more regular, and the period of temporary evolution in the mixed resonance lasts longer. The trajectory is “dragged” into the large chaotic zone via capture into secondary resonance (Fig. 3.44). After the trajectory enters the large chaotic zone, its evolution is very similar to the evolution of the trajectories in the other set, except that the trajectory is temporarily captured into the e_U^2 resonance a few times, causing the somewhat more ragged appearance of the upper eccentricity boundary in the chaotic zone.

As in the circular-inclined problem, the locations of secondary resonances can be analytically estimated. Again, for large δ , the mixed resonance can be approximated as a second-order resonance in $(\sigma_M + \sigma_U)/2$, and $\Sigma_M \approx \Sigma_U$. The average value of $\dot{\sigma}_M + \dot{\sigma}_U$ is zero in the mixed resonance, so the frequency of circulation of the other degree of freedom $\dot{\sigma}_M - \dot{\sigma}_U = 2(C - D)$. Using Eq. 3.22 with parameters appropriate to the eccentricity-type resonances, the ratio of the frequency of low-amplitude libration to the splitting frequency is given by the expression:

$$\frac{\omega_L}{\dot{\sigma}_M - \dot{\sigma}_U} = \frac{\sqrt{2G}}{2(C - D)} \sqrt{\delta - \delta_0} \approx -0.311 \sqrt{\delta - \delta_0}. \quad (3.46)$$

The predicted locations of low-order secondary resonances are given in Table 3.3.

Miranda/Umbriel 3:1 Resonance Energies

Figure 3.42: Trajectories in δ , ΔE parameter space with $\mathcal{Q} = 110$. Solid lines indicate quasiperiodic behavior, dashed lines indicate chaotic behavior. All of these trajectories are temporarily captured into quasiperiodic resonance, then evolve through a chaotic zone and escape with higher eccentricities to large negative ΔE . Diagonal lines indicate maximum e_M in the chaotic zone: a = 0.01, b = 0.02, c = 0.03, d = 0.04, e = 0.05, f = 0.06, g = 0.07, h = 0.08.

Figure 3.43: Evolution of the orbital eccentricities of Miranda (a) and Umbriel (b) during passage through the 3:1 commensurability in the planar-eccentric approximation with $\frac{\dot{a}_M}{a_M} = 6 \times 10^{-12}$ per orbit period and lower initial eccentricities than in the previous figures. The maximum and minimum eccentricities are plotted in intervals $\Delta\delta = 0.002$. After initially being captured into the mixed ($e_M e_U$) resonance, the trajectory enters a large chaotic zone and the orbital eccentricity of Miranda evolves to a high value before the trajectory escapes from the resonance.

Figure 3.44: Surface of section showing the phase space of the trajectory in Fig. 3.42 at $\delta = 1.138$. The trajectory has been captured into a period-3 secondary resonance in the $e_M e_U$ libration region and is about to enter the large chaotic zone. The locations of the secondary resonances agree well with analytic calculations.

Table 3.4: Frequency Ratios: Mixed Resonance

$\omega_L : (\dot{\sigma}_M - \dot{\sigma}_U)$	$\delta - \delta_0$	e_M	e_U
1:4	0.646	0.015	0.003
1:3	1.148	0.020	0.004
1:2	2.585	0.030	0.006

The value of $\delta_0 \approx -16B\Sigma_{U0}$ is 0.854 (see Eq. 3.21) for $e_{U0} = 0.005$ and 0.024 for $e_{U0} = 0.001$. In both ensembles of trajectories studied, only the 1:3 commensurability appears to be important to the evolution in the mixed resonance. Comparing the predicted location of this secondary resonance with the observed location for trajectories with $e_{M0} = e_{U0} = 0.005$ (Fig. 3.23) and $e_{M0} = e_{U0} = 0.001$ (Fig. 3.43), it can be seen that the predictions work well for low initial eccentricities and relatively poorly for high initial values. This is not unexpected, since it is observed that there is much more chaotic behaviour during the initial capture process in the higher-eccentricity case. However, the evolution in both cases is dominated by the large chaotic zone.

3.3.3 Secular Perturbations

In this section we consider the perturbations on the eccentricity-type resonances produced by the secular variations due to the other satellites. To get a qualitative idea of the effects of secular perturbations, we consider only the perturbations on the circular inclined problem due to Ariel as a first approximation, since they are the most significant (see Dermott and Nicholson 1986, Laskar 1986). The development of the Hamiltonian is given in Appendix I. The addition of these perturbations results in the problem no longer being reducible to two degrees of freedom, so it is not possible to study the phase space using surfaces of section.

We define $\phi = \lambda_M - 3\lambda_U$, $p_i \approx e_i\sqrt{L_i} \cos(-\varpi_i)$, and $q_i \approx e_i\sqrt{L_i} \sin(-\varpi_i)$. The momentum conjugate to ϕ is $\Phi = L_M - L_{MR}$. The resonant value L_{MR} is defined at $\frac{\partial \mathcal{H}}{\partial \Phi} = 0$:

$$L_{MR} = \frac{L_U + 3L_M}{3 + (3)^{1/3} \frac{m_U}{m_M}} \quad (3.47)$$

In this model, evolution due to tidal dissipation changes the value of Φ at a rate of $\dot{\Phi} = \dot{L}_M - \dot{L}_{MR}$. The value of Φ can be converted into a value for δ .

Fig. 3.45 shows the orbital eccentricities of Miranda, Ariel, and Umbriel for a trajectory evolved through the 3:1 commensurability using this model. The initial parameters for the trajectory were: $a_M = 4.8631$, $a_A = 7.0846$, $a_U = 10.1179$, $e_M = 0.005$, $e_A = 0.005$, $e_U = 0.005$, $\sigma_M = 0^\circ$, $\sigma_A = 90^\circ$, and $\sigma_U = 270^\circ$. $\dot{a}_M/a_M = 6 \times 10^{-12}$ per orbit period. The secular perturbations on Miranda and Umbriel cause the orbital eccentricities to undergo larger variations before resonance encounter. This type of qualitative change was also observed in the secularly-perturbed circular-inclined problem. As in the planar-eccentric two-satellite problem, the initial stages of resonance encounter are complicated, involving temporary capture into the e_U^2 and mixed resonances, followed by evolution into the large chaotic zone. The presence of Ariel causes the eccentricity variations of Miranda and Umbriel to be more irregular than in the unperturbed case, with less well-defined limits of the eccentricity in the

chaotic zone, however, the evolution is qualitatively similar: the orbital eccentricity of Miranda can jump to a value comparable to the equilibrium value, and then the trajectory may escape from the resonance. The orbital eccentricities of both other satellites vary chaotically, but these variations are relatively insignificant.

(a)

(b)

(c)

Figure 3.45: Orbital eccentricity variations of Miranda (a), Ariel (b), and Umbriel(c) during evolution through the Miranda-Umbriel 3:1 commensurability. The variations of eccentricity are more irregular than in the two-satellite problem, but the orbital eccentricity of Miranda still jumps to a high value before the satellites escape from the resonance.

3.4 The Eccentric-Inclined Problem

In this section, we consider the space Miranda-Umbriel 3:1 resonance problem. The Hamiltonian of this model is developed in Appendix I (Section 7.3.2). We define

$$\begin{aligned}
 \sigma_i &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\varpi_i) \\
 \Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
 \psi_i &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\Omega_i) \\
 \Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
 \Gamma_M &= L_M + \frac{1}{2}(\Sigma_M + \Sigma_U + \Psi_M + \Psi_U) \\
 \Gamma_U &= L_U - \frac{3}{2}(\Sigma_M + \Sigma_U + \Psi_M + \Psi_U)
 \end{aligned} \tag{3.48}$$

This is a four degree of freedom Hamiltonian problem with slow time dependence. There are 3 eccentricity resonances, 3 inclination resonances, and secular interactions. The eccentricities are coupled to the inclinations through the nonlinear terms. δ is defined to be the nonresonant contributions to $3n_U - n_M - \dot{\omega}_M - \dot{\omega}_U - \dot{\Omega}_M - \dot{\Omega}_U$. The expressions for the coefficients and their numerical values are given in Appendix I.

Fig. 3.46 shows the inclinations (*a* and *b*) and eccentricities (*c* and *d*) for a trajectory evolved through the 3:1 resonance using the three-dimensional model. The initial parameters of this trajectory are: $a_M = 4.8610$, $a_U = 10.1179$, $e_M = e_U = 0.005$, $\sigma_M = 90^\circ$, $\sigma_U = 270^\circ$, $i_M = i_U = 0.005$ radians, $\psi_M = 90^\circ$, and $\psi_U = 270^\circ$. Before the resonances are encountered, the secular interactions of the eccentricities and the inclinations are weak. When the trajectory encounters the inclination resonances, it is captured into the i_M^2 resonance. The evolution through this resonance does not appear to be significantly affected by the presence of the nonresonant eccentricity variations. At first, the evolution is quasiperiodic,

showing the same features as were seen in the circular-inclined case. At $\delta \approx 2.4$, the amplitude of oscillation of the orbital inclination of Miranda decreases slightly during passage through the 1:4 commensurability between the degrees of freedom. At $\delta \approx 3.8$, the trajectory is captured into the secondary 1:3 commensurability. As the trajectory further evolves, the oscillation amplitude of the orbital inclination of Miranda increases, while the mean orbital inclination of Umbriel decreases slightly. Between $\delta \approx 5.5$ and $\delta \approx 5.7$, the trajectory becomes chaotic. The trajectory escapes from the inclination resonances with average orbital inclinations of about $3^\circ.4$ for Miranda and about $0^\circ.27$ for Umbriel.

As the trajectory approaches the resonances, the amplitudes of oscillation of the orbital eccentricities increase slightly. During the quasiperiodic phase of evolution through the i_M^2 resonance, the oscillation amplitudes of the orbital eccentricities are approximately constant until the trajectory becomes chaotic at $\delta \approx 5.5$, and then the eccentricities vary irregularly. After the trajectory escapes from the inclination resonances, the eccentricity resonances are encountered. At $\delta \approx 6.2$, the trajectory passes through the resonance involving the orbital pericenter of Umbriel. Between $\delta \approx 6.4$ and $\delta \approx 6.8$, the trajectory passes through the mixed $e_M e_U$ resonance, followed by brief capture into the e_M^2 resonance. The trajectory eventually encounters the large chaotic region associated with the eccentricity resonances. As in the case where secular perturbations due to Ariel were included, the evolution of the eccentricities is more irregular in the chaotic zone than was the case in the unperturbed planar-eccentric problem. The orbital eccentricity of Miranda evolves to a very high value of about 0.08 before the trajectory escapes from the resonance. Note that despite the irregular variations of the mean orbital eccentricity of Miranda in the chaotic zone, the width of the chaotic zone remains relatively constant. For this particular trajectory, the average eccentricity is maintained at a value of about 0.07 for about 10^8 years. The eccentricity variations of Umbriel are significantly

more irregular than in the planar case, but these variations remain relatively small in magnitude.

Due to the nonlinear coupling between the eccentricity resonances and the inclination resonances, the inclinations also become chaotic and vary irregularly. The system eventually escapes from the eccentricity-type resonances, with an orbital eccentricity for Miranda of about 0.07 and an orbital eccentricity for Umbriel of about 0.0065. The final inclinations of the orbits do not change much, despite the presence of the higher dimensional chaos. For the orbit of Miranda, the inclination diffuses from a value of about 3.4° following escape from the i_M^2 resonance to a value of about 4.8° following escape from the eccentricity resonances. The important point is that Miranda retains a high orbital inclination comparable to its current anomalously high value. Other trajectories show similar behaviour. The orbital eccentricity variations of Miranda within the chaotic region are larger and more irregular if the orbital inclination of Miranda is high due to passage through the inclination resonances.

Other trajectories show similar behaviour. The variations in e_M are spectacular, and may have had a significant effect on its thermal history. This issue is dealt with in the next Section.

Figure 3.46: Eccentricity and inclination variations of a trajectory evolved through the 3:1 Miranda-Umbriel mean-motion commensurability ($\dot{a}_M/a_M = 6 \times 10^{-12}$ per orbit period). (a) Orbital inclination of Miranda. (b) Orbital inclination of Umbriel. (c) Orbital eccentricity of Miranda. (d) Orbital eccentricity of Umbriel. The inclination resonances are encountered first, and evolution within them is similar to the evolution in the circular-inclined approximation. There is a large chaotic zone associated with the eccentricity resonances. The orbital eccentricity of Miranda varies significantly within this chaotic zone, and evolves to higher values than in the planar-eccentric case, but the orbital inclination of Miranda remains high.

3.5 Discussion

The 3:1 mean-motion commensurability involving Miranda and Umbriel is a most interesting orbital resonance. It is clear that if the satellites passed through this resonance, their orbits, particularly that of Miranda, may have been substantially affected.

3.5.1 Inclination Resonances

The inclination resonances were encountered first, since $3n_U - n_M$ is negative before the commensurability is reached, and the secular contributions to the $\dot{\Omega}_i$ are retrograde, while those of $\dot{\omega}_i$ are prograde. Therefore, the condition $3n_U - n_M - 2\dot{\Omega}_i = 0$ is met before the condition $3n_U - n_M - 2\dot{\omega}_i = 0$. There was a significant probability that the system was captured into one of the inclination resonances involving the node of Miranda. Evolution within either of these resonances is described well by the single-resonance theory up to the point that the chaotic zone is encountered. The presence of this chaotic zone, along with the existence of commensurabilities between the libration frequencies and other fundamental frequencies in the system, allows the system to escape from the resonance with a high orbital inclination for Miranda. Since the timescale of damping of the inclinations is long, this can explain Miranda's current high inclination. This is a significant new result, underscoring the importance of chaos in the dynamics of the solar system. The integrable theory of passage through isolated mean-motion resonances is useful up to a point, but it certainly cannot be used to predict the interesting features of the passage through this resonance. The secular interaction and the interaction between resonances must be considered in order to adequately describe this problem.

The dynamical evolution of this problem is quite beautiful. Nonlinear dynamical systems have a self-similar structure at all scales (see *e.g.* Hénon 1969), showing resonance within resonance within resonance, *ad infinitum*. The Miranda-Umbriel

3:1 mean-motion commensurability illustrates a physically important manifestation of this seemingly esoteric quality of dynamical systems: during evolution through the inclination resonances in this problem, trajectories will enter the chaotic zone in general only as a result of the slow evolution of the secondary resonances in phase space.

3.5.2 Eccentricity Resonances

Upon escaping from the inclination resonances, the satellites encountered the eccentricity-type resonances. Initially, they may have been temporarily captured into one or more of these resonances, but would soon have entered a large chaotic zone. As the orbits evolved within the chaotic zone, the average orbital eccentricity of Miranda may have increased to a high value of order the equilibrium eccentricity. The eccentricity may have varied significantly about the average value. The eccentricity may have been maintained at this high value for a period of time of order 10^8 years before the satellites escaped from the resonance. During this period of time, the inclination of Miranda's orbit may have diffused to somewhat higher values. The elements of the orbit of Umbriel would not have varied significantly, due to this satellite's much larger mass. Because the orbital eccentricity of Miranda may be quite high upon escape from this resonance, the current anomalously high eccentricity may be explained as a result of resonance passage: in the time since the resonance was encountered, the orbital eccentricity would have decayed to the present value. In addition, the high eccentricities may provide a substantial heat source for the interior of Miranda, perhaps accounting for the significantly modified surface of this satellite.

3.5.3 Tidal Heating of Miranda

The surface of Miranda is very complex (Smith *et al.* 1986). In addition to cratered terrain on the surface, there are regions referred to as "trapezoids",

“banded ovoids”, and “ridged ovoids” by Smith *et al.* (1986), and named “coronae” by Strobell and Masursky (1987). In addition, there are possibly relatively small volcanic flow features. The cratered terrain appears to be the youngest in the Uranian satellite system (Plescia 1988). Could tidal heating of Miranda during passage through the 3:1 resonance have driven the processes that altered the surface of this satellite?

We can estimate the rate at which the interior of Miranda was heated during the high-eccentricity phase of evolution through the 3:1 commensurability with Umbriel, using Eq. 1.20:

$$\frac{dE}{dt} = -\frac{21}{2}k_{2M}\frac{MR_M^5\dot{n}_M^3}{a_M^3Q_M}e_M^2 \approx 7.4 \times 10^{18}e_M^2[\text{ergs s}^{-1}] \quad (3.49)$$

using parameters appropriate for this satellite (Tables 1.1 and 1.2). Tidal heating due to the nonzero obliquity of Miranda in Cassini state 1 would have been negligible, since the obliquity would have been small (see Section 3.2.2).

Because the satellites occupy a large chaotic zone, it is not clear whether an equilibrium eccentricity can be maintained. A reasonable upper limit to the sustained rate of heating of the interior of Miranda is the rate at which energy is added to the orbit of Miranda due to tidal friction in Uranus.

How much energy is available from the expansion of the orbit due to tidal dissipation in Uranus? Since the orbital energy $E_{orbit} \approx GMm_M/2a_M$,

$$\frac{dE}{dt} = -E_{orbit}\frac{\dot{a}_M}{a_M} \quad (3.50)$$

Using Eq. 1.6 with parameters appropriate at this resonance,

$$\frac{dE}{dt} \approx 8.9 \times 10^{19} \frac{1}{Q}[\text{ergs s}^{-1}] \quad (3.51)$$

For $Q = 6600$ and $Q_M = 100$, this predicts a maximum *sustained* tidal heating rate in Miranda of about 1.3×10^{16} ergs s⁻¹. This rate is equivalent to that estimated by Eq. 3.49 for $e_M = 0.042$. It is clear from the numerical experiments

that eccentricities of this magnitude can be attained (see Fig. 3.22). The density of Miranda (see Table 1.2) implies that about 37 percent of the mass of the satellite is composed of rocky materials (see below). If the silicate materials have typical chondritic abundances, and therefore a radiogenic heating rate of about 10^{-7} erg g^{-1} s^{-1} (*e.g.* Kaula 1968), then the above tidal heating rate is about 5 times the current radiogenic heating rate.

We estimate the rise in internal temperature of Miranda due to tidal heating. As a first approximation, we assume that the tidal energy is deposited uniformly throughout the satellite in conductive equilibrium. The temperature as a function of radius in a body with constant thermal conductivity is given by (*e.g.* Stevenson 1984):

$$T(r) = T(R) + \frac{H_v}{6k} R^2 \left\{ 1 - \left(\frac{r}{R} \right)^2 \right\} \quad (3.52)$$

where R is the radius of the satellite, H_v is the volumetric heating rate, and k is the thermal conductivity. We take $T(R_M) \approx 70^\circ K$. The interior of Miranda is most likely a mixture of water ice and silicate materials, with possibly nitrogen- or carbon-containing compounds such as ammonia hydrate ($NH_3 \cdot H_2O$) and/or water clathrates (*e.g.* Stevenson 1982). The thermal conductivity k of water varies approximately linearly with temperature (Hobbs 1974), from about 2.2×10^5 erg cm^{-1} sec^{-1} deg^{-1} at $273^\circ K$ to about 5.3×10^5 erg cm^{-1} sec^{-1} deg^{-1} at $100^\circ K$. The presence of bubbles and imperfections in the ice, silicate particles, and hydrates and clathrates with relatively low conductivities may lower the thermal conductivity of the interior of Miranda by about a factor of two compared to pure water ice (Stevenson 1982). This may be important, since the difference in temperature between the surface and the interior given by Eq. 3.52 depends linearly on the assumed thermal conductivity.

In our very simple model, we consider the interior of Miranda to be homogeneous and undifferentiated, with the thermal conductivity of the interior of Miranda to be

constant with a value equal to $2.2 \times 10^5 \text{ erg cm}^{-1} \text{ sec}^{-1} \text{ deg}^{-1}$, the lowest value for water ice. For an energy input to the interior equal to the energy input to the orbit due to tidal dissipation in Uranus ($e_M = 0.042$) distributed uniformly throughout the volume of the satellite, the central temperature of the satellite will reach a value of about $167^\circ K$. This temperature will certainly not melt water ice, but it approaches the eutectic melting temperature ($175^\circ K$) of $NH_3 \cdot H_2O$, which may make up a significant fraction of the interior.

If the interior is differentiated, with a core of silicate materials with density $\rho_S \approx 3.0 \text{ g cm}^{-3}$ and a mantle mostly made of water ice $\rho_I \approx 0.94 \text{ g cm}^{-3}$, the core radius is given approximately by

$$\frac{R_c}{R} = \left(\frac{\rho_M - \rho_I}{\rho_S - \rho_I} \right)^{1/3} \quad (3.53)$$

from geometric considerations. Since $\rho_M = 1.26 \text{ g cm}^{-3}$, $R_c/R \approx 0.54$. The ice mantle has a lower rigidity than the core, and is therefore more strongly deformed by the tides. If we assume that the energy is deposited uniformly throughout the mantle, for the same parameters used in the homogeneous model, Eq. 3.52 yields a temperature of $152^\circ K$ at the base of the mantle. Again, this temperature approaches the eutectic melting temperature of ammonia hydrate.

Of course, we do not know much about the composition or state of the interior: there are many uncertainties in this problem. The actual temperatures attained in the interior are probably significantly higher than the values given above. Peale and Cassen (1978) find that the tidal heating of a homogeneous, incompressible body is not uniformly distributed throughout the interior: the rate of heating at the center of the body is about three times the globally averaged heating rate. Also, the thermal conductivity of water ice decreases with increasing temperature. Therefore, even at the rate of tidal heating given above, significant melting of materials such as ammonia hydrate may have occurred deep in the interior.

In addition, although the *sustained* rate of tidal heating is constrained by the

rate of tidal dissipation in Uranus, the dynamics allow eccentricities of order twice the value allowed by tidal friction in Uranus alone (see Fig. 3.46), hence resulting in four times the heating rate. This may have raised the internal temperature substantially.

One effect of partial melting of the interior is to decouple a rigid outer shell from the inner layers. The rate of tidal dissipation in a shell may be orders of magnitude higher than in a solid body, due to the greater deformation possible (Peale and Cassen 1978), effectively increasing the Love number k_2 of the satellite. This process is a runaway one, for example causing widespread melting of Io (Peale *et al.* 1979). The ratio of dissipation in a shell to dissipation in a solid body, f , depends on the thickness of the shell and a parameter $\gamma_i = \mu_i / \rho_i g_i R_i$. Squyres *et al.* (1983) evaluated f for Enceladus, which has bulk physical properties very similar to those of Miranda, including a very large γ . They found that the maximum f for Enceladus is about 100. Since Miranda and Enceladus are globally very similar, such high rates of tidal dissipation in Miranda may have been possible. If the orbital eccentricity of the satellite reached a high enough value to begin partial melting of the interior, runaway melting may have been initiated. However, it would first have been necessary to achieve temperatures high enough to melt water ice.

It is possible to estimate the rate of tidal heating necessary to achieve significant melting of water ice in the interior. It is well known (*e.g.* Cassen and Reynolds 1978) that the rate of heat transfer in the interior of a satellite may be affected by solid-state convection. Ice under stress gradually creeps (*e.g.* glaciers) and therefore has an effective viscosity. If there is a sufficiently large temperature difference between the base and top of an ice layer, it may become unstable to solid-state convection, and transfer heat to the outer layers via motion of the material. This process is much more efficient than conduction, therefore “buffering” the heating of water ice. The standard technique for evaluating the stability of an ice layer to convection is

to compute the Rayleigh number Ra of the layer:

$$Ra = \rho g d^3 \alpha \frac{\Delta T}{\kappa \eta} \quad (3.54)$$

where ρ is the density of the layer, g is the gravitational acceleration, d is the thickness of the layer, α is the thermal expansivity of the material, ΔT is the temperature difference between the lower and upper boundaries of the layer, κ is the thermal diffusivity of the material, and η is the effective dynamic viscosity, which is assumed to be constant throughout the layer. Ra relates the bouyant forces on a fluid element to the drag force. A fluid layer becomes unstable to convection if the Rayleigh number exceeds about 10^3 .

For Miranda, $\rho_M = 1.26 \text{ g cm}^{-3}$ and $g = 8.54 \text{ cm s}^{-2}$ at the surface. For water ice, $\rho \approx 0.94 \text{ g cm}^{-3}$, $\alpha \approx 10^{-4} \text{ K}^{-1}$, and $\kappa \approx 10^{-2} \text{ cm}^2 \text{ s}^{-1}$. The viscosity of water ice actually varies strongly with temperature, in a relation of the form (*e.g.* Weertman 1970):

$$\eta(T) = \eta_0 \exp[-a(T_m/T_0 - T_m/T)] \quad (3.55)$$

η_0 is a reference viscosity for some temperature T_0 , and T_m is the melting temperature (273°K). Usually, η_0 is quoted for the melting temperature, having been extrapolated from measured values at lower temperatures. The value of η_0 is about 10^{14} poise [$\text{g cm}^{-1} \text{ s}^{-1}$] at 273°K . The constant a is the ratio between the “activation energy for diffusion” and the thermal energy, and has a value of about 25. The procedure used to evaluate the stability of a layer is as follows (see Cassen and Reynolds 1978). For a given layer of ice with an assumed *linear* temperature gradient and thickness h , evaluate the Rayleigh number for subunits of thickness d from the base, assigning a constant viscosity to the layer which is equal to the temperature at the upper boundary of the layer. If $Ra > 10^3$ for any subunit, the layer is unstable to convection. This method of assigning viscosity tends to overestimate the thickness below which a layer is stable.

By applying this test to the interior of Miranda, we have found that for either a homogeneous or differentiated model, the interior becomes unstable to convection for temperatures *at least* twenty degrees or so below the melting temperature of water ice. Again, this is a very crude calculation, but it indicates strongly that the interior cannot reach the melting temperature of water ice without first becoming convectively unstable. The significance of this is that due to the much greater efficiency of convection in removing heat from the interior, much greater tidal heat input is required to further increase the internal temperature.

The ratio of convective heat flow to conductive heat flow in a fluid layer is given by the Nusselt number (see Cassen and Reynolds 1978): $N \approx 0.167 Ra^{1/3}$. Since the rate of heat transfer through a plane conducting layer is given by $dE/dt = kA\Delta T/d$, where A is the area, the convective heat transfer rate is given approximately by:

$$\frac{dE}{dt} = 0.167 k 4\pi R^2 \left[\frac{\rho g \alpha}{\kappa} \right]^{1/3} \frac{(T_T - T_B)^{4/3}}{\eta \left(\frac{T_T + T_B}{2} \right)^{1/3}} \quad (3.56)$$

where T_T is the temperature of the upper boundary of the ice layer and T_B is the temperature at the base. The effective viscosity of the ice is assigned a constant value calculated at the mean temperature of the layer. In the above equation, it is possible to note two opposing effects of the temperature: dE/dt is proportional to the 4/3 power of the temperature difference, but inverse-exponentially proportional to the mean temperature. This results in a maximum of the heat flow which depends on the temperature at the base and the temperature difference across the layer.

We have applied this model to the interior of Miranda. Fig. 3.47 shows the rate of heat transfer due to convection per unit area for a differentiated model of Miranda with the base held at the melting point of water ice, as a function of the temperature at the top of the convecting layer. As expected, there is a maximum in the function, corresponding to a temperature difference of about 70 degrees across the convecting portion of the mantle. This rate of heat transfer must be matched by the rate of heat conducted across the rigid ice crust, which for the maximum

convecting rate is only about 15 km thick. Integrated over the surface of Miranda, heat can be transferred out of the interior of Miranda by convection at a rate of about 2.5×10^{17} ergs s⁻¹, according to this simplified model. This corresponds to about 20 times the maximum possible *sustained* tidal heating rate for $Q = 6600$. Again, this is a very crude calculation, but the main point is that in order to melt significant amounts of water ice, the tidal heating rate must be much greater than the rate estimated in a solid homogeneous Miranda .

Squyres *et al.* (1983) estimate that the tidal heating enhancement factor f in Enceladus is of order 20 for a rigid crustal thickness of a few tens of kilometers. This indicates that if partial melting of the interior of Miranda occurred out to within a few tens of kilometers of the surface, the tidal heating may have been intense enough to overcome solid-state convection and melt much of the interior. (Note, however, that if there had been significant partial melting of the interior, convective heat transfer may have become much more efficient.)

The eccentricity damping rate, which is proportional to the rate of tidal dissipation, also increases by a factor f , implying that this high rate of heating can last only a short time, since it is likely that the system would be quickly pulled from the chaotic zone into the region of escape (*e.g.* Fig. 3.38).

The energy required to melt most of the interior of Miranda is

$$\Delta E \simeq m_M C_p \Delta T + m_{ice} H_f \quad (3.57)$$

where the first term is due to the increase in temperature of the interior and the second term is overcoming the latent heat of fusion of the ice. C_p is the heat capacity, with a value of about 2×10^7 ergs g⁻¹ K⁻¹ for water ice, and H_f , the latent heat of fusion, is about 3.3×10^9 ergs g⁻¹ for water ice (CRC Handbook, 54th Edition). Therefore, the energy required to melt most of the water ice in Miranda is about 5×10^{32} ergs.

If we are able to begin runaway melting, where does this energy come from? It

Figure 3.47: Rate of convective heat flow per unit area in the interior of Miranda expressed in terms of the temperature at the top of the convecting layer. The base of the layer is maintained at $T = 273^\circ K$, the melting point of water ice. Maximum convective heat flow occurs for a temperature difference across the layer of about 70 degrees.

may come from the orbit: there is a change in orbital energy as the eccentricity damps. Neglecting changes in the inclination, in order to conserve orbital angular momentum, $a_{Mf} = a_{M0}(1 - e_0^2)$ if the final eccentricity is zero. Therefore, the change in orbital energy is

$$\Delta E \approx E_{orbit} e_0^2 \quad (3.58)$$

In order to melt most of Miranda, the orbital eccentricity must damp from a value of at least 0.18 after escaping from the chaotic zone. This is more than 4 times the maximum value allowed by tidal friction in the planet, and more than twice the maximum achieved in any of the numerical experiments. The dynamics do not rule out such high eccentricities, although they are clearly not likely. The rate of tidal dissipation in Uranus cannot sustain such an eccentricity, though, unless $Q = 370$ or lower, which is dynamically implausible (see Chapter 1). However, the behaviour of the system at high eccentricity with dissipation in the satellite included is not now known. It seems, though, that the energy probably could not come from the orbit. (Marcialis and Greeberg (1987) invoke eccentricity damping as an energy source to melt Miranda during a phase of chaotic rotation at some unspecified point in its early history. However, they fail to provide (a): a mechanism to provide Miranda with the orbital eccentricity required both for the energy source and for the presence of a large chaotic zone in the rotational phase space of a body with appropriate hydrostatic moments (b): a mechanism to get Miranda into the chaotic zone in the rotational phase space. Also, for a body in a chaotic tumbling state, one must take into consideration the damping of the “Jacobi integral” due to tidal dissipation, not just the damping of the orbital eccentricity (Wisdom 1987a)).

Another possible source of energy is differentiation. If the satellite approached the resonance in an undifferentiated state, the tidal heating and associated softening of the interior could cause silicate material embedded in the ice to begin falling to the center of the satellite, releasing gravitational potential energy as frictional

heat. This could be a runaway process, since more heat would be generated as more “rocks” fell in, further softening the interior. Plausibly, this runaway could begin if the free-fall time of “rocks” during tidal heating was comparable to the approximately 10^8 year timescale of maximum heating during resonance passage. The bouyancy force on a “rock” is given by

$$F_B \approx g \Delta \rho a^3 \quad (3.59)$$

where g is the local gravitational acceleration, $\Delta \rho$ is the difference in density between the rocks and the ice, and a is the size of the rock. Numerical factors have been ignored. The drag force on the rock as it falls is given by the Stokes drag law:

$$F_D \approx a \eta v \quad (3.60)$$

where η is the effective dynamic viscosity of the ice and v is the velocity of the particle. In free-fall, $F_B = F_D$. Taking into account the variation of gravitational acceleration with distance from the center, the expression for the free-fall time is

$$\Delta t \approx \frac{\eta}{G \rho_M \Delta \rho a^2} \quad (3.61)$$

The maximum possible temperature achieved by the interior before convection sets in is about $250^\circ K$. At this temperature, particles larger than about 10 meters can fall into the center during the time of resonance passage.

How much energy is available? From geometric arguments, one can derive the self-potential of a homogeneous sphere:

$$E_{bh} = -\frac{16}{15} \pi^2 G R^5 \rho_M^2 \quad (3.62)$$

and the self-potential of a differentiated body (see *e.g.* Schubert *et al.* 1986):

$$E_{bd} = E_{bh} \left\{ \left(\frac{\rho_I}{\rho_M} \right)^2 + \left(\frac{3}{2} \frac{\rho_I}{\rho_M} - \frac{\rho_C}{\rho_M} \right) \left(\frac{\rho_I}{\rho_M} - \frac{\rho_C}{\rho_M} \right) \left(\frac{\rho_M - \rho_I}{\rho_C - \rho_I} \right)^{5/3} + \frac{5}{2} \frac{\rho_I}{\rho_M} \left(1 - \frac{\rho_I}{\rho_M} \right) \right\} \quad (3.63)$$

where ρ_C is the density of the core material, for silicate material about 3 g cm^{-3} . For Miranda, $E_{bd} \approx 0.126 E_{bh}$, so about 1/8 of the total gravitational binding energy could be released. This corresponds to about $\Delta E \approx 1.2 \times 10^{30}$ ergs, more than two orders of magnitude too small to melt Miranda.

There are difficulties, then, in melting a great deal of water ice in the interior of Miranda. However, within the uncertainties in our crude models, it is possible that significant partial melting of ammonia hydrate occurred in the interior of Miranda. It is not known exactly how this could affect the surface. Ammonia-water liquid, with a density of approximately 0.946 g cm^{-3} (Croft *et al.* 1988), is neutrally bouyant with respect to a mantle of pure water ice and positively bouyant with respect to an undifferentiated homogeneous mix with the mean density of Miranda. It is therefore possible for the liquid phase to reach the surface and cause some of the resurfacing of Miranda.

However, most of the unusual features on Miranda do not appear to be directly related to resurfacing of liquid material. Janes and Melosh (1988) discuss models proposing that the coronae resulted either from material sinking into or rising out of a viscous mantle overlain by a thin crust. Even without significant melting of the interior, the tidal heating rates in Miranda during passage through this resonance may have been high enough to allow such geological processes to occur.

3.5.4 Q of Uranus

Since the mechanism described in this Chapter is a plausible explanation for the high orbital inclination of Miranda, the requirement that Miranda and Umbriel have passed through the 3:1 resonance also allows us to place an upper limit of 39,000 on the specific dissipation of Uranus (Q). The minimum value of Q which allows for reasonable evolution of the satellite system using the nominal masses of the satellites is $Q = 6600$ (see Section 1.3). Therefore, the specific dissipation

function of Uranus is constrained to within about a factor of 6. However, there is a dynamical “barrier” at some point in the evolution of the satellite system: the 2:1 commensurability involving Ariel and Umbriel, which further constrains Q . This is discussed in Chapter 5.

3.5.5 Conclusions

We conclude that the current anomalously high orbital inclination of Miranda can be accounted for by temporary capture into the 3:1 resonance with Umbriel, and that the high orbital eccentricity attained by Miranda during passage through this commensurability may have damped to the current value during the time since resonance passage. It is therefore likely that the orbits of the satellites have tidally evolved at least enough to have allowed passage through this resonance. If they did, the interior of Miranda may have been significantly affected by tidal heating as the eccentricity was driven to a high value.

Chapter 4

The Miranda-Ariel 5:3 Commensurability

4.1 Introduction

Continuing back in time through the history of the Uranian satellite system, the next major resonant interaction which we find is the 5:3 commensurability involving Miranda and Ariel, which would have been encountered if $Q < 12,000$. As in any second-order mean-motion commensurability, there are three important eccentricity resonances and three important inclination resonances. This commensurability is similar to the 3:1 Miranda-Umbriel commensurability in that the mass of Ariel is much greater than that of Miranda, so the orbital elements of Miranda are more strongly affected by the resonance passage than are those of Ariel. It is similar to the 5:3 Ariel-Umbriel commensurability in that they involve the same combinations of resonance angles for the two satellites, and because the orbits are relatively close together, there is a relatively strong secular interaction between the satellites which significantly affects the orbit of Miranda. However, passage through this resonance differs from the previously considered ones in one major way: the semimajor axis ratio *decreases* due to tidal dissipation in the planet (see Section 1.3). Both the structure of the phase space and the numerical experiments indicate that there is no mechanism of capture into this resonance (see Section 4.2.2).

If the mass of Miranda is at the upper limit of its possible range, then the mean-motion ratio is currently decreasing, and the orbital histories of these two satellites would then have been quite different: they would not have encountered the 5:3 commensurability (Peale 1988). However, any combination of masses which allows passage through this resonance would cause the evolution to be in the same direction and over a similar timescale, in order for the satellites to have ended up in their current orbital configuration (see Fig. 1.3). Therefore, throughout much of the plausible range for Miranda's mass, the dynamical scenario presented here applies, although the point in time at which this resonance was encountered depends on the exact value of the masses (see Eq. 1.16).

In section 4.2, we first consider this resonance in the planar-eccentric approximation, again in an attempt to qualitatively understand the dynamics of the evolution, by studying the phase space using surfaces of section. There is a significant chaotic zone at the 5:3 commensurability. When the satellites encounter this chaotic zone, the maximum eccentricity of Miranda's orbit jumps suddenly to a large value. The satellites may spend a considerable period of time in the chaotic zone, and must eventually escape from the resonance. It has been found that while on average the orbital eccentricity of Miranda is larger after escape from the resonance than before the resonance is encountered, in some cases the mean value decreases due to resonance passage, contrary to the predictions of the single-resonance theory. The qualitative features of resonance passage are found to be essentially preserved when secular perturbations due to Umbriel are added (Section 4.3).

However, there are some qualitative changes in the behaviour when the problem is no longer restricted to the plane (Section 4.4). Both the orbital eccentricity and inclination of Miranda may jump to modestly high values during evolution through the chaotic zone, and then escape from the resonance, *retaining the high values*. Although the evolution of Ariel's orbit is more irregular, it is still relatively

unaffected by the resonance passage.

In the final Section, the importance of this commensurability is discussed. Although the increases in the orbital eccentricity of Miranda are impressive, they appear to not be important in the thermal evolution of this satellite. Since there is a long interval of time between this resonance and the Miranda-Umbriel 3:1 commensurability, e_M probably damped to a relatively low value before this next resonance was encountered. However, an increased orbital inclination of order one degree would be retained during this time interval, and could qualitatively affect evolution through the 3:1 resonance. It is probably the case, though, that the orbital inclination of Miranda still increased to its current high value at the 3:1 resonance with Umbriel.

4.2 The Planar-Eccentric Problem

We begin by studying the 5:3 Miranda-Ariel commensurability in the planar-eccentric approximation. This problem may be reduced to two degrees of freedom, and the phase space can be studied using surfaces of section. In Section 4.2.1, the resonance coordinates are discussed. In Section 4.2.2, we begin by discussing the rate of evolution through the resonance and its effects on the dynamics of resonance passage. It is found that if the rate of evolution through the resonance is too high, the trajectories do not fully explore the phase space of the chaotic zone, and therefore the dynamics are not well represented.

Numerical simulations of the evolution through the 5:3 commensurability reveal interesting and complicated behaviour. At the point of resonance encounter, the system enters a large chaotic zone, and the maximum eccentricity of the orbit of Miranda suddenly shoots up by a factor of three to four. Examination of surfaces of section reveal that at this point, a large chaotic region of phase space becomes available to the trajectory. With further evolution through the resonance, the size of the chaotic zone in phase space diminishes, as does the range within which the eccentricities may vary. The chaotic zone acts as a bridge between the regular region of phase space inside the resonance and a regular region of phase space outside the resonance, into which trajectories eventually must escape. It is found that there are many important secondary commensurabilities between the circulation frequencies of the resonant arguments in these regions, and that these may be important in both causing the trajectory to become chaotic in the first place and in dragging the trajectory out of the chaotic zone and into the quasiperiodic region of escape.

4.2.1 Resonance Coordinates

The Hamiltonian for this resonance is the same as for the Ariel-Umbriel 5:3 commensurability, with appropriate substitutions of parameters. As resonance coordinates we have chosen

$$\begin{aligned}\sigma_M &= \frac{1}{2}(5\lambda_A - 3\lambda_M - 2\varpi_M) \\ \sigma_A &= \frac{1}{2}(5\lambda_A - 3\lambda_M - 2\varpi_A)\end{aligned}\tag{4.1}$$

which together with λ_M and λ_A form a complete set of generalized coordinates. The momenta conjugate to σ_A and σ_M are, in terms of the Delaunay momenta $L_i \approx m_i\sqrt{GMa_i}$ and $G_i = L_i\sqrt{1-e_i^2}$ (see Plummer 1960):

$$\begin{aligned}\Sigma_M &= L_M - G_M \approx \frac{L_M}{2}e_M^2 \\ \Sigma_A &= L_A - G_A \approx \frac{L_A}{2}e_A^2\end{aligned}\tag{4.2}$$

This choice of variables results in two integrals of motion, the momenta conjugate to $\gamma_M = \lambda_M$ and $\gamma_A = \lambda_A$, since in the new variables the resonant Hamiltonian is cyclic in these variables. The resonance integrals are:

$$\begin{aligned}\Gamma_M &= L_M + \frac{3}{2}(\Sigma_M + \Sigma_A) \\ \Gamma_A &= L_A - \frac{5}{2}(\Sigma_M + \Sigma_A)\end{aligned}\tag{4.3}$$

The coefficient A in the Hamiltonian (Eq. 7.5) is approximately $\frac{1}{4}(5n_A - 3n_M)$. Near the 5:3 mean-motion resonance, this quantity is near zero. Due to tidal dissipation in the planet, the mean-motion ratio increases, for the nominal masses of these satellites (see section 1.3). Therefore, the quantity $(5n_A - 3n_M)$ *decreases* as the satellites evolve through the resonance. We define the parameter $\delta = 4A + 2(C + D)$ to be the resonance parameter in this problem. At small eccentricities it is proportional to the non-resonant contribution to $5n_A - 3n_M - \dot{\varpi}_M - \dot{\varpi}_A$. Expressions for the other coefficients and their numerical values are given in Appendix I (Section 7.1.4).

4.2.2 Evolution through the Resonance

Rate of Evolution

Of the various coefficients in the resonance Hamiltonian, only δ depends strongly on the resonant combination of mean-motions; during passage through the resonance, the fractional change in the parameter δ is large compared to the fractional changes in the other coefficients. The other coefficients will therefore be taken to be constant. In addition, the value of δ is influenced by the damping of eccentricities due to tidal dissipation in the satellites. The time rate of change of δ is

$$\dot{\delta} \simeq 4\dot{A} \quad (4.4)$$

From the expansion of A to first order in Σ_i :

$$\begin{aligned} \dot{\delta} &\simeq \frac{d}{dt} [(5n_M - 3n_A) - 16B(\Sigma_M + \Sigma_A)] \\ &= (5\dot{n}_A - 3\dot{n}_M) - 16B(\dot{\Sigma}_M + \dot{\Sigma}_A) \end{aligned} \quad (4.5)$$

The first two terms are due to the tidal expansion of the orbits, and the other terms are due to the exchange of angular momentum between the satellites resulting from dissipation of energy in the satellites (Yoder 1979, see Ch. 5). Note that

$$\dot{\Sigma}_i \simeq \Sigma_i \left[-\frac{3}{4} \frac{\dot{n}_i}{n_i} + 2 \frac{\dot{e}_i}{e_i} \right] \quad (4.6)$$

Tidal friction results in a secular variation of the mean-motions

$$\dot{n}_i = \dot{n}_{ip} + \dot{n}_{is} \quad (4.7)$$

where the first term is due to tides raised on the planet (see Eq. 1.6):

$$\dot{n}_{ip} = -\frac{9}{2} k_2 n_i^2 \frac{m_i}{M} \left(\frac{R}{a_i} \right)^5 \frac{1}{Q} \quad (4.8)$$

The second term is due to tidal dissipation in the satellite, which for a synchronous rotation state must conserve the orbital angular momentum: $\dot{G}_i \simeq d/dt \sqrt{GM a_i (1 - e^2)} =$

0, so $\dot{a}_i \simeq 2a_i e_i \dot{e}_i$. Therefore,

$$\dot{n}_i = \dot{n}_{ip} - 3n_i e_i \dot{e}_i \quad (4.9)$$

Since $D_A = 60.4$ and $D_M = 80.8$ ($Q = 6600$ and $Q_A \approx Q_M \approx 100$), the rates of eccentricity damping due to tidal friction in the satellites dominate the rates of increase in eccentricity due to tidal friction in the planet about two orders of magnitude (see Eqs. 1.9-1.11). Since $\dot{n}_p/n$ and $\dot{e}_p/e$ are of the same order, this means that

$$\dot{\Sigma}_i \approx 2\Sigma_i \frac{\dot{e}_i}{e_i} = L_i e_i \dot{e}_i \quad (4.10)$$

Using numerical values from Table 1.1 with $Q = 6600$ to evaluate relative contributions to $\dot{\delta}$,

$$\frac{5\dot{n}_A}{3\dot{n}_M} = \frac{5m_A}{3m_M} \left(\frac{a_M}{a_A}\right)^8 \approx 1.968 \quad (4.11)$$

so the orbits evolve at comparable rates, but the mean-motion ratio increases. The ratio of the eccentricity dependent terms is

$$\begin{aligned} \frac{\dot{\Sigma}_M}{\dot{\Sigma}_A} &\simeq \frac{k_{2M}}{k_{2A}} \frac{m_A}{m_M} \frac{n_M}{n_A} \left(\frac{R_M}{R_A}\right)^5 \left(\frac{a_A}{a_M}\right)^5 \frac{Q_A}{Q_M} \frac{\Sigma_M}{\Sigma_A} \\ &\approx 0.033 \left(\frac{e_M}{e_A}\right)^2 \end{aligned} \quad (4.12)$$

Since Miranda is much less massive than Ariel, its orbital eccentricity may reach much higher values than that of Ariel, and these terms may be of comparable magnitude.

Eq. 4.5 may be expressed:

$$\begin{aligned} \dot{\delta} = & 5\dot{n}_A \left[1 - \frac{3}{5} \frac{m_M}{m_A} \left(\frac{a_A}{a_M}\right) \right] - 16B(\dot{\Sigma}_M + \dot{\Sigma}_A) + \\ & (5\dot{n}_{Ap} - 3n_A e_A \dot{e}_A) \left[1 - \frac{3}{5} \frac{m_M}{m_A} \left(\frac{a_A}{a_M}\right) \right] - \\ & 16BL_A e_A \dot{e}_A \left[1 + 0.033 \left(\frac{e_M^2}{e_A}\right) \right] \end{aligned} \quad (4.13)$$

Note that (see Appendix I)

$$16B \approx -\frac{n_A}{L_A} \left[\frac{75}{2} + \frac{27}{2} \frac{m_A}{m_M} \left(\frac{a_A}{a_M} \right)^2 \right] \quad (4.14)$$

and

$$\frac{\dot{e}_A}{\dot{n}_A} \simeq \frac{7}{3} D_A \frac{e_A}{n_A} \quad (4.15)$$

Therefore

$$\begin{aligned} \dot{\delta} \approx & 5\dot{n}_A \left\{ \left[1 - \frac{3}{5} \frac{m_M}{m_A} \left(\frac{a_A}{a_M} \right)^8 \right] + \right. \\ & \frac{7}{15} D_A e_A^2 \left[\left(\frac{75}{2} + \frac{27}{2} \frac{m_A}{m_M} \left(\frac{a_A}{a_M} \right)^2 \right) \left(1 + 0.033 \left(\frac{e_M}{e_A} \right)^2 \right) \right] - \\ & \left. 3 \left(1 - \frac{3}{5} \frac{m_M}{m_A} \left(\frac{a_A}{a_M} \right)^8 \right) \right\} \end{aligned} \quad (4.16)$$

With appropriate parameters, and $e_M/e_A \approx 4$ (see Section 4.3), this becomes

$$\dot{\delta} \approx 5\dot{n}_A (0.492 + 22282 e_A^2) \quad (4.17)$$

The first contribution diminishes $\dot{\delta}$ due to the fact that Miranda's orbit evolves at a rate comparable to Ariel's. The second contribution is due to the mutual repulsion between orbits due to dissipation in the satellites. Whether or not this is important depends on the orbital eccentricity of Ariel and the nature of the interaction: if the orbits evolve chaotically, there may be no net torques. However, if for example the orbits are temporarily trapped into a libration state, this may be a significant factor: Ariel may push the relatively small Miranda away quite rapidly. In fact, this factor dominates if the eccentricity of Ariel is greater than about 0.005. Because Ariel is so much more massive than Miranda, its orbital parameters do not vary as much during resonance passage, and for the purposes of this analysis may be taken to be constant; hence, the mutual repulsion to first order increases the tidal evolution rate at a constant value depending on the value of the eccentricity of Ariel approaching the resonance. Note that for this resonance, there is not a stable equilibrium configuration.

For the current eccentricity of Ariel the $\dot{n}_A$ term is dominant in the expression for $\dot{\delta}$. In addition, it is unknown whether the dissipation in the satellites would cause a significant exchange of angular momentum if the orbits were chaotic. We have used only the rate of dissipation due to planet tides in our numerical calculations:

$$\dot{\delta} \approx -0.492 \frac{15G^2 M^2 m_A^3}{2\Gamma_A^3} \frac{\dot{a}_A}{a_A} \quad (4.18)$$

The rates given here may be augmented by the enhancement given in Eq. 4.17.

Tidal friction also introduces explicit frictional terms into the equations of motion, the most important terms arising from the damping of the orbital eccentricities due to dissipation within the satellites. The eccentricity damping timescales of both satellites are about 10^8 years, for $Q_i \approx 100$. The timescale of resonance passage depends on the rate of orbital evolution, which is determined by the specific dissipation function Q of Uranus. For the maximum $Q = 6600$, $\dot{\delta} \approx -6.7 \times 10^{-8} \text{ yr}^{-2}$. Typically, trajectories remain in the chaotic zone $\Delta\delta \approx 5.0 \text{ yr}^{-1}$, indicating a resonance passage timescale of order 10^8 years. Tidal damping of eccentricities may therefore be important in the dynamics of resonance passage. However, in order to understand the orbital dynamics in a Hamiltonian context, we have left out the explicit frictional terms in this initial study.

As in the investigations of the other resonances, we have carried out a study to determine the effect on the dynamics of resonance passage of the effective specific dissipation function of Uranus, defined here, as always, to be script $\mathcal{Q}$. Ensembles of 20 trajectories were evolved through the resonance at different evolution rates. The initial coordinates of the trajectories were determined from an initial state with the following physical parameters: $a_M = 4.6361$, $a_A = 6.5110$, $e_M = 0.005$, $e_A = 0.005$, $\sigma_M = 90^\circ$, and $\sigma_A = 90^\circ$. From this initial state, the 19 other states in the ensemble were determined by integrating the equations of motion forward in time with the mapping without tidal dissipation, spacing successive states 66 mapping iterations ($\Omega = 40.0 \text{ yr}^{-1}$, $\Delta t = 66 \times 2\pi/\Omega = 10.4 \text{ years}$) in time apart. Each of these initial

states was then integrated forward in time with tidal dissipation included.

Fig. 4.1 shows the results of this study of the effect of the rate of tidal evolution on the dynamics. Plotted are the averages of the maximum orbital eccentricities in the chaotic zone as a function of the rate expressed in terms of $\mathcal{Q}$. There is a dependence of the maximum eccentricity of Miranda on $\mathcal{Q}$ for $\mathcal{Q} < 10$ ($\dot{a}_A/a_A \approx 2.6 \times 10^{-10}$ per orbit period); for slower dissipation rates (larger $\mathcal{Q}$), the dynamics in the chaotic zone appear to become independent of the rate of evolution. Since evolution in the chaotic zone appears to conserve an ergodic adiabatic invariant (see Chapter 1) for sufficiently slow evolution, the trajectory must be able to fully explore the chaotic region on timescales much shorter than the evolution timescale. For this resonance, Fig. 4.1 indicates that this is the case for $\mathcal{Q} > 10$, and that therefore the dynamics are well represented.

Numerical Results

Fig. 4.2 shows the evolution of this same ensemble of trajectories in $\delta, \Delta E$ parameter space at the slowest rate shown ($\mathcal{Q} = 110, \dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period). ΔE has the same definition as in Chapter 2 (Eq. 2.5), with coefficients appropriate to this resonance. Note that the scale of δ has been reversed on this plot relative to those in the other Chapters, due to the fact that δ decreases with time, which increases to the right in the plot. Again, the heavily shaded region is the excluded region of parameter space into which trajectories may not evolve. The lightly shaded region represents the zone in parameter space in which large-scale chaos is present in the phase space, numerically determined by computing the rate of growth of distance between nearby trajectories (Lyapunov exponent) for 20 initial conditions equally spaced on the y-axis of the surface of section which plots x_M vs. y_M when $x_A = 0$ for many values of δ and ΔE . For $\Delta E > 0$, there is an excluded region near the origin of the surface of section which plots x_A vs. y_A when $x_M = 0$,

Figure 4.1: Maximum eccentricities of trajectories in the chaotic zone, related to the rate of tidal evolution expressed in terms of the effective specific dissipation function of Uranus Q . For $Q > 10$, trajectories explore the full extent of the chaotic region before escaping from the resonance; for higher rates, the dynamics are not well represented.

which restricts the orbital eccentricity of Ariel to have nonzero values. In the small region near the boundary of the excluded zone, the abovementioned surfaces of section are divided by the energy surface boundary into independent regions in which the resonant arguments librate. Note that this region disappears as δ decreases, indicating that permanent capture into quasiperiodic libration by this mechanism does not occur during passage through this commensurability. Since tidal evolution drives trajectories into the unshaded region of parameter space in the lower right of this figure, where only quasiperiodic circulation of resonant arguments is possible, there is no known mechanism of capture into this resonance.

The trajectories shown in Fig. 4.2 initially evolve to positive ΔE in a regular fashion, then enter a large chaotic zone. Solid lines indicate quasiperiodic behaviour, and dashed lines indicate chaotic behaviour in this figure. The evolution of the energies is also very regular during the chaotic phase, until the trajectory escapes to a quasiperiodic region of phase space and evolves to large negative ΔE . There is not an obvious pattern to the escape process: trajectories “drop out” of the chaotic zone in a seemingly random manner.

Fig. 4.3 shows the distribution of average orbital eccentricities of Miranda after escape from the resonance. For reference, the time-averaged initial value of about 0.0045 is shown by the vertical bar in the upper part of the figure. For most trajectories, the average orbital eccentricity of Miranda is higher than the value approaching the resonance, which is qualitatively in agreement with the predictions of the single-resonance theory. However, a few trajectories evolve to lower eccentricity, which is not predicted by the single-resonance theory. Also, the mean value of the time-averaged orbital eccentricities of Ariel decreases from about 0.005 prior to resonance encounter to about 0.0048 after escape, with a scatter of about 10 percent about this value.

There is a correlation between the orbital eccentricities upon escape from the

Miranda-Ariel 5:3 Resonance Energy

Figure 4.2: Trajectories in $\delta, \Delta E$ space with $\mathcal{Q} = 110$. The mean initial eccentricities of the orbits are $e_M = 0.0045$ and $e_A = 0.0050$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. Evolution of the energy during the chaotic phase of evolution is quite regular. Trajectories escape from the resonance via the chaotic zone at apparently irregular intervals, evolving to large negative ΔE .

Figure 4.3: Distribution of time-averaged orbital eccentricities of Miranda. The vertical bar at the top of the figure indicates the average eccentricity prior to resonance encounter. Most trajectories escape from the resonance with a higher average eccentricity than the initial value, but in a few cases the average eccentricity decreases during resonance passage.

resonance and the length of time spent in the chaotic zone. Figs. 4.4 - 4.6 show the evolution of the orbital eccentricities of three different trajectories in this ensemble. In these figures, the maximum and minimum eccentricity in a small interval of δ ($\Delta\delta \approx 0.005$) is plotted vs. δ . In each case, the eccentricity “envelopes” are smooth and well-defined well before the resonance is encountered. The mean values of the orbital eccentricities oscillate with amplitudes determined by the strength of the secular coupling. As the trajectories approach the resonance, the amplitude of oscillation increases. There may be a sudden small change in the oscillation amplitude, as in Fig. 4.5, just before the large chaotic zone is entered. When the trajectory enters the chaotic zone, the maximum orbital eccentricity of Miranda suddenly jumps to a value approximately four times the maximum prior to resonance encounter. The orbital eccentricity of Miranda varies irregularly between about 0.002 and 0.02. Meanwhile, the orbital eccentricity of Ariel also varies chaotically, but through a much smaller range. As the trajectories tidally evolve through the chaotic zone, the eccentricities continue to vary in an irregular manner, but within well-defined limits which both decrease as δ decreases. Note that while the orbital eccentricity of Miranda extends to zero in the chaotic zone, the orbital eccentricity of Ariel is constrained to be above some limit greater than zero.

Eventually, the trajectories escape from the resonance. The trajectory shown in Fig. 4.4, after only a short time in the chaotic zone, escapes with an average orbital eccentricity of Miranda approximately twice the value prior to resonance encounter. The orbital eccentricity of Ariel also increases a tiny amount. A trajectory which escapes after further evolution (Fig. 4.5) ends up with the eccentricity of Miranda’s orbit about 1.5 times as large as the average value before resonance encounter, and the final orbital eccentricity of Ariel is slightly lower than the initial value. For a trajectory which spends a long time in the chaotic zone, such as the one shown in Fig. 4.6, the average eccentricities of both orbits are lower after escape from the

Figure 4.4: Variations of the orbital eccentricities of Miranda (a) and Ariel (b) during evolution through the 5:3 mean-motion commensurability. $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). Upon entering the chaotic zone, the maximum orbital eccentricity of Miranda suddenly jumps to relatively high values. The final eccentricities of both orbits are higher than the initial values.

Figure 4.5: Variations of the orbital eccentricities of Miranda (a) and Ariel (b) during evolution through the 5:3 mean-motion commensurability. $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). This trajectory remains in the chaotic zone somewhat longer than the trajectory shown in Fig. 4.4; the final orbital eccentricity of Miranda is not as high.

Figure 4.6: Variations of the orbital eccentricities of Miranda (a) and Ariel (b) during evolution through the 5:3 mean-motion commensurability. $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). The final e_M is actually lower than the initial value.

resonance than before the resonance was encountered. In all three cases, escape from the chaotic zone is accompanied by slight sudden changes in the oscillation amplitude of the eccentricities shortly afterwards.

In order to better understand the qualitative features of the dynamics of resonance passage, we have studied the evolution of the trajectory shown in Fig. 4.6 by taking “snapshots” of the phase space by freezing the tidal evolution and computing surfaces of section. The surfaces of section chosen for study plot y_M vs. x_M when $x_A = 0$ (defined as section I) and y_A vs. x_A when $x_M = 0$ (defined to be section II). Again in this problem, the section condition $x_i = 0$ corresponds to 2 or 4 values of the conjugate y_i , due to the quartic nature of the Hamiltonian. These values of y_i are obtained numerically by solving a quartic equation, and are designated $a - d$ in descending numerical order (see Section 2.2.2). For section I, there are two high-eccentricity quartic root “families” and two low-eccentricity root families, as in the surfaces of section studied in Chapter 2. Unfortunately, in order to display all of the features of the phase space, it is necessary to plot surfaces of section corresponding to at least two of these root families, as was the case in the Ariel-Umbriel 5:3 resonance problem. For section II, the phase space is even more complicated due to the appearance of an excluded region near the origin for $\Delta E > 0$. We have chosen here to display sections Ia and Ib, and section IIa only.

Fig. 4.7 shows sections I just after the trajectory in Fig. 4.6 has entered the chaotic zone, at $\delta = 8.337$. Note that on both section Ia and Ib there is a quasiperiodic zone near the origin and a large chaotic zone which extends to the energy surface boundary. The trajectory has entered the chaotic zone from the quasiperiodic zone on section Ib (Fig. 4.7b), and will eventually escape from the resonance into the quasiperiodic zone on section Ia (Fig. 4.7a). The chaotic zone acts as a bridge between these two regions. By studying this figure, it is possible to understand the origin of the sudden jump in orbital eccentricity experienced by Miranda as it enters

the chaotic zone. Prior to resonance encounter, the trajectory is confined to the quasiperiodic region in Fig. 4.7b. The evolution in this region involves the secular interaction between the satellites, and the orbital eccentricity of Miranda never gets above about 0.005. When the trajectory enters the large chaotic zone, a much more extensive region of phase space is available to the trajectory, including the chaotic zone in Fig. 4.7a. The chaotic zone extends all the way out to the energy surface boundary, which constrains the extent of motion. On section 1a, the energy boundary extends to an orbital eccentricity of Miranda of about 0.02, about four times the maximum in the quasiperiodic zone before resonance encounter. In the chaotic zone, the orbital eccentricity varies irregularly between a value corresponding to the maximum extent of the energy surface boundary on Fig. 4.7a and a minimum value corresponding to the outer edge of the quasiperiodic region on Fig. 4.7b. Because the timescale of evolution within the chaotic zone is much shorter than the tidal evolution timescale, we see the sudden jump in the maximum orbital eccentricity of Miranda in Figs. 4.4 - 4.6.

Note that there is a period-4 secondary resonance visible in the quasiperiodic region in Fig. 4.7b. The appearance of this and other secondary resonances at the edge of the chaotic zone may facilitate the transfer of trajectories into the chaotic zone, analogous to the mechanisms of escape from resonance discussed in Section 3.3. Just before the trajectory in Fig. 4.6 enters the large chaotic zone, the orbital eccentricity of Miranda jumps slightly, probably associated with passage through this secondary resonance. Note that the eccentricity undergoes a similar but smaller “glitch” shortly before this, which can be seen more clearly in the eccentricity of Miranda for the trajectory shown in Fig. 4.5. The secondary resonances form in the chaotic region and move inward towards the center of the regular region. As trajectories pass through the separatrix associated with one of these secondary resonances, these “glitches” appear as the action (area enclosed by the trajectory

on the section) changes slightly.

Fig. 4.8 also shows the phase space of the trajectory in Fig. 4.6 at $\delta = 8.336$, in Ariel variables (section IIa). Note that on this surface of section, the energy surface boundary is “doughnut” shaped: there is a region of phase space near the origin on this figure which trajectories may not occupy. Again, there is a large chaotic zone, this time adjacent to the low-eccentricity energy surface boundary, and a quasiperiodic region adjacent to the outer energy surface boundary. The curves in this part of the figure are generated by the same trajectories as generated the curves in the quasiperiodic region of phase space shown in Fig. 4.7a. This figure illustrates an interesting quality of trajectories in this region of $\delta, \Delta E$ parameter space: in the planar-eccentric approximation, the eccentricity of Ariel is constrained by the instantaneous energy surface boundary to values greater than a nonzero minimum.

The surfaces of section in Fig. 4.9 show the phase space of the trajectory in Fig. 4.6 at $\delta = 6.343$. Note that the maximum eccentricity of the energy surface boundary on section Ia (Fig. 4.9a) has decreased to 0.016, in agreement with the observed upper envelope of the eccentricity in Fig. 4.6. There are a number of interesting features in Fig. 4.9. First, there is a period-3 secondary resonance with a visibly chaotic separatrix visible in the regular region of section Ib (Fig. 4.9b). Note that the extent of the quasiperiodic region has shrunk, and in fact no longer encloses the origin: the orbital eccentricity of Miranda in the chaotic zone may be zero, as indeed is observed in Fig. 4.6 at this point in the evolution. A period-2 secondary resonance has just appeared at the inner boundary of the large chaotic zone on section Ia (Fig. 4.9a). The separatrix associated with this resonance is still merged with the large chaotic zone. The relatively large density of points in this region of the chaotic zone indicates that the trajectory may spend relatively long periods of time near the inner edge of the chaotic zone. This supports the idea that the secondary resonances may be connected with the mechanism of escape from

Figure 4.7: Sections I showing the phase space available to the trajectory in Fig. 4.6 just after entering the chaotic zone, at $\delta = 8.337$. The chaotic zone extends from the edge of the quasiperiodic region on section Ib (b) to the energy surface boundary on section Ia (a), explaining the sudden jump in e_M when the chaotic zone is entered.

Miranda/Ariel 5:3 Resonance Surface of Section

Figure 4.8: Section IIa showing the phase space available to the trajectory in Fig. 4.6 just after entering the chaotic zone, at $\delta = 8.337$. There is an excluded region around the origin. The quasiperiodic loops are generated by the same trajectories which generate the regular curves in Fig. 4.7a.

the chaotic zone. The complex network of islands associated with the decoupling of the period-2 separatrix from the chaotic zone may provide a region into which a trajectory may escape from the chaotic zone as it spends some period of time “stuck” to the inner boundary (see also Chapter 3). Another possible mechanism through which trajectory may escape from the chaotic zone via the secondary resonance is as follows: as the period-2 stable islands form in the chaotic zone, there may be some probability that a trajectory may become stuck to the island long enough to become trapped on it as it grows in size. The trajectory would then be “dragged” from the chaotic zone with the secondary resonant structure. As the secondary resonance evolved towards the center of the regular region, and the area enclosed by the librating regions shrunk, the trajectory would eventually be ejected from the period-2 resonance due to the requirement that the action of the (quasiperiodic) trajectory be conserved, and escape would be complete. This dynamical mechanism is akin to the process of capture into secondary resonances described in Chapter 3, but proceeds in the opposite sense. It is interesting to note that the trajectory shown in Fig. 4.5 escapes from the chaotic zone at approximately this value of δ . Notice also that there is a period-3 secondary resonance in the regular region of Fig. 4.9a, which forms the quasiperiodic curves just inside the boundary of the regular zone, but this secondary resonance appears to be much weaker.

Fig. 4.10 shows the phase space available to the trajectory in Fig. 4.6 at $\delta = 5.423$. The regular region on section Ib has almost disappeared, the separatrix associated with a period-2 bifurcation marking the boundary. The maximum eccentricity of the energy surface boundary has decreased to about 0.014. The period-2 secondary resonance on section Ia (Fig. 4.10a) has moved towards the center of the regular region, and although the associated separatrix is still visibly chaotic, it appears detached from the large chaotic zone. (There could conceivably still be a network of very narrow interconnected chaotic zones between them, though). In

Figure 4.9: Sections I showing the phase space available to the trajectory in Fig. 4.6 at $\delta = 6.343$. The regular region on section Ib (b) has shrunk, allowing the minimum e_M in the chaotic zone to reach zero. A period-2 secondary resonance is forming on the inner boundary of the chaotic zone on section Ia (a).

Fig. 4.11 ($\delta = 3.737$), the regular region has disappeared from section Ib. The maximum eccentricity of the energy surface on section Ia is only about 0.01. Note that the period-2 secondary resonance in the quasiperiodic region on section Ia has moved further towards the origin, and that two islands associated with yet another secondary resonance are visible near the inner boundary of the large chaotic zone.

At this point in its evolution, the trajectory shown in Fig. 4.6 is temporarily stuck to a quasiperiodic island, which is visible in the upper left of the Figure. Initially, the trajectory generates many points around the island. Eventually, though, it breaks away and fills the large chaotic zone. Shortly after this point in its evolution, the trajectory escapes from the chaotic zone.

We have also studied the evolution of trajectories with higher initial eccentricities. The initial coordinates of the trajectories were determined from an initial state with the following physical parameters: $a_M = 4.6361$, $a_A = 6.5100$, $e_M = 0.01$, $e_A = 0.01$, $\sigma_M = 90^\circ$, and $\sigma_A = 90^\circ$. As in the set of trajectories described earlier, from this initial state, the 19 other states in the ensemble were determined by integrating the equations of motion forward in time with the mapping without tidal dissipation, spacing successive states 66 mapping iterations ($\Omega = 40.0 \text{ yr}^{-1}$, $\Delta t = 66 \times 2\pi/\Omega = 10.4 \text{ years}$) in time apart. Each of these initial states was then integrated forward in time with tidal dissipation included.

Fig. 4.12 shows the evolution of the Hamiltonian energies of this set of trajectories in $\delta, \Delta E$ parameter space. Note that other than a change of scale, this plot is qualitatively almost identical to Fig. 4.2. Despite the doubling of both initial eccentricities, the qualitative features of the evolution are very similar. Fig. 4.13 shows the distribution of time-averaged orbital eccentricities of Miranda upon escape from the resonance. The vertical bar in the upper center of the plot indicates the time-averaged initial eccentricity $e_M = 0.0091$ for trajectories in this ensemble. Compared to the trajectories in Run 1 (see Fig. 4.3), there is a larger spread in

Figure 4.10: Sections I showing the phase space available to the trajectory in Fig. 4.6 at $\delta = 5.423$. The period-2 secondary resonance appears detached from the large chaotic zone. The quasiperiodic region on the inside of the resonance (b) is becoming unstable.

Figure 4.11: Sections I showing the phase space available to the trajectory in Fig. 4.6 at $\delta = 3.737$. Another secondary resonance is visible near the inner boundary of the chaotic zone on section Ia (a). The large quasiperiodic region has disappeared from section IIb.

escape eccentricities, but also more of the trajectories escape from the resonance with a lower mean e_M than they had approaching the resonance. the average value of e_A dropped from 0.01 prior to resonance encounter to 0.0094 after escape, with about a 10 percent scatter about this value.

Figs. 4.14 - 4.16 illustrate the range of eccentricity behaviour exhibited by this ensemble of trajectories. Upon entering the chaotic zone, the maximum eccentricity of Miranda's orbit in the chaotic zone jumps to a value more than three times the maximum prior to resonance encounter. Again, the orbital eccentricity variations of Ariel, though chaotic, are not very large. As the trajectories evolve through the chaotic zone, the eccentricities vary irregularly between well-defined maximum and minimum values which gradually decrease. Note that in all cases, both evolution into and out of the chaotic zone are apparently accompanied by passage through secondary resonances.

Miranda-Ariel 5:3 Resonance Energy

Figure 4.12: Trajectories in $\delta, \Delta E$ space with $Q = 110$. The mean initial eccentricities of the orbits are $e_M = 0.009$ and $e_A = 0.010$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. The qualitative features of the evolution are virtually identical to those in Fig. 4.2, except for a change in scale.

Figure 4.13: Distribution of time-averaged orbital eccentricities of Miranda. The vertical bar at the top of the figure indicates the average eccentricity prior to resonance encounter. Compare with Fig. 4.3. Again, most trajectories escape from the resonance with a higher average e_M than the initial value, but in more cases the eccentricity decreases.

Figure 4.14: Variations of the orbital eccentricities of Miranda (a) and Ariel (b) during evolution through the 5:3 mean-motion commensurability. $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). Compare with Figs. 4.4 - 4.6. Again, upon entering the chaotic zone, the maximum orbital eccentricity of Miranda suddenly jumps to relatively high values. The final eccentricities of both orbits are higher than the initial values.

Figure 4.15: Variations of the orbital eccentricities of Miranda (a) and Ariel (b) during evolution through the 5:3 mean-motion commensurability. $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). Compare with Figs. 4.4 - 4.6. This trajectory remains in the chaotic zone somewhat longer than the trajectory shown in Fig. 4.14; the final orbital eccentricity of Miranda is not as high.

Figure 4.16: Variations of the orbital eccentricities of Miranda (a) and Ariel (b) during evolution through the 5:3 mean-motion commensurability. $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). Compare with Figs. 4.4 - 4.6. The final e_M is lower than the initial value.

4.3 Secular Perturbations

In this section we consider the perturbations on the eccentricity-type resonances produced by the secular variations due to the other satellites. To get a qualitative idea of the effects of secular perturbations, we consider only the perturbations on the circular inclined problem due to Umbriel as a first approximation, since they are the most significant (see Dermott and Nicholson 1986, Laskar 1986). The development of the Hamiltonian is given in Appendix I (Section 7.2.4), with $\phi = 3\lambda_M - 5\lambda_A$, $p_i \approx e_i\sqrt{L_i}\cos(-\varpi_i)$, and $q_i \approx e_i\sqrt{L_i}\sin(-\varpi_i)$. The momentum conjugate to ϕ is $\Phi = \frac{1}{3}(L_M - L_{MR})$, where the resonant value L_{MR} is defined at $\frac{\partial \mathcal{H}}{\partial \Phi} = 0$:

$$L_{MR} = \frac{3L_A + 5L_M}{5 + 3\left(\frac{5}{3}\right)^{1/3}\frac{m_A}{m_M}} \quad (4.19)$$

In this model, evolution due to tidal dissipation changes the value of Φ at a rate of $\dot{\Phi} = \frac{1}{3}(\dot{L}_M - \dot{L}_{MR})$. The value of Φ can be converted into a value for δ . As in previously considered models of this kind, there are too many degrees of freedom to allow for a detailed examination of the phase space using surfaces of section.

Fig. 4.17 shows the evolution of the orbital eccentricities of Miranda, Ariel and Umbriel for a trajectory evolved through the resonance using this model with $\mathcal{Q} = 22$ ($\dot{a}_M/a_M = 1.2 \times 10^{-10}$ per orbit period). The initial orbital parameters for this trajectory were $a_M = 4.6394$, $a_A = 6.5157$, $a_U = 10.056$, $e_M = e_A = e_U = 0.005$, $\sigma_M = \sigma_A = \sigma_U = 90^\circ$. Numerical simulations of evolution through the resonance were carried out by integrating the averaged equations of motion, using the Bulirsch-Stoer (1966) algorithm with a relative precision of 10^{-9} per stepsize of approximately 0.6 year. The evolution of Miranda's orbital eccentricity is not markedly different than in the planar-eccentric case involving only Miranda and Ariel: the maximum eccentricity jumps suddenly when the evolution becomes chaotic. There still appear to be well-defined limits to the chaotic variations in e_M as the satellites evolve through the resonance. The average orbital eccentricity after escape from

the resonance is about a factor of two larger than the initial value. The evolution of the orbital eccentricity of Ariel appears to have changed qualitatively. Before the resonance is encountered, the variations in e_A are much larger than in the planar-eccentric case involving Miranda and Ariel only, due to secular interactions with Umbriel. The chaotic variations in eccentricity do not appear to be as constrained as in the two-satellite case: there are not well-defined maxima and minima. Apparently, the constraint on the minimum orbital eccentricity of Ariel imposed by the instantaneous energy surface boundary in the planar-eccentric two-satellite problem does not exist when secular perturbations are included: for an equivalent initial value of the eccentricity, the orbital eccentricity of Ariel may reach a significantly lower value in the chaotic zone under the influence of perturbations due to Umbriel. There is also a much larger relative change in the mean orbital eccentricity before and after resonance encounter. Note that away from resonance, there are significant long-period variations in the eccentricity which modulate the maximum and minimum “envelopes”. The orbital eccentricity of Umbriel varies irregularly during evolution through the chaotic zone, but with an amplitude which apparently remains relatively constant. As with Ariel the orbital eccentricity of this satellite after escaping from the resonance may be somewhat different from the average eccentricity before resonant encounter, but not drastically so.

Figure 4.17: Variations of the orbital eccentricities of Miranda (a), Ariel (b), and Umbriel (c) during evolution through the Miranda-Ariel 5:3 mean-motion commensurability. The orbital elements of all three satellites vary chaotically. The evolution of e_M is somewhat more irregular than in the two-satellite problem, but it still attains a high value before the satellites escape from the resonance.

4.4 The Eccentric-Inclined Problem

The Hamiltonian for the eccentric-inclined problem is developed in Appendix I.

We define

$$\begin{aligned}
 \sigma_i &= \frac{1}{2}(5n_A - 3n_A - 2\varpi_i) \\
 \Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
 \psi_i &= \frac{1}{2}(5n_A - 3n_A - 2\Omega_i) \\
 \Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
 \Gamma_M &= L_M + \frac{3}{2}(\Sigma_A + \Sigma_A + \Psi_A + \Psi_A) \\
 \Gamma_A &= L_A - \frac{5}{2}(\Sigma_A + \Sigma_A + \Psi_A + \Psi_A)
 \end{aligned} \tag{4.20}$$

This is a four degree of freedom Hamiltonian problem with slow time dependence. There are 3 eccentricity resonances, 3 inclination resonances, and secular interactions. The eccentricities are coupled to the inclinations through the nonlinear terms. δ is defined to be the nonresonant contributions to $5n_A - 3n_M - \dot{\varpi}_M - \dot{\varpi}_A - \dot{\Omega}_M - \dot{\Omega}_A$. The expressions and numerical values of the coefficients are given in Appendix I.

Two sets each of 10 trajectories were evolved through the resonance. The rate of evolution is $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($\mathcal{Q} = 110$). Initial conditions for the ensembles of trajectories were generated in a manner similar to that used for the models discussed earlier. For Run 1, from an initial state with physical parameters $a_M = 4.6361$, $a_A = 6.5110$, $e_M = 0.003$, $e_A = 0.003$, $\sigma_M = 90^\circ$, $\sigma_A = 270^\circ$, $i_M = 0.003$ radians, $i_U = 0.003$ radians, $\psi_M = 270^\circ$, and $\psi_A = 90^\circ$, the equations of motion for this Hamiltonian were integrated forward in time using the mapping without dissipation 66 mapping periods ($\Omega = 40.0, T = 66 \times 2\pi/\Omega = 10.4$ years) between states, for a total of ten. The trajectories in the ensemble were then mapped forward in time with tidal dissipation included.

Figs. 4.18 - 4.19 show the variations in the orbital eccentricities and inclinations of Miranda and Ariel of two trajectories from Run 1 during evolution through the 5:3 commensurability. In Fig. 4.18, the evolution of the eccentricities is a bit more complicated than it is in the planar-eccentric approximation, but is otherwise qualitatively similar. Again, the maximum orbital eccentricity of Miranda rapidly jumps to a value of order four times the maximum prior to resonance encounter. The variations of Miranda's eccentricity in the chaotic appear more irregular, and do not seem to have well-defined "envelopes". The average eccentricity after escape from the resonance is again of order a factor of two higher than the average initial value. The eccentricity variations of Ariel's orbit, while still relatively small, are much more irregular than in the planar-eccentric approximation. Note that e_A can be zero in the chaotic zone, in contrast to the planar eccentric problem, where it was constrained to be greater than zero. The final eccentricity of Ariel is somewhat higher than the initial value. The variations in the orbital inclinations of the satellites are also irregular, and the average values after escape may be significantly different from the initial values. e_M varies erratically between the initial value of about 0.2 degrees to almost one degree. The variations in e_A are smaller in an absolute sense, but for this trajectory the final value is down a factor of three from the initial value.

Fig. 4.19 shows the variations in orbital elements for a trajectory also from Run 1, which shows quite a different evolution from the trajectory in Fig. 4.18. In this case, both the eccentricity and inclination of Miranda's orbit jump to values of order three to four times higher than the initial values, and then escape from the resonance, *retaining the high values*. In this case, it appears that the quasiperiodic zone of escape, which in the planar-eccentric approximation was confined to a region near the origin, extends to higher eccentricities in the more complicated phase space of this problem. Again, the orbital eccentricity of Ariel varies more irregularly than in the planar-eccentric approximation, but the orbital parameters of this satellite

Figure 4.18: Variations of the orbital eccentricities (a) and (b) and inclinations (c) and (d) of Miranda and Ariel. The rate of evolution is $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). The evolution of the eccentricities is more irregular, but qualitatively similar, to that in the planar-eccentric problem.

change little compared to those of Miranda.

Fig. 4.20 shows the distribution of time-averaged orbital elements of the satellites after escape from the resonance. The time-averaged initial values for the trajectory are indicated by the vertical bars in the upper part of each figure. Note that the orbital elements of Miranda tend to increase substantially during evolution through the resonance, while those of Ariel on average decrease by a much smaller amount.

Fig. 4.21 shows the final orbital elements of the satellites for the trajectories in Run 2. The initial states of these trajectories were generated in the same way as those for Run 1, starting from an initial state with physical parameters $a_M = 4.6361$, $a_A = 6.5110$, $e_M = 0.005$, $e_A = 0.005$, $\sigma_M = 90^\circ$, $\sigma_A = 270^\circ$, $i_M = 0.005$ radians, $i_U = 0.005$ radians, $\psi_M = 270^\circ$, and $\psi_A = 90^\circ$. For this set, as in Run 1, the orbital eccentricity and inclination of Ariel on average decrease during evolution through the resonance. Again, these orbital elements on average increase in magnitude for Miranda, in some cases by impressive amounts: e_M can be of order 0.02 upon escape, and i_M can be over one degree. This may have a significant effect on passage through the Miranda-Umbriel 3:1 commensurability, as discussed in the next Section.

Figure 4.19: Variations of the orbital eccentricities (a) and (b) and inclinations (c) and (d) of Miranda and Ariel. The rate of evolution is $\dot{a}_A/a_A = 4.3 \times 10^{-11}$ per orbit period ($Q = 110$). The eccentricity and inclination of Miranda's orbit jump significantly during resonance passage, and the high values are retained after escape from the resonance.

Figure 4.20: The distribution of time-averaged eccentricities (a) and (b) and inclinations (c) and (d) of Miranda and Umbriel after escape from the 5:3 resonance for Run 1. The average initial values are denoted by the vertical bars. The eccentricity and inclination of the orbit of Miranda may increase significantly.

Figure 4.21: The distribution of time-averaged eccentricities (a) and (b) and inclinations (c) and (d) of Miranda and Umbriel after escape from the 5:3 resonance for Run 2. The average initial values are denoted by the vertical bars. The eccentricity and inclination of the orbit of Miranda may increase significantly.

4.5 Discussion

There is a large chaotic zone associated with the Miranda-Ariel 5:3 mean-motion commensurability, even in the planar approximation. The orbital eccentricities of both satellites may vary chaotically for a considerable period of time. After a sudden increase at the point of entering the chaotic zone, the maximum orbital eccentricity of Miranda gradually decreases until the trajectory escapes from the resonance. The final average orbital eccentricity and inclination of Miranda is most likely, though not necessarily, somewhat higher than the initial value. On average, the average final eccentricity of Ariel is lower than the initial value.

Since the rate of tidal heating depends strongly on the distance to the planet:

$$\frac{dE}{dt} \propto \frac{1}{a^{15/2}} \quad (4.21)$$

a given value of e_M will heat the interior of Miranda more at the 5:3 resonance with Ariel than at the 3:1 resonance with Umbriel. However, the maximum eccentricities achieved at the 5:3 resonance are lower than those achieved at the 3:1 resonance, and the average values of the eccentricity of Miranda's orbit in the chaotic zone are much lower. In addition, tidal evolution within the chaotic zone tends to decrease the orbital eccentricities. Therefore, the tidal heating rate at this resonance was probably insignificant.

Of potentially greater importance is the possibility that Miranda may have left the 5:3 resonance with an orbital inclination of order one degree. The relatively rapid increase in inclination possible during chaotic evolution through this resonance (*e.g.* Fig. 4.19) may have left Miranda with a substantial rotational obliquity, however, this would have quickly damped to the equilibrium Cassini state (see Chapter 3). Therefore, a relatively high inclination of Miranda's orbit could have been retained over the time interval between the 5:3 resonance involving Ariel and the 3:1 resonance involving Umbriel, possibly affecting the evolution through the

latter commensurability.

However, it is quite possible that even if Miranda approached the 3:1 commensurability involving Umbriel with a relatively high orbital inclination, i_M still may have jumped to its present value. Numerical simulations of the evolution of trajectories through the 3:1 Miranda-Umbriel commensurability with $i_M \approx 1^\circ$ approaching the resonance indicate that there is still a substantial probability of capture into the $i_M i_U$ resonance, followed by a large increase in i_M (*e.g.* Fig. 3.15). It has been observed that temporary capture onto the libration zones from the large chaotic zone at the 3:1 resonance is quite common (*e.g.* Fig. 3.4), further supporting the idea that there is a significant probability that i_M would still increase at the 3:1 commensurability.

Chapter 5

The Ariel-Umbriel 2:1 Commensurability

5.1 Introduction

The last of the mean-motion commensurabilities we consider here is the 2:1 resonance involving Ariel and Umbriel. This resonance would have been encountered if $Q < 11,000$. This commensurability is significantly different from those previously considered in that there are two first-order eccentricity-type resonances present, as well as three second-order eccentricity-type resonances, three second-order inclination-type resonances, and a second-order secular interaction. The main effect of these first-order resonances is to greatly increase the probability of capture at low initial eccentricities.

We begin, as usual, by studying the planar-eccentric problem in Section 5.2, in order to understand the dynamics by studying them in detail using surfaces of section. For low initial eccentricities comparable to the current values, both resonant arguments are captured into resonance, and the system reaches its equilibrium eccentricity. Although numerous secondary resonances between the libration frequencies of the resonant arguments are encountered, they only affect the details of the evolution within the resonance, but do not lead to escape. For higher initial eccentricities, these secondary resonances overlap and form significant chaotic zones.

Trajectories may spend considerable periods of time in the chaotic zone before being captured into the resonance. However, escape from the 2:1 commensurability is only highly probable for initial eccentricities of order 0.04, an order of magnitude higher than the current values. Secular perturbations due to Titania (Section 5.3) do not significantly affect this basic picture: for “reasonable” initial orbital eccentricities, trajectories evolve to e_A much higher than the equilibrium value without escaping from the resonance.

In section 5.4, the eccentric inclined problem is considered. It is found that there are interactions between the eccentricities and inclinations which lead to an increase in the orbital inclinations while the amplitudes of libration of the eccentricities decrease. Also, capture into the i_A^2 resonance is likely. However, neither of these mechanisms appears to lead to escape, and both are encountered at e_A greater than the equilibrium value.

In section 5.5, the significance of these results is discussed. Since capture into the Ariel-Umbriel 2:1 commensurability is likely, and because there appears to not be a mechanism to disrupt this resonance once capture occurs, it is probably a dynamical “barrier” to the evolution. Rather special initial conditions are required if the probability of escape from this resonance is to be high. Therefore, the specific dissipation function (Q) of Uranus can be constrained to be greater than 11,000.

5.2 The Planar-Eccentric Problem

For the Ariel-Umbriel 2:1 commensurability, the planar-eccentric problem can be reduced to two degrees of freedom. This allows us to study the phase space in detail using surfaces of section. We begin by discussing the resonant Hamiltonian for this problem in Section 5.2.1. In Section 5.2.2, we discuss evolution through the resonance. We begin by discussing the rate of evolution and its effects on the dynamics, as well as the value of the equilibrium eccentricity. This resonance is quite robust: capture still occurs for rates four orders of magnitude or so higher than the rates at which artifacts occurred in the second-order resonances described in the previous Chapters.

Numerical simulations of the evolution through the commensurability indicate that capture into the resonance is very likely. For initial eccentricities comparable to the current values, it is found that after both resonance arguments are captured into libration, secondary resonances are encountered. Depending on whether or not a particular trajectory is captured into such a resonance, the libration amplitudes of the eccentricities may increase to maximum values defined by the instantaneous energy surface, or may suddenly decrease as the trajectory crosses the separatrix associated with the secondary resonance. However, these secondary resonances appear to affect only the details of resonance passage: they do not lead to escape from the commensurability.

For higher initial eccentricities, the evolution is more complicated: trajectories may be chaotic for a considerable period of time. However, a high probability of escape requires initial eccentricities about an order of magnitude higher than the current values.

5.2.1 Resonance Coordinates

The Hamiltonian for this resonance is developed in Appendix I. As resonance coordinates we have chosen

$$\begin{aligned}\sigma_A &= \frac{1}{2}(4\lambda_U - 2\lambda_A - 2\varpi_A) \\ \sigma_U &= \frac{1}{2}(4\lambda_U - 2\lambda_A - 2\varpi_U)\end{aligned}\tag{5.1}$$

which together with λ_A and λ_U form a complete set of generalized coordinates. The momenta conjugate to σ_A and σ_U are, in terms of the Delaunay momenta $L_i \approx m_i \sqrt{GMa_i}$ and $G_i = L_i \sqrt{1 - e_i^2}$ (see Plummer 1960):

$$\begin{aligned}\Sigma_A &= L_A - G_A \approx \frac{L_A}{2} e_A^2 \\ \Sigma_U &= L_U - G_U \approx \frac{L_U}{2} e_U^2\end{aligned}\tag{5.2}$$

This choice of variables results in two integrals of motion, the momenta conjugate to $\gamma_A = \lambda_A$ and $\gamma_U = \lambda_U$, since in the new variables the resonant Hamiltonian is cyclic in these variables. The resonance integrals are:

$$\begin{aligned}\Gamma_A &= L_A + (\Sigma_A + \Sigma_U) \\ \Gamma_U &= L_U - 2(\Sigma_A + \Sigma_U)\end{aligned}\tag{5.3}$$

The coefficient $A \approx \frac{1}{4}(4n_U - 2n_A)$. At the 2:1 mean-motion resonance, this quantity is near zero. Due to tidal dissipation in the planet, the semimajor axis ratio decreases. (see section 1.3). Therefore, the quantity $(4n_U - 2n_A)$ increases as the satellites evolve through the resonance. We define the parameter $\delta = 4A + 2(C + D)$ to be the resonance parameter in this problem. At small eccentricities it is proportional to the non-resonant contribution to $4n_U - 2n_A - \dot{\varpi}_A - \dot{\varpi}_U$.

5.2.2 Evolution through the Resonance

Rate of Evolution

Of the various coefficients in the resonance Hamiltonian, only δ depends strongly on the resonant combination of mean-motions; during passage through the resonance, the fractional change in the parameter δ is large compared to the fractional changes in the other coefficients. The other coefficients will therefore be taken to be constant. In addition, the value of δ is influenced by the damping of eccentricities due to tidal dissipation in the satellites. The time rate of change of δ is

$$\dot{\delta} \simeq 4\dot{A} \quad (5.4)$$

From the expansion of A to first order in Σ_i :

$$\begin{aligned} \dot{\delta} &\simeq \frac{d}{dt} [(4n_M - 2n_A) - 16B(\Sigma_M + \Sigma_A)] \\ &= (4\dot{n}_A - 2\dot{n}_M) - 16B(\dot{\Sigma}_M + \dot{\Sigma}_A) \end{aligned} \quad (5.5)$$

The first two terms are due to the tidal expansion of the orbits, and the other terms are due to the exchange of angular momentum between the satellites resulting from dissipation of energy in the satellites (Yoder 1979b). Note that

$$\dot{\Sigma}_i \simeq \Sigma_i \left[-\frac{3}{4} \frac{\dot{n}_i}{n_i} + 2 \frac{\dot{e}_i}{e_i} \right] \quad (5.6)$$

Tidal friction results in a secular variation of the mean-motions

$$\dot{n}_i = \dot{n}_{ip} + \dot{n}_{is} \quad (5.7)$$

where the first term is due to tides raised on the planet (*e.g.* Goldreich 1965)

$$\dot{n}_{ip} = -\frac{9}{2} k_2 n_i^2 \frac{m_i}{M} \left(\frac{R}{a_i} \right)^5 \frac{1}{Q} \quad (5.8)$$

The second term is due to tidal dissipation in the satellite, which for a synchronous rotation state must conserve the orbital angular momentum: $\dot{G}_i \simeq d/dt \sqrt{GM a_i (1 - e^2)} =$

0, so $\dot{a}_i \simeq 2a_i e_i \dot{e}_i$. Therefore,

$$\dot{n}_i = \dot{n}_{ip} - 3n_i e_i \dot{e}_i \quad (5.9)$$

Tidal friction also changes the orbital eccentricities (Goldreich, 1963)

$$\dot{e}_i = -\frac{21}{2} k_{2i} n_i \frac{M}{m_i} \left(\frac{R_i}{a_i} \right)^5 \frac{e_i}{Q_i} + \frac{57}{8} k_2 n_i \frac{m_i}{M} \left(\frac{R}{a_i} \right)^5 \frac{e_i}{Q} \quad (5.10)$$

where k_2 and Q are, respectively, the potential Love number and specific dissipation function for Uranus, and the k_{2i} 's and Q_i 's are the same quantities for the satellites. The first term in the expression for $\dot{e}_i$ is the contribution due to tidal dissipation in the satellite, and the second term is due to tidal dissipation in the planet. The relative contributions of these components is

$$\frac{\dot{e}_s}{\dot{e}_p} = -\frac{28}{19} D_i \quad (5.11)$$

where

$$D_i = \frac{k_{2i}}{k_2} \left(\frac{M}{m_i} \right)^2 \left(\frac{R_i}{R} \right)^5 \frac{Q}{Q_i} \quad (5.12)$$

expresses the ratio of the satellite tidal scale factor to that of the planet (*e.g.* Yoder 1979). For Ariel, $D_A = 60.4$ ($Q = 6600$ and $Q_A = 100$), and the satellite dissipation term dominates the planet dissipation term by about two orders of magnitude. This is also the case for the other satellites. Since $\dot{n}_p/n$ and $\dot{e}_p/e$ are of the same order, this means that

$$\dot{\Sigma}_i \approx 2\Sigma_i \frac{\dot{e}_i}{e_i} = L_i e_i \dot{e}_i \quad (5.13)$$

Using numerical values from Table 1.1 with $Q = 6600$ to evaluate relative contributions to $\dot{\delta}$,

$$\frac{4\dot{n}_U}{2\dot{n}_A} \simeq \frac{4m_U}{2m_A} \left(\frac{a_A}{a_U} \right)^8 \approx 0.047 \quad (5.14)$$

so the semimajor axis ratio increases. The ratio of the eccentricity dependent terms is

$$\begin{aligned}\frac{\dot{\Sigma}_U}{\dot{\Sigma}_A} &\simeq \frac{k_{2U} m_A n_U}{k_{2A} m_U n_A} \left(\frac{R_U}{R_A}\right)^5 \left(\frac{a_A}{a_U}\right)^5 \frac{Q_A \Sigma_U}{Q_U \Sigma_A} \\ &\approx 0.06 \left(\frac{e_U}{e_A}\right)^2\end{aligned}\quad (5.15)$$

The most significant terms in Eq. 5.5 are therefore:

$$\dot{\delta} = -2\dot{n}_A - 16B\dot{\Sigma}_A \quad (5.16)$$

Note that (see Appendix I)

$$16B \approx -6 \frac{n_A}{L_A} \left[1 + 4 \frac{m_A}{m_U} \left(\frac{a_A}{a_U}\right)^2 \right] \quad (5.17)$$

and

$$\frac{\dot{e}_A}{\dot{n}_A} \simeq \frac{7}{3} D_A \frac{e_A}{n_A} \quad (5.18)$$

Therefore

$$\dot{\delta} \approx -2\dot{n}_A \left\{ 1 - 14D_A e_A^2 \left[1 + 2 \frac{m_A}{m_U} \left(\frac{a_A}{a_U}\right)^2 \right] \right\} \quad (5.19)$$

δ will be stationary at the equilibrium eccentricity

$$e_A = \sqrt{\frac{1}{14D_A \left[1 + 2 \frac{m_A}{m_U} \left(\frac{a_A}{a_U}\right)^2 \right]}} \quad (5.20)$$

This result is in agreement with the equilibrium eccentricity found for the 2:1 two-body resonances in the Jovian satellite system by Yoder (1979). For the Ariel-Umbriel 2:1 resonance, the equilibrium eccentricity of Ariel is

$$e_A = \sqrt{\frac{1}{25.7D_A}} = 0.025 \quad (5.21)$$

if $Q = 6600$.

For the current eccentricity of Ariel the $\dot{n}_A$ term is dominant in the expression for $\dot{\delta}$. In addition, it is unknown whether the dissipation in the satellites would cause

a significant exchange of angular momentum if the orbits were chaotic. We have used only the rate of dissipation due to planet tides in our numerical calculations:

$$\dot{\delta} \approx \frac{3G^2 M^2 m_A^3}{\Gamma_A^3} \frac{\dot{a}_A}{a_A} \quad (5.22)$$

For e_A in the resonance approaching the equilibrium value, this rate will not be a good approximation.

Tidal friction also introduces explicit frictional terms into the equations of motion, the most important terms arising from the damping of the orbital eccentricities due to dissipation within the satellites. The eccentricity damping timescale of Ariel is about 6×10^7 years and that of Umbriel is about 10^9 years at this resonance, for $Q_i \approx 100$. The timescale of resonance passage depends on the rate of orbital evolution, which is determined by the specific dissipation function Q of Uranus. For the minimum $Q = 6600$, $\dot{\delta} \approx 2.7 \times 10^{-7} \text{ yr}^{-2}$. Typically, escaping trajectories remain in the chaotic zone $\Delta\delta \approx 5.0 \text{ yr}^{-1}$, indicating a resonance passage timescale of order 10^7 years. Since the maximum value of Q which allows the satellites to encounter this resonance is about 11,000, tidal damping of eccentricities is probably not important in the dynamics of resonance passage. In order to understand the orbital dynamics in a Hamiltonian context, we have left out the explicit frictional terms in this initial study.

As in the investigations of the other resonances, we have carried out a study to determine the effect on the dynamics of resonance passage of the effective specific dissipation function of Uranus, defined here, as always, to be script $\mathcal{Q}$. Ensembles of 20 trajectories were evolved through the resonance at different evolution rates. The initial coordinates of the trajectories were determined from an initial state with the physical parameters of Run 2 (see Table 5.1). From this initial state, the 19 other states in the ensemble were determined by integrating the equations of motion forward in time with the mapping without tidal dissipation, spacing successive states 66 mapping iterations ($\Omega = 40.0 \text{ yr}^{-1}$, $\Delta t = 66 \times 2\pi/\Omega = 10.4$ years) in time apart.

Each of these initial states was then integrated forward in time with tidal dissipation included.

For all ensembles integrated through the resonance at a rate slower than about $\dot{a}_A/a_A = 4.5 \times 10^{-6}$ per orbit period, all trajectories were captured into resonance. This rate is about 4 orders of magnitude higher than the rates at which artifacts appeared in the dynamics of the previously considered resonances, indicating that the capture mechanism is very robust. As we will see further on, this appears to be partly a consequence of the fact that for such low eccentricities, trajectories do not cross separatrices associated with the primary mean-motion resonances, and capture occurs via evolution of the energy surface on which a trajectory evolves. The region of escape does not appear until eccentricities are much higher.

Numerical Results

Figs. 5.1 - 5.6 show results of numerical runs in which 5 (Run 1), 20 (Runs 2 - 4), or 10 (Runs 5 and 6) trajectories were numerically integrated through the resonance using the mapping, at a rate of $\mathcal{Q} = 110$ ($\dot{a}_A/a_A = 2.7 \times 10^{-11}$ per orbit period). Table 5.1 gives the physical parameters and initial coordinates from which each ensemble was generated. From these initial coordinates, the equations of motion were mapped forward in time 66 iterations (Runs 1 - 4: $\Omega = 40 \text{ yr}^{-1}$, Runs 5 - 6: $\Omega = 80 \text{ yr}^{-1}$) per initial condition without tidal dissipation, to provide a set of initial coordinates with well-mixed phases, but which had identical actions.

The parameter space shown in Figs. 5.1 - 5.6 is defined in the same way as for the Ariel-Umbriel 5:3 resonance, with coefficients appropriate to this resonance. The heavily shaded region in the upper left part of the Figures is a forbidden region of parameter space: there are no real solutions to the equations which determine initial conditions for the surfaces of section used to study this problem. Again, the lightly shaded region is the parameter space in which large chaotic zones are

Table 5.1: Initial Parameters

Run	ΔE	δ	a_A	e_A	σ_A	x_A	y_A	n
			a_U	e_U	σ_U	x_U	y_U	
1	-7.7232×10^{-5}	-7.8494	6.3100	0.005	90.0	0.0	4.1764×10^{-3}	5
			10.0400	0.005	90.0	0.0	4.5651×10^{-3}	
2	-2.7399×10^{-4}	-6.3608	6.3100	0.01	90.0	0.0	8.3527×10^{-3}	20
			10.0400	0.01	270.0	0.0	-9.1302×10^{-3}	
3	-7.4054×10^{-4}	-6.5631	6.3050	0.015	90.0	0.0	1.2529×10^{-2}	20
			10.0400	0.015	270.0	0.0	-1.3695×10^{-2}	
4	-1.4608×10^{-3}	-5.7724	6.3000	0.02	90.0	0.0	1.6705×10^{-2}	20
			10.0400	0.02	270.0	0.0	-1.8260×10^{-2}	
5	-5.4760×10^{-4}	7.1303	6.3055	0.03	90.0	0.0	2.5058×10^{-2}	10
			10.0400	0.03	270.0	0.0	-2.7391×10^{-2}	
6	3.3903×10^{-3}	21.1875	6.3057	0.04	90.0	0.0	3.3411×10^{-2}	10
			10.0400	0.04	270.0	0.0	-3.6521×10^{-2}	

present in the phase space. The extent of this region was determined by computing the distance function for 20 trajectories with initial conditions which, for each value of δ and ΔE studied, were computed on the surface of section which plots y_A vs. x_A when $y_U = 0$. The points defining the initial conditions on the surface of section were equally spaced along the x_A axis between the limits of motion set by the energy surface boundary. As in previous figures of this kind, the heavy solid line decreasing quadratically in the lower left of the figure marks where the energy of the Hamiltonian, $\mathcal{E}$, is equal to zero. For smaller values of ΔE , there are only two possible solutions of the quartic equation which determines initial conditions on the surfaces of section. For larger values of ΔE , there may be two or four solutions of this equation. For $\Delta E > 0$, the energy surface boundary enclosing the accessible phase space on the surface of section which plots y_U vs. x_U when $x_M = 0$ no longer encloses the origin: for the curves on this surface of section, σ_U librates. It is still possible for the resonant angles to circulate on other surfaces of section and in the full phase space, though. However, for $\Delta E > \mathcal{E}_L$, the boundaries defined

by the energy surface on all of the surfaces of section do not enclose the origin, and both resonant arguments must librate. Both orbits are therefore captured into resonance. Note that capture may occur for lower values of ΔE , but $\mathcal{E}_L$ marks a boundary where all trajectories, even chaotic trajectories, must be captured into resonant libration.

Fig. 5.1 shows the evolution of the Hamiltonian energies of five trajectories with initial orbital eccentricities comparable to the current values (see Table 5.1). In all cases, the trajectories are quickly captured into the resonance, with very little chaotic behaviour present. As we proceed through the series of Figs. 5.2 - 5.6, the behaviour becomes more complex as the initial eccentricities are increased. Trajectories may be chaotic for substantial periods of time before becoming captured. The final energies of different trajectories in an ensemble may be quite different, depending on the evolution within the chaotic zone, however, there is not a significant probability of escape until the initial eccentricities are of order 0.04 (Fig. 5.6). It is clear that if the satellites had approached this resonance with orbital eccentricities even much higher than the current values, capture would have been very likely.

As in the previous commensurabilities, we can study the evolution of the trajectories in more detail. Fig. 5.7 is a plot of the maximum and minimum orbital eccentricities of Ariel and Umbriel in intervals of $\Delta\delta = 0.01$ for a trajectory in Run 1. Both orbits are captured into resonance without displaying any significant chaotic behaviour. However, there are significant “glitches” in the amplitudes of oscillation of the eccentricities. For example, at $\delta \approx 0.5$, the oscillation amplitude of Umbriel’s eccentricity grows considerably, then levels off. Shortly afterwards, at $\delta \approx 4$, the amplitude suddenly decreases. A series of lesser “glitches” in both amplitudes appear at regular intervals during further evolution. Note that the orbital eccentricity of Ariel reaches its equilibrium value (0.025) without any evidence of disruption of the resonance.

Ariel-Umbriel 2:1 Resonance Energy

Figure 5.1: Trajectories in $\delta, \Delta E$ parameter space for Run 1 with $\mathcal{Q} = 110$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. All five trajectories are quickly captured into the resonance.

Figure 5.2: Trajectories in $\delta, \Delta E$ parameter space for Run 2 with $Q = 110$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. All twenty trajectories are captured into the resonance.

Ariel-Umbriel 2:1 Resonance Energy

Figure 5.3: Trajectories in $\delta, \Delta E$ parameter space for Run 3 with $\mathcal{Q} = 110$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. Although significant chaotic behaviour is present, all twenty trajectories are captured into the resonance.

Ariel-Umbriel 2:1 Resonance Energy

Figure 5.4: Trajectories in $\delta, \Delta E$ parameter space for Run 4 with $\mathcal{Q} = 110$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. Significant variations in the evolution of the Hamiltonian energies is apparent, depending on details of evolution in the chaotic zone, but all trajectories are captured into the resonance.

Figure 5.5: Trajectories in $\delta, \Delta E$ parameter space for Run 5 with $Q = 110$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. Most of the evolution of the trajectories after encountering the resonance is chaotic before capture occurs. There is still no significant probability of escape.

Ariel-Umbriel 2:1 Resonance Energy

Figure 5.6: Trajectories in $\delta, \Delta E$ parameter space for Run 6 with $Q = 110$. Solid lines indicate quasiperiodic behaviour, dashed lines indicate chaotic behaviour. For these trajectories, with $e_{A0} \approx e_{U0} \approx 0.04$, there is a significant probability of escape from the commensurability (6 out of 10 trajectories escaped), followed by evolution to large negative ΔE . One of the trajectories escapes from the e_U resonance, but is captured into the e_A resonance (see text).

Figure 5.7: Orbital eccentricity variations of Ariel (a) and Umbriel (b) of a trajectory from Run 1 during evolution through the 2:1 commensurability. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.01$. The evolution is primarily quasiperiodic, but with modulations of the oscillation amplitudes of e_A and e_U . The trajectory attains the equilibrium eccentricity $e_A = 0.025$ without escaping from the resonance.

In order to understand this complicated behaviour, we have again taken snapshots of the phase space using surfaces of section. The section conditions chosen for study are: plotting y_A vs. x_A when $y_U = 0$, defined to be section I, and plotting y_U vs x_U when $y_A = 0$, defined to be section II. Only one quartic root (the one with the lowest value) need be plotted in order to show the relevant structure of the phase space.

Fig. 5.8 shows the phase space inhabited by the trajectory in Fig. 5.7 on section I at $\delta = -7.849$ and $\Delta E = -7.723 \times 10^{-5}$, the initial values for this trajectory. The phase space is characteristic of that for satellites with strong secular interactions, with two secular modes visible on the section (see Section 2.2.2 for a discussion of secular modes). We designate the secular mode dominating section I to be mode I, and that dominating section II to be mode II. The trajectory in Fig. 5.7 generates the second curve from the origin, which with the number of points plotted does not form a fully closed curve. On this and other surfaces of section, other curves are displayed on the surfaces of section in order to show the structure of the phase space through which the trajectory in Fig. 5.7 evolves.

Fig. 5.9 shows the phase space on section I at $\delta = 0.635$. Note that the energy surface boundary only just encloses the origin in this figure: most trajectories have been captured into resonance. A period-2 bifurcation, which corresponds to a 2:1 secondary resonance between the libration frequencies of σ_U and σ_A , has occurred at the fixed point of the secular mode dominating this surface of section (mode II). The trajectory in Fig. 5.7, which generates the closed curve on the section just outside the “figure-8”, is about to be captured into this secondary resonance.

Fig. 5.10 shows the phase space just after capture into this secondary resonance. The separatrix has become visibly chaotic. The trajectory generates the curve which forms the two loops in the upper and lower center of the figure, adjacent to the energy surface boundary. Hence, the secondary resonance has dragged the

Figure 5.8: Surface of section in Umbriel variables (section II), at $\delta = -7.849$, $\Delta E = -7.732 \times 10^{-5}$. The phase space is regular, with two secular modes visible.

Ariel/Umbriel 2:1 Resonance Surface of Section

Figure 5.9: Section II at $\delta = 0.635$. There is a period-2 secondary resonance between the libration frequencies visible on mode II. The trajectory in Fig. 5.7, which generates the curve just outside the separatrix, is about to encounter this secondary resonance.

trajectory away from the center of Mode II, increasing the amplitude of oscillation of the eccentricity about the mean value, as can be seen in Fig. 5.7b between $\delta \approx 0.5$ and $\delta \approx 2$. Note also on that the energy surface boundary no longer encloses the origin: at this point in the evolution, $\mathcal{E} > \mathcal{E}_L$, and both resonant arguments are trapped into libration.

Fig. 5.11 shows the phase space further on, at $\delta = 1.520$. The separatrix associated with the period-2 secondary resonance has disappeared. On this figure, the trajectory in Fig. 5.7 generates the curve forming the loop on the right of the figure adjacent to the energy surface boundary. At this location in phase space, it has about as large an oscillation amplitude as possible, hence, the levelling off of the amplitude visible in Fig. 5.7b.

Fig. 5.12 shows the phase space at $\delta = 3.769$, just before the sudden decrease in oscillation amplitude on Fig. 5.7b. At this point, the trajectory generates the curve which extends from the energy surface boundary at top center to the boundary at bottom center. A period-3 chain of islands, associated with a 3:2 secondary resonance between the libration frequencies of σ_U and σ_A , is moving away from the center of the dominant mode on this section (Mode II), and the trajectory is about to encounter it. In this case, the trajectory passes through the separatrix without being captured onto the islands. As it does so, the action (area enclosed by the trajectory on the surface of section) decreases, and along with it the amplitude of oscillation of e_U , as seen in Fig. 5.7b. As this trajectory further evolves, it encounters other secondary resonances. The higher order resonances appear to be much weaker, and do not significantly affect the libration amplitude.

The slight variations in evolution of the energies seen in Fig. 5.1 are associated with differences in the outcomes of passage through the secondary resonances; for example, a trajectory which is captured into the 3:2 secondary resonance will have a somewhat different energy than one which escapes from the secondary resonance.

Figure 5.10: Section II at $\delta = 1.145$. The trajectory in Fig. 5.7 has been captured by the secondary resonance and dragged to the energy surface boundary, where it forms the two loops at upper and lower center.

Ariel/Umbriel 2:1 Resonance Surface of Section

Figure 5.11: Section II at $\delta = 1.520$. The trajectory in Fig. 5.7 generates the curve nearest the energy surface boundary on the right of the figure.

Ariel/Umbriel 2:1 Resonance Surface of Section

Figure 5.12: Section II at $\delta = 3.769$. The trajectory in Fig. 5.7 generates the curve extending from the energy surface boundary at top center to the boundary at bottom center. It is just about to encounter the 3:2 resonance between the libration frequencies of σ_U and σ_A .

However, it is important to note that in this case, capture into a secondary resonance does not lead to escape from the mean-motion resonance: there is no region of escape into which to evolve. For such low eccentricities, the trajectory does not pass through the separatrices associated with the primary mean-motion resonances. Note that this is also the case in the single-resonance theory (see *e.g.* Peale 1986, and Appendix III) for comparable initial eccentricities. *The existence of secondary resonances does not guarantee the disruption of a mean-motion resonance.* This was also seen in the Miranda-Umbriel 3:1 commensurability for extremely low eccentricities approaching the resonance (see Section 3.3.2).

Fig. 5.13 shows the eccentricity variations of the satellites for a trajectory with similar initial conditions, which was evolved through the commensurability using averaged equations of motion. The integrator used was a Bulirsch-Stoer (1966) algorithm with a relative precision of 10^{-9} per stepsize of about 0.16 years, with $Q = 25$, or $\dot{a}_A/a_A = 1.2 \times 10^{-10}$ per orbit period. This integration confirms that the dynamical features observed in Fig. 5.7 are real.

For higher initial eccentricities, the evolution is qualitatively similar, but more complicated. Fig. 5.14 shows the evolution of the orbital eccentricities of a trajectory from Run 2. This picture is very similar to Fig. 5.7: the evolution is dominated by quasiperiodic behaviour, except between $\delta \approx 0$ and $\delta \approx 4$. Both resonant arguments are captured into libration. “Glitches” in the oscillation amplitude are still present, but the evolution through the resonance proceeds without disruption to e_A much higher than the equilibrium value of 0.025. The phase space through which this trajectory passes at $\delta = 1.316$ is shown in Figs. 5.15 (section I) and 5.16 (section II). A number of secondary resonances is visible on each plot: the 2:1 resonance between the libration frequencies, the 5:3 resonance between libration frequencies, and a period-2 secondary resonance in the region of phase space corresponding to the secular mode dominating section I away from resonance. The secondary

Figure 5.13: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability, integrating the *averaged differential equations*. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.01$. The dynamical features seen on Fig. 5.7 are verified.

resonances are close together, with sizable chaotic separatrices. Clearly, overlap of the separatrices associated with many secondary resonances is occurring, forming a complex structure. (Compare with the overlap of secondary resonances seen in the Miranda-Umbriel 3:1 resonance, *e.g.* Fig. 3.25). At this point in its evolution, the trajectory has just entered the large chaotic zone associated with the period-2 secondary resonance (compare Fig. 5.16 with Fig. 5.9). While in this chaotic zone, the evolution is quite complicated, as can be seen by inspecting Fig. 5.14b between $\delta \approx 2$ and $\delta \approx 4$. Eventually the trajectory enters a quasiperiodic region and capture of both resonant arguments occurs. Other secondary resonances are encountered, but are more widely spaced, and there are not significant chaotic zones.

Fig. 5.17 shows a trajectory with the same initial coordinates given in Table 5.1 for Run 2, but evolved through the commensurability with $\mathcal{Q} = 660$ ($\dot{a}_A/a_A = 4.5 \times 10^{-12}$ per orbit period). The most important features of the evolution are the same as those in Fig. 5.14, indicating that slower rates do not significantly change the evolution. However, the details of passage through the secondary resonances differ, leading to larger oscillation amplitudes of the eccentricities at high eccentricities. This emphasizes the origin of the differences in energy evolution for trajectories in a given ensemble (Figs. 5.1-5.6).

For still higher initial eccentricities, chaotic behaviour becomes more important. Fig. 5.18 shows the evolution of the orbital eccentricities of a trajectory from Run 3. The trajectory is chaotic during much of the evolution, but is still captured into the resonance and evolves to e_A much higher than the equilibrium value without escaping. Fig. 5.19 shows the phase space on section II at $\delta = 2.326$, just after the point that the trajectory first becomes chaotic. Traces of the structure of the secondary resonances remain, but the chaotic zone dominates the surface of section. However, a region into which trajectories may escape from the resonance has not yet appeared. Fig. 5.20 shows the evolution of a orbital eccentricities of a trajectory

Figure 5.14: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability for a trajectory from Run 2. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.01$. The evolution is primarily quasiperiodic, but with changes in the oscillation amplitudes of e_A and e_U . The trajectory attains the equilibrium eccentricity $e_A = 0.025$ without escaping from the resonance.

Ariel/Umbriel 2:1 Resonance Surface of Section

Figure 5.15: Section I at $\delta = 1.316$. The trajectory in Fig. 5.14 has just encountered the large chaotic zone associated with the the 1:2 secondary resonance between the libration frequencies of σ_A and σ_U . Other secondary resonances are also visible.

Ariel/Umbriel 2:1 Resonance Surface of Section

Figure 5.16: Section II at $\delta = 1.316$. The trajectory in Fig. 5.14 has just encountered the large chaotic zone associated with the the 2:1 secondary resonance between libration frequencies of σ_U and σ_A . Compare with Fig. 5.9. Other secondary resonances are also visible.

Figure 5.17: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability for a trajectory from Run 2, but with $Q = 660$. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.01$. Qualitatively, the evolution is the same as in Fig. 5.14, but comparing these two figures shows the effect of slight differences in evolution through the secondary resonances.

from Run 4. The behaviour is qualitatively similar.

As we saw earlier, the initial eccentricities must be quite high, of order 0.04, for the probability of escape from the resonance to become significant. Fig. 5.21 shows the evolution of the orbital eccentricities for a trajectory from Run 6 which escapes from the 2:1 commensurability. Note that apart from the scale, this figure resembles closely the evolution through the 5:3 commensurability (*e.g.* Fig. 2.22.) Fig. 5.22 shows the phase space on section II at $\delta = 28.157$, just before the trajectory escapes from the resonance. This picture looks quite different from the surfaces of section for lower initial eccentricities. At this point in the evolution, the trajectory is still in the large chaotic zone. Note that the mode which dominates section I on the right of the figure is still stable. Two new regions of quasiperiodic behaviour have appeared on the section. The small regular region on the left of the figure in the large chaotic zone is the region into which trajectories escape from the resonance. Note that this is well away from the origin; hence, the orbital eccentricity of Umbriel is relatively high upon escape from the resonance (see Fig. 5.21b), a result which was also found at the 5:3 resonance for a similar reason (see discussion on p. 91). The quasiperiodic region near the origin of the figure is a region into which trajectories may escape from the e_U resonance, but are still captured into the e_A resonance. Fig. 5.23 shows the evolution of the eccentricities of a trajectory which evolved in this manner. The Hamiltonian energy of this trajectory evolves almost horizontally in $\delta, \Delta E$ space (Fig. 5.6).

Figure 5.18: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability for a trajectory from Run 3. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.012$. The trajectory is chaotic for a significant interval of time during evolution through the resonance, but does not escape from the resonance, even at e_A much higher than the equilibrium value.

Figure 5.19: Section II at $\delta = 2.326$. The trajectory in Fig. 5.19 has just encountered the large chaotic zone. Traces of the secondary resonant structure are still visible.

Figure 5.20: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability for a trajectory from Run 4. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.015$. The trajectory is chaotic for a significant interval of time during evolution through the resonance, but does not escape from the resonance, even at e_A much higher than the equilibrium value.

Figure 5.21: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability for a trajectory from Run 6. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.005$. The trajectory is chaotic for a significant interval of time during evolution through the resonance, then escapes from the resonance, with $e_U > e_A$. Compare with Fig. 2.22.

Ariel/Umbriel 2:1 Resonance Surface of Section

Figure 5.22: Section II at $\delta = 28.157$. The trajectory in Fig. 5.21 is still in the large chaotic zone. Trajectories may escape from the resonance into the small quasiperiodic region at the left of the figure and surrounded by the large chaotic zone. The quasiperiodic region near the origin is associated with capture into the e_A resonance only.

Figure 5.23: Orbital eccentricity variations of Ariel (a) and Umbriel (b) during evolution through the 2:1 commensurability for a trajectory from Run 6. Shown are the maximum and minimum eccentricities in intervals of $\Delta\delta = 0.005$. After entering the chaotic zone, this trajectory escapes from the e_U resonance and is captured into the e_A resonance only.

5.3 Secular Perturbations

In this section we consider the perturbations on the eccentricity-type resonances produced by the secular variations due to the other satellites. To get a qualitative idea of the effects of secular perturbations, we consider only the perturbations on the circular inclined problem due to Titania as a first approximation, since they are the most significant (see Dermott and Nicholson 1986, Laskar 1986). The development of the Hamiltonian is given in Appendix I (Section 7.2.5). We define $\phi = \lambda_A - 2\lambda_U$, $p_i \approx e_i \sqrt{L_i} \cos(-\varpi_i)$, and $q_i \approx e_i \sqrt{L_i} \sin(-\varpi_i)$. The momentum conjugate to ϕ is $\Phi = (L_A - L_{AR})$, where the resonant value L_{AR} is defined at $\frac{\partial \mathcal{H}}{\partial \Phi} = 0$:

$$L_{MR} = \frac{L_U + 2L_A}{2 + 2^{1/3} \frac{m_U}{m_A}} \quad (5.23)$$

In this model, evolution due to tidal dissipation changes the value of Φ at a rate of $\dot{\Phi} = (\dot{L}_A - \dot{L}_{AR})$. The value of Φ can be converted into a value for δ . Once again, this problem has too many degrees of freedom for detailed study of the dynamics using surfaces of section.

Fig. 5.24 shows the evolution of the orbital eccentricities of Ariel, Umbriel, and Titania for a trajectory evolved through the resonance using this model with $\mathcal{Q} = 25$ ($\dot{a}_A/a_A = 1.2 \times 10^{-10}$ per orbit period). The initial orbital parameters for this trajectory were $a_A = 6.3100$, $a_U = 10.0400$, $a_T = 16.633$, $e_A = e_U = 0.005$, $e_T = 0.0021$, $\sigma_A = \sigma_U = \sigma_T = 90^\circ$. The equations of motion were integrated using the Bulirsch-Stoer (1966) algorithm with a relative precision of 10^{-9} per stepsize of approximately 0.6 year. The presence of Titania does not affect the evolution much: both resonant arguments are captured into libration as before. Again, as the trajectory evolves, it encounters various secondary resonances which only change the oscillation amplitudes of the eccentricities. The chaotic zones at these secondary resonances are perhaps somewhat wider than in the two-satellite problem with similar initial eccentricities (Run 1), as can be seen from the burst of chaotic behaviour

in the orbital eccentricity of Titania at δ just greater than zero.

Fig. 5.25 shows the evolution of the orbital eccentricities of Ariel, Umbriel, and Titania for a trajectory evolved through the resonance using this model with $\mathcal{Q} = 25$ ($\dot{a}_A/a_A = 1.2 \times 10^{-10}$ per orbit period). The initial orbital parameters for this trajectory were $a_A = 6.3100$, $a_U = 10.0400$, $a_T = 16.633$, $e_A = e_U = 0.01$, $e_T = 0.0021$, $\sigma_A = \sigma_T = 90^\circ$, $\sigma_U = 270^\circ$. The equations of motion were integrated using the Bulirsch-Stoer algorithm with a relative precision of 10^{-9} per stepsize of approximately 0.6 year. Again, the evolution is qualitatively similar to that in the two-satellite problem with similar initial eccentricities (Run 2). Numerous secondary resonances are encountered, including some involving the eccentricity of Titania, but these do not appear to significantly affect capture into the mean-motion resonance.

Figure 5.24: Orbital eccentricity variations of Ariel (a), Umbriel (b), and Titania (c). The maximum and minimum eccentricities in intervals of $\Delta\delta = 0.01$ plotted. The trajectory does not escape from the resonance before e_A reaches its equilibrium value.

Figure 5.25: Orbital eccentricity variations of Ariel (a), Umbriel (b), and Titania (c). The maximum and minimum eccentricities in intervals of $\Delta\delta = 0.01$ plotted. The trajectory does not escape from the resonance before e_A reaches its equilibrium value.

5.4 The Eccentric-Inclined Problem

The resonant Hamiltonian is developed in Appendix I. We define

$$\begin{aligned}
 \sigma_i &= \frac{1}{2}(4n_A - 2n_A - 2\varpi_i) \\
 \Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
 \psi_i &= \frac{1}{2}(4n_A - 2n_A - 2\Omega_i) \\
 \Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
 \Gamma_A &= L_A + (\Sigma_A + \Sigma_U + \Psi_A + \Psi_U) \\
 \Gamma_U &= L_U - 2(\Sigma_A + \Sigma_U + \Psi_A + \Psi_U)
 \end{aligned} \tag{5.24}$$

This is a four degree of freedom Hamiltonian problem with slow time dependence. There are 2 first-order eccentricity resonances, 3 second-order eccentricity resonances, 3 second-order inclination resonances, and second-order secular interactions. The eccentricities are coupled to the inclinations through the nonlinear terms. δ is defined to be the nonresonant contributions to $4n_U - 2n_A - \dot{\varpi}_A - \dot{\varpi}_U - \dot{\Omega}_A - \dot{\Omega}_U$. The expressions for and values of the other coefficients are given in Appendix I.

Fig. 5.26 shows the evolution of the orbital eccentricities and inclinations of Ariel and Umbriel during evolution through the resonance with $\mathcal{Q} = 66$ ($\dot{a}_A/a_A = 4.5 \times 10^{-11}$ per orbit period). The initial orbital parameters for this trajectory were $a_A = 6.3100$, $a_U = 10.0400$, $e_A = e_U = 0.005$, $i_A = i_U = 0.005$ radians, $\sigma_A = \sigma_U = 90^\circ$, and $\psi_A = \psi_U = 90^\circ$. The mapping was used, with $\Omega = 40 \text{ yr}^{-1}$. The evolution through the resonance is qualitatively similar to that in the planar-eccentric approximation with similar initial eccentricities (Run 1), until $\delta \approx 15$, when there is a secondary resonance between the eccentricity frequencies and the inclination frequencies. The effect of this secondary resonance is to increase both inclinations while decreasing the amplitudes of oscillation of the eccentricities. It is only a temporary phenomenon, though, and does not lead to escape from the

eccentricity-type resonances. In addition, note that for this trajectory, it occurs when e_A is greater than the equilibrium value of 0.025.

Fig. 5.27 also shows the evolution of the orbital eccentricities and inclinations of Ariel and Umbriel during evolution through the resonance with $\mathcal{Q} = 66$ ($\dot{a}_A/a_A = 4.5 \times 10^{-11}$ per orbit period), with higher initial values of the eccentricities. The initial orbital parameters for this trajectory were $a_A = 6.3100$, $a_U = 10.0400$, $e_A = e_U = 0.01$, $i_A = i_U = 0.005$ radians, $\sigma_A = \sigma_U = 90^\circ$, and $\psi_A = \psi_U = 270^\circ$. The Bulirsch-Stoer (1966) integrator was used for this simulation, with a relative precision of 10^{-9} per time step of approximately 0.6 years. Again, the evolution through the resonance is qualitatively similar to that in the planar-eccentric approximation with similar initial eccentricities (Run 2), until $\delta \approx 20$, when there is again a secondary resonance between the eccentricity frequencies and the inclination frequencies. The inclinations increase somewhat while the amplitudes of oscillation of the eccentricities decrease. Again, this is a temporary phenomenon, and does not lead to escape from the resonance. At $\delta \approx 40$, the inclination of Ariel is captured into resonance. After this point, i_A increases while e_A and e_U level off. This behaviour has been observed in integrations of trajectories with similar initial conditions using the mapping. In no case did the trajectory escape from the resonances. In addition, note that all of this new behaviour occurs with e_A higher than the equilibrium value, implying that it probably would not be important in the evolution through the 2:1 commensurability anyway.

Figure 5.26: Variations in orbital eccentricity (Ariel: a, Umbriel: b) and inclination (Ariel: c, Umbriel: d) for a trajectory integrated through the 2:1 commensurability using the mapping with $Q = 66$. A secondary resonance at $\delta \approx 15$ significantly affects the inclinations and the amplitudes of oscillation of the eccentricities, but does not lead to escape from the resonance.

Figure 5.27: Variations in orbital eccentricity (Ariel: a, Umbriel: b) and inclination (Ariel: c, Umbriel: d) for a trajectory integrated through the 2:1 commensurability using the mapping with $Q = 66$. A secondary resonance at $\delta \approx 20$ significantly affects the inclinations and the amplitudes of oscillation of the eccentricities, but does not lead to escape from the resonance. At $\delta \approx 40$, i_A is captured into resonance.

5.5 Discussion

For initial eccentricities approaching the 2:1 commensurability which are comparable to or up to about a factor of 10 higher than the current values, capture into this resonance is very likely. Although even at low eccentricities a series of important secondary resonances between the frequencies of libration of σ_U and σ_A are encountered, they affect only the details of resonance passage, *i.e.* the oscillation amplitudes of the eccentricities. Interactions between these frequencies and frequencies associated with the inclination-type resonant arguments or with secular frequencies involving the other Uranian satellites may affect the resonant interaction, but do not appear to increase the likelihood of escape from the eccentricity-type resonances.

A likely scenario for the evolution into this resonance is as follows: since the eccentricity damping timescales of Ariel and Umbriel are, respectively, about 6×10^7 and 10^9 years (for $Q_i = 100$), they would have probably approached the resonance with small free eccentricities. Both eccentricity-type resonant arguments would have been captured into libration, and the orbital eccentricities of both satellites would have increased until e_A reached its equilibrium value of approximately 0.025. Meanwhile, the remaining free eccentricities, expressed as the oscillation amplitudes of the eccentricities in the resonance, would have damped to small values. Most likely, this would have increased the probability of capture into the secondary resonances, so there could be significant fluctuations in these amplitudes as evolution proceeded. Probably, though, they would have established an equilibrium between the rate of expansion of the orbits due to tidal dissipation in Uranus and the rate of shrinkage due to tidal dissipation in the satellites. Both satellites would have evolved outward, maintaining an approximately constant semimajor axis ratio. This is similar to the scenario proposed by Yoder (1979b) for the evolution of Io and Europa in the two body resonance before the Laplace resonance was established. Since the

Ariel-Umbriel 2:1 resonance appears to be stable, the evolution within the resonance probably would have continued to the present time. The fact that the satellites are not currently involved in such a resonance indicates that this is not the most likely scenario.

Could the satellites have encountered the resonance at high eccentricity and escaped? The eccentricities required are of order 0.04, according to the numerical simulations. Are these likely initial values? Using Eqs. 5.9 and 5.18, we can estimate the equilibrium eccentricity of Ariel in the absence of the resonant interaction:

$$e_A = \sqrt{\frac{1}{7D_A}} \approx 0.049 \quad (5.25)$$

At this eccentricity, the secular increase in semimajor axis due to tides on the planet would just be counterbalanced by the decrease due to dissipation in the satellite; hence, the orbit would not be evolving toward the resonance. Therefore, for the satellites to have evolved into the resonance, e_A must have been less than this. This pretty narrowly constrains e_A prior to resonance encounter. It also constrains the time between formation of the satellites and the time of resonance encounter: since the eccentricity damping timescale of Ariel is about 6×10^7 years at this resonance for $Q_A = 100$, an initial $e_A = 0.049$ would have damped to a value of about 0.04 in approximately 10^7 years. If Ariel formed with a higher eccentricity, it may have existed for some number of damping timescales longer than this before beginning to evolve towards the resonance, depending on how extreme you want to get, but the time until resonance encounter still must be short compared to the age of the solar system. Since the corresponding timescales for Umbriel are a factor of 10 longer, it is unlikely that they would have been important. Therefore, there is a severe problem with having the satellites approach the resonance and escape: special initial conditions are required, such as having the satellite form near the 2:1 resonance.

Therefore, we conclude that it is unlikely that Ariel and Umbriel ever encoun-

tered the 2:1 mean-motion commensurability. This implies that the specific dissipation function of Uranus (Q) is greater than about 11,000. This result, combined with the results from Chapter 3, constrains Q to within less than a factor of four, or better than the Q of any other giant planet has been constrained so far (see Burns 1986 for a recent review).

Chapter 6

Summary

The Uranian satellite system has been found to have a rich variety of dynamical behaviour. The principal results of the research leading to this thesis are the following:

- Due to the low oblateness of Uranus ($J_2 = 0.0033$), there are significant chaotic zones at the low-order mean-motion commensurabilities among the Uranian satellites. There are chaotic variations in the eccentricities and/or inclinations for significant intervals of time during evolution through these commensurabilities.
- These chaotic zones have a significant effect on the mechanism of passage through resonance. For example, at both the 5:3 and 2:1 commensurabilities involving Ariel and Umbriel, trajectories may be captured into resonance while they are still chaotic.
- Simulated evolution through the resonances must be extremely slow in order to avoid artifacts: the rate of change in the semimajor axes must be of order $1/a \, da/dt < 10^{-10}$ per orbit period or slower. At these slow rates, there is evidence for ergodic adiabatic invariants during evolution through the chaotic zones.

- For the 5:3 Ariel-Umbriel resonance the eccentricities in the chaotic zone are typically a factor of 2-3 higher than the pre-encounter values. Eccentricities high enough to have a significant effect on the thermal evolution of Ariel have not been found so far in this study: tidal heating rates are of order the current radionuclide heating rate.
- Evolution through the 3:1 Miranda-Umbriel commensurability explains the anomalously high inclination of Miranda. Passage through this resonance involves interesting new dynamical features involving secondary resonances among the fundamental circulation and libration frequencies.
- There are large increases in the orbital eccentricity of Miranda during chaotic evolution through the 3:1 Miranda-Umbriel commensurability. This may have significantly affected the thermal evolution of Miranda, *e.g.* if significant low temperature melting occurred.
- Passage through the 5:3 Miranda-Ariel resonance may have modestly increased the eccentricity and inclination of Miranda.
- Escape from the 2:1 Ariel-Umbriel commensurability is not likely at typical satellite eccentricities, therefore, this probably provides a dynamical barrier to the evolution.
- Since the anomalously high orbital inclination of Miranda is a natural outcome of passage through the 3:1 commensurability with Umbriel, the requirement that the satellites encountered this resonance can be used to constrain the specific dissipation function (Q) of Uranus to be less than 39,000.

- Since capture into resonance at the 2:1 Ariel-Umbriel commensurability is likely at typical satellite orbital eccentricities, the requirement that the satellites did not encounter this resonance can be used to constrain the Q of Uranus to be greater than 11,000.

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Chapter 7

Appendix I: The Averaged Resonant Hamiltonian

7.1 Planar-Eccentric and Circular-Inclined Models

In the following section, we present the derivation of the resonant Hamiltonians for the planar-eccentric and circular-inclined models. These can both be reduced to two degree-of-freedom problems. A detailed derivation of the Ariel-Umbriel 5:3 resonance planar-eccentric model is given first, and expressions for the coefficients and their values at the resonance. Since the Hamiltonians of the other resonances are derived in an analogous way, only the expressions and values of the coefficients are given.

We consider second order resonances involving the combination of mean longitudes $i\lambda_2 - j\lambda_1$, $i - j = 2$, where subscripts 1 and 2 refer to the inner and outer satellites, respectively.

In the case of eccentricity-type resonances, the resonance coordinates are defined to be:

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(i\lambda_2 - j\lambda_1 - 2\varpi_1) \\ \sigma_2 &= \frac{1}{2}(i\lambda_2 - j\lambda_1 - 2\varpi_2)\end{aligned}\tag{7.1}$$

and the conjugate momenta are:

$$\begin{aligned}\Sigma_1 &\approx \frac{L_1}{2}e_1^2 \\ \Sigma_2 &\approx \frac{L_2}{2}e_2^2\end{aligned}\tag{7.2}$$

where $L_i \approx m_i \sqrt{GM a_i}$.

In the case of inclination-type resonances, the resonance coordinates are defined to be:

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(i\lambda_2 - j\lambda_1 - 2\Omega_1) \\ \sigma_2 &= \frac{1}{2}(i\lambda_2 - j\lambda_1 - 2\Omega_2)\end{aligned}\tag{7.3}$$

and the conjugate momenta are:

$$\begin{aligned}\Sigma_1 &\approx \frac{L_1}{2}i_1^2 \\ \Sigma_2 &\approx \frac{L_2}{2}i_2^2.\end{aligned}\tag{7.4}$$

The form of the Hamiltonian for both of these cases is the same:

$$\begin{aligned}\mathcal{H} &= 2A(\Sigma_1 + \Sigma_2) + 4B(\Sigma_1 + \Sigma_2)^2 + 2C\Sigma_1 + 2D\Sigma_2 \\ &+ 2E\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 - \sigma_2) \\ &+ 2F\Sigma_1\cos(2\sigma_1) \\ &+ 2G\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 + \sigma_2) \\ &+ 2H\Sigma_2\cos(2\sigma_2)\end{aligned}\tag{7.5}$$

For the 2:1 resonance, there are the additional terms

$$I\sqrt{2\Sigma_1}\cos\sigma_1 + J\sqrt{2\Sigma_2}\cos\sigma_2\tag{7.6}$$

In terms of the Cartesian elements

$$y_i = \sqrt{2\Sigma_i}\sin(\sigma_i) \approx e_i\sqrt{L_i}\sin(\sigma_i) \text{ or } i_i\sqrt{L_i}\sin(\sigma_i)\tag{7.7}$$

and the conjugate momenta

$$x_i = \sqrt{2\Sigma_i} \cos(\sigma_i) \approx e_i \sqrt{L_i} \cos(\sigma_i) \text{ or } i_i \sqrt{L_i} \sin(\sigma_i) \quad (7.8)$$

the Hamiltonian may be expressed

$$\begin{aligned} \mathcal{H} = & \frac{1}{4}(\delta - 2(C + D))(x_1^2 + y_1^2 + x_2^2 + y_2^2) + B(x_1^2 + y_1^2 + x_2^2 + y_2^2)^2 \\ & + C(x_1^2 + y_1^2) + D(x_2^2 + y_2^2) + E(x_1 x_2 + y_1 y_2) \\ & + F(x_1^2 - y_1^2) + G(x_1 x_2 - y_1 y_2) + H(x_2^2 - y_2^2) \end{aligned} \quad (7.9)$$

with the additional terms

$$I x_1 + J x_2 \quad (7.10)$$

for the 2:1 resonance, and where

$$\delta \equiv 4A - 2(C + D) \approx i n_2 - j n_1 - \dot{\omega}_1 - \dot{\omega}_2 \text{ or } i n_2 - j n_1 - \dot{\Omega}_1 - \dot{\Omega}_2 \quad (7.11)$$

During evolution through the resonance, the quantity $i n_2 - j n_1$ changes by a large amount relative to the other coefficients, which are therefore approximated to be constant. In the evaluation of the coefficients, the unit of mass is the mass of Uranus, the unit of time is one year, and the unit of length is the radius of Uranus.

7.1.1 Ariel-Umbriel 5:3 Planar-Eccentric Problem

The problem to be considered is the planar Ariel-Umbriel system near the 5:3 mean-motion commensurability. This problem will be considered as a model for the two degree-of freedom models used in this Thesis. The Hamiltonian of this system can be written:

$$\mathcal{H} = \mathcal{H}_K + \mathcal{H}_O + \mathcal{H}_S + \mathcal{H}_R \quad (7.12)$$

where $\mathcal{H}_K$ is the sum of the unperturbed Keplerian terms, $\mathcal{H}_O$ is the perturbation due to the oblateness of Uranus, $\mathcal{H}_S$ is the perturbation due to the secular interaction between the satellites, and $\mathcal{H}_R$ is the perturbation due to the resonant interaction between the satellites. In the following development, subscript A refers to Ariel, and subscript U to Umbriel.

Expressing the contributions in terms of Keplerian elements, where a_i is the semimajor axis, e_i is the eccentricity, λ_i is the mean longitude, and ϖ_i is the longitude of perihelion,

$$\mathcal{H}_K = -\frac{GMm_A}{2a_A} - \frac{GMm_U}{2a_U} \quad (7.13)$$

Keeping only the J_2 terms to second order in eccentricity in the expression of the potential for an oblate planet:

$$\begin{aligned} \mathcal{H}_O = & -\frac{GMm_A}{2a_A} J_2 \left(\frac{R}{a_A}\right)^2 \left[1 + \frac{3}{2}e_A^2\right] \\ & -\frac{GMm_U}{2a_U} J_2 \left(\frac{R}{a_U}\right)^2 \left[1 + \frac{3}{2}e_U^2\right] \end{aligned} \quad (7.14)$$

and from the expression for the disturbing function:

$$\begin{aligned} \mathcal{H}_S = & -\frac{Gm_A m_U}{a_U} \left[(1)^{(0)} + (2)^{(0)} \left(\frac{e_A}{2}\right)^2 + (3)^{(0)} \left(\frac{e_U}{2}\right)^2 + \right. \\ & \left. (21)^{(-1)} \frac{e_A}{2} \frac{e_U}{2} \cos(\varpi_A - \varpi_U) \right] \end{aligned} \quad (7.15)$$

and

$$\mathcal{H}_R = -\frac{Gm_A m_U}{a_U} \left[(172)^{(5)} \left(\frac{e_A}{2}\right)^2 \cos(5\lambda_U - 3\lambda_A - 2\varpi_A) + \right.$$

$$\begin{aligned} & (182)^{(4)} \frac{e_A}{2} \frac{e_U}{2} \cos(5\lambda_U - 3\lambda_A - \varpi_A - \varpi_U) + \\ & (192)^{(3)} \left(\frac{e_U}{2} \right)^2 \cos(5\lambda_U - 3\lambda_A - 2\varpi_U) \end{aligned} \quad (7.16)$$

where (see Leverrier, 1855)

$$\begin{aligned} (1)^{(0)} &= \frac{1}{2} b_{1/2}^0(\alpha) \\ (2)^{(0)} &= \alpha \frac{d}{d\alpha} b_{1/2}^0(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^0(\alpha) \\ (3)^{(0)} &= (2)^{(0)} \\ (21)^{(-1)} &= 2b_{1/2}^1(\alpha) - 2\alpha \frac{d}{d\alpha} b_{1/2}^1(\alpha) - \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^1(\alpha) \\ (172)^{(5)} &= \frac{75}{2} b_{1/2}^5(\alpha) + 9\alpha \frac{d}{d\alpha} b_{1/2}^5(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^5(\alpha) \\ (182)^{(4)} &= -72b_{1/2}^4(\alpha) - 18\alpha \frac{d}{d\alpha} b_{1/2}^4(\alpha) - \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^4(\alpha) \\ (192)^{(3)} &= \frac{67}{2} b_{1/2}^3(\alpha) + 9\alpha \frac{d}{d\alpha} b_{1/2}^3(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^3(\alpha) \end{aligned} \quad (7.17)$$

and α is the ratio of semimajor axes, and $b_i^l(\alpha)$ are Laplace coefficients.

Near the resonance, the evolution of e_i and ϖ_i is dominated by the low frequency perturbations, with frequencies associated with changes in the resonant combination of mean longitudes $5\lambda_U - 3\lambda_A$ and the longitudes of pericenter ϖ_i . The high frequency contributions associated with the motions of the mean anomalies and with other non-resonant combinations of the mean-anomalies are removed by averaging. The Delaunay momenta $L_i \approx m_i \sqrt{GM a_i}$ and $G_i = L_i \sqrt{1 - e_i^2}$ are conjugate to the mean anomalies and arguments of pericenter, respectively. Canonical Poincaré elements are defined as the coordinate angles $\lambda_i = l_i + \varpi_i$, which are conjugate to $L_i' = L_i$, and $\omega_{1i} = -\varpi_i$, which are conjugate to $\rho_{1i} = L_i - G_i$. We use the generating function

$$F = \frac{1}{2} (5\lambda_U - 3\lambda_A + 2\omega_{1A}) \Sigma_A + \frac{1}{2} (5\lambda_U - 3\lambda_A + 2\omega_{1U}) \Sigma_U + \lambda_A \Gamma_A + \lambda_U \Gamma_U \quad (7.18)$$

to obtain the new resonance coordinates

$$\sigma_A = \frac{1}{2} (5\lambda_U - 3\lambda_A + 2\omega_{1A})$$

$$\sigma_U = \frac{1}{2}(5\lambda_U - 3\lambda_A + 2\omega_{1U}) \quad (7.19)$$

which together with $\gamma_A = \lambda_A$ and $\gamma_U = \lambda_U$ form a complete set of generalized coordinates.

The momenta conjugate to σ_A and σ_U are

$$\begin{aligned} \Sigma_A &= \rho_{1A} = L'_A - G_A \\ \Sigma_U &= \rho_{1U} = L'_U - G_U \end{aligned} \quad (7.20)$$

The momenta conjugate to γ_A and γ_U form two integrals of motion, since in the new variables the resonant Hamiltonian is cyclic in these variables. These resonance integrals are:

$$\begin{aligned} \Gamma_A &= L'_A + \frac{3}{2}(\Sigma_A + \Sigma_U) \\ \Gamma_U &= L'_U - \frac{5}{2}(\Sigma_A + \Sigma_U) \end{aligned} \quad (7.21)$$

Note that $\Sigma_i \approx \Gamma_i e_i^2/2$. The lowest order terms in the disturbing function which influence the evolution of the eccentricities and pericenter longitudes are of order e_i^2 . The Hamiltonian is now expanded in powers of Σ_i/Γ_i to this order, and constant terms are removed. The resulting expression in terms of the new coordinates is:

$$\begin{aligned} \mathcal{H} &= 2A(\Sigma_A + \Sigma_U) + 4B(\Sigma_A + \Sigma_U)^2 + 2C\Sigma_A + 2D\Sigma_U \\ &+ 2E\sqrt{\Sigma_A\Sigma_U} \cos(\sigma_A - \sigma_U) \\ &+ 2F\Sigma_A \cos(2\sigma_A) \\ &+ 2G\sqrt{\Sigma_A\Sigma_U} \cos(\sigma_A + \sigma_U) \\ &+ 2H\Sigma_U \cos(2\sigma_U) \end{aligned} \quad (7.22)$$

where:

$$A = -\frac{3}{4} \frac{G^2 M^2 m_A^2 m'_A}{\Gamma_A^3} + \frac{5}{4} \frac{G^2 M^2 m_U^2 m'_U}{\Gamma_U^3}$$

$$\begin{aligned}
B &= -\frac{27}{32} \frac{G^2 M^2 m_A^2 m'_A}{\Gamma_A^4} - \frac{75}{32} \frac{G^2 M^2 m_U^2 m'_U}{\Gamma_U^4} \\
C &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{15 m_U^4 m_U'^3}{2 \Gamma_U^7} - \frac{6 m_A^4 m_A'^3}{\Gamma_A^7} \right] \\
&+ \frac{G^2 M m_A m_U^2 m'_U}{4 \Gamma_U^2} \left[\frac{5}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_A} \frac{m'_U}{m'_A} \left(\frac{3 \Gamma_A}{\Gamma_U^2} + \frac{5 \Gamma_A^2}{\Gamma_U^3} \right) - \frac{(2)^{(0)}}{\Gamma_A} \right] \\
D &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{6 m_U^4 m_U'^3}{\Gamma_U^7} - \frac{9 m_A^4 m_A'^3}{2 \Gamma_A^7} \right] \\
&+ \frac{G^2 M m_A m_U^2 m'_U}{4 \Gamma_U^2} \left[\frac{5}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_A} \frac{m'_U}{m'_A} \left(\frac{3 \Gamma_A}{\Gamma_U^2} + \frac{5 \Gamma_A^2}{\Gamma_U^3} \right) - \frac{(3)^{(0)}}{\Gamma_U} \right] \\
E &= -\frac{G^2 M m_A m_U^2 m'_U}{4 \Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (21)^{(-1)} \\
F &= -\frac{G^2 M m_A m_U^2 m'_U}{4 \Gamma_U^2 \Gamma_A} (172)^{(5)} \\
G &= -\frac{G^2 M m_A m_U^2 m'_U}{4 \Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (182)^{(4)} \\
H &= -\frac{G^2 M m_A m_U^2 m'_U}{4 \Gamma_U^3} (192)^{(3)} \tag{7.23}
\end{aligned}$$

where the m'_i are reduced masses in the Jacobi coordinate system (see e.g. Plummer 1960).

Finally, the transformation to canonical coordinates

$$y_i = \sqrt{2 \Sigma_i} \sin(\sigma_i) \approx e_i \sqrt{\Gamma_i} \sin(\sigma_i) \tag{7.24}$$

and the conjugate momenta

$$x_i = \sqrt{2 \Sigma_i} \cos(\sigma_i) \approx e_i \sqrt{\Gamma_i} \cos(\sigma_i) \tag{7.25}$$

is made, resulting in the final form of the Hamiltonian:

$$\begin{aligned}
\mathcal{H} = & \frac{1}{4}(\delta - 2(C + D))(x_A^2 + y_A^2 + x_U^2 + y_U^2) + B(x_A^2 + y_A^2 + x_U^2 + y_U^2)^2 \\
& + C(x_A^2 + y_A^2) + D(x_U^2 + y_U^2) + E(x_A x_U + y_A y_U) \\
& + F(x_A^2 - y_A^2) + G(x_A x_U - y_A y_U) + H(x_U^2 - y_U^2)
\end{aligned} \tag{7.26}$$

which has two degrees of freedom.

The parameter δ , which measures the distance from the resonance, is defined to be $\delta = 4A + 2(C + D)$, or the secular contributions to $5n_U - 3n_A - \dot{\omega}_A - \dot{\omega}_U$. The units chosen are as follows: distance is measured in units of the radius of Uranus R , mass is measured in units of the mass of Uranus M , and time is measured in years. The coefficients were evaluated at $\delta = 0$ for $e_i = 0$ and $a_U = 10.0703R$. (The semi-major axis at resonance encounter is $a_U = 10.1353$. This should properly have been used in the evaluation of the coefficients; the differences are not significant.)

The numerical values for the coefficients thus obtained are:

$$\begin{aligned}
B &= -2632.82 \\
C &= -0.0803237 \\
D &= -0.0477658 \\
E &= 0.0109693 \\
F &= -0.0319666 \\
G &= 0.0797229 \\
H &= -0.0493825
\end{aligned} \tag{7.27}$$

and the integrals of motion are:

$$\begin{aligned}
\Gamma_A &= 0.742492 \\
\Gamma_U &= 0.834860
\end{aligned} \tag{7.28}$$

The state of a system is determined by its coordinates x_i and y_i and the two parameters δ and the numerical value of the Hamiltonian $\mathcal{E}$.

7.1.2 The Miranda-Umbriel 3:1 Circular-Inclined Problem

The subscript M refers to Miranda, and the subscript U to Umbriel in the following development of the Hamiltonian.

The secular contribution to the Hamiltonian is given by

$$\mathcal{H}_S = -\frac{Gm_M m_U}{a_U} \left[(1)^{(0)} + (11)^{(0)} \left\{ \left(\frac{i_M}{2} \right)^2 + \left(\frac{i_U}{2} \right)^2 + \frac{i_M}{2} \frac{i_U}{2} \cos(\Omega_M - \Omega_U) \right\} \right] \quad (7.29)$$

where

$$(11)^{(0)} = -\frac{1}{2} \alpha b_{3/2}^1(\alpha) \quad (7.30)$$

The resonant contribution to the disturbing function involving the inclinations is:

$$\begin{aligned} \mathcal{H}_R = & -\frac{Gm_M m_U}{a_U} (212)^{(3)} \left[\left(\frac{i_M}{2} \right)^2 \cos(3\lambda_U - \lambda_M - 2\Omega_M) \right. \\ & - 2.0 \frac{i_M}{2} \frac{i_U}{2} \cos(3\lambda_U - \lambda_M - \Omega_M - \Omega_U) + \\ & \left. \left(\frac{i_U}{2} \right)^2 \cos(3\lambda_U - \lambda_M - 2\Omega_U) \right] \end{aligned} \quad (7.31)$$

where

$$(212)^{(3)} = \frac{1}{2} \alpha b_{3/2}^2(\alpha) \quad (7.32)$$

α is the ratio of semimajor axes, and $b_i^j(\alpha)$ are Laplace coefficients.

The resonance coordinates are

$$\begin{aligned} \sigma_M &= \frac{1}{2} (3\lambda_U - \lambda_M - 2\Omega_M) \\ \sigma_U &= \frac{1}{2} (3\lambda_U - \lambda_M - 2\Omega_U) \end{aligned} \quad (7.33)$$

and the conjugate momenta are

$$\begin{aligned} \Sigma_M &\approx \frac{L_M}{2} i_M^2 \\ \Sigma_U &\approx \frac{L_U}{2} i_U^2 \end{aligned} \quad (7.34)$$

The momenta conjugate to the cyclic coordinates $\gamma_M = \lambda_M$ and $\gamma_U = \lambda_U$ are

$$\begin{aligned}\Gamma_M &= L_M + \frac{1}{2}(\Sigma_M + \Sigma_U) \\ \Gamma_U &= L_U - \frac{3}{2}(\Sigma_M + \Sigma_U)\end{aligned}\tag{7.35}$$

The coefficients in the Hamiltonian (7.5) are:

$$\begin{aligned}A &= -\frac{1}{4} \frac{G^2 M^2 m_M^2 m'_M}{\Gamma_M^3} + \frac{3}{4} \frac{G^2 M^2 m_U^2 m'_U}{\Gamma_U^3} \\ B &= -\frac{3}{32} \frac{G^2 M^2 m_M^2 m'_M}{\Gamma_M^4} - \frac{27}{32} \frac{G^2 M^2 m_U^2 m'_U}{\Gamma_U^4} \\ C &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{9 m_U^4 m_U^3}{2 \Gamma_U^7} \right] \\ &\quad + \frac{G^2 M m_M m_U^2 m'_U}{4 \Gamma_U^2} \left[\frac{3}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\ &\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_M} \frac{m'_U}{m'_M} \left(\frac{\Gamma_M}{\Gamma_U^2} + \frac{3 \Gamma_M^2}{\Gamma_U^3} \right) - \frac{(11)^{(0)}}{\Gamma_M} \right] \\ D &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{6 m_U^4 m_U^3}{\Gamma_U^7} - \frac{3 m_M^4 m_M^3}{2 \Gamma_M^7} \right] \\ &\quad + \frac{G^2 M m_M m_U^2 m'_U}{4 \Gamma_U^2} \left[\frac{3}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\ &\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_M} \frac{m'_U}{m'_M} \left(\frac{\Gamma_M}{\Gamma_U^2} + \frac{3 \Gamma_M^2}{\Gamma_U^3} \right) - \frac{(11)^{(0)}}{\Gamma_U} \right] \\ E &= \frac{G^2 M m_M m_U^2 m'_U}{2 \Gamma_U^2 \sqrt{\Gamma_M \Gamma_U}} (11)^{(0)} \\ F &= -\frac{G^2 M m_M m_U^2 m'_U}{4 \Gamma_U^2 \Gamma_M} (212)^{(3)} \\ G &= \frac{G^2 M m_M m_U^2 m'_U}{2 \Gamma_U^2 \sqrt{\Gamma_M \Gamma_U}} (212)^{(3)} \\ H &= -\frac{G^2 M m_M m_U^2 m'_U}{4 \Gamma_U^3} (212)^{(3)}\end{aligned}\tag{7.36}$$

The coefficients were evaluated at $\delta = 0$ for $i_i = 0$ and $a_U = 10.1179R$ (the value of the semimajor axis of Umbriel at which the resonance was encountered). This corresponds to $a_M = 4.8645$. The numerical values for the coefficients thus obtained are:

$$\begin{aligned}
B &= -5167.54 \\
C &= 0.043832 \\
D &= -0.121057 \\
E &= -0.0006756 \\
F &= -0.0009765 \\
G &= 0.0003934 \\
H &= -0.00003961
\end{aligned} \tag{7.37}$$

and

$$\begin{aligned}
\Gamma_M &= 0.033946 \\
\Gamma_U &= 0.836831
\end{aligned} \tag{7.38}$$

7.1.3 The Miranda-Umbriel 3:1 Planar-Eccentric Problem

From the expression for the disturbing function (Leverrier, 1855), the resonant contribution is:

$$\begin{aligned} \mathcal{H}_R = & -\frac{Gm_M m_U}{a_U} \left[(172)^{(3)} \left(\frac{e_M}{2} \right)^2 \cos(3\lambda_U - \lambda_M - 2\Omega_M) + \right. \\ & (182)^{(2)} \frac{e_M}{2} \frac{e_U}{2} \cos(3\lambda_U - \lambda_M - \Omega_M - \Omega_U) + \\ & \left. (192)^{(1)} \left(\frac{e_U}{2} \right)^2 \cos(3\lambda_U - \lambda_M - 2\Omega_U) \right] \end{aligned} \quad (7.39)$$

where the subscript M refers to Miranda, and the subscript U to Umbriel, and

$$\begin{aligned} (172)^{(3)} &= \frac{21}{2} b_{1/2}^3(\alpha) + 5\alpha \frac{d}{d\alpha} b_{1/2}^3(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^3(\alpha) \\ (182)^{(2)} &= -20 b_{1/2}^2(\alpha) - 10\alpha \frac{d}{d\alpha} b_{1/2}^2(\alpha) - \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^2(\alpha) \\ (192)^{(1)} &= \frac{17}{2} b_{1/2}^1(\alpha) + 5\alpha \frac{d}{d\alpha} b_{1/2}^1(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^1(\alpha) - \frac{27}{2} \alpha - \frac{3}{2} \frac{1}{\alpha^2} \end{aligned} \quad (7.40)$$

where α is the ratio of semimajor axes, and $b_s^l(\alpha)$ are Laplace coefficients. The last two terms in the expression for $(192)^{(1)}$ are due to indirect terms in the disturbing function, and arise because the planet-centered coordinate system is not an inertial one.

The resonance coordinates are

$$\begin{aligned} \sigma_M &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\varpi_M) \\ \sigma_U &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\varpi_U) \end{aligned} \quad (7.41)$$

and the conjugate momenta are

$$\begin{aligned} \Sigma_M &\approx \frac{L_M}{2} e_M^2 \\ \Sigma_U &\approx \frac{L_U}{2} e_U^2 \end{aligned} \quad (7.42)$$

The momenta conjugate to the cyclic coordinates $\gamma_M = \lambda_M$ and $\gamma_U = \lambda_U$ are

$$\begin{aligned} \Gamma_M &= L_M + \frac{1}{2}(\Sigma_M + \Sigma_U) \\ \Gamma_U &= L_U - \frac{3}{2}(\Sigma_M + \Sigma_U) \end{aligned} \quad (7.43)$$

The coefficients in the Hamiltonian Eq. 7.5 are:

$$\begin{aligned}
A &= -\frac{1}{4} \frac{G^2 M^2 m_M^2 m'_M}{\Gamma_M^3} + \frac{3}{4} \frac{G^2 M^2 m_U^2 m'_U}{\Gamma_U^3} \\
B &= -\frac{3}{32} \frac{G^2 M^2 m_M^2 m'_M}{\Gamma_M^4} - \frac{27}{32} \frac{G^2 M^2 m_U^2 m'_U}{\Gamma_U^4} \\
C &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{9m_U^4 m_U'^3}{2\Gamma_U^7} - \frac{3m_M^4 m_M'^3}{\Gamma_M^7} \right] \\
&\quad + \frac{G^2 M m_M m_U^2 m'_U}{4\Gamma_U^2} \left[\frac{3}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_M} \frac{m'_U}{m'_M} \left(\frac{\Gamma_M}{\Gamma_U^2} + \frac{3\Gamma_M^2}{\Gamma_U^3} \right) - \frac{(2)^{(0)}}{\Gamma_M} \right] \\
D &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{3m_U^4 m_U'^3}{\Gamma_U^7} - \frac{3m_M^4 m_M'^3}{2\Gamma_M^7} \right] \\
&\quad + \frac{G^2 M m_M m_U^2 m'_U}{4\Gamma_U^2} \left[\frac{3}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_M} \frac{m'_U}{m'_M} \left(\frac{\Gamma_M}{\Gamma_U^2} + \frac{3\Gamma_M^2}{\Gamma_U^3} \right) - \frac{(3)^{(0)}}{\Gamma_U} \right] \\
E &= -\frac{G^2 M m_M m_U^2 m'_U}{4\Gamma_U^2 \sqrt{\Gamma_M \Gamma_U}} (21)^{(-1)} \\
F &= -\frac{G^2 M m_M m_U^2 m'_U}{4\Gamma_U^2 \Gamma_M} (172)^{(3)} \\
G &= -\frac{G^2 M m_M m_U^2 m'_U}{4\Gamma_U^2 \sqrt{\Gamma_M \Gamma_U}} (182)^{(2)} \\
H &= -\frac{G^2 M m_M m_U^2 m'_U}{4\Gamma_U^3} (192)^{(1)} \tag{7.44}
\end{aligned}$$

The coefficients were evaluated at $\delta = 0$ for $e_i = 0$ and $a_U = 10.1179R$ (the value of the semimajor axis of Umbriel at which the resonance was encountered). This corresponds to a semimajor axis of Miranda $a_M = 4.86595$. The numerical values

for the coefficients thus obtained are:

$$\begin{aligned}
 B &= -5164.7 \\
 C &= -0.31297 \\
 D &= -0.14827 \\
 E &= 0.00039381 \\
 F &= -0.0070667 \\
 G &= 0.0052590 \\
 H &= 0.00060138
 \end{aligned}
 \tag{7.45}$$

and

$$\begin{aligned}
 \Gamma_M &= 0.033952 \\
 \Gamma_U &= 0.836831.
 \end{aligned}
 \tag{7.46}$$

7.1.4 The Miranda-Ariel 5:3 Planar-Eccentric Problem

The Hamiltonian for this resonance is the same as for the Ariel-Umbriel 5:3 commensurability, with appropriate substitutions of parameters:

$$\begin{aligned} \mathcal{H}_R = & -\frac{Gm_M m_A}{a_A} \left[(172)^{(5)} \left(\frac{e_M}{2} \right)^2 \cos(5\lambda_A - 3\lambda_M - 2\varpi_M) + \right. \\ & (182)^{(4)} \frac{e_A}{2} \frac{e_U}{2} \cos(5\lambda_A - 3\lambda_M - \varpi_M - \varpi_A) + \\ & \left. (192)^{(3)} \left(\frac{e_U}{2} \right)^2 \cos(5\lambda_A - 3\lambda_M - 2\varpi_A) \right] \end{aligned} \quad (7.47)$$

where

$$\begin{aligned} (172)^{(5)} &= \frac{75}{2} b_{1/2}^5(\alpha) + 9\alpha \frac{d}{d\alpha} b_{1/2}^5(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^5(\alpha) \\ (182)^{(4)} &= -72 b_{1/2}^4(\alpha) - 18\alpha \frac{d}{d\alpha} b_{1/2}^4(\alpha) - \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^4(\alpha) \\ (192)^{(3)} &= \frac{67}{2} b_{1/2}^3(\alpha) + 9\alpha \frac{d}{d\alpha} b_{1/2}^3(\alpha) + \frac{1}{2} \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^3(\alpha) \end{aligned} \quad (7.48)$$

and α is the ratio of semimajor axes, and $b_s^l(\alpha)$ are Laplace coefficients.

The resonance coordinates are

$$\begin{aligned} \sigma_M &= \frac{1}{2}(5\lambda_A - 3\lambda_M - 2\varpi_M) \\ \sigma_A &= \frac{1}{2}(5\lambda_A - 3\lambda_M - 2\varpi_A) \end{aligned} \quad (7.49)$$

and the conjugate momenta are

$$\begin{aligned} \Sigma_M &\approx \frac{L_M}{2} e_M^2 \\ \Sigma_A &\approx \frac{L_A}{2} e_A^2 \end{aligned} \quad (7.50)$$

The momenta conjugate to the cyclic coordinates $\gamma_M = \lambda_M$ and $\gamma_A = \lambda_A$ are

$$\begin{aligned} \Gamma_M &= L_M + \frac{3}{2}(\Sigma_M + \Sigma_A) \\ \Gamma_A &= L_A - \frac{5}{2}(\Sigma_M + \Sigma_A) \end{aligned} \quad (7.51)$$

The expressions for the coefficients are:

$$\begin{aligned}
A &= -\frac{3}{4} \frac{G^2 M^2 m_M^2 m'_M}{\Gamma_M^3} + \frac{5}{4} \frac{G^2 M^2 m_A^2 m'_A}{\Gamma_A^3} \\
B &= -\frac{27}{32} \frac{G^2 M^2 m_M^2 m'_M}{\Gamma_M^4} - \frac{75}{32} \frac{G^2 M^2 m_A^2 m'_A}{\Gamma_A^4} \\
C &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{15 m_A^4 m_A'^3}{2 \Gamma_A^7} - \frac{6 m_M^4 m_M'^3}{\Gamma_M^7} \right] \\
&\quad + \frac{G^2 M m_M m_A^2 m'_A}{4 \Gamma_A^2} \left[\frac{5}{\Gamma_A} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_A}{m_M} \frac{m'_A}{m'_M} \left(\frac{3 \Gamma_M}{\Gamma_A^2} + \frac{5 \Gamma_M^2}{\Gamma_A^3} \right) - \frac{(2)^{(0)}}{\Gamma_M} \right] \\
D &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{6 m_A^4 m_A'^3}{\Gamma_A^7} - \frac{9 m_M^4 m_M'^3}{2 \Gamma_M^7} \right] \\
&\quad + \frac{G^2 M m_M m_A^2 m'_A}{4 \Gamma_A^2} \left[\frac{5}{\Gamma_A} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_A}{m_M} \frac{m'_A}{m'_M} \left(\frac{3 \Gamma_M}{\Gamma_A^2} + \frac{5 \Gamma_M^2}{\Gamma_A^3} \right) - \frac{(3)^{(0)}}{\Gamma_A} \right] \\
E &= -\frac{G^2 M m_M m_A^2 m'_A}{4 \Gamma_A^2 \sqrt{\Gamma_M \Gamma_A}} (21)^{(-1)} \\
F &= -\frac{G^2 M m_M m_A^2 m'_A}{4 \Gamma_A^2 \Gamma_M} (172)^{(5)} \\
G &= -\frac{G^2 M m_M m_A^2 m'_A}{4 \Gamma_A^2 \sqrt{\Gamma_M \Gamma_A}} (182)^{(4)} \\
H &= -\frac{G^2 M m_M m_A^2 m'_A}{4 \Gamma_A^3} (192)^{(3)} \tag{7.52}
\end{aligned}$$

The coefficients were evaluated at $\delta = 0$ for $e_i = 0$ and $a_M = 4.6361R$, with a corresponding value of $a_A = 6.5157$. The numerical values for the coefficients thus

obtained are:

$$\begin{aligned}B &= -49208.56 \\C &= -0.5183112 \\D &= -0.3589054 \\E &= 0.0051029 \\F &= -0.0648220 \\G &= 0.0370799 \\H &= -0.0052682\end{aligned}\tag{7.53}$$

and

$$\begin{aligned}\Gamma_M &= 0.033140 \\ \Gamma_A &= 0.708087\end{aligned}\tag{7.54}$$

7.1.5 The Ariel-Umbriel 2:1 Planar-Eccentric Problem

The Hamiltonian for this resonance differs in that there are resonant terms proportional to the first power in eccentricity:

$$\begin{aligned}
 \mathcal{H}_R = & -\frac{Gm_A m_U}{a_U} \left[(172)^{(4)} \left(\frac{e_A}{2} \right)^2 \cos(4\lambda_U - 2\lambda_A - 2\varpi_A) + \right. \\
 (182)^{(3)} & \frac{e_U}{2} \frac{e_U}{2} \cos(4\lambda_U - 2\lambda_A - \varpi_A - \varpi_U) + \\
 (192)^{(2)} & \left(\frac{e_U}{2} \right)^2 \cos(4\lambda_U - 2\lambda_A - 2\varpi_U) \\
 (50)^{(2)} & \left(\frac{e_A}{2} \right) \cos(2\lambda_U - \lambda_A - \varpi_A) \\
 (70)^{(1)} & \left. \left(\frac{e_U}{2} \right) \cos(2\lambda_U - \lambda_A - \varpi_U) \right] \tag{7.55}
 \end{aligned}$$

where

$$\begin{aligned}
 (172)^{(4)} &= 22b_{1/2}^4(\alpha) + 7\alpha \frac{d}{d\alpha} b_{1/2}^4(\alpha) + \frac{1}{2}\alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^4(\alpha) \\
 (182)^{(3)} &= -42b_{1/2}^3(\alpha) - 14\alpha \frac{d}{d\alpha} b_{1/2}^3(\alpha) - \alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^3(\alpha) \\
 (192)^{(2)} &= 19b_{1/2}^2(\alpha) + 7\alpha \frac{d}{d\alpha} b_{1/2}^2(\alpha) + \frac{1}{2}\alpha^2 \frac{d^2}{d\alpha^2} b_{1/2}^2(\alpha) \\
 (50)^{(2)} &= -4b_{1/2}^2(\alpha) - \alpha \frac{d}{d\alpha} b_{1/2}^2(\alpha) \\
 (70)^{(1)} &= 3b_{1/2}^1(\alpha) + \alpha \frac{d}{d\alpha} b_{1/2}^1(\alpha) - 4\alpha - \frac{1}{\alpha^2} \tag{7.56}
 \end{aligned}$$

and α is the ratio of semimajor axes, and $b_s^l(\alpha)$ are Laplace coefficients. The last two terms in the expression for $(70)^{(1)}$ are indirect terms, which arise due to the fact that the planet-centered coordinate system is not an inertial one.

In terms of the resonance coordinates

$$\begin{aligned}
 \sigma_A &= \frac{1}{2}(4\lambda_U - 2\lambda_A - 2\varpi_A) \\
 \sigma_U &= \frac{1}{2}(4\lambda_U - 2\lambda_A - 2\varpi_U) \tag{7.57}
 \end{aligned}$$

the momenta conjugate to σ_A and σ_U

$$\begin{aligned}
 \Sigma_A &\approx \frac{L_A}{2} e_A^2 \\
 \Sigma_U &\approx \frac{L_U}{2} e_U^2 \tag{7.58}
 \end{aligned}$$

The momenta conjugate to the cyclic coordinates $\gamma_A = \lambda_A$ and $\gamma_U = \lambda_U$ are:

$$\begin{aligned}\Gamma_A &= L_A + (\Sigma_A + \Sigma_U) \\ \Gamma_U &= L_U - 2(\Sigma_A + \Sigma_U)\end{aligned}\tag{7.59}$$

and the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= 2A(\Sigma_A + \Sigma_U) + 4B(\Sigma_A + \Sigma_U)^2 \\ &+ 2C\Sigma_A + 2D\Sigma_U + 2E\sqrt{\Sigma_A\Sigma_U}\cos(\sigma_A - \sigma_U) \\ &+ 2F\Sigma_A\cos(2\sigma_A) + 2G\sqrt{\Sigma_A\Sigma_U}\cos(\sigma_A + \sigma_U) + 2H\Sigma_U\cos(2\sigma_U) \\ &+ I\sqrt{2\Sigma_A}\cos(\sigma_A) + J\sqrt{2\Sigma_U}\cos(\sigma_U)\end{aligned}\tag{7.60}$$

$$A = -\frac{1}{2}\frac{G^2M^2m_A^2m'_A}{\Gamma_A^3} + \frac{G^2M^2m_U^2m'_U}{\Gamma_U^3}$$

$$B = -\frac{3}{8}\frac{G^2M^2m_A^2m'_A}{\Gamma_A^4} - \frac{12}{8}\frac{G^2M^2m_U^2m'_U}{\Gamma_U^4}$$

$$\begin{aligned}C &= \frac{1}{2}G^4M^4R^2J_2\left[\frac{6m_U^4m_U'^3}{\Gamma_U^7} - \frac{9m_A^4m_A'^3}{2\Gamma_A^7}\right] \\ &+ \frac{G^2Mm_Am_U^2m'_U}{4\Gamma_U^2}\left[\frac{2}{\Gamma_U}b_{1/2}^0(\alpha) + \right. \\ &\quad \left. \frac{d}{d\alpha}b_{1/2}^0(\alpha)\frac{m_U}{m_A}\frac{m'_U}{m'_A}\left(\frac{\Gamma_A}{\Gamma_U^2} + \frac{2\Gamma_A^2}{\Gamma_U^3}\right) - \frac{(2)^{(0)}}{\Gamma_A}\right]\end{aligned}$$

$$\begin{aligned}D &= \frac{1}{2}G^4M^4R^2J_2\left[\frac{9m_U^4m_U'^3}{2\Gamma_U^7} - \frac{3m_A^4m_A'^3}{\Gamma_A^7}\right] \\ &+ \frac{G^2Mm_Am_U^2m'_U}{4\Gamma_U^2}\left[\frac{2}{\Gamma_U}b_{1/2}^0(\alpha) + \right. \\ &\quad \left. \frac{d}{d\alpha}b_{1/2}^0(\alpha)\frac{m_U}{m_A}\frac{m'_U}{m'_A}\left(\frac{\Gamma_A}{\Gamma_U^2} + \frac{2\Gamma_A^2}{\Gamma_U^3}\right) - \frac{(3)^{(0)}}{\Gamma_U}\right]\end{aligned}$$

$$E = -\frac{G^2Mm_Am_U^2m'_U}{4\Gamma_U^2\sqrt{\Gamma_A\Gamma_U}}(21)^{(-1)}$$

$$\begin{aligned}
F &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^2 \Gamma_A} (172)^{(4)} \\
G &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (182)^{(3)} \\
H &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^3} (192)^{(2)} \\
I &= -\frac{G^2 M m_A m_U^2 m'_U}{2\Gamma_U^2 \sqrt{\Gamma_A}} (50)^{(2)} \\
J &= -\frac{G^2 M m_A m_U^2 m'_U}{2\sqrt{\Gamma_U^5}} (70)^{(1)}
\end{aligned} \tag{7.61}$$

The coefficients were evaluated at $\delta = 0$ for $e_i = 0$ and $a_U = 10.0400$, with a corresponding value of $a_A = 6.3256$. The numerical values for the coefficients thus obtained are:

$$\begin{aligned}
B &= -1616.96 \\
C &= -0.133852 \\
D &= -0.076645 \\
E &= 0.00549096 \\
F &= -0.01767825 \\
G &= 0.047365 \\
H &= -0.0313493 \\
I &= 0.0103638 \\
J &= 0.00661595
\end{aligned} \tag{7.62}$$

and the integrals of motion are:

$$\begin{aligned}\Gamma_M &= 0.697682 \\ \Gamma_A &= 0.833603\end{aligned}\tag{7.63}$$

In terms of the cartesian coordinates

$$y_i = \sqrt{2\Sigma_i} \sin(\sigma_i) \approx e_i \sqrt{\Gamma_i} \sin(\sigma_i)\tag{7.64}$$

and the conjugate momenta

$$x_i = \sqrt{2\Sigma_i} \cos(\sigma_i) \approx e_i \sqrt{\Gamma_i} \cos(\sigma_i)\tag{7.65}$$

the Hamiltonian is expressed

$$\begin{aligned}\mathcal{H} &= \frac{1}{4}(\delta - 2(C + D))(x_A^2 + y_A^2 + x_U^2 + y_U^2) + B(x_A^2 + y_A^2 + x_U^2 + y_U^2)^2 \\ &+ C(x_A^2 + y_A^2) + D(x_U^2 + y_U^2) + E(x_A x_U + y_A y_U) \\ &+ F(x_A^2 - y_A^2) + G(x_A x_U - y_A y_U) + H(x_U^2 - y_U^2) \\ &+ I x_A + J x_U.\end{aligned}\tag{7.66}$$

The state of the system is determined by its coordinates x_i and y_i and the parameter δ .

7.2 Secular Perturbations

In order to include the secular perturbations due to other satellites, the development of the Hamiltonian is somewhat different, in order to keep secular variations distinct from resonant variations (see Wisdom 1982). We define as coordinates

$$\begin{aligned}\phi &= j\lambda_1 - i\lambda_2 \\ p_i &\approx e_i\sqrt{L_i}\cos(-\varpi_i) \text{ or } i_i\sqrt{L_i}\cos(-\Omega_i) \\ q_i &\approx e_i\sqrt{L_i}\sin(-\varpi_i) \text{ or } i_i\sqrt{L_i}\sin(-\Omega_i)\end{aligned}\tag{7.67}$$

and expand the Hamiltonian about the resonant value of L_1 (pendulum approximation). The momentum conjugate to ϕ is $\Phi = \frac{1}{j}(L_1 - L_{1R})$, where the resonant value of L_{1R} is defined by setting $\frac{\partial \mathcal{H}}{\partial \Phi} = 0$.

The Hamiltonian for the two-body resonance problem can then be expressed:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\alpha\Phi^2 \\ &+ C'(p_1^2 + q_1^2) + D'(p_2^2 + q_2^2) + E(p_1p_2 + q_1q_2) \\ &+ F((p_1^2 - q_1^2)\cos\phi + 2p_1q_1\sin\phi) \\ &+ G((p_1p_2 - q_1q_2)\cos\phi + (p_1q_2 + q_1p_2)\sin\phi) \\ &+ H((p_2^2 - q_2^2)\cos\phi + 2p_2q_2\sin\phi)\end{aligned}\tag{7.68}$$

where $\alpha \approx 32B$, and where C' and D' contain only secular contributions. For the 2:1 resonance, there are the additional terms:

$$I(x_1\cos\phi + y_1\sin\phi) + J(x_2\cos\phi + y_2\sin\phi)\tag{7.69}$$

To include the perturbations due to a third satellite, we add the terms in the secular part of the disturbing function involving the third satellite and its interactions with the first two. We end up with the following expression for the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\alpha\Phi^2$$

$$\begin{aligned}
& + C''(p_1^2 + q_1^2) + D''(p_2^2 + q_2^2) + E(p_1 p_2 + q_1 q_2) \\
& + F((p_1^2 - q_1^2) \cos \phi + 2p_1 q_1 \sin \phi) \\
& + G((p_1 p_2 - q_1 q_2) \cos \phi + (p_1 q_2 + q_1 p_2) \sin \phi) \\
& + H((p_2^2 - q_2^2) \cos \phi + 2p_2 q_2 \sin \phi) \\
& + U(p_3^2 + q_3^2) + V(p_1 p_3 + q_1 q_3) + W(p_3 p_2 + q_3 q_2) \quad (7.70)
\end{aligned}$$

where the perturbing satellite is designated by subscript 3. For the 2:1 resonance, there are the additional terms:

$$I(x_1 \cos \phi + y_1 \sin \phi) + J(x_2 \cos \phi + y_2 \sin \phi) \quad (7.71)$$

The expressions for the coefficients $E - H$ (and I and J for the 2:1 resonance) are the same as those in the two-satellite problem, and $\alpha \approx 32B$, except that in the expressions for the coefficients, L_i should be substituted for Γ_i . The coefficients C'', D'', U, V , and W are given for the different resonances below.

7.2.1 Ariel-Umbriel 5:3 Planar-Eccentric Problem

The most significant perturbations on this resonance are due to Titania. For this problem,

$$\begin{aligned}\phi &= 3\lambda_A - 5\lambda_U \\ \Phi &= \frac{1}{3}(L_A - L_{AR})\end{aligned}\tag{7.72}$$

where

$$L_{AR} = \frac{3L_U + 5L_A}{5 + 3\left(\frac{5}{3}\right)^{1/3} \frac{m_U}{m_A}}\tag{7.73}$$

The new coefficients are:

$$\begin{aligned}C'' &= -\frac{3}{4} \frac{G^4 M^4 m_A^4 m_A'^3 R^2 J_2}{L_A^7} \\ &\quad - \frac{G^2 M m_A m_U^2 m_U'}{4L_U^2} \frac{(2)_{AU}^{(0)}}{L_A} \\ &\quad - \frac{G^2 M m_A m_T^2 m_T'}{4L_T^2} \frac{(2)_{AT}^{(0)}}{L_A}\end{aligned}$$

$$\begin{aligned}D'' &= -\frac{3}{4} \frac{G^4 M^4 m_U^4 m_U'^3 R^2 J_2}{L_U^7} \\ &\quad - \frac{G^2 M m_A m_U^2 m_U'}{4L_U^2} \frac{(3)_{AU}^{(0)}}{L_U} \\ &\quad - \frac{G^2 M m_U m_T^2 m_T'}{4L_T^2} \frac{(2)_{UT}^{(0)}}{L_U}\end{aligned}$$

$$\begin{aligned}U &= -\frac{3}{4} \frac{G^4 M^4 m_T^4 m_T'^3 R^2 J_2}{L_T^7} \\ &\quad - \frac{G^2 M m_A m_T^2 m_T'}{4L_T^2} \frac{(3)_{AT}^{(0)}}{L_T} \\ &\quad - \frac{G^2 M m_U m_T^2 m_T'}{4L_T^2} \frac{(3)_{UT}^{(0)}}{L_T}\end{aligned}$$

$$V = -\frac{G^2 M m_A m_T^2 m_T'}{4L_T^2 \sqrt{L_A L_T}} (21)_{AT}^{(-1)}$$

$$W = -\frac{G^2 M m_U m_T^2 m'_T}{4 L_T^2 \sqrt{L_U L_T}} (21)_{UT}^{(-1)} \quad (7.74)$$

Using the physical parameters given in Section 1.1, the numerical values of the coefficients were evaluated at the resonance semimajor axes $a_A = 7.2105$, $a_U = 10.1353$, and $a_T = 16.6500$, with $e_i = 0.0$, yielding:

$$\begin{aligned} \alpha &= -8.3173 \times 10^4 \\ C'' &= -0.053292 \\ D'' &= -0.024290 \\ E &= 0.010864 \\ F &= -0.031660 \\ G &= 0.078959 \\ H &= -0.048909 \\ U &= -0.0041012 \\ V &= 0.00088761 \\ W &= 0.0034782 \end{aligned} \quad (7.75)$$

7.2.2 Miranda-Umbriel 3:1 Circular-Inclined Problem

The most significant perturbations on this resonance are due to Ariel. For this problem,

$$\begin{aligned}\phi &= \lambda_M - 3\lambda_U \\ \Phi &= (L_M - L_{MR})\end{aligned}\tag{7.76}$$

where

$$L_{MR} = \frac{L_U + 3L_M}{3 + 3^{1/3} \frac{m_U}{m_M}}\tag{7.77}$$

The new coefficients are:

$$\begin{aligned}C'' &= \frac{3 G^4 M^4 m_M^4 m_M'^3 R^2 J_2}{4 L_M^7} \\ &- \frac{G^2 M m_M m_A^2 m_A' (11)_{MA}^{(0)}}{4 L_A^2 L_M} \\ &- \frac{G^2 M m_M m_U^2 m_U' (11)_{MU}^{(0)}}{4 L_U^2 L_M}\end{aligned}$$

$$\begin{aligned}D'' &= \frac{3 G^4 M^4 m_U^4 m_U'^3 R^2 J_2}{4 L_U^7} \\ &- \frac{G^2 M m_M m_U^2 m_U' (11)_{MU}^{(0)}}{4 L_U^2 L_U} \\ &- \frac{G^2 M m_A m_U^2 m_U' (11)_{AU}^{(0)}}{4 L_U^2 L_U}\end{aligned}$$

$$\begin{aligned}U &= \frac{3 G^4 M^4 m_A^4 m_A'^3 R^2 J_2}{4 L_A^7} \\ &- \frac{G^2 M m_M m_A^2 m_A' (11)_{MA}^{(0)}}{4 L_A^2 L_A} \\ &- \frac{G^2 M m_A m_U^2 m_U' (11)_{AU}^{(0)}}{4 L_U^2 L_A}\end{aligned}$$

$$V = -\frac{G^2 M m_M m_A^2 m_A' (11)_{MA}^{(0)}}{4 L_A^2 \sqrt{L_M L_A}}$$

$$W = -\frac{G^2 M m_A m_U^2 m'_U}{4 L_U^2 \sqrt{L_A L_U}} (11)_{AU}^{(0)} \quad (7.78)$$

The numerical values of the coefficients are evaluated at $a_M = 4.8642$, $a_A = 7.0846$, and $a_U = 10.1179$, with all inclinations zero, yielding:

$$\begin{aligned} \alpha &= -165377.33 \\ C'' &= 0.189147 \\ D'' &= 0.0193986 \\ E &= -0.0006755 \\ F &= -0.0009763 \\ G &= 0.00039327 \\ H &= -0.0000396 \\ U &= 0.054396 \\ V &= -0.00451298 \\ W &= -0.012133 \end{aligned} \quad (7.79)$$

7.2.3 Miranda-Umbriel 3:1 Planar-Eccentric Problem

The most significant perturbations on this resonance are due to Ariel. For this problem,

$$\begin{aligned}\phi &= \lambda_M - 3\lambda_U \\ \Phi &= (L_M - L_{MR})\end{aligned}\tag{7.80}$$

where

$$L_{MR} = \frac{L_U + 3L_M}{3 + 3^{1/3} \frac{m_U}{m_M}}\tag{7.81}$$

The new coefficients are:

$$\begin{aligned}C'' &= -\frac{3}{4} \frac{G^4 M^4 m_M^4 m_M'^3 R^2 J_2}{L_M^7} \\ &- \frac{G^2 M m_M m_A^2 m_A' (2)_{MA}^{(0)}}{4L_A^2 L_M} \\ &- \frac{G^2 M m_M m_U^2 m_U' (2)_{MU}^{(0)}}{4L_U^2 L_M}\end{aligned}$$

$$\begin{aligned}D'' &= -\frac{3}{4} \frac{G^4 M^4 m_U^4 m_U'^3 R^2 J_2}{L_U^7} \\ &- \frac{G^2 M m_M m_U^2 m_U' (3)_{MU}^{(0)}}{4L_U^2 L_U} \\ &- \frac{G^2 M m_A m_U^2 m_U' (3)_{AU}^{(0)}}{4L_U^2 L_U}\end{aligned}$$

$$\begin{aligned}U &= -\frac{3}{4} \frac{G^4 M^4 m_A^4 m_A'^3 R^2 J_2}{L_A^7} \\ &- \frac{G^2 M m_M m_A^2 m_A' (3)_{MA}^{(0)}}{4L_A^2 L_A} \\ &- \frac{G^2 M m_A m_U^2 m_U' (2)_{AU}^{(0)}}{4L_U^2 L_A}\end{aligned}$$

$$V = -\frac{G^2 M m_M m_A^2 m_A'}{4L_A^2 \sqrt{L_M L_A}} (21)_{MA}^{(-1)}$$

$$W = -\frac{G^2 M m_A m_U^2 m'_U}{4 L_U^2 \sqrt{L_A L_U}} (21)_{AU}^{(-1)} \quad (7.82)$$

The numerical values of the coefficients are evaluated at $a_M = 4.8658$, $a_A = 7.0846$, and $a_U = 10.1179$, with all eccentricities zero, yielding:

$$\begin{aligned} \alpha &= -165281.64 \\ C'' &= -0.188964 \\ D'' &= -0.0193987 \\ E &= 0.0003938 \\ F &= -0.0070660 \\ G &= 0.00525854 \\ H &= 0.00060146 \\ U &= -0.054396 \\ V &= 0.00361009 \\ W &= 0.00984307 \end{aligned} \quad (7.83)$$

7.2.4 Miranda-Ariel 5:3 Planar-Eccentric Problem

The most significant perturbations on this resonance are due to Umbriel. For this problem,

$$\begin{aligned}\phi &= 3\lambda_M - 5\lambda_A \\ \Phi &= \frac{1}{3}(L_M - L_{MR})\end{aligned}\tag{7.84}$$

where

$$L_{MR} = \frac{3L_A + 5L_M}{5 + 3\left(\frac{5}{3}\right)^{1/3} \frac{m_A}{m_M}}\tag{7.85}$$

The new coefficients are:

$$\begin{aligned}C'' &= -\frac{3}{4} \frac{G^4 M^4 m_M^4 m_M'^3 R^2 J_2}{L_M^7} \\ &\quad - \frac{G^2 M m_M m_A^2 m_A' (2)_{MA}^{(0)}}{4L_A^2 L_M} \\ &\quad - \frac{G^2 M m_M m_U^2 m_U' (2)_{MU}^{(0)}}{4L_U^2 L_M}\end{aligned}$$

$$\begin{aligned}D'' &= -\frac{3}{4} \frac{G^4 M^4 m_A^4 m_A'^3 R^2 J_2}{L_A^7} \\ &\quad - \frac{G^2 M m_M m_A^2 m_A' (3)_{MA}^{(0)}}{4L_A^2 L_A} \\ &\quad - \frac{G^2 M m_A m_U^2 m_U' (2)_{AU}^{(0)}}{4L_U^2 L_A}\end{aligned}$$

$$\begin{aligned}U &= -\frac{3}{4} \frac{G^4 M^4 m_U^4 m_U'^3 R^2 J_2}{L_U^7} \\ &\quad - \frac{G^2 M m_M m_U^2 m_U' (3)_{MU}^{(0)}}{4L_U^2 L_U} \\ &\quad - \frac{G^2 M m_A m_U^2 m_U' (3)_{AU}^{(0)}}{4L_U^2 L_U}\end{aligned}$$

$$V = -\frac{G^2 M m_M m_U^2 m_U'}{4L_U^2 \sqrt{L_M L_U}} (21)_{MU}^{(-1)}$$

$$W = -\frac{G^2 M m_A m_U^2 m_U'}{4 L_U^2 \sqrt{L_A L_U}} (21)_{AU}^{(-1)} \quad (7.86)$$

Using the physical parameters given in Section 1.1, the numerical values of the coefficients were evaluated at the resonance semimajor axes $a_M = 4.6361$, $a_A = 6.5157$, and $a_U = 10.0560$, with $e_i = 0.0$, yielding:

$$\begin{aligned} \alpha &= -1.5747 \times 10^6 \\ C'' &= -0.22521 \\ D'' &= -0.068809 \\ E &= 0.0051031 \\ F &= -0.064824 \\ G &= 0.037081 \\ H &= -0.052683 \\ U &= -0.017831 \\ V &= 0.00033903 \\ W &= 0.0063445 \end{aligned} \quad (7.87)$$

7.2.5 Ariel-Umbriel 2:1 Planar-Eccentric

The most significant perturbations on this resonance are due to Titania. For this problem,

$$\begin{aligned}\phi &= \lambda_A - 2\lambda_U \\ \Phi &= (L_A - L_{AR})\end{aligned}\tag{7.88}$$

where

$$L_{MR} = \frac{L_U + 2L_A}{2 + 2^{1/3} \frac{m_U}{m_A}}\tag{7.89}$$

The new coefficients are:

$$\begin{aligned}C'' &= -\frac{3}{4} \frac{G^4 M^4 m_A^4 m_A'^3 R^2 J_2}{L_A^7} \\ &\quad - \frac{G^2 M m_A m_U^2 m_U'}{4L_U^2} \frac{(2)_{AU}^{(0)}}{L_A} \\ &\quad - \frac{G^2 M m_A m_T^2 m_T'}{4L_T^2} \frac{(2)_{AT}^{(0)}}{L_A}\end{aligned}$$

$$\begin{aligned}D'' &= -\frac{3}{4} \frac{G^4 M^4 m_U^4 m_U'^3 R^2 J_2}{L_U^7} \\ &\quad - \frac{G^2 M m_A m_U^2 m_U'}{4L_U^2} \frac{(3)_{AU}^{(0)}}{L_U} \\ &\quad - \frac{G^2 M m_U m_T^2 m_T'}{4L_T^2} \frac{(2)_{UT}^{(0)}}{L_U}\end{aligned}$$

$$\begin{aligned}U &= -\frac{3}{4} \frac{G^4 M^4 m_T^4 m_T'^3 R^2 J_2}{L_T^7} \\ &\quad - \frac{G^2 M m_A m_T^2 m_T'}{4L_T^2} \frac{(3)_{AT}^{(0)}}{L_T} \\ &\quad - \frac{G^2 M m_U m_T^2 m_T'}{4L_T^2} \frac{(3)_{UT}^{(0)}}{L_T}\end{aligned}$$

$$V = -\frac{G^2 M m_A m_T^2 m_T'}{4L_T^2 \sqrt{L_A L_T}} (21)_{AT}^{(-1)}$$

$$W = -\frac{G^2 M m_U m_T^2 m_T'}{4 L_T^2 \sqrt{L_U L_T}} (21)_{UT}^{(-1)} \quad (7.90)$$

Using the physical parameters given in Section 1.1, the numerical values of the coefficients were evaluated at the resonance semimajor axes $a_A = 6.3256$, $a_U = 10.0400$, and $a_T = 16.6330$, with $e_i = 0.0$, yielding:

$$\begin{aligned} \alpha &= -5.1742 \times 10^4 \\ C'' &= -0.075839 \\ D'' &= -0.021761 \\ E &= 0.0054912 \\ F &= -0.017679 \\ G &= 0.047366 \\ H &= -0.031350 \\ I &= 0.010364 \\ J &= 0.0066157 \\ U &= -0.0039395 \\ V &= 0.00056777 \\ W &= 0.0033443 \end{aligned} \quad (7.91)$$

7.3 Eccentric-Inclined Models

In order to include the effects of three-dimensional motion on the resonant interactions, we can develop a resonant Hamiltonian similar to that developed for the planar-eccentric case, but generalized to include the inclination resonant terms. We define

$$\begin{aligned}
\sigma_i &= \frac{1}{2}(i\lambda_2 - j\lambda_1 - 2\varpi_i) \\
\Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
\psi_i &= \frac{1}{2}(5\lambda_2 - 3\lambda_1 - 2\Omega_i) \\
\Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
\Gamma_1 &= L_1 + \frac{j}{2}(\Sigma_1 + \Sigma_2 + \Psi_1 + \Psi_2) \\
\Gamma_2 &= L_2 - \frac{i}{2}(\Sigma_1 + \Sigma_2 + \Psi_1 + \Psi_2)
\end{aligned} \tag{7.92}$$

In all cases, the inclination part of the secular contribution to the Hamiltonian is given by

$$\mathcal{H}_S = -\frac{Gm_1m_2}{a_2} \left[(1)^{(0)} + (11)^{(0)} \left\{ \left(\frac{i_1}{2} \right)^2 + \left(\frac{i_2}{2} \right)^2 + \frac{i_1}{2} \frac{i_2}{2} \cos(\Omega_1 - \Omega_2) \right\} \right] \tag{7.93}$$

where

$$(11)^{(0)} = -\frac{1}{2}\alpha b_{3/2}^1(\alpha) \tag{7.94}$$

In terms of these coordinates, the Hamiltonian is

$$\begin{aligned}
\mathcal{H} &= 2A(\Sigma_1 + \Sigma_2 + \Psi_1 + \Psi_2) + 4B(\Sigma_1 + \Sigma_2 + \Psi_1 + \Psi_2)^2 \\
&+ 2C\Sigma_1 + 2D\Sigma_2 + 2E\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 - \sigma_2) \\
&+ 2F\Sigma_1\cos(2\sigma_1) + 2G\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 + \sigma_2) + 2H\Sigma_2\cos(2\sigma_2) \\
&+ 2K\Psi_1 + 2L\Psi_2 + 2M\sqrt{\Psi_1\Psi_2}\cos(\psi_1 - \psi_2) \\
&+ 2N\Psi_1\cos(2\psi_1) + 2O\sqrt{\Psi_1\Psi_2}\cos(\psi_1 + \psi_2) + 2P\Psi_2\cos(2\psi_2)
\end{aligned} \tag{7.95}$$

For the 2:1 resonance, there are also the terms

$$I\sqrt{2\Sigma_1}\cos\sigma_1 + J\sqrt{2\Sigma_2}\cos\sigma_2 \quad (7.96)$$

Note that in the above expression, the inclination and eccentricity terms are only coupled through the nonlinear terms. δ is defined to be the nonresonant contributions to $in_2 - jn_1 - \dot{\omega}_1 - \dot{\omega}_2 - \dot{\Omega}_1 - \dot{\Omega}_2$.

7.3.1 Ariel-Umbriel 5:3 Resonance

We define

$$\begin{aligned}
\sigma_i &= \frac{1}{2}(5\lambda_U - 3\lambda_A - 2\varpi_i) \\
\Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
\psi_i &= \frac{1}{2}(5\lambda_U - 3\lambda_A - 2\Omega_i) \\
\Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
\Gamma_A &= L_A + \frac{3}{2}(\Sigma_A + \Sigma_U + \Psi_A + \Psi_U) \\
\Gamma_U &= L_U - \frac{5}{2}(\Sigma_A + \Sigma_U + \Psi_A + \Psi_U)
\end{aligned} \tag{7.97}$$

The contributions of the inclination resonances to the Hamiltonian are given by (see Leverrier, 1855):

$$\begin{aligned}
\mathcal{H}_R &= -\frac{Gm_A m_U}{a_U} (212)^{(5)} \left[\left(\frac{i_A}{2} \right)^2 \cos(5\lambda_U - 3\lambda_A - 2\Omega_A) \right. \\
&\quad \left. - 2\frac{i_A}{2}\frac{i_U}{2} \cos(5\lambda_U - 3\lambda_A - \Omega_A - \Omega_U) + \right. \\
&\quad \left. \left(\frac{i_U}{2} \right)^2 \cos(5\lambda_U - 3\lambda_A - 2\Omega_U) \right]
\end{aligned} \tag{7.98}$$

where

$$(212)^{(5)} = \frac{1}{2}\alpha b_{1/2}^4(\alpha) \tag{7.99}$$

α is the ratio of semimajor axes, and $b_s^l(\alpha)$ are Laplace coefficients.

The coefficients $B - H$ are defined as before in Section 7.1.1.

$$\begin{aligned}
K &= \frac{1}{2}G^4 M^4 R^2 J_2 \left[\frac{-15m_U^4 m_U^3}{2\Gamma_U^7} + \frac{3m_A^4 m_A^3}{\Gamma_A^7} \right] \\
&\quad + \frac{G^2 M m_A m_U^2 m_U'}{4\Gamma_U^2} \left[\frac{5}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_A} \frac{m_U'}{m_A'} \left(\frac{3\Gamma_A}{\Gamma_U^2} + \frac{5\Gamma_A^2}{\Gamma_U^3} \right) - \frac{(11)^{(0)}}{\Gamma_A} \right] \\
L &= \frac{1}{2}G^4 M^4 R^2 J_2 \left[\frac{-9m_U^4 m_U^3}{\Gamma_U^7} + \frac{9m_A^4 m_A^3}{2\Gamma_A^7} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^2} \left[\frac{5}{\Gamma_U} b_{1/2}^0(\alpha) + \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_A} \frac{m'_U}{m'_A} \left(\frac{3\Gamma_A}{\Gamma_U^2} + \frac{5\Gamma_A^2}{\Gamma_U^3} \right) - \frac{(11)^{(0)}}{\Gamma_U} \right] \\
M &= \frac{G^2 M m_A m_U^2 m'_U}{2\Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (11)^{(0)} \\
N &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^2 \Gamma_A} (212)^{(5)} \\
O &= \frac{G^2 M m_A m_U^2 m'_U}{2\Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (212)^{(5)} \\
P &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^3} (212)^{(5)} \tag{7.100}
\end{aligned}$$

The coefficients are defined at $\delta = 0$ for $i_i = 0$, $e_i = 0$, and $a_U = 10.1353$ (which corresponds to $a_A = 7.2104$, slightly changing the values of the coefficients B through H). The numerical values of the coefficients for this system are:

$$\begin{aligned}
B &= -2599.18 \\
C &= -0.078125 \\
D &= -0.04628 \\
E &= 0.01086 \\
F &= -0.03166 \\
G &= 0.07895 \\
H &= -0.048906 \\
K &= 0.025132 \\
L &= -0.0067122 \\
M &= -0.01322 \\
N &= -0.003561
\end{aligned}$$

$$\begin{aligned}
O &= 0.0067164 \\
P &= -0.003167
\end{aligned}
\tag{7.101}$$

and

$$\begin{aligned}
\Gamma_A &= 0.74488 \\
\Gamma_U &= 0.83755
\end{aligned}
\tag{7.102}$$

7.3.2 Miranda-Umbriel 3:1 Resonance

We define

$$\begin{aligned}
\sigma_i &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\varpi_i) \\
\Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
\psi_i &= \frac{1}{2}(3\lambda_U - \lambda_M - 2\Omega_i) \\
\Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
\Gamma_M &= L_M + \frac{1}{2}(\Sigma_M + \Sigma_U + \Psi_M + \Psi_U) \\
\Gamma_U &= L_U - \frac{3}{2}(\Sigma_M + \Sigma_U + \Psi_M + \Psi_U)
\end{aligned} \tag{7.103}$$

The expressions for the coefficients $B - H$ are given in Section 7.1.3, those for $K - P$ are given in Section 7.1.2.

The coefficients are defined at $\delta = 0$ for $i_i = 0$, $e_i = 0$, and $a_U = 10.1353$ (which corresponds to $a_A = 7.2104$, slightly changing the values of the coefficients B through H). The numerical values of the coefficients for this system are:

$$B = -5164.15$$

$$C = -0.312892$$

$$D = -0.148229$$

$$E = -0.0003939$$

$$F = -0.007068$$

$$G = 0.0052597$$

$$H = 0.0006012$$

$$K = 0.043833$$

$$L = -0.1208288$$

$$M = -0.00067632$$

$$N = -0.0009778$$

$$O = 0.0003939$$

$$P = 0.00003967 \quad (7.104)$$

and

$$\Gamma_M = 0.03395267$$

$$\Gamma_U = 0.836831 \quad (7.105)$$

7.3.3 Miranda-Ariel 5:3 Resonance

We define

$$\begin{aligned}
\sigma_i &= \frac{1}{2}(5n_A - 3n_M - 2\varpi_i) \\
\Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
\psi_i &= \frac{1}{2}(5n_A - 3n_M - 2\Omega_i) \\
\Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
\Gamma_M &= L_M + \frac{3}{2}(\Sigma_M + \Sigma_A + \Psi_M + \Psi_A) \\
\Gamma_A &= L_A - \frac{5}{2}(\Sigma_M + \Sigma_A + \Psi_M + \Psi_A)
\end{aligned} \tag{7.106}$$

The contributions of the inclination resonances to the Hamiltonian are given by (see Leverrier, 1855):

$$\begin{aligned}
\mathcal{H}_R = & -\frac{Gm_M m_A}{a_A} (212)^{(5)} \left[\left(\frac{i_M}{2} \right)^2 \cos(5\lambda_A - 3\lambda_M - 2\Omega_M) \right. \\
& - 2 \frac{i_M}{2} \frac{i_A}{2} \cos(5\lambda_A - 3\lambda_M - \Omega_M - \Omega_A) + \\
& \left. \left(\frac{i_A}{2} \right)^2 \cos(5\lambda_A - 3\lambda_M - 2\Omega_A) \right]
\end{aligned} \tag{7.107}$$

where

$$(212)^{(5)} = \frac{1}{2} \alpha b_{1/2}^4(\alpha) \tag{7.108}$$

α is the ratio of semimajor axes, and $b_s^l(\alpha)$ are Laplace coefficients.

The coefficients $B - H$ are defined as before in Section 7.1.4.

$$\begin{aligned}
K &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{-15m_A^4 m_A'^3}{2\Gamma_A^7} + \frac{3m_M^4 m_M'^3}{\Gamma_M^7} \right] \\
&+ \frac{G^2 M m_M m_A^2 m_A'}{4\Gamma_A^2} \left[\frac{5}{\Gamma_A} b_{1/2}^0(\alpha) + \right. \\
&\quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_A}{m_M} \frac{m_A'}{m_M'} \left(\frac{3\Gamma_M}{\Gamma_A^2} + \frac{5\Gamma_M^2}{\Gamma_A^3} \right) - \frac{(11)^{(0)}}{\Gamma_M} \right] \\
L &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{-9m_A^4 m_A'^3}{\Gamma_A^7} + \frac{9m_M^4 m_M'^3}{2\Gamma_M^7} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{G^2 M m_M m_A^2 m'_A}{4\Gamma_A^2} \left[\frac{5}{\Gamma_A} b_{1/2}^0(\alpha) + \right. \\
& \quad \left. \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_A}{m_M} \frac{m'_A}{m'_M} \left(\frac{3\Gamma_M}{\Gamma_A^2} + \frac{5\Gamma_M^2}{\Gamma_A^3} \right) - \frac{(11)^{(0)}}{\Gamma_A} \right] \\
M &= \frac{G^2 M m_M m_A^2 m'_A}{2\Gamma_A^2 \sqrt{\Gamma_M \Gamma_A}} (11)^{(0)} \\
N &= -\frac{G^2 M m_M m_A^2 m'_A}{4\Gamma_A^2 \Gamma_M} (212)^{(5)} \\
O &= \frac{G^2 M m_M m_A^2 m'_A}{2\Gamma_A^2 \sqrt{\Gamma_M \Gamma_A}} (212)^{(5)} \\
P &= -\frac{G^2 M m_M m_A^2 m'_A}{4\Gamma_A^3} (212)^{(5)} \tag{7.109}
\end{aligned}$$

The coefficients are defined at $\delta = 0$ for $i_i = 0$, $e_i = 0$, and $a_A = 6.5152$ (which corresponds to $a_M = 4.6361$, slightly changing the values of the coefficients B through H). The numerical values of the coefficients for this system are:

$$\begin{aligned}
B &= -49209.1 \\
C &= -0.518229 \\
D &= -0.35883 \\
E &= 0.005106 \\
F &= -0.064858 \\
G &= 0.037098 \\
H &= -0.0052705 \\
K &= -0.0708403 \\
L &= -0.2302364 \\
M &= -0.0062127 \\
N &= -0.00729905
\end{aligned}$$

$$O = 0.0031582$$

$$P = -0.00034162$$

(7.110)

and

$$\Gamma_A = 0.03314$$

$$\Gamma_U = 0.70806$$

(7.111)

7.3.4 Ariel-Umbriel 2:1 Resonance

We define

$$\begin{aligned}
 \sigma_i &= \frac{1}{2}(4\lambda_U - 2\lambda_A - 2\varpi_i) \\
 \Sigma_i &\approx \frac{\Gamma_i}{2}e_i^2 \\
 \psi_i &= \frac{1}{2}(4\lambda_U - 2\lambda_A - 2\Omega_i) \\
 \Psi_i &\approx \frac{\Gamma_i}{2}i_i^2 \\
 \Gamma_A &= L_A + (\Sigma_A + \Sigma_U + \Psi_A + \Psi_U) \\
 \Gamma_U &= L_U - 2(\Sigma_A + \Sigma_U + \Psi_A + \Psi_U)
 \end{aligned} \tag{7.112}$$

The contributions of the inclination resonances to the Hamiltonian are given by (see Leverrier, 1855):

$$\begin{aligned}
 \mathcal{H}_R = & -\frac{Gm_A m_U}{a_U} (212)^{(4)} \left[\left(\frac{i_A}{2} \right)^2 \cos(4\lambda_U - 2\lambda_A - 2\Omega_A) \right. \\
 & - 2 \frac{i_A}{2} \frac{i_U}{2} \cos(4\lambda_U - 2\lambda_A - \Omega_A - \Omega_U) + \\
 & \left. \left(\frac{i_U}{2} \right)^2 \cos(4\lambda_U - 2\lambda_A - 2\Omega_U) \right]
 \end{aligned} \tag{7.113}$$

where

$$(212)^{(4)} = \frac{1}{2} \alpha b_{1/2}^3(\alpha) \tag{7.114}$$

α is the ratio of semimajor axes, and $b_s^l(\alpha)$ are Laplace coefficients.

The coefficients $B - J$ are defined as before in Section 7.1.5.

$$\begin{aligned}
 K &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{-6m_U^4 m_U^3}{\Gamma_U^7} + \frac{3m_A^4 m_A^3}{2\Gamma_A^7} \right] \\
 &+ \frac{G^2 M m_A m_U^2 m_U'}{4\Gamma_U^2} \left[\frac{4}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
 &\quad \left. 2 \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_A} \frac{m_U'}{m_A'} \left(\frac{\Gamma_A}{\Gamma_U^2} + \frac{2\Gamma_A^2}{\Gamma_U^3} \right) - \frac{(11)^{(0)}}{\Gamma_A} \right] \\
 L &= \frac{1}{2} G^4 M^4 R^2 J_2 \left[\frac{-15m_U^4 m_U^3}{2\Gamma_U^7} + \frac{3m_A^4 m_A^3}{\Gamma_A^7} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^2} \left[\frac{4}{\Gamma_U} b_{1/2}^0(\alpha) + \right. \\
& \quad \left. 2 \frac{d}{d\alpha} b_{1/2}^0(\alpha) \frac{m_U}{m_A} \frac{m'_U}{m'_A} \left(\frac{\Gamma_A}{\Gamma_U^2} + \frac{2\Gamma_A^2}{\Gamma_U^3} \right) - \frac{(11)^{(0)}}{\Gamma_U} \right] \\
M &= \frac{G^2 M m_A m_U^2 m'_U}{2\Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (11)^{(0)} \\
N &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^2 \Gamma_A} (212)^{(4)} \\
O &= \frac{G^2 M m_A m_U^2 m'_U}{2\Gamma_U^2 \sqrt{\Gamma_A \Gamma_U}} (212)^{(4)} \\
P &= -\frac{G^2 M m_A m_U^2 m'_U}{4\Gamma_U^3} (212)^{(4)} \tag{7.115}
\end{aligned}$$

The coefficients are defined at $\delta = 0$ for $i_i = 0$, $e_i = 0$, and $a_U = 10.0400$ (which corresponds to $a_A = 6.3256$, slightly changing the values of the coefficients B through J). The numerical values of the coefficients for this system are:

$$\begin{aligned}
B &= -1616.95 \\
C &= -0.130254 \\
D &= -0.073048 \\
E &= 0.00549116 \\
F &= -0.0176788 \\
G &= 0.0473665 \\
H &= -0.03134998 \\
I &= 0.010364 \\
J &= 0.0061657 \\
K &= 0.0189338 \\
L &= -0.0382723
\end{aligned}$$

$$\begin{aligned}
M &= -0.0073938 \\
N &= -0.0021373 \\
O &= 0.0039106 \\
P &= -0.00178881
\end{aligned}
\tag{7.116}$$

and

$$\begin{aligned}
\Gamma_A &= 0.69768 \\
\Gamma_U &= 0.83360
\end{aligned}
\tag{7.117}$$

Chapter 8

Appendix II: Mapping for Two Degree-of-Freedom Models

The mapping explicitly given here may be used to examine any of the two degree of freedom models for second-order resonances derived in Appendix I. With slight modifications, it may also be used for the first-order 2:1 resonance discussed in Chapter 5. It may also be generalized to include the inclination terms. The mappings used to explore the secularly perturbed problems are similar to those developed by Wisdom (1982, 1983).

In developing the algebraic mapping, equation (7.5) is rewritten

$$\begin{aligned}\mathcal{H} = & 2A(\Sigma_1 + \Sigma_2) + 4B(\Sigma_1 + \Sigma_2)^2 + 2C\Sigma_1 + 2D\Sigma_2 \\ & + 2E\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 - \sigma_2) \sum_{n=-\infty}^{\infty} \cos n(\Omega t - \phi_1) \\ & + 2F\Sigma_1\cos(2\sigma_1) \sum_{n=-\infty}^{\infty} \cos n(\Omega t - \phi_2) \\ & + 2G\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 + \sigma_2) \sum_{n=-\infty}^{\infty} \cos n(\Omega t - \phi_3) \\ & + 2H\Sigma_2\cos(2\sigma_2) \sum_{n=-\infty}^{\infty} \cos n(\Omega t - \phi_4)\end{aligned}\tag{8.1}$$

Expressing the cosine terms using the Fourier representation of the Dirac delta function yields

$$\mathcal{H} = 2A(\Sigma_1 + \Sigma_2) + 4B(\Sigma_1 + \Sigma_2)^2 + 2C\Sigma_1 + 2D\Sigma_2$$

$$\begin{aligned}
& + 2E\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 - \sigma_2)2\pi\delta_T(\Omega t - \phi_1) \\
& + 2F\Sigma_1\cos(2\sigma_1)2\pi\delta_T(\Omega t - \phi_2) \\
& + 2G\sqrt{\Sigma_1\Sigma_2}\cos(\sigma_1 + \sigma_2)2\pi\delta_T(\Omega t - \phi_3) \\
& + 2H\Sigma_2\cos(2\sigma_2)2\pi\delta_T(\Omega t - \phi_4)
\end{aligned} \tag{8.2}$$

This form allows us to integrate the angle-dependent terms at singular points, and integrate the contributions from the other terms between them. The delta functions are periodic in $T = \frac{2\pi}{\Omega}$, which becomes one cycle of the mapping. During one mapping cycle, the delta functions are integrated at times related to the phases ϕ_i , and between them the secular terms are integrated. These integrations are all performed analytically.

Since the new high-frequency terms are not directly related to those removed by averaging, the choice of the mapping frequency Ω and the phases ϕ_i is largely arbitrary. However, for the mapping to be valid, Ω must be much higher than the long-period frequencies, which are of order 10^{-4} of the orbital frequencies. The values $\Omega = 20 \text{ yr}^{-1}$ and 40 yr^{-1} have been found to be suitable for most purposes. These are of order a few hundred times the secular frequencies in this problem. We have chosen $\phi_1 = 0$, $\phi_2 = \phi_4 = \pi/4$ (since these two are independent), and $\phi_3 = \pi/2$.

A full mapping cycle involves the following steps, starting with coordinates $x_1^{(0)}, y_1^{(0)}, x_2^{(0)}$ and $y_2^{(0)}$:

Step 1: Integrate across first delta function at $t = t_1$

$$\begin{aligned}
x_1^{(1)} &= x_1^{(0)} \cos\left(\frac{2\pi E}{\Omega}\right) - y_2^{(0)} \sin\left(\frac{2\pi E}{\Omega}\right) \\
y_1^{(1)} &= y_1^{(0)} \cos\left(\frac{2\pi E}{\Omega}\right) + x_2^{(0)} \sin\left(\frac{2\pi E}{\Omega}\right) \\
x_2^{(1)} &= x_2^{(0)} \cos\left(\frac{2\pi E}{\Omega}\right) - y_1^{(0)} \sin\left(\frac{2\pi E}{\Omega}\right) \\
y_2^{(1)} &= y_2^{(0)} \cos\left(\frac{2\pi E}{\Omega}\right) + x_1^{(0)} \sin\left(\frac{2\pi E}{\Omega}\right)
\end{aligned} \tag{8.3}$$

Step 2: Integrate secular part from $t = t_1$ to $t = t_2 = t_1 + \frac{\phi_2}{\Omega}$

$$\begin{aligned}
x_1^{(2)} &= x_1^{(1)} \cos \left\{ \left[\frac{\delta}{2} + (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
&\quad - y_1^{(1)} \sin \left\{ \left[\frac{\delta}{2} + (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
y_1^{(2)} &= y_1^{(1)} \cos \left\{ \left[\frac{\delta}{2} + (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
&\quad + x_1^{(1)} \sin \left\{ \left[\frac{\delta}{2} + (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
x_2^{(2)} &= x_2^{(1)} \cos \left\{ \left[\frac{\delta}{2} - (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
&\quad - y_2^{(1)} \sin \left\{ \left[\frac{\delta}{2} - (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
y_2^{(2)} &= y_2^{(1)} \cos \left\{ \left[\frac{\delta}{2} - (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\} \\
&\quad + x_2^{(1)} \sin \left\{ \left[\frac{\delta}{2} - (C - D) + 4B(x_1^{(1)2} + y_1^{(1)2} + x_2^{(1)2} + y_2^{(1)2}) \right] \left(\frac{\phi_2 - \phi_1}{\Omega} \right) \right\}
\end{aligned} \tag{8.4}$$

Step 3: Integrate across second and fourth delta function at $t = t_2 = \frac{\phi_2}{\Omega}$

$$\begin{aligned}
x_1^{(3)} &= x_1^{(2)} \frac{1}{2} \left(e^{4\pi F/\Omega} + e^{-4\pi F/\Omega} \right) + y_1^{(2)} \frac{1}{2} \left(e^{4\pi F/\Omega} - e^{-4\pi F/\Omega} \right) \\
y_1^{(3)} &= y_1^{(2)} \frac{1}{2} \left(e^{4\pi F/\Omega} + e^{-4\pi F/\Omega} \right) + x_1^{(2)} \frac{1}{2} \left(e^{4\pi F/\Omega} - e^{-4\pi F/\Omega} \right) \\
x_2^{(3)} &= x_2^{(2)} \frac{1}{2} \left(e^{4\pi H/\Omega} + e^{-4\pi H/\Omega} \right) + y_2^{(2)} \frac{1}{2} \left(e^{4\pi H/\Omega} - e^{-4\pi H/\Omega} \right) \\
y_2^{(3)} &= y_2^{(2)} \frac{1}{2} \left(e^{4\pi H/\Omega} + e^{-4\pi H/\Omega} \right) + x_2^{(2)} \frac{1}{2} \left(e^{4\pi H/\Omega} - e^{-4\pi H/\Omega} \right)
\end{aligned} \tag{8.5}$$

Step 4: Integrate secular part between t_2 and $t = t_3 = t_2 + \frac{\phi_3 - \phi_2}{\Omega}$

Step 5: Integrate across third delta function at $t = t_3 = \frac{\phi_3}{\Omega}$

$$\begin{aligned}
x_1^{(5)} &= x_1^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} + e^{-4\pi G/\Omega} \right) + y_2^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} - e^{-4\pi G/\Omega} \right) \\
y_1^{(5)} &= y_1^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} + e^{-4\pi G/\Omega} \right) + x_2^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} - e^{-4\pi G/\Omega} \right) \\
x_2^{(5)} &= x_2^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} + e^{-4\pi G/\Omega} \right) + y_1^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} - e^{-4\pi G/\Omega} \right) \\
y_2^{(5)} &= y_2^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} + e^{-4\pi G/\Omega} \right) + x_1^{(4)} \frac{1}{2} \left(e^{4\pi G/\Omega} - e^{-4\pi G/\Omega} \right)
\end{aligned} \tag{8.6}$$

Step 6: Integrate secular part between t_3 and $t = t_4 = t_0 + \frac{2\pi}{\Omega}$. This full cycle of the mapping iterates all coordinates forward in time by one mapping period $T = \frac{2\pi}{\Omega}$.

If tidal dissipation is included, the mapping becomes explicitly time dependent through $\delta(t)$, and also through a slightly more complicated secular evolution. The secular evolution is integrated assuming a linear dependence of δ on time.

Since the mapping includes high frequency perturbations, the averaged Hamiltonian is no longer a conserved quantity. We require a conserved quantity corresponding to the energy for the analysis of the evolution of trajectories in δ , ΔE parameter space. In addition, if the mapping is used to compute a surface of section according to the standard criteria for the averaged Hamiltonian, there will be a slight distortion due to the difference between the mapping coordinates and the averaged coordinates. For these reasons, we have used perturbation theory to transform from mapping coordinates to averaged coordinates, by removing the new high-frequency perturbations (Wisdom and Tittlemore 1988).

We use a Von Zeipel transformation to go from mapping coordinates to averaged coordinates, using the generating function

$$F = \sum_i \Sigma'_i \sigma_i + \mu S(\Sigma'_i, \sigma_i, t) \quad (8.7)$$

and the transformation to first order in μ

$$\begin{aligned} \Sigma_i &= \Sigma'_i + \mu \frac{\partial S}{\partial \sigma_i} \\ \sigma'_i &= \sigma_i + \mu \frac{\partial S}{\partial \Sigma'_i} \end{aligned} \quad (8.8)$$

The Hamiltonian (8.1) can be expressed

$$\mathcal{H} = \mathcal{H}_0(\Sigma_i) + \mu \bar{\mathcal{H}}_1(\Sigma_i, \sigma_i) + \mu \tilde{\mathcal{H}}_1(\Sigma_i, \sigma_i, t) \quad (8.9)$$

where $\tilde{\mathcal{H}}_1$ contains time-dependent terms ($n \neq 0$) in Eq. (8.1).

To first order in μ , this Hamiltonian becomes

$$\mathcal{H}' = \mathcal{H}_0(\Sigma_i(\Sigma'_i)) + \mu \bar{\mathcal{H}}_1(\Sigma'_i, \sigma'_i) + \mu \tilde{\mathcal{H}}_1(\Sigma'_i, \sigma'_i, t) + \mu \frac{\partial S}{\partial t}$$

$$\begin{aligned}
&= \chi_0(\Sigma'_i) + \mu \bar{\chi}_1(\Sigma'_i, \sigma'_i) + \mu \tilde{\chi}_1(\Sigma'_i, \sigma'_i, t) + \mu \frac{\partial S}{\partial t} \\
&+ \mu \frac{\partial S}{\partial \sigma'_1} \left[\frac{\delta}{2} + (C - D) + 8B(\Sigma'_1 + \Sigma'_2) \right] \\
&+ \mu \frac{\partial S}{\partial \sigma'_2} \left[\frac{\delta}{2} - (C - D) + 8B(\Sigma'_1 + \Sigma'_2) \right]
\end{aligned} \tag{8.10}$$

S is chosen to eliminate time-dependent terms, so that

$$\begin{aligned}
\mu \tilde{\chi}_1(\Sigma'_i, \sigma'_i, t) &+ \mu \frac{\partial S}{\partial t} \\
&+ \mu \frac{\partial S}{\partial \sigma'_1} \left[\frac{\delta}{2} + (C - D) + 8B(\Sigma'_1 + \Sigma'_2) \right] \\
&+ \mu \frac{\partial S}{\partial \sigma'_2} \left[\frac{\delta}{2} - (C - D) + 8B(\Sigma'_1 + \Sigma'_2) \right] = 0
\end{aligned} \tag{8.11}$$

Since

$$\begin{aligned}
\tilde{\chi}_1 &= \sum_{ij} A_{ij} \cos(i\sigma_1 + j\sigma_2) \sum_{n \neq 0} \cos n(\Omega t - \phi_{ij}) \\
&= \sum_{ij} A_{ij} \sum_{n \neq 0} \cos(i\sigma_1 + j\sigma_2 + n(\Omega t - \phi_{ij}))
\end{aligned} \tag{8.12}$$

then

$$S = - \sum_{ij} A_{ij} \sum_{n \neq 0} \frac{\sin(i\sigma_1 + j\sigma_2 + n(\Omega t + \phi_{ij}))}{Z + n\Omega} \tag{8.13}$$

or

$$\begin{aligned}
S = &- 2E\sqrt{\Sigma'_1 \Sigma'_2} [K(1, -1) \sin(\sigma'_1 - \sigma'_2) + L(1, -1) \cos(\sigma'_1 - \sigma'_2)] \\
&- 2F\Sigma'_1 [K(2, 0) \sin(2\sigma'_1) + L(2, 0) \cos(2\sigma'_1)] \\
&- 2G\sqrt{\Sigma'_1 \Sigma'_2} [K(1, 1) \sin(\sigma'_1 + \sigma'_2) + L(1, 1) \cos(\sigma'_1 + \sigma'_2)] \\
&- 2H\Sigma'_2 [K(0, 2) \sin(2\sigma'_2) + L(0, 2) \cos(2\sigma'_2)]
\end{aligned} \tag{8.14}$$

with the functions

$$\begin{aligned}
K(i, j) &= \frac{\pi \cos \frac{Z}{\Omega}(\Omega t - \phi_{ij} - \pi)}{\Omega \sin \frac{Z}{\Omega} \pi} - \frac{1}{Z} \\
L(i, j) &= -\frac{\pi \sin \frac{Z}{\Omega}(\Omega t - \phi_{ij} - \pi)}{\Omega \sin \frac{Z}{\Omega} \pi}
\end{aligned} \tag{8.15}$$

evaluated at $t = \frac{2\pi}{\Omega}$, where

$$\begin{aligned} Z &= i \left[\frac{\delta}{2} + (C - D) + 8B(\Sigma'_1 + \Sigma'_2) \right] \\ &\quad - j \left[\frac{\delta}{2} - (C - D) + 8B(\Sigma'_1 + \Sigma'_2) \right] \end{aligned} \quad (8.16)$$

and i and j refer respectively to the coefficients of σ_1 and σ_2 . ϕ_{ij} are the phases from equation (8.1).

To first order in μ , the transformation from map coordinates to averaged coordinates is just

$$\begin{aligned} x_i^{av} &= \left(1 - \frac{1}{2} \mu \frac{\partial S / \partial \sigma_i}{\Sigma_i} \right) \left[x_i^m \cos \left(\mu \frac{\partial S}{\partial \Sigma_i} \right) - y_i^m \sin \left(\mu \frac{\partial S}{\partial \Sigma_i} \right) \right] \\ y_i^{av} &= \left(1 - \frac{1}{2} \mu \frac{\partial S / \partial \sigma_i}{\Sigma_i} \right) \left[y_i^m \cos \left(\mu \frac{\partial S}{\partial \Sigma_i} \right) + x_i^m \sin \left(\mu \frac{\partial S}{\partial \Sigma_i} \right) \right] \end{aligned} \quad (8.17)$$

which is a slight rotation and change of scale.

To illustrate the usefulness of this transformation, we have included surfaces of section from the Ariel-Umbriel 5:3 planar-eccentric problem in Fig. 8.1. (a) shows the phase space of a low-inclination trajectory just before capture into resonance, computed using the *untransformed* mapping coordinates. (b) shows the same surface of section in transformed coordinates. The untransformed surface of section is not bad, but note that the general pattern is slightly rotated clockwise in the figure relative to (b): the stable equilibria are not on the y-axis, and some of the points fall outside the energy surface boundary. This is simply a result of the mapping coordinates having a different definition than the averaged coordinates. Including the transformation greatly improves the appearance of the surface of section.

Figure 8.1: Surfaces of section computed with the planar mapping for the Ariel-Umbriel 5:3 resonance, at $\delta = -0.320$, $\Delta E = -1.0 \times 10^{-10}$. (a) shows the phase space in *untransformed* mapping coordinates, (b) shows the phase space after the transformation. The untransformed section is rotated slightly relative to the axes.

Chapter 9

Appendix III: Standard Theory of Evolution through Resonances

In this appendix we shall examine evolution into two of the second-order resonances at the 5:3 Ariel-Umbriel commensurability, and two of the first-order resonances at the 2:1 Ariel-Umbriel commensurability. We treat for the moment these resonances as isolated resonances in order to verify that our numerical methods are in satisfactory agreement with the results of the isolated single resonance model when the model is applicable. We emphasize that this is a being undertaken only to verify our methods; the actual dynamics near these commensurabilities are much more complicated, with the presence of large chaotic zones in the phase space, and the single resonance model is not applicable. However, the capture probabilities derived here, with appropriate substitutions of coordinates and coefficients, may be used with some confidence in predicting the initial behaviour of the orbits of Miranda and Umbriel upon encountering the 3:1 resonance at low eccentricity and/or inclination, due to the extremely small widths of the chaotic zones (see Chapter 3).

9.1 Adiabatic Invariance

The theory of adiabatic invariance can be used to completely specify the evolution of a one degree of freedom Hamiltonian problem with a slowly varying parameter δ , in the absence of infinite period orbits (separatrices) and/or chaotic

zones. (Because of the time dependence of the parameter δ the problem is non-autonomous and separatrices and chaotic zones are strictly speaking practically everywhere, though most are too small to be important. See Chirikov 1979). For a one degree of freedom problem the adiabatic invariant is, to first order, the action J , which is defined for fixed δ in terms of generalized canonical coordinates p and q as

$$J = \oint p dq \quad (9.1)$$

or the area on the phase plane enclosed by the trajectory. For fixed δ the action is strictly conserved since trajectories follow level curves of the Hamiltonian for a one degree of freedom system. In action angle variables, the Hamiltonian is cyclic in the angle conjugate to the action:

$$\mathcal{H}(p, q; \delta) \Rightarrow \mathcal{H}(J; \delta) \quad (9.2)$$

Evolution of the parameter δ at some slow rate $\dot{\delta} = \epsilon$ can be considered a perturbation on the motion. By transforming to action-angle coordinates and averaging over the period of motion on the trajectory, which is in general much shorter than the timescale of the evolution of δ , it can be shown that the action is to first order invariant (see e.g. Born 1960, Arnold 1978, Landau and Lifshitz 1978). It has also been shown that there is an adiabatic invariant to all orders in ϵ , if ϵ is sufficiently small (Lenard 1959, Kruskal 1962, Arnold 1963). To first order then the evolution of the energy depends only on the form in which δ appears in the Hamiltonian in action angle variables:

$$\mathcal{H}(t) = \mathcal{H}(\delta(t)) \quad (9.3)$$

The evolution of trajectories defined by the Hamiltonian as δ varies can be viewed as slow changes in the level curves of $\mathcal{H}$, maintaining constant area on the phase plane.

The standard theory of evolution through resonances (Goldreich and Peale,

1966, Yoder 1979a, Henrard 1982 and 1988, Henrard and Lemaître 1983, Borderies and Goldreich 1984, Lemaître 1984, Peale 1986) treats the resonances arising from a commensurability independently. This approximation may be valid for mean-motion resonances, for instance, if the J_2 of the planet is large, and has been successfully applied to J:J+1 resonant motions in the satellite system of Saturn (Yoder 1979a, Peale 1976, Henrard and Lemaître 1983).

9.2 Second-Order Resonances

We consider here two of the second order eccentricity resonances at the 5:3 Ariel-Umbriel commensurability: the resonance proportional to the square of the eccentricity of Ariel, and the resonance proportional to the square of the eccentricity of Umbriel. With appropriate substitutions of coordinates and coefficients, these functions can be applied to other second-order resonance. The Hamiltonians for these two resonances, artificially treated as though they are isolated, are:

$$\begin{aligned}\mathcal{H}_A &= \left[\frac{\delta}{2} + (C - D) \right] \Sigma_A + 4B\Sigma_A^2 + 2F\Sigma_A \cos(2\sigma_A) \\ &= \frac{1}{2} \left[\frac{\delta}{2} + (C - D) \right] (x_A^2 + y_A^2) + B(x_A^2 + y_A^2)^2 + F(x_A^2 - y_A^2)\end{aligned}\quad (9.4)$$

and

$$\begin{aligned}\mathcal{H}_U &= \left[\frac{\delta}{2} - (C - D) \right] \Sigma_U + 4B\Sigma_U^2 + 2H\Sigma_U \cos(2\sigma_U) \\ &= \frac{1}{2} \left[\frac{\delta}{2} - (C - D) \right] (x_U^2 + y_U^2) + B(x_U^2 + y_U^2)^2 + H(x_U^2 - y_U^2).\end{aligned}\quad (9.5)$$

These can be derived by removing the coupling terms from the full Hamiltonian (Eqs. 7.5 and 7.9). These are both one degree of freedom problems, and are therefore integrable. The mixed second order resonance could of course be treated in a similar manner, but will not be considered here.

The structure of the level curves of the Hamiltonian for second order resonances is well known. Fig. 9.1 illustrates the level curves of these systems for various fixed

values of δ , computed with the mapping. For large negative δ , or before the resonance is encountered, the curves in x_i, y_i space are approximately concentric circles surrounding a stable fixed point at the origin, and the eccentricity is approximately constant. As the value of δ is increased, curves near the origin become distorted in the y -direction. At a critical value of δ_{c1} a period-2 bifurcation of the fixed point occurs, and it becomes unstable. There are now two librating islands surrounding stable fixed points on the y -axis, bounded by a separatrix with a “figure 8” shape. As δ increases, these islands become larger. At a second critical value of δ_{c2} , another bifurcation occurs at the origin. The fixed point at the origin becomes stable again, and is surrounded by concentric level curves. Two unstable equilibria are emitted from the origin. The separatrix now resembles two “bananas” joined at the “ends.” Further evolution of δ results in these islands moving further from the origin, and the trajectories near the origin become more circular. Critical values of δ and coordinates of stable and unstable fixed points are summarized in Table 9.1.

For the physical system trajectories encounter the resonance as δ passes from large negative values through values near zero. The semimajor axis of the inner satellite Ariel is increasing more rapidly than that of Umbriel. The ratio of their semimajor axes, a_A/a_U is increasing. For this direction of passage through an isolated second order resonance the trajectory may be either captured into resonance or escape from resonance. If the separatrix is encountered with $\delta_{c1} \leq \delta \leq \delta_{c2}$ then capture is certain. This occurs for trajectories with small eccentricity. If the separatrix is encountered with larger values of δ then the trajectory may either be captured or escape. The adiabatic invariance theory breaks down as the trajectory approaches the separatrix, since the period of motion on the separatrix is infinite, and it is no longer proper to average the evolution over the libration period. The action (area inside a trajectory) changes suddenly on passage through the separatrix, which can be seen on inspection of Fig. 9.1. Trajectories which escape have lower eccentricities

Figure 9.1: Level curves for the Ariel-Umbriel 5:3 e_A^2 resonance, computed with the uncoupled mapping. (a): $\delta = -0.5$, (b): $\delta = 0.15$, (c): $\delta = 0.5$.

Table 9.1: Single Resonances
Ariel Resonance

$$\delta_{c1} = -2(C - D) + 4F \approx -0.065$$

$$\delta_{c2} = -2(C - D) - 4F \approx 0.193$$

Unstable Equilibria:

$$x_1^2 = 0.0, \delta_{c1} \leq \delta < \delta_{c2}$$

$$x_1^2 = -\frac{[\delta + 2(C - D) + 4F]}{8B}, \delta \geq \delta_{c2}$$

Stable Equilibria:

$$y_1^2 = -\frac{[\delta + 2(C - D) - 4F]}{8B}, \delta \geq \delta_{c1}$$

Umbriel Resonance

$$\delta_{c1} = 2(C - D) + 4H \approx -0.263$$

$$\delta_{c2} = 2(C - D) - 4H \approx 0.133$$

Unstable Equilibria:

$$x_2^2 = 0.0, \delta_{c1} \leq \delta < \delta_{c2}$$

$$x_2^2 = -\frac{[\delta - 2(C - D) + 4H]}{8B}, \delta \geq \delta_{c2}$$

Stable Equilibria:

$$y_2^2 = -\frac{[\delta - 2(C - D) - 4H]}{8B}, \delta \geq \delta_{c1}$$

than they had approaching the resonance. The eccentricities of captured trajectories increase until the damping rate due to tidal dissipation in the satellite becomes comparable to the input of energy to the orbits.

The evolution of the energy (the numerical value of the Hamiltonian) of a trajectory can be estimated from the eccentricity behavior of the various cases, summarized in Table 9.1. Interior to the resonance ($\delta \ll 0$), the level curves are roughly circular, and adiabatic invariance of the area enclosed by the curves results in the eccentricities remaining approximately constant as δ changes. Therefore $\mathcal{H}_A$ and $\mathcal{H}_U$ are linearly proportional to δ , and the energies increase linearly with time. The same is true for trajectories which escape. The change in eccentricity at the point of escape causes the slope of $\mathcal{H}_i$ to decrease discontinuously at the value of δ where the separatrix is crossed. For captured trajectories, the eccentricity increases approximately as $\sqrt{\delta}$. The functions $\mathcal{H}_A$ and $\mathcal{H}_U$ are quartic in eccentricity. Therefore, the energy increase is roughly quadratic in time.

Evolution of trajectories in $\delta, \Delta E$ space for the single resonance Hamiltonians are much simpler than for the full resonant Hamiltonian. Since only two fates are possible, trajectories will follow one of only two branches in $\delta, \Delta E$ space for values of δ larger than that at which the separatrix is crossed. If the trajectory is captured into the resonance, ΔE will increase linearly. If it escapes, ΔE will decrease quadratically.

The energies of the separatrices for $\mathcal{H}_A$ and $\mathcal{H}_U$ are shown in Fig. 2.1. Expressions for these energies have been obtained by substituting the coordinates of the unstable fixed points as a function of δ (see Table 9.1) in equations (9.4) and (9.5).

9.2.1 Capture Probabilities

The determination of the fate of a given trajectory upon an encounter with an isolated resonance has been developed in a probabilistic approach based on the

“balance of energy” as the trajectory crosses the separatrix (Goldreich and Peale 1966, Counsellman and Shapiro 1970, Yoder 1979, Henrard 1982, 1988, Henrard and Lemaitre 1983, Borderies and Goldreich 1984, Lemaitre 1984). The balance of energy is defined as the difference between the change in energy of the trajectory and the change in energy of the separatrix in one cycle of the trajectory.

The analytic theory of capture into resonance assumes a uniform distribution of trajectory energies just before separatrix crossing. These energies lie within a range corresponding to the energy balance of the outer separatrix B_o . The probabilities of capture and escape depend on the magnitude of the energy balance before transition across the separatrix B_i . A trajectory loses energy relative to the separatrix as it approaches, since the separatrix gains energy quadratically with δ and the trajectory gains energy only linearly with δ . Therefore, B_o is less than zero. B_i can be either positive or negative. If it is negative, the total relative change in energy is negative, and the trajectory continues losing energy with respect to the separatrix, and escapes. If $B_i > 0$, and $B_i + B_o < 0$, the same thing happens. However, if $B_i + B_o > 0$, and there is a net gain in relative energy, capture will occur. The outcome will depend on both the phase and energy of the trajectory relative to the separatrix just before crossing. Since these are generally unknown only an estimate of the likelihood of various outcomes can be made. Probabilities are assigned under the assumption that the energy of the physical system is equally likely to have been anywhere within the balance of energy. Equivalently, probabilities are assigned by assuming the actual system belongs to an ensemble of similar systems whose energies are distributed uniformly over a range of energies comparable to the balance of energy. Probability of capture is then

$$P_c = \frac{B_o + B_i}{B_o} \quad (9.6)$$

where B_o and B_i are estimated at a given value of δ by approximating the motion as motion on the separatrix. By distributing the energies of trajectories uniformly

through the range B_o , one should obtain a representative sample of trajectories with both fates.

The balance of energy has been approximated by Henrard (1988) as the rate of increase of the area in phase space enclosed by the separatrix at the value of δ at which the trajectory crosses the separatrix:

$$B \simeq -\dot{\delta} \frac{\partial J^*}{\partial \delta} \quad (9.7)$$

where J^* is the area enclosed by the separatrix. The probability of capture, then, is the ratio of the rate of increase of the combined area of the libration islands to the rate of increase of the area enclosed by the outer boundary of the separatrix. This has a nice intuitive interpretation in terms of Liouville's theorem.

When the standard theory is applied to the two J:J+2 resonances in our problem treated individually, the probabilities of capture are:

$$P_A = \frac{2}{1 + \frac{\pi}{2 \sin^{-1} \left(\frac{-2F}{\sqrt{-F(\delta + 2(C-D))}} \right)}} \quad (9.8)$$

for the Ariel resonance and

$$P_U = \frac{2}{1 + \frac{\pi}{2 \sin^{-1} \left(\frac{-2H}{\sqrt{-H(\delta + 2(C-D))}} \right)}} \quad (9.9)$$

for the Umbriel resonance. The value of δ is that at which the trajectory crosses the separatrix.

For $-2(C-D) + 4F \leq \delta \leq -2(C-D) - 4F$, $P_A = 1$, and for $2(C-D) + 4H \leq \delta \leq 2(C-D) - 4H$, $P_U = 1$. For larger values of δ , the dependence of probability on eccentricity far from resonance can be found. The action of a trajectory is to first order adiabatically invariant until just before it crosses the separatrix. At this point, the action is approximately equal to the area enclosed by the outer separatrix, for which analytic expressions can be found:

$$J_A^* = 2\sqrt{\frac{-F(\delta + 2(C-D) - 4F^2)}{16B^2} - \frac{(\delta + 2(C-D))}{4B}} \left[\frac{\pi}{2} + \sin^{-1} \left(\frac{-2F}{\sqrt{-F(\delta + 2(C-D))}} \right) \right] \quad (9.10)$$

for the Ariel resonance and

$$J_U^* = 2\sqrt{\frac{-H(\delta - 2(C - D) - 4H^2)}{16B^2} - \frac{(\delta - 2(C - D))}{4B}} \left[\frac{\pi}{2} + \sin^{-1} \left(\frac{-2H}{\sqrt{-H(\delta - 2(C - D))}} \right) \right] \quad (9.11)$$

for the Umbriel resonance.

By considering the trajectories far from resonance to be approximately circular with actions $J_i \simeq 2\pi\Sigma_{i0}$, and equating these with the above expressions, the dependence of capture probability on eccentricity far from resonance can be determined. Using the above expressions, $e_A < 0.00404$ and $e_U < 0.00474$ for certain capture, in agreement with the calculations of Peale (1988).

9.2.2 Numerical Results

We have applied the techniques used to study the full resonant problem to a numerical check of the formulae for the standard theory of passage through isolated resonances. We have numerically computed the probability function for a number of sets of 100 trajectories, the trajectories in each set having the same initial eccentricities, but different phases. Different sets have different initial eccentricities. The mapping (Eq. 8.1) with all coupling terms set to zero was used. We have taken a number of steps to ensure that various sources of error are under control.

The values predicted for B_o are estimates. In order to avoid possible “edge effects” by under- or over-sampling ranges of B_o which result in capture or escape, and to ensure that the distribution of phases just before crossing the separatrix is uniform, we have randomly distributed initial energies and phases far from resonance such that near the separatrix the spread in energy is uniform across approximately 100 times the energy balance of the outer separatrix at the value of δ at which it is predicted to cross. This has taken into consideration the approximately linear spread in energies of trajectories with slight initial differences over the range of δ between the initial value and the value at separatrix crossing.

This initial spread in energies corresponds to a spread of initial eccentricities. Because of the variation of capture probability with initial eccentricity in the single resonance model, trajectories at opposite ends of the range of initial eccentricity have slightly different capture probabilities. This contributes some error to the estimate of the probability:

$$\Delta P_c \simeq \Delta e_o \times \left. \frac{\partial P_c}{\partial e} \right|_{\langle e_o \rangle} \quad (9.12)$$

Since the probability function is known, the expected statistical fluctuation at any point on the function can be estimated using binomial statistics:

$$\Delta P_c = \frac{\sigma}{N} = \sqrt{\frac{P_c(1 - P_c)}{N}} \quad (9.13)$$

There is some uncertainty in exact values of energy due to fluctuations in the mapping integral, which are approximately inversely proportional to Ω through the K and L functions (Eqs. 8.15). $\Omega = 80 \text{ yr}^{-1}$ has been chosen to keep the error less than about 1%.

Based on the above discussion, the expected probability errors have been estimated. These are plotted in Fig. 9.2 for the variation over the probability functions and for the statistical fluctuations. The energy error is less than about 1%. The statistical fluctuations dominate over most of the range except at the steepest parts of the probability functions.

Numerical measurements of the capture probabilities for both resonances were computed by running sets of 100 trajectories through the resonance using the map, each with the aforementioned energy and phase spreads and $\langle e_o \rangle = 0.0025, 0.005, 0.0075, 0.01, 0.0125, 0.015, 0.0175, \text{ and } 0.02$. Results are plotted in Fig. 9.3, along with the predictions of the analytic theory. The agreement with the analytical predictions is very good. Measured errors are somewhat less than expected on the steep part of the probability function.

Figure 9.2: Expected probability variations in Ariel-Umbriel 5:3 single resonance capture probabilities. (a): e_A^2 resonance, (b): e_U^2 resonance. Solid line: expected statistical fluctuation for 100 trajectories. Points: initial spread of eccentricities multiplied by the slope of the probability function.

(a)

(b)

Figure 9.3: Ariel-Umbriel 5:3 single resonance capture probabilities. (a): e_A^2 resonance, (b): e_U^2 resonance. Solid lines are theoretical predictions from the single-resonance theory. Data points are measured capture probabilities for ensembles of 100 trajectories evolved through the resonance with the uncoupled mapping. The mapping method properly reproduces the capture probabilities within the range of eccentricities below the equilibrium value (see Section 2.2).

The agreement between our numerical results for these single resonance problems and the expectations from the single resonance model give us confidence that our methods can be used to reliably examine the passage through the more complicated resonances considered in this paper.

9.3 First-Order Resonances

A first order resonance may be described by a Hamiltonian of the following form:

$$\mathcal{H} = \alpha\Sigma + \beta\Sigma^2 + \epsilon\sqrt{2\Sigma}\cos\sigma \quad (9.14)$$

The problem of computing capture probabilities for such a resonance becomes more tractable when the momentum is scaled via the following transformation (Henrard and Lemaitre 1983):

$$R = \left| \frac{2\beta}{\epsilon} \right|^{2/3} \Sigma \quad (9.15)$$

yielding the transformed Hamiltonian

$$H' = -3(\delta' + 1)\dot{R} + R^2 - 2\sqrt{2R}\cos r \quad (9.16)$$

where

$$\delta' + 1 = \frac{1}{3} \left(\frac{4}{\beta\epsilon^2} \right)^{1/3} \alpha \quad (9.17)$$

The probability of capture is then given by the relation (Borderies and Goldreich 1984):

$$P = \frac{2}{1 + \frac{\pi}{2\sin^{-1}(\pi/2)^{-s/2}}} \quad (9.18)$$

where

$$\begin{aligned} s &= \sqrt{\delta' + 1} \\ \Delta &= \frac{1}{3} \cos^{-1} \left(\frac{1}{s^3} \right) \\ t &= \cos \Delta + \sqrt{3} \sin \Delta \end{aligned} \quad (9.19)$$

The area of the outer separatrix is given by

$$J^* = 6s^2 \left[\frac{\pi}{2} + \sin^{-1}(st)^{-3/2} \right] + \frac{6}{st} [(st)^3 - 1]^{1/2} \quad (9.20)$$

which can be equated to the scaled action of the trajectory well before resonance encounter: $2\pi R_0$. Therefore, it is possible to relate the eccentricity of an orbit well before resonance encounter to the probability of capture into the resonance in the single-resonance approximation.

If we consider them independently (not a good approximation), the first order resonances at the Ariel-Umbriel 2:1 commensurability may be described by the following Hamiltonian functions:

$$\mathcal{H}_1 = \frac{1}{2}(\delta - \delta_1)\Sigma_A + 4B\Sigma_A^2 + I\sqrt{2\Sigma_A} \cos \sigma_A \quad (9.21)$$

where $\delta_1 = -2(C - D) + 4F - 16B\Sigma_U$, and

$$\mathcal{H}_2 = \frac{1}{2}(\delta - \delta_2)\Sigma_U + 4B\Sigma_U^2 + J\sqrt{2\Sigma_U} \cos \sigma_U \quad (9.22)$$

where $\delta_1 = 2(C - D) + 4H - 16B\Sigma_A$. These functions are readily adapted to the transformation described above. Fig. 9.4 shows the capture probabilities for these two resonances as a function of initial eccentricity. Note that the capture probability is certain even for quite high eccentricities. The initial eccentricity e_0 below which capture is certain in the single-resonance approximation may be easily calculated. Inspection of Eq. 9.18 indicates that the capture probability is certain if $st = 1$. For this to be true, both s and t must equal unity, therefore $J^* = 6\pi$ and $R_0 = 3$. Therefore, the certain capture eccentricity is given by the following expression:

$$e_c = \sqrt{\frac{6}{L}} \left| \frac{\epsilon}{2\beta} \right|^{1/3} \quad (9.23)$$

where $L \approx m\sqrt{GMa}$. At the Ariel-Umbriel 2:1 commensurability, $e_A \leq 0.027$ and $e_U \leq 0.021$ for certain capture into the respective first-order resonances.

Qualitatively, these results agree with the results of Chapter 5: escape from the 2:1 resonance is not likely at low eccentricities. However, we must emphasize that

Figure 9.4: Analytically predicted probabilities of capture into the e_A (a) and e_U (b) resonances at the Ariel-Umbriel 2:1 mean-motion commensurability.

the single-resonance model does not really describe the behaviour at this commensurability well, even at low eccentricities, despite the dominance of quasiperiodic behaviour. Fig. 9.5 shows the evolution of the eccentricities of Ariel and Umbriel during evolution through this resonance in the single-resonance approximation. The initial conditions for this trajectory are similar to those for the trajectory in Fig. 5.7, but the integration was done without any coupling between the resonances. Note that there are significant differences between the two evolutions: the uncoupled trajectory evolves to higher eccentricities more rapidly than the fully coupled model, and the oscillations of the eccentricities after capture into the resonance are smaller.

Note that for a mixed second order resonance where one satellite is much more massive than the other, the variations of the orbital elements of the more massive satellite may be ignored at the point of capture (Peale 1988). The first-order single-resonance model may be used to evaluate the capture probability, with appropriate substitution of coefficients in Eq. 9.15, assuming that the motion is regular at the point of capture and that the libration region of the mixed resonance is not strongly distorted by nearby resonances (see Chapter 3).

Figure 9.5: Evolution of the orbital eccentricities of Ariel (a) and Umbriel (b) through the 2:1 commensurability, with all coupling terms removed. There are significant differences between the evolution in this figure and that of the trajectory in Fig. 5.7, which has similar initial conditions.

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