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TOPOLOGICAL AUTOMORPHIC FORMS ON $U(1,1)$

MARK BEHRENS AND TYLER LAWSON

ABSTRACT. The homotopy type and homotopy groups of some spectra TAF_{GU} of topological automorphic forms associated to a unitary similitude group GU of type $(1,1)$ are explicitly described in quasi-split cases. The spectrum TAF_{GU} is shown to be closely related to the spectrum TMF in these cases, and homotopy groups of some of these spectra are explicitly computed.

1. INTRODUCTION

Let F be a quadratic imaginary extension of \mathbb{Q} , and p be a fixed prime which splits as $u\bar{u}$ in F . Let GU be the group of unitary similitudes of a Hermitian form on an F -vector space of signature $(1, n-1)$, and suppose that $K \subset GU(\mathbb{A}^{p,\infty})$ is a compact open subgroup of the finite adèle points of GU away from p . Associated to this data is a unitary Shimura variety $\mathrm{Sh}(K)$ whose complex points are given by an adelic quotient

$$\mathrm{Sh}(K)(\mathbb{C}) \cong GU(\mathbb{Q}) \backslash GU(\mathbb{A}) / K \cdot K_p \cdot K_\infty.$$

Here $K_p \subset GU(\mathbb{Q}_p)$ is maximal compact, and $K_\infty \subset GU(\mathbb{R})$ is maximal compact modulo center. The Shimura variety admits a p -integral model as a moduli stack of certain n -dimensional polarized abelian varieties A with complex multiplication by F and level structure dependent on K . Over a p -complete ring, the complex multiplication decomposes the formal completion of A as

$$\widehat{A} \cong \widehat{A}_u \oplus \widehat{A}_{\bar{u}}$$

and the summand \widehat{A}_u is required to be 1-dimensional. We refer the reader to [HT01], [Kot92], and [BL] for a detailed exposition of these moduli stacks.

In [BL], the authors used a theorem of Jacob Lurie to construct a p -complete E_∞ -ring spectrum $\mathrm{TAF}_{GU}(K)$ of topological automorphic forms “realizing” the formal groups \widehat{A}_u associated to the Shimura varieties $\mathrm{Sh}(K)$. The height of the formal groups \widehat{A}_u varies between 1 and n , and the resulting spectra are v_n -periodic. In [BL, Ch. 15], the authors showed that in the case of $n=1$, for appropriate choices of GU and K , the associated spectrum of topological automorphic forms is essentially

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a product of copies of K -theory indexed by the class group of F . Thus the $n = 1$ case of the theory reduces to a well-understood v_1 -periodic cohomology theory.

The purpose of this paper is to demystify the case of $n = 2$, and show that, at least in some cases, the resulting cohomology theory $\mathrm{TAF}_{GU}(K)$ is closely related to TMF , the Goerss-Hopkins-Miller theory of topological modular forms. Thus in these cases, the $n = 2$ case of the theory of topological automorphic forms reduces to the well-understood theory of topological modular forms in a manner analogous to the way in which the $n = 1$ theory reduces to K -theory.

In the case of $n = 2$, the p -completion of the associated moduli stack $\mathrm{Sh}(K)$ consists of certain polarized abelian surfaces A with complex multiplication by F with $\dim \widehat{A}_u = 1$. In Section 2, we use the Honda-Tate classification of isogeny classes of abelian varieties over $\overline{\mathbb{F}}_p$ to show that every F -linear abelian surface A with $\dim A_u = 1$ is isogenous to a product of elliptic curves, and that there is a bijective correspondence between such isogeny classes of abelian surfaces and the isogeny classes of elliptic curves. Although this material serves as motivation for the constructions of the later sections, the remainder of the paper is independent of Section 2.

We concentrate solely on the case where the Hermitian form is isotropic, and study the case of two compact open subgroups K_0 and K_1 that are, in some sense, extremal examples of maximal compact open subgroups of $GU(\mathbb{A}^{p,\infty})$. A detailed account of the initial data, the moduli functor represented by $\mathrm{Sh}(K)$, and the associated cohomology theory $\mathrm{TAF}_{GU}(K)$ is given in Section 3.

Section 4 gives a description of the moduli stack $\mathrm{Sh}(K_0)$. The subgroup K_0 is the stabilizer of a non-self-dual lattice chosen such that the moduli stack $\mathrm{Sh}(K_0)$ admits a complete uniformization by copies of the moduli stack \mathcal{M}_{ell} of elliptic curves. We show that there is an equivalence:

$$\prod_{\mathrm{Cl}(F)} \mathcal{M}_{ell, \mathbb{Z}_p} \cong \mathrm{Sh}(K_0) \quad (\text{Theorem 4.9})$$

except in the cases where $F = \mathbb{Q}(i)$ or $\mathbb{Q}(\omega)$, where a slight modification is given. In Section 5, the associated spectra of topological automorphic forms are computed to be

$$\mathrm{TAF}_{GU}(K_0) \simeq \prod_{\mathrm{Cl}(F)} \mathrm{TMF}_p \quad (\text{Theorem 5.2})$$

except in the cases mentioned above. These cases are analyzed separately.

In Section 6 we study the moduli stack $\mathrm{Sh}(K_1)$. The subgroup K_1 is defined to be the stabilizer of a self-dual lattice, and the resulting moduli stack may be taken to be the moduli stack of *principally* polarized abelian surfaces with complex multiplication by F . We are only able to give a description of a connected component $\mathrm{Sh}(K_1)_0$ of $\mathrm{Sh}(K_1)$. Let N be such that $F = \mathbb{Q}(\sqrt{-N})$, and let $\mathcal{M}_0(N)$ be the moduli stack of elliptic curves with $\Gamma_0(N)$ -structure. We show that there is an equivalence

$$\mathcal{M}_0(N)_{\mathbb{Z}_p} // \langle w \rangle \cong \mathrm{Sh}(K_1)_0 \quad (\text{Theorem 6.4})$$

where w is the Fricke involution, unless $N = 1$ or 3 , where slightly different descriptions must be given. The homotopy groups of the corresponding summand

of the spectrum $\text{TAF}_{GU}(K_1)$ is analyzed in Section 7. In general, one takes the fixed points of modular forms for $\Gamma_0(N)$ with respect to an involution. Complete computations are given in the cases $N = 1, 2, 3$.

2. HONDA-TATE THEORY OF F -LINEAR ABELIAN SURFACES

In this section we analyze the isogeny classes of abelian surfaces A over $\overline{\mathbb{F}}_p$ with complex multiplication

$$i : F \rightarrow \text{End}^0(A),$$

using Honda-Tate theory. The splitting

$$\mathcal{O}_{F,p} \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

induces a splitting of the formal group

$$\widehat{A} \cong \widehat{A}_u \times \widehat{A}_{\bar{u}}.$$

We will always assume $\dim \widehat{A}_u = 1$. This is equivalent to assuming that the summand $A(u)$ of the p -divisible group $A(p)$ is 1-dimensional. The analysis of this section is independent of the rest of the paper, but serves to motivate some of the constructions in later sections.

We recall from [BL, Theorem 2.2.3] that the Honda-Tate classification of simple abelian varieties implies that isogeny classes of simple abelian varieties over $\overline{\mathbb{F}}_p$ are in one-to-one correspondence with *minimal p -adic types*.

Let M be a CM field (a field with a complex conjugation c whose fixed field is totally real), and for any prime x over p we let f_x be the degree of the residue field extension and e_x the ramification index. A p -adic type $(M, (\eta_x))$ consists of such a CM field, together with positive rational numbers η_x for all primes x over p of M , satisfying the relation

$$\eta_x/e_x + \eta_{c(x)}/e_x = 1$$

for all x . The associated simple abelian variety A has $M = \text{center}(\text{End}^0(A))$ and dimension $\frac{1}{2}[M : \mathbb{Q}]m$, where $m = [\text{End}^0(A) : M]^{1/2}$. The p -divisible group of A breaks up as the sum of simple p -divisible groups $A(x)$, each with height $[M_x : \mathbb{Q}_x]m$, dimension $\eta_x f_x m$, and pure slope η_x/e_x . The p -completion of the endomorphism ring of A is a product over x of division algebras $\text{End}^0(A(x))$ with center M_x and invariant $\eta_x f_x$.

A map of CM fields $M \hookrightarrow M'$ takes a p -adic type $(M, (\eta_x))$ to $(M', (\eta_x e_{x'/x}))$, where $e_{x'/x}$ is the ramification degree of the prime x' over the prime x . The p -adic type is minimal if it is not in the image of such a non-identity map.

Simple F -linear abelian varieties are classified by initial objects of the subcategory of p -adic types $(L, (\eta_x))$ under F . Such F -linear abelian varieties are isotypical, with simple type given by the minimal p -adic type over L . The associated simple F -linear abelian variety A has $L = \text{center}(\text{End}_F^0(A))$ and dimension $\frac{1}{2}[L : \mathbb{Q}]t$, where $t = [\text{End}_F^0(A) : L]^{1/2}$. The p -completion of the endomorphism ring of A is a product over x of division algebras $\text{End}_{F_x}(A(x))$ with center L_x and invariant $\eta_x f_x$.

Let E be an elliptic curve over $\bar{\mathbb{F}}_p$. Choosing a basis of \mathcal{O}_F gives an inclusion

$$\mathcal{O}_F \hookrightarrow M_2(\mathbb{Z}).$$

We associate to E an F -linear abelian surface

$$E \otimes \mathcal{O}_F := E \times E$$

with complex multiplication given by the composite

$$\mathcal{O}_F \hookrightarrow M_2(\mathbb{Z}) \hookrightarrow M_2(\text{End}(E)) \cong \text{End}(E \times E).$$

We now classify the isogeny classes $[(A, i)]$ of F -linear abelian surfaces (A, i) with $\dim \hat{A}_u = 1$.

Case 1: (A, i) is simple. Associated to (A, i) is a minimal p -adic type (L, η) under F with

$$2 = \dim A = \frac{1}{2}[L : \mathbb{Q}]t.$$

Since L contains F , $[L : \mathbb{Q}]$ is divisible by 2. We therefore have two possibilities, corresponding to $t = 1$ and $t = 2$.

Subcase 1a: $t = 1$. In this case, $L = \text{End}_F^0(A)$ is a CM-field which is quadratic extension of F . In particular, L is totally complex, and possesses a unique involution c for which $L^{(c)}$ is totally real, and which restricts to conjugation on F . Since L is a quadratic extension of F , there is also an involution σ of L satisfying $L^{(\sigma)} = F$. Since they give distinct fixed fields, the involutions σ and c must be distinct, and we deduce that L is Galois over \mathbb{Q} with Galois group

$$\text{Gal}(L/\mathbb{Q}) \cong C_2 \times C_2 = \langle c, \sigma \rangle.$$

The prime u of F lying over p is either split, ramified, or inert in L . It is easy to see that if p is ramified or inert, any p -adic type η associated to L must come from one on F , and so will not be minimal under F . Therefore, for (L, η) to be minimal under F , u must split as $v\sigma(v)$ in L . Then \bar{u} splits as $c(v)c\sigma(v)$. We must have

$$\eta_{\sigma^\epsilon v} + \eta_{c(\sigma^\epsilon v)} = 1$$

for $\epsilon \in \{0, 1\}$. Since A is 2-dimensional, we deduce $\eta_{\sigma^\epsilon v} \in \{0, 1\}$. Since we are assuming $\dim A(u) = 1$, and $A(u) = A(v_1) \oplus A(v_2)$, one of the $\eta_{\sigma^\epsilon v}$ must equal 1 and the other must be 0. Without loss of generality, assume $\eta_v = 1$. It follows that we must have

$$\begin{aligned} \eta_{c(v)} &= 0, \\ \eta_{\sigma(v)} &= 0, \\ \eta_{\sigma c(v)} &= 1. \end{aligned}$$

Although L is a minimal p -adic type under F , it is not minimal under \mathbb{Q} . Indeed, letting F' be the quadratic imaginary of \mathbb{Q} given by $L^{\sigma c=1}$, with conjugation

$$c' = c|_{F'} = \sigma|_{F'},$$

the prime p must split as $wc'(w)$ in F' . The prime w splits as $v\sigma c(v)$ in L , and the prime $c'(w)$ splits as $c(v)\sigma(v)$ in L . The p -adic type (L, η) is induced from a p -adic type (F', η') , where:

$$\begin{aligned}\eta'_w &= 1, \\ \eta'_{c'(w)} &= 0.\end{aligned}$$

The isogeny class of simple abelian varieties associated to (F', η') is an elliptic curve E with complex multiplication by F' such that $E(w)$ is 1-dimensional, and the isogeny class of F -linear abelian varieties containing (A, i) is given by

$$[(A, i)] = [E \otimes \mathcal{O}_F].$$

Subcase 1b: $t = 2$. In this case $L = F$. Since we are assuming $\dim A(u) = 1$, we deduce that $\eta_u = 1/2$. We must therefore have $\eta_{\bar{u}} = 1/2$.

The p -adic type (F, η) is not minimal under \mathbb{Q} : it is induced from the p -adic type (\mathbb{Q}, η') where $\eta_p = 1/2$. The isogeny class corresponding to (\mathbb{Q}, η') is the isogeny class of a supersingular elliptic curve E . Thus the F -linear isogeny class (A, i) contains $E \otimes \mathcal{O}_F$ as a representative.

Case 2: (A, i) is not simple. Then we must have

$$[(A, i)] = [(A_1, i_1)] \oplus [(A_2, i_2)]$$

where (A_j, i_j) are 1-dimensional F -linear abelian varieties. Thus the abelian varieties A_j are elliptic curves with complex multiplication by F . There is one isogeny class $[E]$ of elliptic curves with complex multiplication by F , and a representative E admits 2 conjugate complex multiplications. There result two distinct F -linear isogeny classes:

$$[(E, i_0)] \quad \text{and} \quad [(\bar{E}, \bar{i}_0)].$$

Here we use E to denote the F -linear elliptic curve with $\dim E(u) = 1$, and \bar{E} to be the same elliptic curve with conjugate complex multiplication, so that $\dim \bar{E}(u) = 0$. Since $\dim A(u) = 1$, we must have an F -linear isogeny $A \simeq E \times \bar{E}$. The diagonal embedding of abelian varieties

$$E \hookrightarrow E \times \bar{E}$$

extends to an F -linear quasi-isogeny

$$E \otimes \mathcal{O}_F \xrightarrow{\simeq} E \times \bar{E}$$

where \mathcal{O}_F acts on $E \otimes \mathcal{O}_F$ on the second factor only. Thus the isogeny class is computed to be $[(A, i)] = [E \otimes \mathcal{O}_F]$.

Proposition 2.1. *The F -linear isogeny classes of (A, i) over $\bar{\mathbb{F}}_p$ with $\dim A(u) = 1$ are given by*

- Case 1a:** $[E \otimes \mathcal{O}_F]$, where E is an elliptic curve with complex multiplication by a quadratic imaginary extension $F' \neq F$ in which p splits.
- Case 1b:** $[E \otimes \mathcal{O}_F]$, where E is a supersingular elliptic curve.
- Case 2:** $[E \otimes \mathcal{O}_F]$, where E is an elliptic curve with complex multiplication by F .

Corollary 2.2. *The construction*

$$[E] \mapsto [E \otimes \mathcal{O}_F]$$

gives a bijection between isogeny classes of elliptic curves over $\overline{\mathbb{F}}_p$, and isogeny classes of F -linear abelian surfaces A over $\overline{\mathbb{F}}_p$ with $\dim A(u) = 1$.

3. OVERVIEW OF THE SHIMURA STACK

We first review the moduli problem represented by the Shimura stacks under consideration. A more complete description, with motivation, can be found in [BL]. Fix a prime p and consider the following initial data:

- $F =$ quadratic imaginary extension of \mathbb{Q} in which p splits as $u\bar{u}$,
- $\mathcal{O}_F =$ ring of integers of F ,
- $V = F$ -vector space of dimension 2,
- $\langle -, - \rangle = \mathbb{Q}$ -valued non-degenerate hermitian alternating form.

We require that, for a complex embedding of F , the signature of $\langle -, - \rangle$ on V is $(1, 1)$. When necessary, we will regard

$$F = \mathbb{Q}(\delta) \quad \text{where} \quad \delta^2 = -N$$

for a positive square-free integer N . Let ι denote the involution on $\text{End}_F(V)$, defined by $\langle \alpha v, w \rangle = \langle v, \alpha^\iota w \rangle$.

Let $GU = GU_V$ be the associated unitary similitude group over \mathbb{Q} , with R -points

$$\begin{aligned} GU(R) &= \{g \in \text{End}_F(V) \otimes_{\mathbb{Q}} R : \langle gv, gw \rangle = \nu(g) \langle v, w \rangle, \nu(g) \in R^\times\} \\ &= \{g \in \text{End}_F(V) \otimes_{\mathbb{Q}} R : g^\iota g \in R^\times\}. \end{aligned}$$

Let $\widehat{\mathbb{Z}}^p$ denote the product $\prod_{\ell \neq p} \mathbb{Z}_\ell$, and let $\mathbb{A}^{p, \infty}$ be the finite adeles away from p , so that we have

$$\mathbb{A}^{p, \infty} = \widehat{\mathbb{Z}}^p \otimes \mathbb{Q}.$$

We let $V^{p, \infty}$ denote $V \otimes \mathbb{A}^{p, \infty}$. For an abelian variety A/k , where k is an algebraically closed field of characteristic 0 or p , we define

$$V^p(A) = \left(\prod_{\ell \neq p} T_\ell(A) \right) \otimes \mathbb{Q}$$

where

$$T_\ell(A) = \varprojlim_i A(k)[\ell^i]$$

denotes the covariant ℓ -adic Tate module of A .

For every compact open subgroup

$$K \subset GU(\mathbb{A}^{p, \infty})$$

there is a Deligne-Mumford stack $\text{Sh}(K)/\text{Spec}(\mathbb{Z}_p)$. For a locally noetherian connected \mathbb{Z}_p -scheme S , and a geometric point s of S , the S -points of $\text{Sh}(K)$ are the groupoid whose objects are tuples $(A, i, \lambda, [\eta]_K)$, with:

A , an abelian scheme over S of dimension 2,
 $\lambda: A \rightarrow A^\vee$, a $\mathbb{Z}_{(p)}$ -polarization,
 $i: \mathcal{O}_{F,(p)} \hookrightarrow \text{End}(A)_{(p)}$, an inclusion of rings, such that the λ -Rosati involution is compatible with conjugation,
 $[\eta]_K$, a $\pi_1(S, s)$ -invariant K -orbit of F -linear similitudes:
 $\eta: (V^{p,\infty}, \langle -, - \rangle) \xrightarrow{\cong} (V^p(A_s), \langle -, - \rangle_\lambda)$,

subject to the following condition:

(3.1) the coherent sheaf $\text{Lie } A \otimes_{\mathcal{O}_{F,p}} \mathcal{O}_{F,u}$ is locally free of rank 1.

Here, since S is a \mathbb{Z}_p -scheme, the action of $\mathcal{O}_{F,(p)}$ on $\text{Lie } A$ factors through the p -completion $\mathcal{O}_{F,p}$.

The morphisms

$$(A, i, \lambda, \eta) \rightarrow (A', i', \lambda', \eta')$$

of the groupoid of S -points of $\text{Sh}(K)$ are the prime-to- p quasi-isogenies of abelian schemes

$$\alpha: A \xrightarrow{\cong} A'$$

such that

$$\begin{aligned} \lambda &= r\alpha^\vee \lambda' \alpha, & r &\in \mathbb{Z}_{(p)}^\times, \\ i'(z)\alpha &= \alpha i(z), & z &\in \mathcal{O}_{F,(p)}, \\ [\eta']_K &= [\eta \circ \alpha_*]_K. \end{aligned}$$

The p -completion $\text{Sh}(K)_p^\wedge / \text{Spf}(\mathbb{Z}_p)$ is determined by the S -points of $\text{Sh}(K)$ on which p is locally nilpotent. On such schemes, the abelian surface A has a 2-dimensional, height 4 p -divisible group $A(p)$, and the composite

$$\mathcal{O}_{F,(p)} \xrightarrow{i} \text{End}(A)_{(p)} \rightarrow \text{End}(A(p))$$

factors through the p -completion

$$\mathcal{O}_{F,p} \cong \mathcal{O}_{F,u} \times \mathcal{O}_{F,\bar{u}} \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

Therefore, the action of \mathcal{O}_F naturally splits $A(p)$ into two summands, $A(u)$ and $A(\bar{u})$, both of height 2. For such schemes S , Condition (3.1) is equivalent to the condition that $A(u)$ is 1-dimensional. This forces the formal group of A to split into two 1-dimensional formal summands.

Remark 3.2. We pause to relate this to the moduli discussed in [BL]. There, the moduli at chromatic height 2 actually consisted of 4-dimensional abelian varieties with an action of an order \mathcal{O}_B in a 2-dimensional central simple algebra B over F with involution of the second kind. The moduli problem we consider here is the case where this order is $\mathcal{O}_B = M_2(\mathcal{O}_F)$, with involution being conjugate-transpose; any abelian variety A with such an action is canonically isomorphic to A_0^2 for some abelian surface A_0 with complex multiplication by \mathcal{O}_F , together with a polarization $A_0^2 \rightarrow (A_0^\vee)^2$ which is a product of two copies of the same polarization.

A theorem of Jacob Lurie [BL, Thm. 8.1.4] associates to a 1-dimensional p -divisible group \mathbb{G} over a locally noetherian separated Deligne-Mumford stack $X/\text{Spec}(\mathbb{Z}_p)$ which is locally a universal deformation of all of its mod p points, a (Jardine fibrant)

presheaf of E_∞ -ring spectra $\mathcal{E}_{\mathbb{G}}$ on the site $(X_p^\wedge)_{et}$. The construction is functorial in (X, \mathbb{G}) : given another pair (X', \mathbb{G}') , a morphism $g: X \rightarrow X'$ of Deligne-Mumford stacks over $\text{Spec}(A)$, and an isomorphism of p -divisible groups $\alpha: \mathbb{G} \xrightarrow{\cong} g^*\mathbb{G}'$, there is an induced map of presheaves of E_∞ -ring spectra

$$(g, \alpha)^*: g^* \mathcal{E}_{\mathbb{G}'} \rightarrow \mathcal{E}_{\mathbb{G}}.$$

A proof of this theorem has not yet appeared in print.

If $(\mathbf{A}, \mathbf{i}, \boldsymbol{\lambda}, [\boldsymbol{\eta}])$ is the universal tuple over $\text{Sh}(K)$, then the p -divisible group $\mathbf{A}(u)$ satisfies the hypotheses of Lurie's theorem [BL, Sec. 8.3]. The associated sheaf will be denoted

$$\mathcal{E}_{GU} := \mathcal{E}_{\mathbf{A}(u)}.$$

The E_∞ -ring spectrum of topological automorphic forms is obtained by taking the global sections:

$$\text{TAF}_{GU}(K) := \mathcal{E}_{GU}(\text{Sh}(K)_p^\wedge).$$

For the remainder of this paper, we fix $V = F^2$, with alternating form:

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \text{Tr}_{F/\mathbb{Q}} \left([x_1 \ x_2] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} \right).$$

Let $GU = GU_V$ be the associated unitary similitude group. This will be the only case we shall consider in this paper.

4. A TENSORING CONSTRUCTION

In this section, let L be the lattice $\mathcal{O}_F^2 \subset F^2 = V$, and let K_0 denote the compact open subgroup of $GU(\mathbb{A}^{p,\infty})$ given by

$$K_0 = \{g \in GU(\mathbb{A}^{p,\infty}) : g(\widehat{L}^p) = \widehat{L}^p\}$$

where $\widehat{L}^p := L \otimes \widehat{\mathbb{Z}}^p$. In this section we will show that the associated Shimura variety $\text{Sh}(K_0)$ is closely related to the moduli stack of elliptic curves.

Let $V_0 = \mathbb{Q}^2$. Let $\mathcal{M}_{ell, \mathbb{Z}_p}$ be the base-change to $\text{Spec}(\mathbb{Z}_p)$ of the moduli stack of elliptic curves. For a locally noetherian connected scheme S , and a geometric point s of S , the S -points may be taken to be the groupoid whose objects are pairs (E, η) with:

$$\begin{aligned} E, & \quad \text{an elliptic scheme over } S, \\ [\eta]_{GL_2(\widehat{\mathbb{Z}}^p)}, & \quad \text{a } \pi_1(S, s)\text{-invariant } GL_2(\widehat{\mathbb{Z}}^p)\text{-orbit of linear isomorphisms:} \\ & \quad \eta: V_0^{p,\infty} \xrightarrow{\cong} V^p(E_s). \end{aligned}$$

The morphisms

$$(E, [\eta]) \rightarrow (E', [\eta'])$$

of the groupoid of S -points of $\mathcal{M}_{ell, \mathbb{Z}_p}$ are the prime-to- p quasi-isogenies of elliptic schemes over S

$$\alpha: E \xrightarrow{\cong} E'$$

such that

$$[\eta']_{GL_2(\widehat{\mathbb{Z}}^p)} = [\eta \circ \alpha_*]_{GL_2(\widehat{\mathbb{Z}}^p)}.$$

A brief explanation of why this groupoid of elliptic schemes over S with level structure up to quasi-isogeny is equivalent to the groupoid of elliptic schemes over S up to isomorphism is given in [Beh09, Sec. 3].

Let $I \subset F$ be a fractional ideal of F . We define I^\vee to be the fractional ideal

$$I^\vee = \{z \in F : \text{Tr}_{F/\mathbb{Q}}(z\bar{w}) \in \mathbb{Z} \text{ for all } w \in I\}$$

We let

$$F^* = \text{Hom}_{\mathbb{Q}}(F, \mathbb{Q})$$

and define a lattice

$$I^* = \{\alpha \in F^* : \alpha(z) \in \mathbb{Z} \text{ for all } z \in I\}.$$

The bilinear pairing

$$\begin{aligned} F \otimes_{\mathbb{Q}} F &\rightarrow \mathbb{Q} \\ z \otimes w &\mapsto \text{Tr}_{F/\mathbb{Q}}(z\bar{w}) \end{aligned}$$

induces an isomorphism

$$\alpha_I : I^\vee \rightarrow I^*.$$

Lemma 4.1. *Let $[I] \in \text{Cl}(F)$ be an ideal class. Then there exists a representative I such that:*

- (1) we have $I_{(p)} = (I^\vee)_{(p)} = \mathcal{O}_{F,(p)} \subset F$,
- (2) we have $I \subset I^\vee$.

Proof. Note that for $a \in F^\times$, $(aI)^\vee = \bar{a}^{-1}(I^\vee)$. The lemma is easily proven using the weak approximation theorem [Mil97, Thm. 6.3]. \square

Given an elliptic scheme E/S and a fractional ideal $I \subset F$, we associate a 2-dimensional abelian scheme $E \otimes I/S$. Any chosen isomorphism $I \rightarrow \mathbb{Z}^2$ gives rise to an isomorphism $E \otimes I \rightarrow E \times E$. Such an isomorphism gives a composite map

$$\mathcal{O}_F \hookrightarrow M_2(\mathbb{Z}) \rightarrow \text{End}(E \times E) \xleftarrow{\sim} \text{End}(E \otimes I)$$

that is independent of the choice of isomorphism. This construction is natural in the elliptic curve and \mathcal{O}_F -modules I . In particular, the action of \mathcal{O}_F on I gives $E \otimes I$ a canonical complex multiplication

$$i_I : \mathcal{O}_F \rightarrow \text{End}(E \otimes I).$$

We have

$$\text{Lie}(E \otimes I) \otimes_{\mathcal{O}_{F,p}} \mathcal{O}_{F,u} \cong \text{Lie } E \otimes_{\mathbb{Z}_p} I_u.$$

In particular, Condition 3.1 is satisfied.

Remark 4.2. The authors learned from the referee that the construction $E \otimes I$ has been around for some time. See, for example, [Mil72, Sec. 2].

The elliptic curve E comes equipped with a canonical principal polarization $\lambda: E \rightarrow E^\vee$. If I satisfies Lemma 4.1(1)-(2), then one can define an induced prime-to- p polarization:

$$\lambda_I: E \otimes I \rightarrow E \otimes I^\vee \xrightarrow{\lambda \otimes (\alpha_I)_*} E^\vee \otimes I^* \cong (E \otimes I)^\vee.$$

Note that this polarization is never principal, since the non-triviality of the different ideal of F implies $I^\vee \neq I$. Replacing the ideal I with the ideal aI for any non-zero $a \in \mathcal{O}_F$ gives an isogenous abelian variety with polarization rescaled by the positive integer $N_{F/\mathbb{Q}}(a)$. Hence the isogeny class of weakly polarized abelian surface $E \otimes I$ associated to I depends only on the ideal class represented by I .

Let $\langle -, - \rangle_0$ be the alternating form on $V_0 = \mathbb{Q}^2$ given by

$$(4.3) \quad \langle (x_1, x_2), (y_1, y_2) \rangle_0 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let ι_0 be the induced involution on $M_2(\mathbb{Q}) = \text{End}(V_0)$, defined by

$$\langle \alpha v, w \rangle_0 = \langle v, \alpha^{\iota_0} w \rangle_0.$$

Explicitly, we have

$$(4.4) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\iota_0} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Let $\langle -, - \rangle_0^F$ be the induced \mathbb{Q} -valued alternating hermitian form on $V_0 \otimes F$ determined by

$$\langle x \otimes z, y \otimes w \rangle_0^F = \langle x, y \rangle_0 \cdot \text{Tr}_{F/\mathbb{Q}}(z\bar{w}).$$

There is a canonical isometry

$$\omega: (V_0 \otimes F, \langle -, - \rangle_0^F) \xrightarrow{\cong} (V, \langle -, - \rangle)$$

given by

$$\omega((x_1, x_2) \otimes z) = (x_1 z, x_2 z).$$

For any \mathbb{Q} -algebra R , and any $\alpha \in GL_2(R)$, we have

$$\alpha^{\iota_0} \alpha = \det \alpha \in R^\times.$$

Therefore, any level structure

$$\eta: V_0 \otimes \mathbb{A}^{p,\infty} \xrightarrow{\cong} V^p(E)$$

is a similitude between $\langle -, - \rangle_0$ and $\langle -, - \rangle_E$, where the latter is the Weil pairing on $V^p(E_s)$. The inclusion $I \hookrightarrow F$ induces an isomorphism

$$(V^p(E_s \otimes I), \langle -, - \rangle_{\lambda_I}) \cong (V^p(E_s) \otimes F, \langle -, - \rangle_E^F).$$

The abelian variety $E_s \otimes I$ admits an induced level structure:

$$\eta_I: V \otimes \mathbb{A}^{p,\infty} \cong V_0 \otimes \mathbb{A}^{p,\infty} \otimes F \xrightarrow{\eta \otimes 1} V^p(E_s) \otimes F \cong V^p(E_s \otimes I).$$

Clearly, the K_0 -orbit $[\eta_I]_{K_0}$ depends only on the $GL_2(\widehat{\mathbb{Z}}^p)$ -orbit $[\eta]_{GL_2(\widehat{\mathbb{Z}}^p)}$.

Fix a fractional ideal $I \subset F$ satisfying Lemma 4.1(1)-(2). We define a morphism of stacks

$$\Phi_I: \mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \text{Sh}(K_0)$$

by

$$\Phi_I(E, [\eta]) = (E \otimes I, i_I, \lambda_I, [\eta_I]).$$

Choosing representatives of each element of $\text{Cl}(F)$ satisfying Lemma 4.1(1)-(2), we get a morphism of stacks

$$\coprod_{[I]} \Phi_I : \prod_{\text{Cl}(F)} \mathcal{M}_{\text{ell}, \mathbb{Z}_p} \rightarrow \text{Sh}(K_0).$$

Lemma 4.5. *Suppose S is connected, let $(E, [\eta])$ be an S -object of $\mathcal{M}_{\text{ell}, \mathbb{Z}_p}$, and let I be a fractional ideal satisfying Lemma 4.1(1)-(2). Then there is an isomorphism*

$$\text{Aut}_{\text{Sh}(K_0)}(\Phi_I(E, [\eta])) \cong \text{Aut}_{\mathcal{M}_{\text{ell}, p}}(E, [\eta]) \times_{\{\pm 1\}} \mathcal{O}_F^\times.$$

Remark 4.6. Since F is a quadratic imaginary extension of \mathbb{Q} , the group \mathcal{O}_F^\times is cyclic of order 4 if $F = \mathbb{Q}(i)$, cyclic of order 6 if $F = \mathbb{Q}(\omega)$, and isomorphic to $\{\pm 1\}$ otherwise. Thus, except for two exceptions, Φ_I preserves automorphism groups. Note that both $\mathbb{Q}(i)$ and $\mathbb{Q}(\omega)$ have class number 1.

Proof of Lemma 4.5. Let

$$\text{End}^0(-) = \text{End}(-) \otimes \mathbb{Q}$$

denote the ring of quasi-endomorphisms. Note that

$$\text{End}_{\mathcal{O}_F}^0(E \otimes I) = \text{End}^0(E) \otimes F.$$

The λ_I -Rosati involution \dagger_I on $\text{End}_{\mathcal{O}_F}^0(E \otimes I)$ under this identification is given by

$$(\alpha \otimes z)^{\dagger_I} = \alpha^\vee \otimes \bar{z}$$

where α^\vee is the dual isogeny (the image of α under the Rosati involution on $\text{End}^0(E)$) corresponding to the unique weak polarization on E). An F -linear quasi-isogeny

$$f : E \otimes I \rightarrow E \otimes I$$

preserves the weak polarization λ_I if and only if $f^{\dagger_I} f \in \mathbb{Q}^\times$. If we write a general element

$$f = \alpha \otimes 1 + \beta \otimes \delta,$$

where $\alpha, \beta \in \text{End}^0(E)$, we either have $\alpha = 0$, or $\beta = 0$, or both α and β are non-zero. Assume we are in the last case. We find the following:

$$\begin{aligned} f^{\dagger_I} f &= (\alpha^\vee \otimes 1 - \beta^\vee \otimes \delta)(\alpha \otimes 1 + \beta \otimes \delta) \\ &= (\alpha^\vee \alpha + N\beta^\vee \beta) \otimes 1 + (\alpha^\vee \beta - \beta^\vee \alpha) \otimes \delta \end{aligned}$$

Hence the requirement is that $\alpha^\vee \beta$, and hence $\alpha\beta^{-1}$ or $\beta\alpha^{-1}$, are self-dual isogenies. The only such endomorphisms of an elliptic curve are those which are locally scalar multiplication, and since S is connected we find $r\alpha = s\beta$ for some integers r, s . This forces the element to be of the form $\phi \otimes z$ for some $z \in F$, and $\phi \in \text{End}^0(E)$. Therefore, the group of \mathcal{O}_F -linear quasi-isogenies $E \rightarrow E$ which preserve the weak polarization is the group of elements

$$\alpha \otimes z \in \text{End}^0(E) \otimes F$$

with $\alpha \in \text{End}^0(E)^\times$ and $z \in F^\times$.

A quasi-isogeny of elliptic curves

$$\alpha : E \rightarrow E$$

is prime-to- p if and only if the associated quasi-isogeny of p -divisible groups

$$\alpha_* : E(p) \rightarrow E(p)$$

is an isomorphism. An \mathcal{O}_F -linear quasi-isogeny

$$\beta : E \otimes I \rightarrow E \otimes I$$

is prime-to- p if and only if the associated quasi-isogeny of p -divisible groups

$$\beta_* : (E \otimes I)(p) \rightarrow (E \otimes I)(p)$$

is an isomorphism. Since there is a canonical isomorphism $(E \otimes I)(p) \cong E(p) \otimes I$, we deduce that a quasi-isogeny $\alpha \otimes z$ of $E \otimes I$ is prime-to- p , for $\alpha \in \text{End}^0(E)$ and $z \in F$, if and only if α is prime to p and $z \in \mathcal{O}_{F,(p)}^\times$. (Here, $E(p) \otimes I$ denote the analogous tensoring construction applied to the p -divisible group $E(p)$.)

Finally, suppose that $\alpha \otimes z$ preserves the level structure $[\eta_I]_{K_0}$. This happens if and only if there exists a $g \in K_0$ such that the following diagram commutes.

$$\begin{array}{ccc} V_0^p \otimes F & \xrightarrow{\eta \otimes 1} & V^p(E_s) \otimes F \\ g \downarrow & & \downarrow \alpha_* \otimes z \\ V_0^p \otimes F & \xrightarrow{\eta \otimes 1} & V^p(E_s) \otimes F \end{array}$$

This will happen if and only if

$$\eta^{-1} \alpha_* \eta \otimes z \in K_0$$

which in turn happens if and only if for the lattice

$$L_0 = \mathbb{Z}^2 \subset \mathbb{Q}^2 = V_0$$

we have

$$\begin{aligned} \alpha_*(\eta(\widehat{L}_0^p)) &= \eta(\widehat{L}_0^p), \\ z &\in \widehat{\mathcal{O}}_F^p. \end{aligned}$$

The first condition is equivalent to asserting that the quasi-isogeny α preserves the level structure $[\eta]_{GL_2(\widehat{\mathbb{Z}}^p)}$.

Putting this all together, we conclude that $\alpha \otimes z$ represents an automorphism of $\Phi_I(E, [\eta])$ in $\text{Sh}(K_0)$ if and only if α is an automorphism of $(E, [\eta])$ in $\mathcal{M}_{ell, \mathbb{Z}_p}$ and $z \in \mathcal{O}_F^\times$. The lemma follows from the fact that

$$(-\alpha) \otimes z = \alpha \otimes (-z).$$

□

Lemma 4.7. *Suppose that I and I' are two ideals satisfying Lemma 4.1(1)-(2), let $(E, [\eta])$ and $(E', [\eta'])$ be objects of $\mathcal{M}_{ell, \mathbb{Z}_p}(S)$. Then $\Phi_I(E)$ is isomorphic to $\Phi_{I'}(E')$ in $\text{Sh}(K_0)(S)$ if and only if $(E, [\eta])$ is isomorphic to $(E', [\eta'])$ in $\mathcal{M}_{ell, \mathbb{Z}_p}(S)$ and $[I] = [I'] \in \text{Cl}(F)$.*

Proof. Clearly if $(E, [\eta])$ is isomorphic to $(E', [\eta'])$ then $\Phi_I(E, [\eta])$ is isomorphic to $\Phi_I(E', [\eta'])$. Moreover, if $[I'] = [I]$, then there is an $a \in F^\times$ such that $I' = aI$. Since $I_{(p)} = I'_{(p)}$, we deduce that $a \in \mathcal{O}_{F,(p)}$. The mapping

$$1 \otimes a : E \otimes I \rightarrow E \otimes I'$$

is a prime-to- p , \mathcal{O}_F -linear, quasi-isogeny that preserves the weak polarization and level structure.

Conversely, suppose that

$$f : E \otimes I \rightarrow E' \otimes I'$$

gives an isomorphism between $\Phi_I(E)$ and $\Phi_{I'}(E')$. There is an isomorphism

$$(4.8) \quad \mathrm{Hom}_{\mathcal{O}_F}^0(E \otimes I, E' \otimes I') \cong \mathrm{Hom}^0(E, E') \otimes F.$$

The pair of polarizations $\lambda_I, \lambda_{I'}$ induces a homomorphism

$$\dagger_{I,I'} : \mathrm{Hom}_{\mathcal{O}_F}^0(E \otimes I, E' \otimes I') \rightarrow \mathrm{Hom}_{\mathcal{O}_F}^0(E' \otimes I', E \otimes I)$$

such that an \mathcal{O}_F -linear quasi-isogeny g preserves the weak polarization if and only if

$$g^{\dagger_{I,I'}} \circ g \in \mathbb{Q}^\times.$$

Under the isomorphism (4.8), we have

$$(\beta \otimes w)^{\dagger_{I,I'}} = \beta^\vee \otimes \bar{w}.$$

Thus similar arguments as given in the proof of Lemma 4.5 imply that there exists a prime-to- p isogeny

$$\alpha : E \rightarrow E'$$

and $z \in \mathcal{O}_{F,(p)}^\times$ so that $f = \alpha \otimes z$. Since f preserves level structures, we deduce that:

$$\begin{aligned} [\alpha_*\eta] &= [\eta'], \\ zI_\ell &= I'_\ell, \quad \text{for all } \ell \neq p. \end{aligned}$$

Since $I_p = I'_p$, we conclude that $I' = zI$. □

The following theorem gives a complete description of $\mathrm{Sh}(K_0)$ in terms of the moduli stack of elliptic curves.

Theorem 4.9.

- (1) If $F = \mathbb{Q}(i)$, the map $\mathcal{M}_{\mathrm{ell}, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$ is a degree 2 Galois cover of Deligne-Mumford stacks.
- (2) If $F = \mathbb{Q}(\omega)$, the map $\mathcal{M}_{\mathrm{ell}, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$ is a degree 3 Galois cover of Deligne-Mumford stacks.
- (3) In all other cases, the map $\coprod_{\mathrm{Cl}(F)} \mathcal{M}_{\mathrm{ell}, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$ is an equivalence.

We remark that in the first two cases the Galois group acts trivially on the underlying coarse moduli object, but acts via nontrivial automorphisms on the points of the stack. A generic point of the stack $\mathcal{M}_{\mathrm{ell}, \mathbb{Z}_p}$ has automorphism group $\mathbb{Z}/2 \cong \mathbb{Z}^\times$, whereas a generic point of $\mathrm{Sh}(K_0)$ associated to the quadratic imaginary field F has automorphism group \mathcal{O}_F^\times .

The remainder of this section will be devoted to proving Theorem 4.9. We will first establish the following weaker version of Theorem 4.9.

Lemma 4.10.

- (1) *If $F = \mathbb{Q}(i)$, the map $\mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$ is a degree 2 Galois cover of a connected component.*
- (2) *If $F = \mathbb{Q}(\omega)$, the map $\mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$ is a degree 3 Galois cover of a connected component.*
- (3) *In all other cases, the map $\coprod_{\mathrm{Cl}(F)} \mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$ is an inclusion of a set of connected components.*

Proof. We know that these maps are étale, as both moduli are locally universal deformations of the associated p -divisible groups. Given Lemmas 4.5 and 4.7, to finish the justification of this statement, we must prove that these maps are proper.

Suppose that we are given a discrete valuation ring R over $(\mathcal{O}_F)_{(p)}$ with fraction field K , an elliptic curve E over K , and an extension of $E \otimes I$ to a polarized abelian variety A over R with \mathcal{O}_F -action serving as an R -point of the moduli. Then A is a Néron model for $E \otimes I$, and in particular the universal mapping property allows the direct product decomposition $E \otimes I \cong E \times E$ to extend uniquely to A , together with the \mathcal{O}_F -action. The same holds true for A^\vee and the polarization. \square

We are left with showing that the map

$$\Phi : \coprod_{\mathrm{Cl}(F)} \mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$$

is surjective on π_0 . Assume this is not true. Then there is a connected component Y of $\mathrm{Sh}(K_0)$ that is disjoint from the image of Φ . Since $\mathrm{Sh}(K_0)$ possesses an étale cover by a quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}_p)$, it follows that Y must have an \mathbb{F}_p -point y_0 . By Serre-Tate theory, there exists a lift of this point to a \mathbb{Q}_p^{nr} -point y . Choosing an isomorphism $\mathbb{Q}_p \cong \mathbb{C}$, we see that y corresponds to a \mathbb{C} -point of $\mathrm{Sh}(K_0)$ which is not in the image of Φ . To arrive at a contradiction, and hence prove Theorem 4.9, it suffices to demonstrate that Φ is surjective on \mathbb{C} -points.

Lemma 4.10 implies that Φ surjects onto a set of connected components of $\mathrm{Sh}(K_0)_{\mathbb{C}}$. Therefore we simply must prove that the induced map

$$(4.11) \quad \Phi_* : \pi_0 \left(\coprod_{\mathrm{Cl}(F)} \mathcal{M}_{ell, \mathbb{C}} \right) \rightarrow \pi_0(\mathrm{Sh}(K_0)_{\mathbb{C}})$$

is an isomorphism. Since $\mathcal{M}_{ell, \mathbb{C}}$ is connected, the left-hand-side of (4.11) is isomorphic to

$$\mathrm{Cl}(F) = F^\times \setminus (\mathbb{A}_F^\infty)^\times / \widehat{\mathcal{O}}_F^\times$$

whereas Theorem 9.3.5 and Remark 9.3.6 of [BL] shows that the right-hand-side is isomorphic to

$$GU(\mathbb{Q}) \setminus GU(\mathbb{A}^\infty) / K_0.$$

The map

$$(4.12) \quad \pi_0 \Phi : F^\times \setminus (\mathbb{A}_F^\infty)^\times / \widehat{\mathcal{O}}_F^\times \rightarrow GU(\mathbb{Q}) \setminus GU(\mathbb{A}^\infty) / K_0$$

induced by Φ under these isomorphisms is the map of adelic quotients induced by the inclusion of the center $\text{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ of GU . This can be seen as follows: given a fractional ideal $I \subset F$, picking a generator $z_{\mathfrak{p}}$ of $I_{\mathfrak{p}}$ for each prime \mathfrak{p} of F , we get an element $(a_{\mathfrak{p}}) \in \mathbb{A}_F^{\times}$. Given a complex elliptic curve $C = \mathbb{C}/\Lambda$ associated to a lattice Λ with basis (e_1, e_2) , the abelian variety $\Phi(C)$ is given by $\Phi(C) = \mathbb{C}^2/\Lambda'$ where Λ' has basis $(a_{\mathfrak{p}}e_1, a_{\mathfrak{p}}e_2)$.

Lemma 4.13. *The map $\pi_0\Phi$ of (4.12) is an isomorphism.*

Proof. The map is easily seen to be a monomorphism. Thus it suffices to show that

$$h(GU) := |GU(\mathbb{Q}) \backslash GU(\mathbb{A}^{\infty})/K_0| = h(F)$$

where $h(F)$ is the class number of F . Shimura [Shi64, Thm. 5.24(ii)] computed the class numbers of indefinite unitary similitude groups. In our particular case his formula gives

$$h(GU) = h(\mathbb{Q})[\mathfrak{C} : \mathfrak{C}_0][\mathfrak{E}, f(\mathcal{O}_F^{\times})].$$

Here $h(\mathbb{Q}) = 1$ is the class number of \mathbb{Q} . The group \mathfrak{C} is the class group of F , and \mathfrak{C}_0 denotes the subgroup generated by ideals which are invariant under conjugation. However, in [Shi64, Thm. 5.24(i)], Shimura argues that $[\mathfrak{C} : \mathfrak{C}_0]$ is the class number for a unitary group associated to a hermitian form on an odd dimensional F -vector space. Specializing this result to the 1-dimensional case, we deduce

$$[\mathfrak{C} : \mathfrak{C}_0] = h(T)$$

where $h(T)$ is the class number of the torus

$$T = \ker(N_{F/\mathbb{Q}} : \text{Res}_{F/\mathbb{Q}}\mathbb{G}_m \rightarrow \mathbb{G}_m).$$

Just prior to [Shi64, Thm. 5.24], the group \mathfrak{E} is defined, and shown to be isomorphic to the product $(\mathbb{Z}/2)^v$, where v denotes the number of primes ℓ which ramify in F for which the corresponding local hermitian form has non-trivial anisotropic subspace. Since our hermitian form is actually isotropic, the the number v is equal to u , the number of primes which ramify in F . The map f maps \mathcal{O}_F^{\times} into \mathfrak{E} with kernel $(\mathcal{O}_F^{\times})^2$. We therefore deduce that

$$[\mathfrak{E} : f(\mathcal{O}_F^{\times})] = 2.$$

The class number $h(T)$ is given by

$$h(T) = h(F)/2^{u-1},$$

(see, for instance, p.375, Equation (16) of [Shy77]) and this give the desired result. \square

5. COMPUTATION OF $\text{TAF}_{GU}(K_0)$

In this section we continue to take $V, \langle -, - \rangle, GU$, and K_0 as in Section 4. In this section we give a complete description of the spectrum TAF_{GU} for this choice of initial data.

Let $(\mathbf{E}, [\eta])$ be the universal elliptic curve over $\mathcal{M}_{ell, \mathbb{Z}_p}$. Let

$$\mathcal{E}_{GL_2} = \mathcal{E}_{\mathbf{E}(p)}$$

be the sheaf of E_∞ -ring spectra associated to the p -divisible group $\mathbf{E}(p)$. The global sections give the p -completion of the spectrum of topological modular forms:

$$\mathrm{TMF}_p = \mathcal{E}_{GL_2}((\mathcal{M}_{ell})_p^\wedge).$$

The pullback of the p -divisible group $\mathbf{A}(u)$ associated to the universal abelian scheme over $\mathrm{Sh}(K_0)$ under the map

$$\Phi : \prod_{\mathrm{Cl}(F)} \mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \mathrm{Sh}(K_0)$$

is given by

$$\Phi^* \mathbf{A}(u) \cong \mathbf{E}(p).$$

We deduce that there is an isomorphism of presheaves

$$(5.1) \quad \nabla^* \mathcal{E}_{GL_2} \cong \Phi^* \mathcal{E}_{GU}$$

where

$$\nabla : \prod_{\mathrm{Cl}(F)} \mathcal{M}_{ell, \mathbb{Z}_p} \rightarrow \mathcal{M}_{ell, \mathbb{Z}_p}$$

is the codiagonal.

Theorem 5.2. *We have the following equivalences of E_∞ -ring spectra.*

(1) *If $F = \mathbb{Q}(i)$, there is an equivalence*

$$\mathrm{TAF}_{GU}(K_0) \simeq \mathrm{TMF}_p^{hC_2}.$$

(2) *If $F = \mathbb{Q}(\omega)$, there is an equivalence*

$$\mathrm{TAF}_{GU}(K_0) \simeq \mathrm{TMF}_p^{hC_3}.$$

(3) *In all other cases, we have*

$$\mathrm{TAF}_{GU}(K_0) \cong \prod_{\mathrm{Cl}(F)} \mathrm{TMF}_p.$$

Proof. (3) follows immediately from applying the global sections functor to Equation 5.1. (1) and (2) are established by noting that since the presheaf \mathcal{E}_{GU} is Jardine fibrant, it satisfies descent with respect to the Galois cover Φ . \square

We pause to give a precise description of the group actions of Theorem 5.2 on the spectrum TMF_p . In [Beh06, 1.2.1], the first author described certain operations

$$\psi_{[k]}^* : \mathrm{TMF}_p \rightarrow \mathrm{TMF}_p$$

for k coprime to p which are analogs of the Adams operations on K -theory. These operations give an action of $\mathbb{Z}_{(p)}^\times$ on TMF_p . The functoriality of the sheaf $\mathcal{E}_{\mathbb{G}}$ with respect to the p -divisible group \mathbb{G} implies that the central action of \mathbb{Z}_p^\times on the p -divisible group \mathbb{G} by isomorphisms induces an extension of the $\mathbb{Z}_{(p)}^\times$ -action on TMF_p to \mathbb{Z}_p^\times . The action factors through $\mathbb{Z}_p^\times / \{\pm 1\}$, since for any elliptic curve E , the isogeny

$$[-1] : E \rightarrow E$$

is an isomorphism. Since p is assumed to split in F , there is an inclusion

$$\mathcal{O}_F^\times \hookrightarrow \mathcal{O}_{F,u}^\times \cong \mathbb{Z}_p^\times$$

and hence an action of $\mathcal{O}_F^\times / \{\pm 1\}$ on TMF_p .

Corollary 5.3. *The homotopy groups of the spectra of topological automorphic forms $\mathrm{TAF}_{GU}(K_0)$ are computed as follows.*

(1) *If $F = \mathbb{Q}(i)$, there is an isomorphism*

$$\mathrm{TAF}_{GU}(K_0)_* \cong \mathbb{Z}_p[c_4, c_6^2, \Delta^{-1}]_p^\wedge \subset \mathbb{Z}_p[c_4, c_6, \Delta^{-1}]_p^\wedge.$$

(2) *If $F = \mathbb{Q}(\omega)$, there is an isomorphism*

$$\mathrm{TAF}_{GU}(K_0)_* \cong \mathbb{Z}_p[c_4^3, c_6, \Delta^{-1}]_p^\wedge \subset \mathbb{Z}_p[c_4, c_6, \Delta^{-1}]_p^\wedge.$$

(3) *In all other cases, there is an isomorphism*

$$\mathrm{TAF}_{GU}(K_0)_* \cong \prod_{\mathrm{Cl}(F)} \pi_* \mathrm{TMF}_p.$$

Here, c_4 , c_6 , and Δ are the standard integral modular forms.

Proof. Case (3) follows immediately from Theorem 5.2. Suppose we are in cases (1) or (2), and that G is the Galois group of the cover Φ , so that G is either C_2 or C_3 , respectively. Note that since p is assumed to split in F , cases (1) and (2) of Theorem 5.2 only occur when the prime p does not divide 6. Therefore, the associated homotopy fixed point spectral sequence

$$H^s(G, \pi_t \mathrm{TMF}_p) \Rightarrow \pi_{t-s} \mathrm{TAF}_{GU}(K_0)$$

collapses to the 0-line. Since p does not divide 6, we have

$$\pi_* \mathrm{TMF}_p \cong \mathbb{Z}_p[c_4, c_6, \Delta^{-1}]_p^\wedge.$$

The proof of the corollary is completed by identifying the action of the group G . The p -divisible summand $A(u)$ is $E(p) \otimes_{\mathbb{Z}_p} I_u^\wedge$, with \mathcal{O}_F acting via its image in $(\mathcal{O}_F)_u^\wedge = \mathbb{Z}_p$. Therefore, the roots of unity in \mathcal{O}_F^\times act on the p -divisible group (and hence on invariant 1-forms in the formal part) by multiplication by roots of unity in \mathbb{Z}_p^\times . As a result, the action on forms of weight k is through the k 'th power map. Thus, in case (1), i acts trivially on c_4 and by negation on c_6 ; in case (2), ω acts trivially on c_6 and by multiplication by ω on c_4 . \square

Remark 5.4. The existence of these summands of TMF can be derived directly. The previously described action of \mathbb{Z}_p^\times on the p -divisible group of $\mathcal{M}_{ell, \mathbb{Z}_p}$ gives rise to an action by a group of $(p-1)$ 'st roots of unity $\mu_{p-1} \subset \mathbb{Z}_p^\times$ via scalar multiplication, and hence this group acts on the associated spectrum TMF_p . The invariants under this action form a summand analogous to the Adams summand, generalizing the summands for $\mathbb{Q}(i)$ and $\mathbb{Q}(\omega)$.

6. A QUOTIENT CONSTRUCTION

Let $\mathfrak{d} \subset F$ be the different of F , let L' be the \mathcal{O}_F -lattice

$$L' = \mathcal{O}_F \oplus \mathfrak{d}^{-1} \subset F^2 = V.$$

Let K_1 denote the compact open subgroup of $GU(\mathbb{A}^{p, \infty})$ given by

$$K_1 = \{g \in GU(\mathbb{A}^{p, \infty}) : g(\widehat{L}'^p) = \widehat{L}'^p\}.$$

The significance of the lattice L' is that, unlike the lattice L of Section 4, the lattice L' is self-dual with respect to $\langle -, - \rangle$, in the sense that we have

$$L' = \{x \in V : \langle x, L' \rangle \subseteq \mathbb{Z}\}.$$

This implies that the associated Shimura stack $\mathrm{Sh}(K_1)$ admits a moduli interpretation where all of the points are represented by principally polarized abelian schemes with complex multiplication by F (see Remark 6.1). This should be contrasted with the moduli interpretation of $\mathrm{Sh}(K_0)$ developed in Section 4, where none of the polarized abelian schemes $(E \otimes I, \lambda_I)$ were principally polarized.

In this section we will identify one connected component of the associated Shimura variety $\mathrm{Sh}(K_1)$ with the p -completion of a quotient of the moduli stack of elliptic curves with $\Gamma_0(N)$ -structure. Here, as always in this paper, $F = \mathbb{Q}(-N)$, where N is a positive square-free integer relatively prime to p .

We recall that any elliptic curve E has a canonical principal polarization λ , and each isogeny f of elliptic curves has a dual $f^\vee = \lambda^{-1} f^\vee \lambda$. The composite $f^\vee f$ is multiplication by the degree of the isogeny.

In [BL, Sec. 6.4], the authors observed that the moduli interpretation of $\mathrm{Sh}(K_1)$ as a moduli stack of polarized F -linear abelian schemes up to *isogeny* with level structure could be replaced with a moduli interpretation as a moduli stack of polarized abelian schemes up to *isomorphism* without level structure.

Specifically, for a locally noetherian connected \mathbb{Z}_p -scheme S , the S -points of $\mathrm{Sh}(K_1)$ is the groupoid whose objects are tuples (A, i, λ) , with:

$$\begin{array}{ll} A, & \text{an abelian scheme over } S \text{ of dimension 2,} \\ \lambda: A \rightarrow A^\vee, & \text{a } \mathbb{Z}_{(p)}\text{-polarization,} \\ i: \mathcal{O}_{F,(p)} \hookrightarrow \mathrm{End}(A)_{(p)}, & \text{an inclusion of rings, such that the } \lambda\text{-Rosati involu-} \\ & \text{tion is compatible with conjugation.} \end{array}$$

subject to the following two conditions:

- (1) the coherent sheaf of \mathcal{O}_S -modules $\mathrm{Lie} A \otimes_{\mathcal{O}_{F,p}} \mathcal{O}_{F,u}$ is locally free of rank 1,
- (2) for a geometric point s of S , there exists a $\pi_1(S, s)$ -invariant \mathcal{O}_F -linear similitude:

$$\eta: (\widehat{L}^p, \langle -, - \rangle) \xrightarrow{\cong} (T^p(A_s), \langle -, - \rangle_\lambda).$$

(Here, $T^p(A_s)$ is the Tate module of A_s away from p .)

The morphisms

$$(A, i, \lambda) \rightarrow (A', i', \lambda')$$

of the groupoid of S -points of $\mathrm{Sh}(K_1)$ are isomorphisms of abelian schemes

$$\alpha: A \xrightarrow{\cong} A'$$

such that

$$\begin{aligned} \lambda &= r \alpha^\vee \lambda' \alpha, & r &\in \mathbb{Z}_{(p)}^\times, \\ i'(z) \alpha &= \alpha i(z), & z &\in \mathcal{O}_{F,(p)}. \end{aligned}$$

Remark 6.1. Observe that the tuple (A, i, λ) only depends on the weak polarization class of λ . There is a unique similitude class of $\widehat{\mathcal{O}}_F^p$ -lattice in $V^{p,\infty}$ which is self-dual. Therefore, Condition (2) above is *equivalent* to the condition that the weak polarization class of λ contains a representative which is principal. We may and will restrict ourselves to principal polarizations in this section.

Let $\mathcal{M}_0(N)$ denote the moduli stack (over $\mathbb{Z}[1/N]$) whose S -points are the groupoid whose objects are pairs (E, H) where E is a (nonsingular) elliptic scheme over S , and $H \leq E$ is a $\Gamma_0(N)$ -structure, i.e. a cyclic subgroup of order N . The morphisms of the groupoid of S -points consist of isomorphisms of elliptic curves which preserve the level structure. Note that we have $\mathcal{M}_{ell} = \mathcal{M}_0(1)$. We may interpret the S -points of this moduli as being isogenies $q: E \rightarrow \bar{E}$ of elliptic curves whose kernel is cyclic of order N .

We will construct a morphism

$$\begin{aligned} \Phi' : \mathcal{M}_0(N)_{\mathbb{Z}_p} &\rightarrow \mathrm{Sh}(K_1), \\ (E, H) &\mapsto (\Phi'(E), i_E, \lambda_E). \end{aligned}$$

We break the construction down into two cases.

Case I: $-N \equiv 2, 3 \pmod{4}$. In this case, the ring of integers is given by

$$\mathcal{O}_F \cong \mathbb{Z}[x]/(x^2 + N).$$

We define our abelian scheme to be

$$\Phi(E) := E \times \bar{E},$$

with polarization the component-wise principal polarization

$$\lambda_E = \lambda \times \lambda.$$

We define a complex multiplication

$$i_E : \mathcal{O}_F \rightarrow \mathrm{End}(E \times \bar{E})$$

by the map

$$x \mapsto \tau \circ (q, -q^\vee).$$

Here, τ is the twist map $\bar{E} \times E \rightarrow E \times \bar{E}$. The dual of this element is

$$x^\vee = (q^\vee, -q) \circ \tau = \tau \circ (-q, q^\vee) = -x,$$

so the Rosati involution induces complex conjugation on \mathcal{O}_F .

As p splits in F , let $a \in \mathbb{Z}_p^\times$ be the image of x corresponding to the prime u , satisfying $a^2 + N = 0$. The canonical rank 1 summand of the coherent sheaf $\mathrm{Lie}(E \times \bar{E})$ is the image of $\mathrm{Lie} E$ under the map

$$\left(1 \times \frac{q}{a}\right) \circ \Delta : \mathrm{Lie} E \rightarrow \mathrm{Lie}(E \times \bar{E}).$$

Case II: $-N \equiv 1 \pmod{4}$. In this case, the ring of integers \mathcal{O}_F is $\mathbb{Z}[y]/(y^2 + y + \frac{N+1}{4})$. We would like to define our abelian variety as in Case I; however, this definition would not allow an action of the full ring of integers \mathcal{O}_F .

Instead, we take

$$\Phi'(E) = \frac{E \times \bar{E}}{E[2]}$$

where we have taken quotients by the image of the composite

$$E[2] \xrightarrow{\Delta} E[2] \times E[2] \hookrightarrow E \times E \xrightarrow{1 \times q} E \times \bar{E}.$$

(Note that in this case, the kernel of q has order prime to 2.) The polarization

$$2\lambda \times 2\lambda : E \times \bar{E} \rightarrow E^\vee \times \bar{E}^\vee$$

descends to the quotient to give a principal polarization

$$\lambda_E : \Phi'(E) \rightarrow \Phi'(E)^\vee.$$

The action of the order

$$\mathbb{Z} + \mathbb{Z}(2y) \subset F$$

on $E \times \bar{E}$ given in Case I descends to the quotient to give complex multiplication

$$i_E : \mathcal{O}_F \rightarrow \text{End}(\Phi'(E))$$

over the resulting quotient.

We note that the endomorphism

$$\begin{bmatrix} 2 & 0 \\ -q & 1 \end{bmatrix} \in \text{End}(E \times \bar{E})$$

factors through the quotient $\Phi'(E)$ to give an isomorphism

$$\Phi'(E) \xrightarrow{\cong} E \times \bar{E}.$$

We will use this to display explicit formulas.

On $E \times \bar{E}$, the induced polarization is defined by

$$\lambda_E = (\lambda, \lambda) \circ A = (\lambda, \lambda) \circ \begin{bmatrix} \frac{1+N}{2} & q^\vee \\ q & 2 \end{bmatrix} : E \times \bar{E} \rightarrow E^\vee \times \bar{E}^\vee$$

The matrix A is symmetric with respect to transpose dual \dagger , and is positive definite, as required to define a polarization.

We define complex multiplication

$$i_E : \mathcal{O}_F \rightarrow \text{End}(E \times \bar{E})$$

by

$$y \mapsto \begin{bmatrix} \frac{-N-1}{2} & q^\vee \\ \frac{N+1}{4}q & \frac{N-1}{2} \end{bmatrix}.$$

This endomorphism satisfies the equation $y^2 + y + \frac{N+1}{4}$ and conjugate-commutes with the polarization. This last follows from the identity $y^\dagger A = A(-1 - y)$, or

$$\begin{bmatrix} \frac{-N-1}{2} & \frac{N+1}{4}q^\vee \\ q & \frac{N-1}{2} \end{bmatrix} \begin{bmatrix} \frac{1+N}{2} & q^\vee \\ q & 2 \end{bmatrix} = \begin{bmatrix} \frac{1+N}{2} & q^\vee \\ q & 2 \end{bmatrix} \begin{bmatrix} \frac{N-1}{2} & -q^\vee \\ \frac{-N-1}{4}q & \frac{-N-1}{2} \end{bmatrix}.$$

As in the previous case, one can check that the summand of $\text{Lie } \Phi'(E)$ is canonically isomorphic to the rank 1 summand of $\text{Lie}(E \times \bar{E})$.

Isomorphisms of objects. We now consider when two such abelian varieties $(E \times \bar{E})$ and $(E' \times \bar{E}')$ can become isomorphic in the moduli. This proceeds in a way similar to Section 4.

We note that after rationalizing Hom-sets, both cases become isomorphic to the abelian variety $E \times \bar{E}$ as in Case I. The rationalized set of maps $E \times \bar{E} \rightarrow E' \times \bar{E}'$ is the set of matrices

$$\left\{ \begin{bmatrix} \alpha & \beta q^\vee \\ q\gamma & q\delta q^\vee \end{bmatrix} \mid \alpha, \beta, \gamma, \delta \in \text{Hom}(E, E') \otimes \mathbb{Q} \right\}.$$

The set of such elements that commute with $\begin{bmatrix} 0 & -q^\vee \\ q & 0 \end{bmatrix}$ is

$$\left\{ \begin{bmatrix} \alpha & \beta q^\vee \\ -q\beta & \frac{1}{N} q\alpha q^\vee \end{bmatrix} \mid \alpha, \beta \in \text{Hom}(E, E') \otimes \mathbb{Q} \right\}.$$

The set of such elements f that preserve the polarization, i.e. such that $f^\dagger f$ is scalar, are the elements satisfying $\alpha^\vee \beta = \beta^\vee \alpha$, or $\alpha^\vee \beta$ is symmetric. As in Section 4, the only symmetric endomorphisms of an elliptic curve are scalar. We find that there is an isogeny ϕ such that $\alpha = a\phi$ and $\beta = b\phi$ for some endomorphism ϕ and some scalars a, b . In order for f to additionally be an *isomorphism*, we must have $f^\vee f = 1$, implying $\alpha^\vee \alpha + (q\beta)^\vee (q\beta) = 1$, and hence

$$\deg(\alpha) + \deg(q\beta) = \deg(a\phi) + N \deg(b\phi) = 1.$$

In Case I, as the Hom-set embeds into its rationalization, we must have such a 2×2 matrix whose entries are genuine homomorphisms. As $g^\vee g$ is the degree of the isogeny g , this can only occur if one of $\alpha, q\beta$ is an isomorphism and the other is zero.

In Case II, we note that since the abelian variety differs from $E \times \bar{E}$ in this expression by a subgroup of 2-torsion, the entries of the matrix may not be genuine homomorphisms, but multiplying any of them by 2 is. Therefore, we find that 2α and $2q\beta$ are homomorphisms with $\deg(2\alpha) + \deg(2q\beta) = 4$. There are only the following possibilities:

- (1) $\deg(2\alpha) = 4, \deg(q\beta) = 0$. Such elements come from isomorphisms $E \rightarrow E'$ which respect the level structure.
- (2) $\deg(\alpha) = 0, \deg(2q\beta) = 4$. Such elements are compositions of isomorphisms $E \rightarrow E'$ with the “twist” map $(1, -1) \circ \Delta: E \times \bar{E} \rightarrow \bar{E} \times E$.
- (3) $\deg(2\alpha) = 2, \deg(2q\beta) = 2$. This would force $N = 1$, which is not in case II.
- (4) $\deg(2\alpha) = 1, \deg(2q\beta) = 3$. This forces $N = 3$, and such elements are compositions of isomorphisms $E \rightarrow E'$ with the matrix $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}q^\vee \\ -\frac{1}{2}q & -\frac{1}{2} \end{bmatrix}$ from $E \times \bar{E}$ to itself. This element is precisely a third root of unity from \mathcal{O}_F^\times .

- (5) $\deg(2\alpha) = 3$, $\deg(2q\beta) = 1$. This forces $N = 3$, and such elements are compositions of isomorphisms $E \rightarrow E'$, the twist map $E \times \bar{E} \rightarrow \bar{E} \times E$, and matrices of the previous type.

Let

$$w : \mathcal{M}_0(N) \rightarrow \mathcal{M}_0(N)$$

be the Fricke involution, which on S -points is given by

$$(E, q) \mapsto (\bar{E}, q^\vee).$$

Clearly, $w^2 = \text{Id}$. Let $\mathcal{M}_0(N) // \langle w \rangle$ denote the stack quotient by the action of w .

Observe that the map

$$\tau' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : E \times \bar{E} \xrightarrow{\cong} \bar{E} \times E.$$

induces an isomorphism between the points $\Phi'(E, q)$ and $\Phi'(\bar{E}, q^\vee)$ of $\text{Sh}(K_1)$. Therefore, the map Φ' factors through the quotient by the Fricke involution to give a map

$$\Phi' : \mathcal{M}_0(N)_{\mathbb{Z}_p} // \langle w \rangle \rightarrow \text{Sh}(K_1).$$

Remark 6.2. One must treat the case where $N = 1$ as exceptional, where

$$E \times \bar{E} = E \times E$$

has complex multiplication by $\mathcal{O}_F = \mathbb{Z}[i]$, and the isomorphism τ' corresponds to the action by i . The Fricke involution is the identity on the moduli, because there is no level structure, but the rescaled involution on the p -divisible group is nontrivial. The compact open subgroup K_1 is actually conjugate to K_0 in this case, so there is an isomorphism of Deligne-Mumford stacks

$$\text{Sh}(K_1) \cong \text{Sh}(K_0).$$

(This is the only case where the subgroups K_0 and K_1 are conjugate.)

Remark 6.3. The natural 1-dimensional summand of the p -divisible group, and similarly the Lie algebra, of $\Phi'(E)$ are identified with the p -divisible group and Lie algebra of E . Under this identification, the Lie algebra of $w(E, q) = (\bar{E}, q^\vee)$ is identified with that of E via a rescaling of the isogeny

$$q : \text{Lie } E \rightarrow \text{Lie } \bar{E}$$

by dividing by $a = \sqrt{-N} \in \mathbb{Z}_p$.

Theorem 6.4.

- (1) If $F = \mathbb{Q}(i)$, then the map

$$\Phi' : \mathcal{M}_0(1)_{\mathbb{Z}_p} = \mathcal{M}_{\text{ell}, \mathbb{Z}_p} \rightarrow \text{Sh}(K_1)$$

is a degree 2 Galois cover.

- (2) If $F = \mathbb{Q}(\omega)$, the map

$$\Phi' : \mathcal{M}_0(3)_{\mathbb{Z}_p} // \langle w \rangle \rightarrow \text{Sh}(K_1)$$

is a degree 3 Galois cover onto a connected component.

(3) *In all other cases, the map*

$$\Phi' : \mathcal{M}_0(N)_{\mathbb{Z}_p} // \langle w \rangle \rightarrow \mathrm{Sh}(K_1)$$

is an inclusion of a connected component.

Proof. Comparing the morphisms in the induced map on groupoids of S -points, we see from our Case I analysis that an isomorphism

$$E \times \bar{E} \rightarrow E' \times \bar{E}'$$

is either of the form

$$\alpha \times \frac{1}{N} q' \alpha q^\vee$$

for an isomorphism

$$\alpha : (E, q) \xrightarrow{\cong} (E', q')$$

in $\mathcal{M}_0(N)(S)$ or of the form

$$\left(\alpha \times \frac{1}{N} q' \alpha q\right) \circ \tau'$$

for an isomorphism

$$\alpha : (\bar{E}, q^\vee) \xrightarrow{\cong} (E', q')$$

in $\mathcal{M}_0(N)(S)$. (The case $F = \mathbb{Q}(i)$ is an exception, as noted in Remark 6.2.)

The Case II situation is analogous, since the isomorphisms above descend through the diagonal quotient by the 2-torsion of the elliptic curve. The only exception is the case where $F = \mathbb{Q}(\omega)$, where, as noted in the Case II analysis, one gets additional automorphisms from composition with the complex multiplication by cube roots of unity.

The verification that Φ' is étale and surjective on a connected component is by the same methods outlined in the proof of Theorem 4.9. The equivalence of the formal moduli functors at mod p -points implies the map is étale, and properness is established through the use of Néron models. \square

7. COMPUTATION OF $\mathrm{TAF}_{GU}(K_1)$

The analysis of the moduli in the previous section now allows us to make calculations in homotopy.

Theorem 7.1. *If $p > 3$ and $N \neq 1, 3$, the topological automorphic forms spectrum $\mathrm{TAF}_{GU}(K_1)$ has a factor E whose homotopy groups are given by the p -completion of the subring*

$$E_{2*} \subseteq M_*(\Gamma_0(N))_{\mathbb{Z}_p}[\Delta^{-1}]$$

of the ring of modular forms for $\Gamma_0(N)$ over \mathbb{Z}_p , consisting of those elements invariant under an involution.

Proof. If $p > 3$, the Adams-Novikov spectral sequence is concentrated on the zero-line, and collapses to give an isomorphism

$$\pi_{2*} \mathrm{TMF}_0(N)_p \cong M_*(\Gamma_0(N))_{\mathbb{Z}_p}[\Delta^{-1}]_p^\wedge.$$

In this generic case $N \neq 1, 3$, the description of the allowable isomorphisms asserts that the map $\mathcal{M}_0(N)_{\mathbb{Z}_p} \rightarrow Y$ is a Galois cover with Galois group $\mathbb{Z}/2$, as we have pullback diagram of Deligne-Mumford stacks

$$\begin{array}{ccc} \mathcal{M}_0(N)_{\mathbb{Z}_p} \amalg \mathcal{M}_0(N)_{\mathbb{Z}_p} & \longrightarrow & \mathcal{M}_0(N)_{\mathbb{Z}_p} \\ \downarrow & & \downarrow \Phi' \\ \mathcal{M}_0(N)_{\mathbb{Z}_p} & \xrightarrow{\Phi'} & Y \end{array}$$

where Y is the image component of Φ' in $\mathrm{Sh}(K_1)$. Here the two factors in the coproduct correspond to the identity morphism and the twist morphism. We get a descent spectral sequence with E_2 -term

$$H^s(\mathbb{Z}/2; \pi_t \mathrm{TMF}_0(N)_p) \Rightarrow E_{t-s}.$$

The group $\mathbb{Z}/2$ acts via the Fricke involution on the moduli. The lift of the $\mathbb{Z}/2$ -action to the line bundle of invariant 1-forms is the involution obtained by rescaling the natural isogeny by $a \in \mathbb{Z}_p$. If $p > 2$, the higher cohomology vanishes and we find that the homotopy of the result consists of the involution-invariant elements in the ring of modular forms. \square

A complete description of this ring rests on a complete description of the ring of p -integral modular forms for $\Gamma_0(N)$, together with the action of the canonical involution on the p -divisible group. Thus a more detailed computation requires a case-by-case analysis.

The rest of the section will be devoted to such an analysis for the cases where $N \leq 3$.

Proposition 7.2. *If $N = 1$, the spectrum $\mathrm{TAF}_{GU}(K_1)$ has homotopy groups given by the subring*

$$\pi_* \mathrm{TAF}_{GU}(K_1) \cong \mathbb{Z}_p[c_4, c_6^2, \Delta^{-1}]_p^\wedge \subset \mathbb{Z}_p[c_4, c_6, \Delta^{-1}]_p^\wedge$$

of the p -completed ring of modular forms.

Proof. In this case, we may (up to natural isomorphism) take the isogeny $q: E \rightarrow \bar{E}$ to be the identity map. As we observed in Remark 6.2, we are simply restating the $\mathbb{Q}(i)$ case of Theorem 5.2. \square

Theorem 7.3. *If $p \neq 3$ and $N = 2$, the topological automorphic forms spectrum $\mathrm{TAF}_{GU}(K_1)$ has a factor E whose homotopy groups are given by the subring*

$$\pi_* E \cong \mathbb{Z}_p[q_2, D^{\pm 1}]_p^\wedge \subset \pi_* \mathrm{TMF}_0(2)_p$$

of the p -completed ring of modular forms of level 2, , where $|q_2| = 4$ and $|D| = 8$.

Proof. In the case $N = 2$, the constraint $p > 3$ is forced by the requirement that p splits in $\mathbb{Q}(\sqrt{-2})$ and $p \neq 3$. From Theorem 7.1, we have that $\mathrm{TAF}_{GU}(K_1)$ has a summand whose homotopy consists of the invariants in the p -completed ring of modular forms of level 2 invariant under the involution.

See [Beh06] for a proof of following descriptions. The p -completed ring of $\Gamma_0(2)$ -modular forms with the discriminant inverted is

$$\mathbb{Z}_p[q_2, q_4, \Delta^{-1}]_p^\wedge,$$

where $\Delta = q_4^2(16q_2^2 - 64q_4)$. The self-map t satisfying $t^2 = [2]$ that gives rise to the involution is given on homotopy by

$$\begin{aligned} t^*(q_2) &= -2q_2, \\ t^*(q_4) &= q_2^2 - 4q_4. \end{aligned}$$

The involution itself is then given by

$$\begin{aligned} w(q_2) &= q_2, \\ w(q_4) &= \frac{1}{4}q_2^2 - q_4. \end{aligned}$$

We formally define the element r_4 as $8q_4 - q_2^2$. The elements q_2, r_4 , and Δ^{-1} thus generate the ring of modular forms, and the involution w negates r_4 . In this expression, we have the following identities.

$$\begin{aligned} \Delta &= \frac{1}{8}(q_2^2 + r_4)^2(q_2^2 - r_4) \\ \Delta w(\Delta) &= \frac{1}{64}(q_2^4 - r_4^2)^3. \end{aligned}$$

The subring of the ring of modular forms invariant under the involution is then generated by

$$q_2, (q_2^4 - r_4^2), (q_2^4 - r_4^2)^{-1},$$

as desired. \square

Theorem 7.4. *If $N = 3$, the topological automorphic forms spectrum $\mathrm{TAF}_{GU}(K_1)$ has a summand E whose homotopy is a subring*

$$\mathbb{Z}_p[a_1^6, D^{\pm 1}]_p^\wedge \subset \pi_*\mathrm{TMF}_0(3)_p$$

of the p -completed ring of modular forms of level 3, where $|a_1^6| = |D| = 12$.

Proof. In the case $N = 3$, the constraint $p \equiv 1 \pmod{3}$ is forced by the requirement that p splits in $\mathbb{Q}(\sqrt{-3})$. The map $\mathcal{M}_0(3)_{\mathbb{Z}_p} \rightarrow Y$ is a Galois cover, as we have the following pullback diagram of moduli.

$$\begin{array}{ccc} \coprod^6 \mathcal{M}_0(3)_{\mathbb{Z}_p} & \longrightarrow & \mathcal{M}_0(3)_{\mathbb{Z}_p} \\ \downarrow & & \downarrow \Phi' \\ \mathcal{M}_0(3)_{\mathbb{Z}_p} & \longrightarrow & Y \end{array}$$

Here Y is the image component of $\mathrm{Sh}(K_1)$. The six factors in the coproduct correspond to compositions of the action of the third roots of unity and the involution. The involution commutes with the action of the roots of unity. We get a descent spectral sequence of the form

$$H^s(\mathbb{Z}/6; \pi_t\mathrm{TMF}_0(3)) \Rightarrow E_{t-s},$$

As $p > 3$, the higher cohomology vanishes. In this case, the ring of modular forms for level 3 structures, together with the Fricke involution, is known; see Mahowald and Rezk [MR]. We list the result at primes away from 6.

$$\mathrm{TMF}_0(3)[1/6]_* \cong \mathbb{Z}[1/6][a_1^2, a_1 a_3, a_3^2, \Delta^{-1}]$$

where $\Delta = a_1^3 a_3^3 - 27a_3^4$. The self-map t satisfying $t^2 = [3]$ that gives rise to the involution is given on homotopy by

$$\begin{aligned} t^*(a_1^2) &= -3a_1^2, \\ t^*(a_1 a_3) &= \frac{1}{3}a_1^4 - 9a_1 a_3, \\ t^*(a_3^2) &= -\frac{1}{27}a_1^6 + 2a_1^3 a_3 - 27a_3^2. \end{aligned}$$

The involution itself is then given by

$$\begin{aligned} w(a_1^2) &= a_1^2, \\ w(a_1 a_3) &= \frac{1}{27}a_1^4 - a_1 a_3, \\ w(a_3^2) &= \frac{1}{27^2}a_1^6 - \frac{2}{27}a_1^3 a_3 + a_3^2. \end{aligned}$$

We formally define the element d_2 as $(54\frac{a_3}{a_1} - a_1^2)$. The elements a_1^2 and d_2 thus generate a larger ring where the involution w negates a_1^2 and negates d_2 . In this expression, we have the identity

$$\Delta w(\Delta) = \frac{1}{28318}(a_1^6 - a_1^2 d_2^2)^4 = \frac{1}{28318}D^4.$$

The subring of the ring of modular forms invariant under the involution is then generated by elements $a_1^{2k} d_2^{2l} D^{-m}$, where $2k \geq 2l$. This subring is generated by the elements

$$a_1^2, D, D^{-1}.$$

As in Theorem 5.2, the third root of unity ω acts on modular forms of weight k by multiplication by ω^k . Therefore, the subring of elements invariant under the action of $\mathbb{Z}/3$ consists precisely of those elements of total degree divisible by 6. This subring is generated by the algebraically independent elements a_1^6 and D , together with D^{-1} . \square

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