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Spherically symmetric gravitationally bound, oscillating scalar lumps (boson stars and oscillatons) are considered in Einstein's gravity coupled to massive scalar fields in $1 + D$ -dimensional de Sitter-type inflationary space-times. We show that due to inflation bosons stars and oscillatons lose mass through scalar radiation, but at a rate that is exponentially small when the expansion rate is slow.

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I. INTRODUCTION

Current physical field theories explaining the presently known fundamental interactions are remarkably successful to account for phenomena both on particle physics (i.e. 10^{-18} m) and on cosmological (10^{26} m) scales, usually incorporate various fundamental scalar fields (Higgs bosons, axion, supersymmetric partners of fermions, inflaton, waterfall fields etc.). It has been observed since a long time ago that self-interacting scalar fields admit at least up to $D = 4$ spatial dimensions spatially localized, oscillating lumps, nowadays called oscillons [1–3]. Oscillons are reminiscent of spatially localized, time-periodic breathers known in one-dimensional Sine-Gordon theory. There is, however, a fundamental difference, in that oscillons are only *approximately* time periodic since they radiate, thereby losing continuously their energy, albeit very slowly [4–6]. Moreover oscillon-type objects exist in a broad class of field theories containing in addition to scalar, vector and possibly other fields [7,8]. A prominent theory admitting classical oscillons is the bosonic sector of the standard model [9–11]. Einstein's gravity coupled to a free, massive real Klein-Gordon field, also possesses oscillon-type solutions, which go under the name of oscillatons [12–17].

An important problem is to investigate the possible pertinence of oscillatons for cosmology where according to current theories some scalar fields are necessarily involved. In recent investigations the evolution of oscillons in an inflating background has been investigated [18–23]. It has been found that an inflating background also induces a radiative tail in oscillons, *distinct* from the one present in flat space-times, leading to an additional leakage of energy. However, this decay rate is also exponentially suppressed, so inflating oscillons may survive for an exponentially long time provided that the horizon is far larger than the width of an oscillon. As recently shown, due to quantum effects, the decay rate of oscillons becomes power law like and in many (but not in all) cases dominates over the classical decay rate [24]. There are indications that oscillons can be copiously produced from thermal initial conditions and a substantial fraction of the energy of the system will be

stored by them [18,23,25]. Therefore, it is important to investigate the physical rôle of such long-living localized objects. In a cosmological setting it is necessary to take into account the influence of gravity, i.e. instead of simply considering oscillons in a background metric, to investigate solutions of the *coupled Einstein-scalar system—oscillatons*—with an expanding, asymptotically homogeneous metric.

In this paper we undertake a thorough investigation of localized, oscillaton-type solutions in Einstein's gravity coupled to a massive Klein-Gordon field, when space-time undergoes inflation, driven by a negative pressure cosmological fluid—i.e. including a cosmological constant, $\Lambda > 0$. (For recent reviews on inflation theory, see e.g. [26–28].)

Our results show that once admitting massive scalar fields, both in present day cosmology and in the early Universe, spatially localized, very long-living oscillatons exist. The inflaton field generates an energy density of the order of $(10^{16} \text{ GeV})^4$ and negative pressure which can be described by an effective (and slowly varying) cosmological parameter. Using the standard slow roll approximation for the inflaton, the metric during the inflationary era can be approximated very well with a de Sitter one. The order of magnitude of the cosmological constant during the inflationary epoch is $\Lambda \approx 10^{-11}$ (in Planck units). If other massive scalar fields are present during inflation such as the waterfall-field in hybrid inflationary scenarios [29,30], oscillaton production is expected to be significant. We find that oscillatons may survive on cosmological time scales only if the mass of the scalar field is larger than $10^{15} \text{ GeV}/c^2$. The mass loss of an oscillaton during Hubble time is completely determined by the ratio μ_c/μ_Λ , where μ_Λ is the energy density due to Λ and μ_c is that of the oscillaton in its center. For example, when $\mu_c/\mu_\Lambda \approx 55$ the mass loss of such a “waterfall” oscillaton is of the order of 1% during Hubble time. This mass loss is exponentially decreasing for increasing values of the ratio μ_c/μ_Λ .

Another interesting scenario for oscillaton production is during reheating, assuming that the inflaton field undergoes

oscillations near the minimum of its potential. These oscillations store a considerable amount of energy and due to parametric resonances induced by quantum effects (for oscillons this has been shown in Ref. [24]) they may influence the fluctuation spectrum.

The present value of $\Lambda \approx 10^{-122}$ (in Planck units) being extremely small, its effect on the mass loss of generic oscillations is completely negligible on a cosmological time scale. For example, if the mass of the scalar field is of the order of $m \approx 10^{-15}$ eV and the amplitude of the oscillation is approximatively 10^{-17} , the effect of the cosmological constant results in a roughly 1% mass loss during the lifetime of the Universe.

We also consider (spherically symmetric) boson stars [31–34] in this setting, i.e. when a complex scalar field is coupled to gravity with a nonzero cosmological constant. Adopting our small-amplitude expansion techniques developed for oscillons and oscillations, we obtain boson star solutions in a systematic way. Remarkably, this method yields very good results even for near maximal mass boson stars. Spatially well localized boson star-type solutions with a static metric, familiar in the case $\Lambda \leq 0$ no longer exist when $\Lambda > 0$. There exist, however, radiating boson stars with a time dependent metric loosing continually their mass due to emission of scalar waves. We obtain their mass loss using our methods developed for oscillons resp. oscillations, with the result that this mass loss is exponentially small, and it is negligible for sufficiently small values of Λ .

II. FIELD EQUATIONS

We consider a real or complex scalar field coupled to gravity. The total Lagrangian density is

$$\mathcal{L} = \frac{1}{16\pi} \mathcal{L}_G + \mathcal{L}_M, \quad (1)$$

where the Einstein Lagrangian density in the case of a cosmological constant Λ is

$$\mathcal{L}_G = \sqrt{-g}(R - 2\Lambda). \quad (2)$$

We work in a $D + 1$ -dimensional curved space-time with metric g_{ab} and assume metric signature $(- + + + \dots)$. Unless explicitly noted, we use Planck units $G = c = \hbar = 1$. In some cases, it is advantageous to use natural units, where $c = \hbar = 1$ (but $G \neq 1$) and all quantities are measured in various powers of electron volts. Because of the different notation in the potential term we consider the real and the complex field case separately. Variation of the action (1) with respect to g^{ab} yields Einstein equations,

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}. \quad (3)$$

A. Boson stars

We consider a complex scalar field Φ with a self-interaction potential $U(\Phi^*\Phi)$. For a free field with mass

m , the potential is $U(\Phi^*\Phi) = m^2\Phi^*\Phi$. The Lagrangian density belonging to the scalar field is

$$\mathcal{L}_M = -\frac{\sqrt{-g}}{2}[g^{ab}\Phi_{,a}^*\Phi_{,b} + U(\Phi^*\Phi)]. \quad (4)$$

Variation of the action (1) with respect to Φ^* yields the wave equation

$$g^{ab}\Phi_{;ab} - \Phi U'(\Phi^*\Phi) = 0, \quad (5)$$

where the prime denotes derivative with respect to $\Phi^*\Phi$. The stress-energy tensor in this case is

$$T_{ab} = \frac{1}{2}[\Phi_{,a}^*\Phi_{,b} + \Phi_{,b}^*\Phi_{,a} - g_{ab}[g^{cd}\Phi_{,c}^*\Phi_{,d} + U(\Phi^*\Phi)]]. \quad (6)$$

We shall assume that the self-interaction potential, $U(\Phi^*\Phi)$, has a minimum $U(\Phi^*\Phi) = 0$ at $\Phi = 0$, and expand its derivative as

$$U'(\Phi^*\Phi) = m^2\left[1 + \sum_{k=1}^{\infty} u_k(\Phi^*\Phi)^k\right], \quad (7)$$

where m is the scalar field mass and u_k are constants.

In order to get rid of the 8π factors in the equations we introduce a rescaled scalar field and potential by

$$\phi = \sqrt{8\pi}\Phi, \quad \bar{U}(\phi^*\phi) = 8\pi U(\Phi^*\Phi). \quad (8)$$

Then

$$\bar{U}'(\phi^*\phi) = m^2\left[1 + \sum_{k=1}^{\infty} \bar{u}_k(\phi^*\phi)^k\right], \quad (9)$$

with

$$\bar{u}_k = \frac{u_k}{(8\pi)^k}. \quad (10)$$

B. Oscillations

In this case, we consider a real scalar field Φ with a self-interaction potential $U(\Phi)$. For a free field with mass m , the potential is $U(\Phi) = m^2\Phi^2/2$. The Lagrangian density belonging to the scalar field is

$$\mathcal{L}_M = -\sqrt{-g}\left(\frac{1}{2}\Phi_{,a}\Phi^{,a} + U(\Phi)\right). \quad (11)$$

Variation of the action (1) with respect to Φ yields the wave equation

$$g^{ab}\Phi_{;ab} - U'(\Phi) = 0, \quad (12)$$

where the prime now denotes derivative with respect to Φ . The stress-energy tensor is

$$T_{ab} = \Phi_{,a}\Phi_{,b} - g_{ab}\left(\frac{1}{2}\Phi_{,c}\Phi^{,c} + U(\Phi)\right). \quad (13)$$

We shall assume that $U(\Phi)$ has a minimum $U(\Phi) = 0$ at $\Phi = 0$, and expand its derivative as

$$U'(\Phi) = m^2 \left(\Phi + \sum_{k=2}^{\infty} u_k \Phi^k \right), \quad (14)$$

where m is the scalar field mass and u_k are constants.

In order to get rid of the 8π factors in the equations we introduce a rescaled scalar field and potential by

$$\phi = \sqrt{8\pi} \Phi, \quad \bar{U}(\phi) = 8\pi U(\Phi). \quad (15)$$

Then

$$\bar{U}'(\phi) = m^2 \left(\phi + \sum_{k=2}^{\infty} \bar{u}_k \phi^k \right), \quad (16)$$

with

$$\bar{u}_k = \frac{u_k}{(8\pi)^{(k-1)/2}}. \quad (17)$$

C. Scaling properties

In both the real and the complex case, if the pair $\phi(x^c)$ and $g_{ab}(x^c)$ solves the field equations with a potential \bar{U} and cosmological constant Λ , then

$$\hat{\phi}(x^c) = \phi(\gamma x^c), \quad \hat{g}_{ab}(x^c) = g_{ab}(\gamma x^c) \quad (18)$$

for any positive constant γ , is a solution with a rescaled potential $\gamma^2 \bar{U}$ and rescaled cosmological constant $\gamma^2 \Lambda$. This scaling property may be used to set the scalar field mass to any prescribed value, for example, to make $m = 1$.

If $D = 1$ then, by definition, the Einstein tensor is traceless, and from the trace of the Einstein equations it follows that the potential U is constant. Hence, we assume that $D > 1$.

D. Spherically symmetric $D + 1$ -dimensional space-time

We consider a spherically symmetric $D + 1$ -dimensional space-time using isotropic coordinates $x^\mu = (t, r, \theta_1, \dots, \theta_{D-1})$. In this coordinate system the metric is diagonal and its spatial part is in conformally flat form, with components

$$g_{tt} = -A, \quad g_{rr} = B, \quad g_{\theta_1 \theta_1} = r^2 B, \quad (19)$$

$$g_{\theta_n \theta_n} = r^2 B \prod_{k=1}^{n-1} \sin^2 \theta_k,$$

where A and B are functions of temporal coordinate t and radial coordinate r . In the case of oscillatons, the periodic change in the acceleration of the constant radius observers is much smaller in isotropic coordinates than in the more commonly used Schwarzschild coordinate system [16,35]. It is also more convenient to study the Newtonian limit of boson stars using isotropic coordinates [36]. The components of the Einstein tensor are

$$G_{tt} = \frac{D-1}{2} \left[\frac{D}{4B^2} (B_{,t})^2 - \frac{A}{r^{D-1} B^{(D+2)/4}} \left(\frac{r^{D-1} B_{,r}}{B^{(6-D)/4}} \right)_{,r} \right], \quad (20)$$

$$G_{rr} = \frac{D-1}{2} \left[\frac{(D-2)(r^2 B)_{,r}}{4r^4 A^{2/(D-2)} B^2} (r^2 A^{2/(D-2)} B)_{,r} - \frac{1}{A^{1/2} B^{(D/4)-1}} \left(\frac{B^{(D/4)-1} B_{,t}}{A^{1/2}} \right)_{,t} - \frac{D-2}{r^2} \right], \quad (21)$$

$$G_{tr} = -\frac{D-1}{2} A^{1/2} \left(\frac{B_{,t}}{A^{1/2} B} \right)_{,r}, \quad (22)$$

$$G_{\theta_1 \theta_1} = r^2 G_{rr} + \frac{r^3 B}{2A^{1/2}} \left(\frac{A_{,r}}{rA^{1/2} B} \right)_{,r} + \frac{D-2}{2} r^3 B^{1/2} \left(\frac{B_{,r}}{rB^{3/2}} \right)_{,r}. \quad (23)$$

The potential independent term in the wave Eq. (5) and (12) is

$$g^{ab} \phi_{;ab} = \frac{\phi_{,rr}}{B} - \frac{\phi_{,tt}}{A} - \frac{\phi_{,t}}{2B^D} \left(\frac{B^D}{A} \right)_{,t} + \frac{\phi_{,r}}{2r^{2D-2} AB^{D-1}} (r^{2D-2} AB^{D-2})_{,r}. \quad (24)$$

E. De Sitter space-time

If the cosmological constant Λ is positive, far from the central boson star or oscillaton the space-time should approach the de Sitter metric. In static Schwarzschild coordinates on a $D + 1$ -dimensional space-time this metric can be written as

$$ds^2 = -(1 - H^2 \bar{r}^2) dt^2 + \frac{d\bar{r}^2}{1 - H^2 \bar{r}^2} + \bar{r}^2 d\Omega_{D-1}^2, \quad (25)$$

where $d\Omega_{D-1}^2$ is the metric on the unit sphere S^{D-1} . The Hubble constant H is related to the cosmological constant Λ by

$$H^2 = \frac{2\Lambda}{D(D-1)}. \quad (26)$$

In isotropic coordinates the de Sitter metric takes the form

$$ds^2 = -\frac{(1 - \frac{H^2 r^2}{4})^2}{(1 + \frac{H^2 r^2}{4})^2} dt^2 + \frac{dr^2 + r^2 d\Omega_{D-1}^2}{(1 + \frac{H^2 r^2}{4})^2}, \quad (27)$$

where the two radial coordinates are related by

$$\bar{r} = \frac{r}{1 + \frac{H^2 r^2}{4}}. \quad (28)$$

The cosmological horizon, which is at $\bar{r} = 1/H$ in the Schwarzschild coordinate system, is at $r_h = 2/H$ in isotropic coordinates.

III. SMALL-AMPLITUDE EXPANSION

The small-amplitude expansion procedure has been applied successfully to describe the core region of flat background oscillons [1,4,37,38]. The method was generalized in [35] to the case when the scalar field is coupled to Einstein gravity, and in [39] to a very similarly behaving scalar-dilaton system. In this section, we expand oscillatons in the case when the cosmological constant is positive.

Newtonian boson stars have already been investigated in the paper of Ruffini and Bonazzola [32], obtaining a system of two coupled differential equations, which nowadays are generally called Schrödinger-Newton equations in the literature [40–43]. The same system was obtained as weak gravity limit of general relativistic boson stars in [44,45]. The leading order results of the small-amplitude expansion of boson stars, that we present in this section, when restricted for $\Lambda = 0$, also yield the same equations.

The repulsive effect of the cosmological constant is taken into account in a similar way as in [20–22]. Since we intend to investigate localized objects, where at large distances the scalar field is negligible, by the generalized Birkhoff's theorem [46–48], the metric should approach the Schwarzschild-de Sitter metric. This also means that in the asymptotic de Sitter region the expansion rate (Hubble parameter) is constant.

A. Choice of coordinates

We are looking for spatially localized bounded solutions of the field equations for which ϕ is small and the metric is close to de Sitter. The smaller the amplitude of a boson star or an oscillaton is, the larger its spatial extent becomes. Therefore, we introduce a new radial coordinate ρ by

$$\rho = \varepsilon m r, \quad (29)$$

where ε denotes the small-amplitude parameter. Motivated by the scaling property (18) we have also included a m factor into the definition of ρ .

Long-living localized solutions are expected to exist only if their size is considerably smaller than the distance to the cosmological horizon. We introduce a rescaled Hubble constant h by

$$H = \varepsilon^2 m h, \quad (30)$$

and assume that even h is reasonably small. This way we ensure that the typical size of boson stars or oscillatons, which is of the order $1/(\varepsilon m)$ using the r coordinates, remains smaller than the radius of the cosmological horizon, $r_h = 2/H = 2/(\varepsilon^2 m h)$.

We will see in Sec. III E that the energy density μ of the scalar field at the central region of small-amplitude oscillons or boson stars is proportional to ε^4 . On the other hand, the cosmological constant can be interpreted as a fluid with energy density,

$$\mu_\Lambda = \frac{\Lambda}{8\pi} \quad (31)$$

and pressure $p_\Lambda = -\mu_\Lambda$. Using (26) and (30) μ_Λ can be written as

$$\mu_\Lambda = \frac{D(D-1)}{16\pi} \varepsilon^4 m^2 h^2. \quad (32)$$

The smallness of h guaranties that μ_Λ remains smaller than energy density μ of the scalar field even in the small-amplitude limit.

We expand ϕ and the metric functions in powers of ε as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^{2k} \phi_{2k}, \quad (33)$$

$$A = \frac{(1 - \frac{\varepsilon^2 h^2 \rho^2}{4})^2}{(1 + \frac{\varepsilon^2 h^2 \rho^2}{4})^2} + \sum_{k=1}^{\infty} \varepsilon^{2k} A_{2k}, \quad (34)$$

$$B = \frac{1}{(1 + \frac{\varepsilon^2 h^2 \rho^2}{4})^2} + \sum_{k=1}^{\infty} \varepsilon^{2k} B_{2k}. \quad (35)$$

The first terms in A and B represent the de Sitter background, according to (27). Since we intend to use asymptotically de Sitter coordinates, we look for functions ϕ_{2k} , A_{2k} and B_{2k} that tend to zero when ρ is large. One could initially include odd powers of ε into the expansions (33)–(35). However, for oscillatons it can be shown by the method presented below, that the coefficients of those terms necessarily vanish when we are looking for configurations that remain bounded in time. It can also be verified directly, that the small-amplitude expansion of boson stars do not include odd powers of ε . This is in contrast to flat background oscillons, where the leading order behavior of the amplitude is proportional to ε .

The frequency of boson stars and oscillatons also depends on their amplitude. The smaller the amplitude is, the closer the frequency becomes to the threshold m . Hence we introduce a rescaled time coordinate τ by

$$\tau = \omega t, \quad (36)$$

and expand the square of the ε dependent factor ω as

$$\omega^2 = m^2 \left(1 + \sum_{k=1}^{\infty} \varepsilon^{2k} \omega_{2k} \right). \quad (37)$$

It is possible to allow odd powers of ε into the expansion of ω^2 , but the coefficients of those terms turn out to be zero when solving the equations arising from the small-amplitude expansion. There is a considerable freedom in choosing different parametrizations of the small-amplitude states, changing the actual form of the function ω . The physical parameter is not ε but the frequency of the periodic states that will be given by ω . Similarly to the asymptotically flat case in [35] and to the dilaton model

in [39], it turns out that for spatial dimensions $2 < D < 6$ the parametrization of the small-amplitude states can be fixed by setting $\omega = m\sqrt{1 - \varepsilon^2}$.

The field equations we intend to solve using the τ and ρ coordinates are the Einstein Eqs. (3), substituting the Einstein tensor components from (20)–(23), together with the wave Eq. (5) or (12). We note that these equations are not independent. The (τ, ρ) component of the Einstein equations is a constraint, and the wave equation is a consequence of the contracted Bianchi identity.

B. Boson stars

In the case of boson stars the metric is static, hence A_k and B_k are time independent. The complex scalar field ϕ oscillates with frequency ω ,

$$\phi = \psi e^{i\omega t}, \quad (38)$$

where ψ is real and depends only on the radial coordinate ρ . The small-amplitude expansion coefficients of ϕ satisfy

$$\phi_k = \psi_k e^{i\omega t} = \psi_k e^{i\tau}, \quad (39)$$

where ψ_k are real functions of ρ . Substituting the expansion (33)–(35) into the field equations, if $D \neq 2$, then to leading ε^4 order the (ρ, ρ) component gives

$$B_2 = \frac{A_2}{2 - D}. \quad (40)$$

Since for $D = 2$ there is no solution representing a localized object we assume $D > 2$. Then the (τ, τ) component and the wave equation yield to leading order a coupled system of differential equations for A_2 and ψ_2 ,

$$\frac{d^2 A_2}{d\rho^2} + \frac{D-1}{\rho} \frac{dA_2}{d\rho} = 2 \frac{D-2}{D-1} \psi_2^2, \quad (41)$$

$$\frac{d^2 \psi_2}{d\rho^2} + \frac{D-1}{\rho} \frac{d\psi_2}{d\rho} = \psi_2 (A_2 - \omega_2 - h^2 \rho^2). \quad (42)$$

Introducing the functions s and S by

$$s = \omega_2 - A_2, \quad S = \psi_2 \sqrt{2 \frac{D-2}{D-1}}, \quad (43)$$

Eqs. (41) and (42) can be written into the form

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + (s + h^2 \rho^2) S = 0, \quad (44)$$

$$\frac{d^2 s}{d\rho^2} + \frac{D-1}{\rho} \frac{ds}{d\rho} + S^2 = 0. \quad (45)$$

Since the same equations will be obtained for small-amplitude oscillatons in the next subsection, we postpone the discussion of this system to Sec. III D.

Proceeding to higher orders, the ε^6 components give a system of lengthy differential equations linear in A_4 , B_4 ,

ψ_4 and their derivatives. These equations have nonlinear source terms containing A_2 , B_2 and ψ_2 . They will also involve the first coefficient u_1 from the series expansion of the potential U . To leading order, the structure of small-amplitude boson stars only depends on the mass parameter m .

C. Oscillatons

In this case the scalar field is real, and the metric components are also oscillating. We are looking for solutions that remain bounded as time passes. It turns out, that these configurations are necessarily periodically oscillating in time.

From the ε^2 components of the field equations follows that B_2 only depends on ρ and that

$$\phi_2 = p_2 \cos(\tau + \delta), \quad (46)$$

where two new functions, p_2 and δ are introduced, depending only on ρ . From the ε^4 part of the (τ, ρ) component of the Einstein equations it follows that B_4 can remain bounded as time passes only if δ is a constant. Then by a shift in the time coordinate, we set

$$\delta = 0. \quad (47)$$

This shows that the scalar field oscillates simultaneously, with the same phase at all radii.

From the ε^4 component of the difference of the (ρ, ρ) and (θ_1, θ_1) Einstein equations follows that A_2 also depends only on the radial coordinate ρ , showing that the metric is static to order ε^2 . This is the main advantage of using isotropic coordinates over Schwarzschild coordinates. But even in this system the coefficients A_4 and B_4 will already contain $\cos(2\tau)$ terms. From the ε^4 component of the (τ, ρ) Einstein equations follows that

$$B_4 = b_4 - \frac{p_2^2}{4(D-1)} \cos(2\tau), \quad (48)$$

where b_4 is a function of ρ . The (ρ, ρ) component shows that if $D \neq 2$, then

$$B_2 = \frac{A_2}{2 - D}. \quad (49)$$

If $D = 2$, then $A_2 = 0$, and there are no nontrivial localized regular solutions for B_2 and p_2 , so we assume $D > 2$ from now. The (τ, τ) component yields

$$\frac{d^2 A_2}{d\rho^2} + \frac{D-1}{\rho} \frac{dA_2}{d\rho} = \frac{D-2}{D-1} p_2^2. \quad (50)$$

The wave equation provides an equation that determines the time dependence of ϕ_4 . In order to keep ϕ_4 bounded, the resonance terms proportional to $\cos\tau$ must vanish, yielding

$$\frac{d^2 p_2}{d\rho^2} + \frac{D-1}{\rho} \frac{dp_2}{d\rho} = p_2(A_2 - \omega_2 - h^2 \rho^2). \quad (51)$$

Equations (50) and (51) do not depend on the coefficients \bar{u}_k of the potential $\bar{U}(\phi)$. This means that the leading order small-amplitude behavior of oscillatons is always the same as for the Klein-Gordon case.

Introducing the functions s and S by

$$s = \omega_2 - A_2, \quad S = p_2 \sqrt{\frac{D-2}{D-1}}, \quad (52)$$

Eqs. (50) and (51) can be written into the form (44) and (45) already obtained at the small-amplitude expansion of boson stars.

D. Schrödinger-Newton equations

The equations describing both small-amplitude boson stars and oscillatons on an expanding background has been written into a form, which for $h = 0$ reduces to the time-independent Schrödinger-Newton (SN) equations [40–43]

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + (s + h^2 \rho^2)S = 0, \quad (53)$$

$$\frac{d^2 s}{d\rho^2} + \frac{D-1}{\rho} \frac{ds}{d\rho} + s^2 = 0. \quad (54)$$

If $S(\rho)$ and $s(\rho)$ are solutions of (53) and (54), then the transformed functions

$$\tilde{S}(\rho) = \lambda^2 S(\lambda\rho), \quad \tilde{s}(\rho) = \lambda^2 s(\lambda\rho) \quad (55)$$

solve the equations

$$\frac{d^2 \tilde{S}}{d\rho^2} + \frac{D-1}{\rho} \frac{d\tilde{S}}{d\rho} + (\tilde{s} + \tilde{h}^2 \rho^2)\tilde{S} = 0, \quad (56)$$

$$\frac{d^2 \tilde{s}}{d\rho^2} + \frac{D-1}{\rho} \frac{d\tilde{s}}{d\rho} + \tilde{s}^2 = 0, \quad (57)$$

which are obtained from the SN equations by replacing the constant h by $\tilde{h} = \lambda^2 h$.

If the scalar field, and consequently S , tends to zero for $\rho \rightarrow \infty$, then (54) gives the following approximation for s :

$$s = s_0 + s_1 \rho^{2-D}. \quad (58)$$

Since we are looking for solutions for which $A_2 = \omega_2 - s$ also approaches zero for large ρ , necessarily $\omega_2 = s_0$. In order to make the solution of the SN Eqs. (53) and (54) unique we demand

$$\omega_2 = s_0 = -1. \quad (59)$$

Setting $\omega_k = 0$ for $k \geq 3$ this also fixes the connection between the frequency ω and the small-amplitude parameter ε , giving

$$\omega = m\sqrt{1 - \varepsilon^2}. \quad (60)$$

Substituting into (53) we get

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + (h^2 \rho^2 - 1 + s_1 \rho^{2-D})S = 0, \quad (61)$$

determining the large ρ behavior of S .

E. Energy density

The energy density of the scalar field is $\mu = T_{ab}u^a u^b$, where the unit timelike vector u^a has the components $(1/\sqrt{A}, 0, \dots, 0)$. In terms of the rescaled scalar field ϕ , the energy density of oscillatons can be written as

$$\mu = \frac{1}{16\pi} \left[\frac{1}{A} \left(\frac{d\phi}{dt} \right)^2 + \frac{1}{B} \left(\frac{d\phi}{dr} \right)^2 + 2\bar{U}(\phi) \right], \quad (62)$$

and for boson stars

$$\mu = \frac{1}{16\pi} \left[\frac{1}{A} \frac{d\phi^*}{dt} \frac{d\phi}{dt} + \frac{1}{B} \frac{d\phi^*}{dr} \frac{d\phi}{dr} + \bar{U}(\phi^* \phi) \right]. \quad (63)$$

In the case of boson stars, using (33), (39), and (43), to leading order in ε we get

$$\mu = \frac{1}{16\pi} \varepsilon^4 \frac{D-1}{D-2} m^2 S^2. \quad (64)$$

For oscillatons, we obtain the same expression for μ using (33), (46), and (52). Measuring the scalar field mass m in eV/c^2 units, the energy density at the symmetry center for $D = 3$ can be written as

$$\mu_c = \varepsilon^4 \left(\frac{mc^2}{\text{eV}} \right)^2 1.436 \times 10^{39} \frac{\text{kg}}{\text{m}^3}. \quad (65)$$

(We use Roman m for meters in order to distinguish from the scalar field mass m .) Here we have used the numerically obtained central value of S , which is $S_c = 1.0215$ for $D = 3$.

The effective energy density μ_Λ corresponding to the present value of the cosmological constant is given in ordinary units by (D3) in Appendix D. Comparing to (65), one can see that the energy density of the scalar field is much higher than μ_Λ unless ε or m is extremely small. Using (32), the ratio of μ_Λ to the central density μ_c can be written as

$$\frac{\mu_\Lambda}{\mu_c} = \frac{D(D-2)}{S_c^2} h^2, \quad (66)$$

providing an important physical interpretation for the rescaled Hubble constant h . In Table I we give the value of S_c and the coefficient of h^2 in μ_Λ/μ_c for the relevant dimensions. We will see in the following sections that h turns out to be the essential parameter determining the energy loss rate of boson stars and oscillatons.

TABLE I. The numerically calculated values of the function S at the center and the coefficient of h^2 in μ_Λ/μ_c for $D = 3, 4$, and 5 spatial dimensions.

	$D = 3$	$D = 4$	$D = 5$
S_c	1.0215	3.5421	14.020
$D(D-2)/S_c^2$	2.8751	0.63761	0.076313

F. Outer core region

If $2 < D < 6$, and the cosmological constant is zero, i.e. $h = 0$, then assuming that $s_0 = -1$ the SN equations have a unique localized nodeless solution. Solutions with nodes have higher energy and are unstable. For large ρ , the function S satisfies the equation

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + (-1 + s_1 \rho^{2-D})S = 0, \quad (67)$$

which has exponentially decaying solutions

$$S = S_t \frac{e^{-\rho}}{\rho^{1-s_1/2}} \left[1 + \mathcal{O}\left(\frac{1}{\rho}\right) \right], \quad D = 3, \quad (68)$$

$$S = S_t \frac{e^{-\rho}}{\rho^{(D-1)/2}} \left[1 + \mathcal{O}\left(\frac{1}{\rho}\right) \right], \quad D > 3. \quad (69)$$

The numerically determined values of the constants s_1 and S_t for the case $h = 0$ are given in Table II.

Even if h is nonzero, we assume that it is small enough such that there is a region of ρ which is well outside the core region, but where the influence of the cosmological constant is still negligible. In this region S satisfies (67), and consequently (68) and (69) are good approximations together with the values of the constants belonging to the $h = 0$ case given in Table II.

G. Oscillating tail region

If h is nonzero, at very large distances $\rho \gg 1/h$, and the behavior of S is determined by the equation

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} + h^2 \rho^2 S = 0. \quad (70)$$

The function S has an oscillating standing wave tail in this domain,

$$S = \frac{\alpha}{\rho^{D/2}} \cos\left(\frac{h\rho^2}{2} + \beta\right), \quad (71)$$

 TABLE II. The numerical values of the constants s_1 and S_t for $h = 0$ in 3, 4, and 5 spatial dimensions.

	$D = 3$	$D = 4$	$D = 5$
s_1	3.50533	7.69489	10.4038
S_t	3.49513	88.2419	23.3875

where α and β are constants, describing the amplitude and phase. We can interpret this standing wave as the superposition of an outgoing wave carrying energy out from the central object and an artificial ingoing wave component that is added to keep the solution exactly periodic. We will relate the amplitude of the tail to the energy loss rate of dynamical boson stars or oscillatons.

When solving the SN Eqs. (53) and (54) by a numerical method, the amplitude α depends on the central values of s and S given at $\rho = 0$. We intend to determine the minimal tail amplitude, $\alpha = \alpha_{\min}$, assuming that s tends to $s_0 = -1$ for $\rho \rightarrow \infty$. We are interested in solutions which has no nodes in the central region where the influence of h is negligible. If $h > 0$, then α_{\min} is nonzero. In order to show the influence of h on the form of the function S , on Fig. 1 we plot the radial dependence of S for $h = 0.1016$ and for $h = 0$.

Assuming that h is small, we calculate the amplitude α by WKB analysis. Our aim is to continue the function S given by (68) and (69) into the very large distances where the cosmological term $h^2 \rho^2$ dominates. We will express the amplitude α in terms of the constant S_t . To the function which is exponentially decaying when going outwards at the outer core region belongs an oscillating tail with a certain amplitude and phase. Adding the mode which exponentially grows when going outwards does not change the function noticeably in the outer core region but adds an oscillating tail with a phase shift $\pi/2$. The resulting tail amplitude is minimal if this second mode is not present. Hence the WKB method will give the minimum amplitude $\alpha = \alpha_{\min}$. In Sec. IV F, we will compare the WKB amplitude to results obtained by direct numerical integration of the SN equations. Besides indicating the correctness of the WKB results, this will also give information on how large

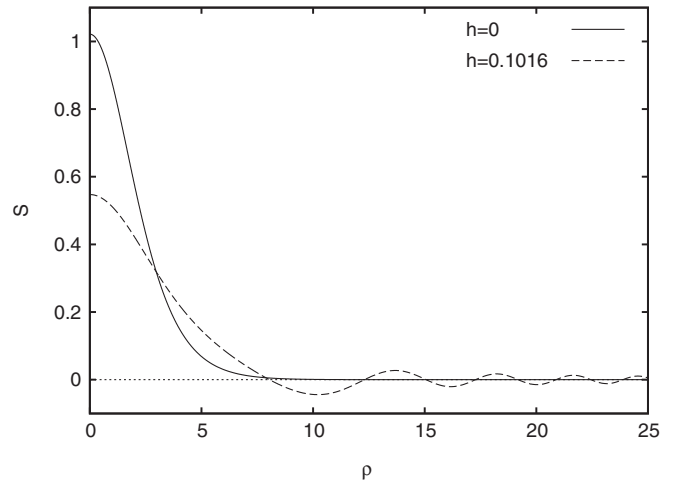


FIG. 1. The radial dependence of the function S for a large value of h is compared to the $h = 0$ flat background case. The function S for $h = 0$ decays exponentially, while for $h > 0$ curve starts to oscillate. For $h = 0.1016$, the first change in signature is at $\rho_t = 8.04$.

h can be chosen such that the WKB result can be still considered a valid approximation.

IV. WKB FORMALISM

A. Schrödinger form

Introducing a new radial coordinate

$$y = h\rho, \quad (72)$$

we transform (61) into a form corresponding to a Schrödinger equation with h playing the role of the Planck's constant \hbar ,

$$h^2 \left(\frac{d^2 S}{dy^2} + \frac{D-1}{y} \frac{dS}{dy} \right) + \left(y^2 - 1 + \frac{h^{D-2}}{y^{D-2}} s_1 \right) S = 0. \quad (73)$$

Introducing

$$S = y^{(1-D)/2} Z, \quad (74)$$

we can transform this into a one-dimensional form,

$$h^2 \frac{d^2 Z}{dy^2} + p^2 Z = 0, \quad (75)$$

where

$$p^2 = y^2 - 1 + \frac{h^{D-2}}{y^{D-2}} s_1 - \frac{h^2}{4y^2} (D-1)(D-3). \quad (76)$$

The outer domain of the core region is represented now by $0 < y \ll 1$, where

$$Z = S_i h^{1-s_1/2} y^{s_1/2} e^{-y/h} \left[1 + \mathcal{O}\left(\frac{h}{y}\right) \right], \quad D = 3, \quad (77)$$

$$Z = S_i h^{(D-1)/2} e^{-y/h} \left[1 + \mathcal{O}\left(\frac{h}{y}\right) \right], \quad D > 3. \quad (78)$$

B. WKB approximation

Introducing a new function f by

$$Z = e^{if/h} \quad (79)$$

and substituting into (75) we get

$$ih \frac{d^2 f}{dy^2} - \left(\frac{df}{dy} \right)^2 + p^2 = 0. \quad (80)$$

We write f as a power series in h ,

$$f = \sum_{k=0}^{\infty} f_k h^k. \quad (81)$$

Differently from the standard WKB approach, now p^2 also has to be expanded in h ,

$$p^2 = \sum_{k=0}^{\infty} p_k^2 h^k, \quad (82)$$

where the nonvanishing coefficients are

$$p_0^2 = y^2 - 1, \quad (83)$$

$$p_2^2 = -\frac{(D-1)(D-3)}{4y^2}, \quad (84)$$

$$p_{D-2}^2 = \frac{s_1}{y^{D-2}}. \quad (85)$$

Substituting into (80), to leading order we get

$$\left(\frac{df_0}{dy} \right)^2 = p_0^2, \quad \frac{df_0}{dy} = \pm p_0. \quad (86)$$

The order h components yield

$$i \frac{d^2 f_0}{dy^2} + p_1^2 = 2 \frac{df_0}{dy} \frac{df_1}{dy}. \quad (87)$$

C. WKB for $D > 3$

If $D > 3$, then $p_1^2 = 0$,

$$f_1 = \frac{i}{2} \ln \left(\frac{df_0}{dy} \right) = \frac{i}{2} \ln(\pm p_0), \quad (88)$$

and we get the standard WKB result,

$$Z = \frac{A_{\pm}}{\sqrt{|p_0|}} \exp \left(\pm \frac{i}{h} \int_1^y p_0 dy \right), \quad 0 < y < 1, \quad (89)$$

$$Z = \frac{B_{\pm}}{\sqrt{p_0}} \exp \left(\pm \frac{i}{h} \int_1^y p_0 dy \right), \quad y > 1. \quad (90)$$

The general solution for $0 < y < 1$ is a sum of two terms with proportionality constants A_+ and A_- , and similarly for the $y > 1$ case with constants B_+ and B_- . For the $0 < y < 1$ case $|p_0| = \sqrt{1 - y^2}$. Substituting (83), it is possible to perform the integral in the exponentials. For $0 < y < 1$,

$$\begin{aligned} \frac{i}{h} \int_1^y p_0 dy &= -\frac{1}{h} \int_1^y \sqrt{1 - y^2} dy \\ &= -\frac{1}{2h} \left(y \sqrt{1 - y^2} + \arcsin y - \frac{\pi}{2} \right), \end{aligned} \quad (91)$$

while for $y > 1$,

$$\begin{aligned} \frac{i}{h} \int_1^y p_0 dy &= \frac{i}{h} \int_1^y \sqrt{y^2 - 1} dy \\ &= \frac{i}{2h} [y \sqrt{y^2 - 1} - \ln(y + \sqrt{y^2 - 1})]. \end{aligned} \quad (92)$$

In the region $0 < y \ll 1$, substituting the expansion of (91), to leading order (89) takes the form

$$Z = A_{\pm} \exp \left[\mp \frac{1}{h} \left(y - \frac{\pi}{4} \right) \right]. \quad (93)$$

Comparing with (78), follows that for $D > 3$

$$A_+ = S_t h^{(D-1)/2} \exp\left(-\frac{\pi}{4h}\right), \quad A_- = 0. \quad (94)$$

D. WKB for $D = 3$

Because of the term proportional to h in p^2 , the $D = 3$ case has to be treated separately. Then (87) takes the form

$$\frac{i}{2} \frac{d}{dy} [\ln(\pm p_0)] \pm \frac{s_1}{2yp_0} = \frac{df_1}{dy}. \quad (95)$$

Using (83),

$$\frac{i}{2} \frac{d}{dy} [\ln(\pm \sqrt{y^2 - 1})] \pm \frac{s_1}{2iy\sqrt{1 - y^2}} = \frac{df_1}{dy}. \quad (96)$$

Integrating, for $0 < y < 1$,

$$f_1 = \frac{i}{2} \ln(\sqrt{y^2 - 1}) \pm \frac{s_1}{2i} [\ln y - \ln(1 + \sqrt{1 - y^2})] + c_1, \quad (97)$$

and for $y > 1$,

$$f_1 = \frac{i}{2} \ln(\sqrt{y^2 - 1}) \mp \frac{s_1}{2} \arctan\left(\frac{1}{\sqrt{y^2 - 1}}\right) + c_2. \quad (98)$$

Substituting into (79), for $0 < y < 1$,

$$Z = \frac{A_{\pm}}{\sqrt{|p_0|}} \left(\frac{y}{1 + \sqrt{1 - y^2}} \right)^{\pm s_1/2} \exp\left(\pm \frac{i}{h} \int_1^y p_0 dy\right), \quad (99)$$

and for $y > 1$,

$$Z = \frac{B_{\pm}}{\sqrt{p_0}} \exp\left\{\mp \frac{is_1}{2} \left[\arctan\left(\frac{1}{\sqrt{y^2 - 1}}\right) - \frac{\pi}{2} \right]\right\} \times \exp\left(\pm \frac{i}{h} \int_1^y p_0 dy\right). \quad (100)$$

For $0 < y \ll 1$, the expansion of (99) gives

$$Z = A_{\pm} \left(\frac{y}{2}\right)^{\pm s_1/2} \exp\left[\mp \frac{1}{h} \left(y - \frac{\pi}{4}\right)\right]. \quad (101)$$

Comparing with (77), follows that for $D = 3$

$$A_+ = 2^{s_1/2} S_t h^{1-s_1/2} \exp\left(-\frac{\pi}{4h}\right), \quad A_- = 0. \quad (102)$$

E. Connection formulae

The next step is to relate the amplitudes A_{\pm} in the $y < 1$ region to the amplitudes B_{\pm} for $y > 1$. The expressions (89) and (90) for $D > 3$ are just the standard WKB solutions, so one can apply the appropriate connection formulae to express B_{\pm} in terms of A_{\pm} . In our case, we need the one where the field is exponentially increases when going away from the turning point (see, for example, (34.18) of [49])

$$\frac{1}{\sqrt{p_0}} \cos\left(\frac{1}{h} \int_1^y p_0 dy + \frac{\pi}{4}\right) \rightarrow \frac{1}{\sqrt{|p_0|}} \exp\left(\frac{1}{h} \int_y^1 |p_0| dy\right), \quad (103)$$

from $y > 1$ to $y < 1$. This is equivalent to

$$A_- = 0, \quad B_{\pm} = \frac{A_+}{2} \exp\left(\pm \frac{i\pi}{4}\right). \quad (104)$$

The expressions (99) and (100) for $D = 3$ differ from the standard WKB solutions by factors involving s_1 . However, since both of these factors take the value 1 at the $y = 1$ turning point, the formulae (104) relating the amplitudes at the two sides hold in this case too. A detailed derivation of (104) is given in Appendix A.

F. Schrödinger-Newton tail amplitude

For $y \gg 1$, the expression (92) can be approximated by

$$\frac{i}{h} \int_1^y p_0 dy = \frac{iy^2}{2h}. \quad (105)$$

In the $D > 3$ case to leading order (90) yields

$$Z = \frac{B_{\pm}}{\sqrt{y}} \exp\left(\pm \frac{iy^2}{2h}\right). \quad (106)$$

Substituting B_{\pm} from (104) gives

$$Z = \frac{A_+}{\sqrt{y}} \cos\left(\frac{y^2}{2h} + \frac{\pi}{4}\right), \quad D > 3. \quad (107)$$

For $D = 3$, to leading order (100) yields

$$Z = \frac{B_{\pm}}{\sqrt{y}} \exp\left[\pm i\left(\frac{y^2}{2h} + \frac{\pi s_1}{4}\right)\right], \quad (108)$$

and substituting B_{\pm} from (104) gives

$$Z = \frac{A_+}{\sqrt{y}} \cos\left[\frac{y^2}{2h} + \frac{\pi}{4}(s_1 + 1)\right], \quad D = 3. \quad (109)$$

Using (74) we can express the original S variable in the SN equations using the coordinate $\rho = y/h$, obtaining the same asymptotic formula as in (71),

$$S = \frac{\alpha}{\rho^{D/2}} \cos\left(\frac{h\rho^2}{2} + \beta\right), \quad (110)$$

where the amplitude is

$$\alpha = \frac{A_+}{h^{D/2}} \quad (111)$$

for any D , and the phase is

$$\beta = \begin{cases} \frac{\pi}{4}(s_1 + 1) & \text{if } D = 3, \\ \frac{\pi}{4} & \text{if } D > 3. \end{cases} \quad (112)$$

Substituting the value of A_+ from (94) and (102) gives our final WKB result for the minimal amplitude

TABLE III. For several choices of h the numerically calculated minimal tail amplitude $\alpha_{\min}^{\text{num}}$ and its relative difference from the WKB value (113) is listed assuming $D = 3$. We also give the ratio of central values of the functions s and S to the flat background values $s_c^{h=0} = 0.938\,322\,84$ and $S_c^{h=0} = 1.021\,493\,04$. In order to show where the tail region starts, we also list the radius ρ_t , where S first crosses the value zero.

h	$s_c/s_c^{h=0}$	$S_c/S_c^{h=0}$	ρ_t	$\alpha_{\min}^{\text{num}}$	$\frac{\alpha_{\min}^{\text{num}}}{\alpha_{\min}^{\text{WKB}}} - 1$
0.036 111	0.962 95	0.970 43	28.65	7.7377×10^{-6}	0.033
0.046 769	0.935 39	0.948 30	22.04	6.2645×10^{-4}	0.054
0.0645 93	0.861 23	0.888 02	15.73	0.033 101	0.120
0.071 112	0.817 42	0.851 71	14.13	0.085 425	0.177
0.076 502	0.765 80	0.808 27	12.90	0.169 98	0.268
0.085 350	0.631 80	0.694 20	10.62	0.490 07	0.614
0.088 000	0.588 03	0.657 20	9.94	0.645 53	0.727
0.091 400	0.537 09	0.614 43	9.24	0.850 85	0.778
0.094 800	0.495 18	0.579 76	8.72	1.025 07	0.709
0.101 600	0.439 60	0.535 76	8.04	1.252 56	0.402

$$\alpha_{\min} = \begin{cases} \frac{S_c}{\sqrt{h}} \left(\frac{2}{h}\right)^{s_1/2} \exp\left(-\frac{\pi}{4h}\right) & \text{if } D = 3, \\ \frac{S_c}{\sqrt{h}} \exp\left(-\frac{\pi}{4h}\right) & \text{if } D > 3. \end{cases} \quad (113)$$

It is instructive to compare the above analytical result to the minimal amplitude obtained by the numerical solution of the SN Eqs. (53) and (54). For a chosen h , one has to minimize the oscillating tail for those solutions for which the field s tends to $s_0 = -1$ asymptotically, while S has no nodes in the core region. The solutions depend very strongly on the central values s_c and S_c . The results for various h for $D = 3$ spatial dimensions are listed in Table III. The amplitudes α_{\min} for $D = 3, 4$, and 5 dimensions are plotted on Fig. 2, and the relative difference from the analytically obtained WKB value is displayed on Fig. 3. From this figure we read off that the WKB analysis gives less than 10% error, if $h < 0.06$ in the case of $D = 3$, and if $h < 0.1$ when $D = 4$ or 5.

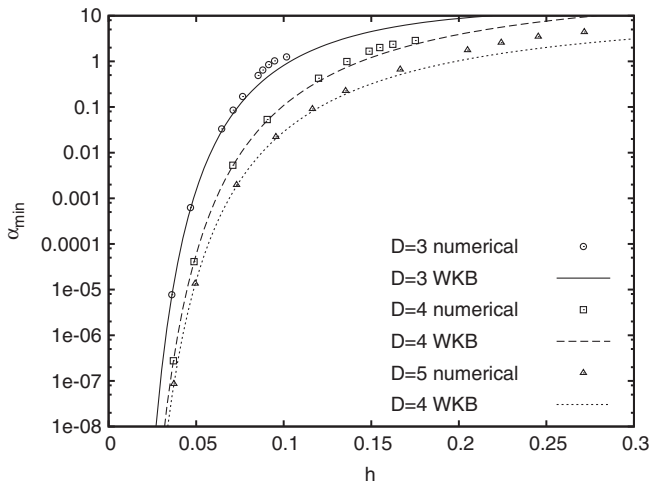


FIG. 2. The curves represent the WKB tail amplitude, while the points the numerically calculated values, as a function of h , for $D = 3, 4$, and 5 spatial dimensions.

G. Scalar field tail amplitude

For oscillatons, using (33), (46), (52), and (110), the real scalar in the tail region behaves like

$$\begin{aligned} \phi &= \varepsilon^2 \phi_2 = \varepsilon^2 p_2 \cos \tau = \varepsilon^2 S \sqrt{\frac{D-1}{D-2}} \cos \tau \\ &= \frac{\phi_A}{r^{D/2}} \cos(\omega t) \cos\left(\frac{h \varepsilon^2 m^2 r^2}{2} + \beta\right), \end{aligned} \quad (114)$$

where the amplitude is

$$\phi_A = \frac{\varepsilon^{2-(D/2)}}{m^{D/2}} \sqrt{\frac{D-1}{D-2}} \alpha_{\min}. \quad (115)$$

Applying the identity

$$\cos a \cos b = \frac{1}{2} [\cos(a-b) + \cos(a+b)], \quad (116)$$

it is apparent that (114) is a sum of an ingoing and outgoing

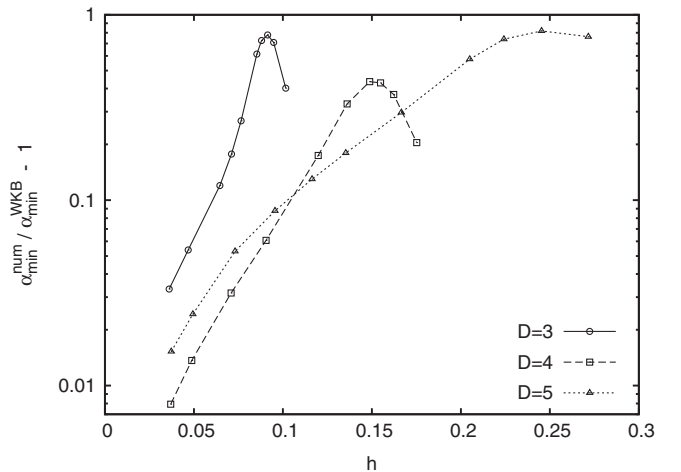


FIG. 3. Relative difference of the numerically calculated minimal tail amplitude from the analytically calculated WKB values.

wave, both with amplitude $\phi_A/2$. One might be tempted to add an ingoing wave with an opposite amplitude to obtain a purely outgoing solution, and to conclude that the remaining outgoing component has the amplitude $\phi_A/2$. However, an ingoing wave contains a $\cos(h\varepsilon^2 m^2 r^2/2 + \beta)$ component, which would change the amplitude of the WKB mode which is increasing when going away from the turning point. What one can do instead is adding a standing wave proportional to $\sin(h\varepsilon^2 m^2 r^2/2 + \beta)$, corresponding to the suppressed WKB mode. If the time dependence is also phase shifted, then we obtain the minimal amplitude outgoing wave,

$$\phi = \frac{\phi_A}{r^{D/2}} \cos\left(\frac{h\varepsilon^2 m^2 r^2}{2} + \beta - \omega t\right). \quad (117)$$

In the case of boson stars, using (33), (39), (43), and (110), the complex scalar in the tail region behaves like

$$\begin{aligned} \phi &= \varepsilon^2 \phi_2 = \varepsilon^2 \psi_2 e^{i\tau} = \varepsilon^2 \frac{S}{\sqrt{2}} \sqrt{\frac{D-1}{D-2}} e^{i\tau} \\ &= \frac{\phi_A}{\sqrt{2} r^{D/2}} e^{i\omega t} \cos\left(\frac{h\varepsilon^2 m^2 r^2}{2} + \beta\right). \end{aligned} \quad (118)$$

Now one can obtain the minimal amplitude outgoing wave by adding a term

$$-i \frac{\phi_A}{\sqrt{2} r^{D/2}} e^{i\omega t} \sin\left(\frac{h\varepsilon^2 m^2 r^2}{2} + \beta\right), \quad (119)$$

obtaining

$$\phi = \frac{\phi_A}{\sqrt{2} r^{D/2}} \exp\left[-i\left(\frac{h\varepsilon^2 m^2 r^2}{2} + \beta - \omega t\right)\right]. \quad (120)$$

H. Mass loss rate

Even if $\rho \gg 1/h$ in the oscillating tail region, according to (34) and (35), for small ε the metric functions A and B are still very close to 1. The mass loss rate of the system can be calculated by applying the expression (B11) for the time derivative of the mass function from Appendix B,

$$\frac{d\hat{m}}{dt} = \frac{2\pi^{D/2} r^{D-1}}{\Gamma(\frac{D}{2})} T_{tr}. \quad (121)$$

Substituting the stress-energy tensor from (6) or (13),

$$\begin{aligned} \frac{d\hat{m}}{dt} &= \frac{\pi^{D/2} r^{D-1}}{\Gamma(\frac{D}{2})} (\Phi_{,t}^* \Phi_{,r} + \Phi_{,r}^* \Phi_{,t}) \\ &= \frac{\pi^{(D/2)-1} r^{D-1}}{8\Gamma(\frac{D}{2})} (\phi_{,t}^* \phi_{,r} + \phi_{,r}^* \phi_{,t}), \end{aligned} \quad (122)$$

which expression is valid for both real and complex fields. We have to evaluate this expression at large r in the tail region. Substituting the outgoing wave form (117) and averaging for one oscillation period, or substituting (120), for both the oscillaton and boson star case, we obtain

$$\frac{d\hat{m}}{dt} = -\frac{\pi^{(D/2)-1} h \varepsilon^2 m^3 \phi_A^2}{8\Gamma(\frac{D}{2})}. \quad (123)$$

Here, since we are interested in the leading ε order result, we substituted $\omega = m$, and for large r we neglected the term arising when taking the derivative of the $r^{-D/2}$ factor. For large radius \hat{m} agrees with the total mass M of the oscillon or boson star. Using (113) and (115) for the mass loss rate and substituting $h = H/(m\varepsilon^2)$, for $D > 3$, we obtain

$$\frac{dM}{dt} = -\frac{\pi^{(D/2)-1} (D-1) S_t^2 \varepsilon^{6-D}}{8m^{D-3} (D-2) \Gamma(\frac{D}{2})} \exp\left(-\frac{\pi m \varepsilon^2}{2H}\right). \quad (124)$$

If $D = 3$, there is an extra term involving s_1 ,

$$\frac{dM}{dt} = -\frac{1}{2} S_t^2 \varepsilon^3 \left(\frac{2m\varepsilon^2}{H}\right)^{s_1} \exp\left(-\frac{\pi m \varepsilon^2}{2H}\right). \quad (125)$$

For small-amplitude configurations, the parameter ε can be expressed by the mass M using (B14). Because of the complexity of the resulting expression we only give the result for $D = 3$, when $M = \varepsilon s_1/(2m)$

$$\frac{dM}{dt} = -4 S_t^2 \frac{m^3 M^3}{s_1^3} \left(\frac{8m^3 M^2}{H s_1^2}\right)^{s_1} \exp\left(-\frac{2\pi m^3 M^2}{H s_1^2}\right). \quad (126)$$

For specific m and H , this can be integrated numerically in order to obtain the change of the mass as a function of time.

Using the leading order expression (B14) for the total mass and dividing by M we can obtain the relative mass loss rate. The Hubble time $T_H = 1/H$ describes the time scale of the expansion of the Universe. The value of T_H obtained from the present value of the cosmological constant is given by (D7) in Appendix D. Dividing by both M and H , we obtain the relative mass loss extrapolated for a period corresponding to the Hubble time. For $D > 3$,

$$\frac{T_H}{M} \frac{dM}{dt} = -\frac{S_t^2}{(D-2)s_1 h} \exp\left(-\frac{\pi}{2h}\right), \quad (127)$$

and for $D = 3$,

$$\frac{T_H}{M} \frac{dM}{dt} = -\frac{S_t^2}{s_1 h} \left(\frac{2}{h}\right)^{s_1} \exp\left(-\frac{\pi}{2h}\right), \quad (128)$$

where the values of the constants s_1 , S_t are given in Table II. This result is independent on the units used to measure the time and mass, and it depends on H , m and ε only through the rescaled cosmological constant $h = H/(m\varepsilon^2)$. According to (66) the square of the parameter h is proportional to the ratio of the central density μ_c to the effective energy density corresponding to the cosmological constant μ_Λ . On Fig. 4 and on Table IV we give $\frac{T_H}{M} \frac{dM}{dt}$ as a function of the density ratio $\frac{\mu_c}{\mu_\Lambda}$.

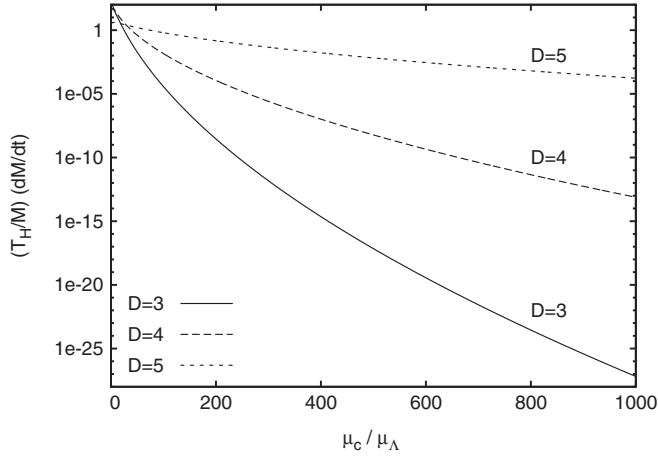


FIG. 4. The relative mass loss during a Hubble time period, $\frac{T_H}{M} \frac{dM}{dt}$, as a function of $\frac{\mu_c}{\mu_\Lambda}$ for $D = 3, 4$, and 5 spatial dimensions.

TABLE IV. The rescaled cosmological constant h and the density ratio $\frac{\mu_c}{\mu_\Lambda}$ belonging to various relative mass loss during a Hubble time period.

$\frac{T_H}{M} \frac{dM}{dt}$	$D = 3$		$D = 4$		$D = 5$	
	h	$\frac{\mu_c}{\mu_\Lambda}$	h	$\frac{\mu_c}{\mu_\Lambda}$	h	$\frac{\mu_c}{\mu_\Lambda}$
1	0.118 07	24.95	0.200 53	39.00	0.421 36	73.81
0.1	0.094 60	38.87	0.150 73	69.03	0.237 93	231.5
10^{-2}	0.079 85	54.55	0.121 39	106.4	0.169 98	453.5
10^{-3}	0.069 51	71.99	0.101 88	151.1	0.133 27	737.8
10^{-4}	0.061 76	91.19	0.087 91	202.9	0.109 99	1083
10^{-5}	0.055 69	112.1	0.077 39	261.9	0.093 82	1489

V. MASS LOSS RATE OF OSCILLATONS IN THE $\Lambda = 0$ CASE

If the cosmological constant is zero and the space-time is asymptotically flat, then the metric of boson stars is static and their mass is independent of time. In contrast, oscillatons are very slowly losing mass by scalar field radiation even in the $\Lambda = 0$ case [16]. This mass loss is nonperturbatively small in the amplitude parameter ε . According to [35], for $D = 3$ this mass loss rate depends only on the amplitude parameter, and it is given by

$$\frac{dM}{dt} = -\frac{c_1}{\varepsilon^2} \exp\left(-\frac{c_2}{\varepsilon}\right). \quad (129)$$

where the numerical values of the constants are

$$c_1 = 30.0, \quad c_2 = 22.4993. \quad (130)$$

For small ε , this is exponentially small, while the mass loss induced by the cosmological constant, given by (125), tends to zero only polynomially when $\varepsilon \rightarrow 0$. This implies that if $\Lambda \neq 0$ then for small ε the mass loss (125) originating from the cosmological constant dominates, while for

large ε the expression (129) can be applied. Apart from the amplitude parameter ε the mass loss (125) depends only on the ratio H/m . It can be checked numerically that for a concrete choice of H/m there is only one ε value where the two type of mass loss rates are equal, which we denote by ε_e . For low H/m values, the function ε_e can be expanded as

$$\varepsilon_e = c_3 \left(\frac{H}{m}\right)^{1/3} (1 + \varepsilon_1 + \varepsilon_2 + \dots), \quad (131)$$

with

$$\varepsilon_1 = \frac{R}{3c_3^2} \left(\frac{H}{m}\right)^{1/3}, \quad \varepsilon_2 = \frac{c_4 R}{9c_3^4} \left(\frac{H}{m}\right)^{2/3}, \quad (132)$$

$$R = \frac{2}{\pi} \ln\left(\frac{2^{s_1-1} S_1^2}{c_1}\right) + c_4 \ln c_3 + \left(\frac{c_4}{3} - \frac{2s_1}{\pi}\right) \ln\left(\frac{H}{m}\right), \quad (133)$$

where the newly introduced two constants are

$$c_3 = \left(\frac{2c_2}{\pi}\right)^{1/3}, \quad c_4 = \frac{2}{\pi} (2s_1 + 5). \quad (134)$$

On Fig. 5 we plot ε_e as a function of H/m . To illustrate how well the above analytic approximation works we plot for relatively large values of H/m , although the solutions for $\varepsilon > \varepsilon_{\max} \approx 0.5$ are unstable. The quantity H/m is originally assumed to be in Planck units, but it has the same simple form in natural units if the Hubble constant is expressed in electron volts.

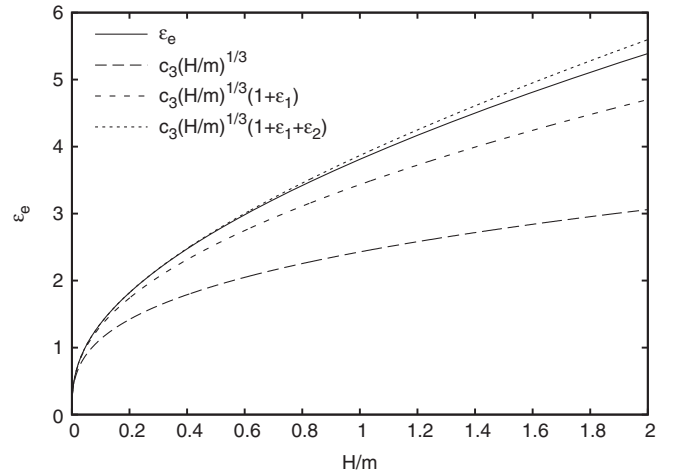


FIG. 5. The values of ε_e , for which the oscillaton mass loss rates (125) and (129) are equal are plotted as a function of the ratio of the Hubble constant H and the scalar field mass m (solid line). Above the line the asymptotically flat result (129) is dominant, while below (125) induced by the cosmological constant gives a larger contribution. The first three orders of the analytical approximation given by (131) are also shown.

VI. POSSIBLE EFFECTS IN THE CURRENT UNIVERSE

From the definition (30) of the rescaled cosmological constant we can express the square of the small-amplitude parameter ε . The amplitude of the scalar field is proportional to ε^2 to leading order. Measuring the cosmological constant in $1/s$ units, and the scalar field mass in eV/c^2 units, (30) can be written as

$$\varepsilon^2 = \frac{6.58 \times 10^{-16}}{h} Hs \frac{\text{eV}}{mc^2}. \quad (135)$$

According to (127) and (128), the part of the mass lost during a Hubble time period only depends on the value of the rescaled Hubble constant h . For example, we can look for configurations where this mass loss is 1%. According to Table IV, for the physically interesting $3 + 1$ -dimensional case this corresponds to the value of $h_{0.01} = 0.07985$. Using the present value of the Hubble constant, which is given by (D6),

$$\varepsilon_{0.01}^2 = 1.60 \times 10^{-32} \frac{\text{eV}}{mc^2}. \quad (136)$$

Oscillatons or boson stars are generally stable if the amplitude parameter ε , which for larger amplitudes can be defined from the frequency ω by $\varepsilon = \sqrt{1 - \omega^2/m^2}$, is smaller than $\varepsilon_{\text{max}} \approx 0.5$. From (136) it is apparent that significant mass loss for the larger amplitude configurations with $\varepsilon \approx \varepsilon_{\text{max}}$ can happen only for extremely small scalar field masses. Very small ε values are not excluded, but since for $D = 3$, according to (B16), the total mass of the oscillaton or boson star is proportional to ε , these configurations tend to have extremely small masses. Similarly, since by (C6) the radius is inversely proportional to ε , small ε values belong to spatially extended configurations. It is obvious that the influence of the cosmological constant is larger for large radius objects. In Table V we list the values of ε , the total mass and the radius for various

TABLE V. Oscillaton or boson star configurations for which 1% of the mass is lost during Hubble time, assuming the present value of the cosmological constant. For each choice of the scalar field mass m the amplitude parameter ε , the total mass M and the radius containing 95% of the mass is given.

mc^2/eV	$\varepsilon_{0.01}$	M/kg	r_{95}/m
10^{10}	1.27×10^{-21}	5.90×10^{-11}	6.95×10^4
10^5	4.01×10^{-19}	1.87×10^{-3}	2.20×10^7
1	1.27×10^{-16}	5.90×10^4	6.95×10^9
10^{-5}	4.01×10^{-14}	1.87×10^{12}	2.20×10^{12}
10^{-10}	1.27×10^{-11}	5.90×10^{19}	6.95×10^{14}
10^{-15}	4.01×10^{-9}	1.87×10^{27}	2.20×10^{17}
10^{-20}	1.27×10^{-6}	5.90×10^{34}	6.95×10^{19}
10^{-25}	4.01×10^{-4}	1.87×10^{42}	2.20×10^{22}
10^{-30}	1.27×10^{-1}	5.90×10^{49}	6.95×10^{24}

choices of the scalar field mass m , assuming that the mass loss extrapolated for a Hubble time period is 1%. Keeping the same scalar mass, for larger ε , and consequently, for larger total masses and smaller radii, the mass loss rate is smaller than 1%. For all configurations listed in Table V, the $\Lambda = 0$ oscillaton mass loss rate given by (129) is negligible, many orders of magnitude smaller than 1%.

To show some concrete examples, we examine three specific choices for the scalar field mass. First, for an axion with $m = 10^{-5} \text{ eV}$ the mass of the oscillaton for which $\frac{T_H}{M} \frac{dM}{dt} = 0.01$ is about that of a very small asteroid, and its radius is about 15 astronomical units. Second, if $m = 9.55 \times 10^{-18} \text{ eV}/c^2$, then the total mass is about a solar mass, $M_\odot = 2.00 \times 10^{30} \text{ kg}$, which is distributed in a region of radius of 238 light-years. Third, for $m = 10^{-25}$ the corresponding mass is about the mass of the Milky Way (including dark matter), and the radius is 2×10^6 light-years, which is about 4 times the radius of the stellar disk. According to Table IV and (D3), for all the above configurations the central density is $\mu_c = 54.55 \mu_\Lambda = 3.7 \times 10^{-22} \text{ kg/m}^3$.

The maximal-amplitude boson stars or oscillatons, with $\varepsilon \approx \varepsilon_{\text{max}}$ generally have much bigger masses and smaller sizes than the states with 1% mass loss discussed in the previous paragraphs. For example, the $\varepsilon = 0.5$ oscillaton for the axion case has a mass $M = 2.3 \times 10^{25} \text{ kg}$ (about four Earth masses) and radius $r_{95} = 18 \text{ cm}$. Only considering the mass loss (125) induced by the cosmological constant, the maximal-amplitude boson stars or oscillatons have extremely long lifetime. For the axion case, using (135), we get $h = 5.1 \times 10^{-28}$, and the relative mass loss during a Hubble time period is of the order $10^{-10^{27}}$, certainly negligible. For boson stars formed by complex fields, this is the only mechanism to lose mass, consequently, close to maximal mass boson stars have practically constant mass. However, oscillatons lose mass even in the $\Lambda = 0$ case by (129) which gives $\frac{T_H}{M} \frac{dM}{dt} = -3.1 \times 10^{10}$ for $\varepsilon = 0.5$, showing that this state is unstable on the cosmological time scale. However, the mass loss decreases exponentially with ε . An initially maximal mass oscillaton created in the early Universe loses mass relatively quickly, and its amplitude parameter decreases to about $\varepsilon = 0.31$ during a Hubble time period, where according to (129) the relative mass loss rate is $\frac{T_H}{M} \frac{dM}{dt} = -0.14$. For more details and different scalar field masses, see Table VIII of [35].

VII. INFLATIONARY ERA

The energy density of the inflaton field during the inflationary epoch can be estimated as $\mu_\Lambda = (10^{16} \text{ GeV})^4$ (see e.g. [26]). This expression is valid in natural units, where $c = \hbar = 1$ (but $G \neq 1$) and the energy is measured in electron volts. In ordinary units this corresponds to the mass density $\mu_\Lambda = 2.3 \times 10^{84} \text{ kg/m}^3$. Assuming a de Sitter geometry, the Hubble constant is

$$H = \sqrt{\frac{8\pi G\mu_\Lambda}{3}} = 3.6 \times 10^{37} \frac{1}{s} = 2.4 \times 10^{13} \text{ GeV}, \quad (137)$$

in ordinary and natural units. By (26) this belongs to a cosmological constant

$$\Lambda = \frac{3H^2}{c^2} = 4.3 \times 10^{58} \frac{1}{m^2} = 1.1 \times 10^{-11}, \quad (138)$$

in ordinary and Planck units, respectively.

If in addition to the inflaton field there is another scalar field χ on this de Sitter background, and it is massive, then it is likely to form localized oscillaton configurations (or boson stars in the case of a complex field). This second field may be, for example, the waterfall field in a hybrid inflation theory [29,30]. If the mass parameter of the field χ is m , then we can apply (135) to relate the small-amplitude parameter ε to the rescaled Hubble constant h . The parameter h is the essential parameter determining the mass loss rate of the object. For example, if the mass loss extrapolated for a Hubble time period is assumed to be 1%, then $h = h_{0.01} = 0.07985$, and for the inflationary era, we obtain

$$\varepsilon_{0.01}^2 = 3.0 \times 10^{14} \frac{\text{GeV}}{mc^2}. \quad (139)$$

Since oscillatons and boson stars are stable only if the amplitude parameter satisfies $\varepsilon < \varepsilon_{\max} \approx 0.5$, states with 1% mass loss can only exist for $m > 1.2 \times 10^{15} \text{ GeV}/c^2$. For objects formed from fields with $m < 1.2 \times 10^{15} \text{ GeV}/c^2$, necessarily $\varepsilon < \varepsilon_{\max} < \varepsilon_{0.01}$. Then from (135) it follows that $h > h_{0.01}$, which implies that the mass loss during the Hubble time is always larger than 1%. Oscillatons and boson stars formed from scalar fields with mass $m < 10^{15} \text{ GeV}/c^2$ are very short living during the inflationary era. Another way to see why this lower limit is necessary is to calculate the central density μ_c by (65), and observe that for small scalar field mass $\mu_c < \mu_\Lambda$, and hence long-living oscillatons are not expected to exist.

For scalar field masses $m > 1.2 \times 10^{15} \text{ GeV}/c^2$, we list the properties of some oscillatons with 1% mass loss in

TABLE VI. Oscillaton or boson star configurations for which 1% of the mass is lost during a Hubble time period in the inflationary era. For each scalar field mass m the amplitude parameter ε , the total mass M , and the radius containing 95% of the mass is listed. In the last column we give the relation to the $\varepsilon = \varepsilon_{\max}$ state, which has maximal mass and minimal radius.

$\frac{mc^2}{\text{GeV}}$	$\varepsilon_{0.01}$	$\frac{M}{\text{kg}}$	$\frac{r_{95}}{\text{m}}$	$\frac{M_{\max}}{M} = \frac{r_{95}^{\max}}{r_{95}^{\min}}$
1.2×10^{15}	0.50	1.9×10^{-4}	1.5×10^{-30}	1.0
10^{16}	0.17	8.0×10^{-6}	5.1×10^{-31}	2.9
10^{17}	0.054	2.5×10^{-7}	1.6×10^{-31}	9.2
10^{18}	0.017	8.0×10^{-9}	5.1×10^{-32}	29
10^{19}	0.0054	2.5×10^{-10}	1.6×10^{-32}	92

TABLE VII. Relative mass loss rates extrapolated for a Hubble time period of maximal-amplitude configurations during the inflationary era. For each choice of scalar field mass m the rescaled Hubble constant h , the ratio μ_c/μ_Λ , the mass loss rate induced by the cosmological constant, and the $\Lambda = 0$ oscillaton mass loss rate are listed.

$\frac{mc^2}{\text{GeV}}$	h	$\frac{\mu_c}{\mu_\Lambda}$	$\frac{T_H}{M} \frac{dM}{dt}$	$\frac{T_H}{M} \frac{dM}{dt} _{\Lambda=0}$
10^{15}	0.095	38	0.10	1.7×10^{-16}
2×10^{15}	0.047	150	1.5×10^{-7}	3.3×10^{-16}
3×10^{15}	0.032	350	5.8×10^{-14}	5.0×10^{-16}
4×10^{15}	0.024	620	1.4×10^{-20}	6.6×10^{-16}
5×10^{15}	0.019	970	2.3×10^{-27}	8.3×10^{-16}

Table VI. All these states have the central density $\mu_c = 54.55\mu_\Lambda = (2.7 \times 10^{16} \text{ GeV})^4 = 1.3 \times 10^{86} \text{ kg/m}^3$, and the $\Lambda = 0$ mass loss is much smaller than 1% for them.

The mass loss rate induced by the cosmological constant decreases very quickly when considering higher-amplitude configurations. The behavior of the $\Lambda = 0$ oscillaton mass loss is just the opposite. The two kind of relative mass loss extrapolated for a Hubble time period of some maximal-amplitude states belonging to $\varepsilon = \varepsilon_{\max} \approx 0.5$ are given in Table VII.

VIII. REHEATING

Oscillatons are likely to form after the end of inflation, in the reheating era, when the inflaton field is oscillating around the potential minimum. These oscillatons are likely to influence the efficiency of how the inflaton's energy is transferred to other fields. Since oscillatons are concentrated energy lumps, their influence on the formation of inhomogeneities may also be important. At the reheating stage the equation of state can be well approximated by a pressureless fluid, and hence the effective cosmological constant can be taken to be zero. For the lifetime of oscillatons in this era, one can apply the considerations given in [35] which are valid in the $\Lambda = 0$ asymptotically flat case. The mass of the inflaton field when it is near the vacuum value can be estimated as $m = 10^{13} \text{ GeV}/c^2$ (see e.g. [28]). The maximal-amplitude stable oscillaton which can be formed by this field, when $\varepsilon = \varepsilon_{\max} \approx 0.5$, has the total mass $M = 0.023 \text{ kg}$ and radius $r_{95} = 1.8 \times 10^{-28} \text{ m}$. The mass and radius of smaller-amplitude oscillatons formed by this field can be easily obtained by using the fact that the total mass is proportional to ε and the radius is proportional to $1/\varepsilon$.

If the cosmological constant is zero, then a natural time scale is the oscillation period, $T_\omega = 2\pi/\omega$. According to (60), for small ε we can replace ω by m . Taking the total mass from (B15), the part of the mass lost during an oscillation period is

$$\frac{T_\omega}{M} \frac{dM}{dt} = -\frac{4\pi c_1}{s_1 \varepsilon^3} \exp\left(-\frac{c_2}{\varepsilon}\right). \quad (140)$$

This expression is independent of the scalar field mass m and on the units that we use for measuring time. Even for the maximal-amplitude oscillaton with $\varepsilon = 0.5$ this gives a very tiny value, $\frac{T_\omega}{M} \frac{dM}{dt} = -2.5 \times 10^{-17}$. This shows that oscillatons formed in the reheating period perform a large number of oscillations before the mass loss by emitting classical scalar field radiation becomes apparent. However, since their oscillation period is extremely short, $T_\omega = 4.1 \times 10^{-37}$ s, they lose about half of their mass in 8.3×10^{-21} s. On the other hand, since this mass loss depends exponentially on ε , even during a period corresponding to the lifetime of the Universe the amplitude parameter does not decrease much below $\varepsilon = 0.166$, where $\frac{T_H}{M} \times \frac{dM}{dt} = -0.13$.

IX. CONCLUSIONS

We have constructed spherically symmetric, spatially localized time dependent solutions in a large class of scalar theories coupled to Einstein's theory of gravitation and a positive cosmological constant, Λ . Examples include boson star-type objects (for a complex field) and oscillatons (for a real field), considered previously for vanishing cosmological constant. A positive constant has important qualitative effects on boson stars, in that due to the repulsion induced by Λ only radiating solutions exist, leading to a mass loss. For sufficiently small values of Λ this effect is negligible, so boson stars with almost unchanged mass may be present in our Universe. Even when $\Lambda = 0$ oscillatons radiate slowly, and the consequent mass loss is maximal for large amplitudes. There is an additional mass loss of oscillatons induced by a nonzero $\Lambda > 0$, which dominates for small amplitudes (corresponding to large sizes). The mass loss of an oscillaton during a Hubble time period, is completely determined by the ratio μ_c/μ_Λ , where μ_Λ is the energy density due to Λ and μ_c is that of the oscillaton in its center. Our computations are valid in spatial dimensions $2 < D < 6$, although the case $D = 3$ has been given special attention. An important cosmological application of our results is for "waterfall"-type oscillatons appearing in hybrid inflation models.

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APPENDIX A: CONNECTION FORMULAE

In this appendix we derive the formulae (104) connecting the amplitudes in the $y < 1$ and $y > 1$ regions. We follow the method described in [50]. Since the expressions for Z obtained by the WKB method are singular at $y = 1$, it is necessary to find a solution of (75) which is valid in a region around this point. Let us introduce a new radial

coordinate by

$$y = 1 + x. \quad (\text{A1})$$

We approximate p^2 for small h by

$$p^2 \approx y^2 - 1 = x(x + 2) \approx 2x. \quad (\text{A2})$$

Introducing a rescaled coordinate z by

$$x = -z \frac{h^{2/3}}{2^{1/3}}, \quad (\text{A3})$$

(75) takes the form

$$\frac{d^2 Z}{dz^2} - zZ = 0. \quad (\text{A4})$$

The solution can be written in terms of Airy functions

$$Z = a \text{Ai}z + b \text{Bi}z. \quad (\text{A5})$$

For $z \gg 0$, this has the asymptotic form

$$Z = \frac{1}{\sqrt{\pi}z^{1/4}} \left[\frac{a}{2} \exp\left(-\frac{2}{3}z^{3/2}\right) + b \exp\left(\frac{2}{3}z^{3/2}\right) \right], \quad (\text{A6})$$

and for $z \ll 0$

$$Z = \frac{1}{\sqrt{\pi}(-z)^{1/4}} \left\{ a \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] + b \cos\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] \right\}. \quad (\text{A7})$$

We consider two matching regions surrounding $y = 1$, where $|z| \gg 1$ and at the same time $|x| \ll 1$. Both of these can hold simultaneously if h is small enough. First we continue through the matching region with $x < 0$. Substituting $y = 1 + x$ into (91) and expanding in power series around $x = 0$, we obtain

$$\frac{i}{h} \int_1^y p_0 dy = \frac{1}{3h} (-2x)^{3/2} + \mathcal{O}(x^{5/2}). \quad (\text{A8})$$

This result can also be obtained by simply approximating p_0 by $\sqrt{2x}$. In this region the influence of the s_1 term is subleading, so for $3 \leq D \leq 5$ dimensions, we obtain both from (89) and (99)

$$Z = \frac{A_+}{(-2x)^{1/4}} \exp\left[\frac{1}{3h} (-2x)^{3/2}\right]. \quad (\text{A9})$$

Comparing with (A6), we get

$$a = 0, \quad b = \frac{\sqrt{\pi}}{(2h)^{1/6}} A_+. \quad (\text{A10})$$

Continuing into the second matching region with $x > 0$, (A7) yields

$$Z = \frac{b}{2\sqrt{\pi}(-z)^{1/4}} \left\{ \exp\left(\frac{i\pi}{4}\right) \exp\left[i\frac{2}{3}(-z)^{3/2}\right] + \exp\left(-\frac{i\pi}{4}\right) \exp\left[-i\frac{2}{3}(-z)^{3/2}\right] \right\}. \quad (\text{A11})$$

For small positive x (92) gives

$$\frac{i}{h} \int_1^y p_0 dy = \frac{i}{3h} (2x)^{3/2} + \mathcal{O}(x^{5/2}). \quad (\text{A12})$$

To leading order both (90) and (100) can be written as

$$Z = \frac{B_{\pm}}{(2x)^{1/4}} \exp\left[\pm \frac{i}{3h} (2x)^{3/2}\right]. \quad (\text{A13})$$

Comparing with (A11) gives

$$B_{\pm} = \frac{b(2h)^{1/6}}{2\sqrt{\pi}} \exp\left(\pm \frac{i\pi}{4}\right). \quad (\text{A14})$$

Substituting b from (A10) yields the desired formulae (104).

APPENDIX B: MASS IN SPHERICALLY SYMMETRIC ASYMPTOTICALLY DE SITTER SPACE-TIMES

Since the scalar field tends to zero exponentially, at large distances the metric approaches the static Schwarzschild-de Sitter metric. Considering $D + 1$ -dimensional space-times, in Schwarzschild area coordinates the Schwarzschild-de Sitter metric has the form

$$ds^2 = -F(\bar{r})dt^2 + \frac{1}{F(\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega_{D-1}^2, \quad (\text{B1})$$

where

$$F(\bar{r}) = 1 - \frac{r_0^{D-2}}{\bar{r}^{D-2}} - H^2 \bar{r}^2 \quad (\text{B2})$$

and r_0 is a constant related to the mass M by

$$M = \frac{(D-1)\pi^{D/2}}{8\pi\Gamma(\frac{D}{2})} r_0^{D-2}. \quad (\text{B3})$$

In general, spherically symmetric space-times there is a naturally defined radius function \hat{r} , defined in terms of the area of the symmetry spheres. In Schwarzschild coordinates $\hat{r} = \bar{r}$, while in the isotropic coordinates (19) $\hat{r} = r\sqrt{B}$. In terms of this radius function one can define the Misner-Sharp energy (or local mass) function \hat{m} [51,52], which can be defined for arbitrary dimensions by

$$\hat{m} = \frac{(D-1)\pi^{D/2}}{8\pi\Gamma(\frac{D}{2})} \hat{r}^{D-2} (1 - g^{ab} \hat{r}_{,a} \hat{r}_{,b} - H^2 \hat{r}^2). \quad (\text{B4})$$

For the Schwarzschild-de Sitter metric $\hat{m} = M$. At infinity the mass function \hat{m} agrees with the Abbott-Deser mass [53,54], which is a Killing energy defined in static asymptotically de Sitter space-times.

The Misner-Sharp energy \hat{m} also agrees with a conserved energy that can be defined using the Kodama vector. The Kodama vector [55,56] is defined by

$$K^a = \epsilon^{ab} \hat{r}_{,b}, \quad (\text{B5})$$

where ϵ_{ab} is the volume form in the (t, r) plane. Choosing the orientation such that $\epsilon_{rt} = \sqrt{AB}$ makes K^a future pointing, with nonvanishing components

$$K^t = \frac{\hat{r}_{,r}}{\sqrt{AB}}, \quad K^r = -\frac{\hat{r}_{,t}}{\sqrt{AB}}. \quad (\text{B6})$$

It can be checked that, in general, the Kodama vector is divergence free, $K^a{}_{;a} = 0$. Since contracting with the Einstein tensor, $G^{ab} K_{a;b} = 0$, the current

$$J_a = T_{ab} K^b \quad (\text{B7})$$

is also divergence free, $J^a{}_{;a} = 0$, it defines a conserved charge. Integrating on a constant t hypersurface with a future oriented unit normal vector n^a , the conserved charge is

$$E = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^r \hat{r}^{D-1} \sqrt{B} n^a J_a dr \\ = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^r \frac{\hat{r}^{D-1}}{A} (T_{tt} \hat{r}_{,r} - T_{tr} \hat{r}_{,t}) dr. \quad (\text{B8})$$

It can be checked that the derivative of the mass function \hat{m} can be expressed in terms of the current J_a ,

$$\hat{m}_{,a} = -\frac{2\pi^{D/2} \hat{r}^{D-1}}{\Gamma(\frac{D}{2})} \epsilon_{ab} J^b. \quad (\text{B9})$$

For the radial derivative, it follows that

$$\hat{m}_{,r} = \frac{2\pi^{D/2} \hat{r}^{D-1}}{\Gamma(\frac{D}{2}) A} (T_{tt} \hat{r}_{,r} - T_{tr} \hat{r}_{,t}), \quad (\text{B10})$$

which, comparing with (B8), gives $E = \hat{m}$.

The time derivative of the mass function is

$$\hat{m}_{,t} = \frac{2\pi^{D/2} \hat{r}^{D-1}}{\Gamma(\frac{D}{2}) B} (T_{rt} \hat{r}_{,r} - T_{rr} \hat{r}_{,t}). \quad (\text{B11})$$

This equation describes the mass loss caused by the outward energy current of the massive scalar field.

For large radii the function \hat{m} tends to the total mass M of the oscillon or boson star. Substituting the small-amplitude expansions (34) and (35) into the definition (B4) of the mass function \hat{m} , to leading order in ε , we obtain

$$\hat{m} = -\varepsilon^{4-D} \frac{\pi^{D/2} (D-1) \rho^{D-1}}{8\pi m^{D-2} \Gamma(\frac{D}{2})} \frac{dB_2}{d\rho}. \quad (\text{B12})$$

We have seen in Sec. III B and III C that for both the boson star and oscillon case $B_2 = A_2/(2-D)$ and $A_2 = -1 - s$, giving

$$\hat{m} = -\frac{\varepsilon^{4-D} \pi^{D/2} (D-1) \rho^{D-1}}{8\pi(D-2)m^{D-2}\Gamma(\frac{D}{2})} \frac{ds}{d\rho}. \quad (\text{B13})$$

Since the asymptotic behavior of s is given by (63),

$$M = \varepsilon^{4-D} \frac{(D-1)\pi^{D/2}}{8\pi m^{D-2}\Gamma(\frac{D}{2})} s_1. \quad (\text{B14})$$

For 3 + 1 dimensional space-time,

$$M_{(D=3)} = \varepsilon \frac{s_1}{2m}. \quad (\text{B15})$$

For small h , the numerical value of the constant s_1 can be approximated by the $h = 0$ value given in Table II. Measuring the mc^2 belonging to the scalar field in electron volts and the mass of the boson star or oscillaton in kilograms,

$$M_{(D=3)} = 4.657 \times 10^{20} \text{ kg} \frac{\text{eV}}{mc^2} \varepsilon. \quad (\text{B16})$$

APPENDIX C: PROPER MASS AND CORE RADIUS

The proper mass inside a sphere of radius r is defined by the D -dimensional volume integral of the energy density μ ,

$$M_p(r) = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^r dr \mu B^{D/2} r^{D-1}. \quad (\text{C1})$$

Substituting the leading order expression (64) for μ , and the leading order value $B = 1$ from (35), using the rescaled radial coordinate $\rho = \varepsilon mr$,

$$M_p(\rho) = \frac{\varepsilon^{4-D} \pi^{D/2} (D-1)}{8\pi(D-2)m^{D-2}\Gamma(\frac{D}{2})} \int_0^\rho d\rho \rho^{D-1} S^2. \quad (\text{C2})$$

The integral can be performed using the SN Eq. (54),

$$M_p(\rho) = -\frac{\varepsilon^{4-D} \pi^{D/2} (D-1) \rho^{D-1}}{8\pi(D-2)m^{D-2}\Gamma(\frac{D}{2})} \frac{ds}{d\rho}. \quad (\text{C3})$$

To leading order in ε the calculated proper mass M_p agrees with the mass function \hat{m} given by (B13). However, at higher order the proper mass is expected to be larger by an amount corresponding to the binding energy.

Oscillatons and boson stars do not have a definite outer surface. A natural definition for their size is to take the radius r_n inside which n percentage of the mass energy can be found. It is usual to take, for example, $n = 95$ or 99.9 . The mass energy inside a given radius r can be defined either by the integral (C1) or by taking the local mass function \hat{m} in (B4). To leading order in ε both definitions give (C3). The rescaled radius ρ_n can be defined by

$$\frac{M_p(\rho_n)}{M_p(\infty)} = \frac{n}{100}. \quad (\text{C4})$$

The numerical values of ρ_n for various n in $D = 3, 4$, and 5

TABLE VIII. The radius inside which given percentage of the mass energy is contained for various spatial dimensions.

	$D = 3$	$D = 4$	$D = 5$
ρ_{50}	2.240	1.778	1.317
ρ_{90}	3.900	3.013	2.284
ρ_{95}	4.471	3.455	2.652
ρ_{99}	5.675	4.410	3.478
$\rho_{99.9}$	7.239	5.692	4.634

dimensions are listed in Table VIII. The physical radius is

$$r_n = \frac{\rho_n}{\varepsilon m}. \quad (\text{C5})$$

In ordinary units, measuring the scalar field mc^2 in electron volts and r_n in meters (Roman m),

$$r_n = 1.97 \times 10^{-7} \text{ m} \frac{\rho_n}{\varepsilon} \frac{\text{eV}}{mc^2}. \quad (\text{C6})$$

We note that for complex field boson stars the current

$$j^a = \frac{i}{2} g^{ab} (\Phi^* \Phi_{,b} - \Phi_{,b}^* \Phi) \quad (\text{C7})$$

can be used to define a conserved quantity. The conserved particle number N can be defined by integrating the time component j^0 for a spacelike slice. To leading order in ε the function j^0 is proportional to the density μ given in (64). Hence, to this order, the use of the particle number instead of the mass energy yields the same result for the radius of boson stars.

APPENDIX D: PRESENT VALUE OF THE COSMOLOGICAL CONSTANT

For the present observational value of the Hubble constant, we take

$$H_0 = 70.4 \frac{\text{km}}{\text{s Mpc}} = 2.28 \times 10^{-18} \frac{1}{\text{s}} \quad (\text{D1})$$

according to the combined 7-year Wilkinson Microwave Anisotropy Probe (WMAP) data [57]. To this corresponds a critical density

$$\mu_{cr} = \frac{3H_0^2}{8\pi G} = 9.31 \times 10^{-24} \frac{\text{kg}}{\text{m}^3}. \quad (\text{D2})$$

The fraction of this belonging to the dark energy provided by the cosmological constant is given by $\Omega_\Lambda = 0.728$, according to WMAP results [57]. Consequently, the energy density belonging to the cosmological constant is

$$\mu_\Lambda = \mu_{cr} \Omega_\Lambda = 6.78 \times 10^{-24} \frac{\text{kg}}{\text{m}^3}. \quad (\text{D3})$$

According to (31), this corresponds to a cosmological constant of

$$\Lambda = \frac{8\pi G\mu_\Lambda}{c^2} = 1.26 \times 10^{-52} \frac{1}{\text{m}^2}. \quad (\text{D4})$$

In Planck units, since the Planck length is $l_P = 1.616252 \times 10^{-35} \text{ m}$,

$$\Lambda = 3.30 \times 10^{-122}. \quad (\text{D5})$$

Since we are interested in the effect of the cosmological constant, we neglect the ordinary and dark matter content of the Universe. Then the space-time can be described by a de Sitter metric with Hubble constant H given by (26). In Planck, ordinary and natural units

$$\begin{aligned} H &= 1.05 \times 10^{-61} = 1.95 \times 10^{-18} \frac{1}{\text{s}} \\ &= 1.28 \times 10^{-33} \text{ eV}. \end{aligned} \quad (\text{D6})$$

The Hubble time is then

$$T_H = \frac{1}{H} = 5.14 \times 10^{17} \text{ s} = 16.3 \text{ Gyr}. \quad (\text{D7})$$

Even though the ordinary and dark matter content have been neglected, this value is still not too far from the age of the Universe, which according to WMAP data is 13.75 Gyr assuming the Λ CDM model [57].

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- [1] R. F. Dashen, B. Hasslacher, and A. Neveu, *Phys. Rev. D* **11**, 3424 (1975).
 - [2] I. L. Bogolyubskii, and V. G. Makhan'kov, *JETP Lett.* **25**, 107 (1977).
 - [3] E. J. Copeland, M. Gleiser, and H.-R. Müller, *Phys. Rev. D* **52**, 1920 (1995).
 - [4] H. Segur and M. D. Kruskal, *Phys. Rev. Lett.* **58**, 747 (1987).
 - [5] G. Fodor, P. Forgács, Z. Horváth, and M. Mezei, *Phys. Rev. D* **79**, 065002 (2009).
 - [6] G. Fodor, P. Forgács, Z. Horváth, and M. Mezei, *Phys. Lett. B* **674**, 319 (2009).
 - [7] M. Gleiser and J. Thorarinson, *Phys. Rev. D* **76**, 041701 (R) (2007).
 - [8] M. Gleiser and J. Thorarinson, *Phys. Rev. D* **79**, 025016 (2009).
 - [9] E. Farhi, N. Graham, V. Khemani, R. Markov, and R. Rosales, *Phys. Rev. D* **72**, 101701(R) (2005).
 - [10] N. Graham, *Phys. Rev. Lett.* **98**, 101801 (2007).
 - [11] N. Graham, *Phys. Rev. D* **76**, 085017 (2007).
 - [12] E. Seidel and W.-M. Suen, *Phys. Rev. Lett.* **66**, 1659 (1991).
 - [13] E. Seidel and W.-M. Suen, *Phys. Rev. Lett.* **72**, 2516 (1994).
 - [14] L. A. Ureña-López, T. Matos, and R. Becerril, *Classical Quantum Gravity* **19**, 6259 (2002).
 - [15] M. Alcubierre, R. Becerril, F. S. Guzmán, T. Matos, D. Núñez, and L. A. Ureña-López, *Classical Quantum Gravity* **20**, 2883 (2003).
 - [16] D. N. Page, *Phys. Rev. D* **70**, 023002 (2004).
 - [17] J. Balakrishna, R. Bondarescu, G. Daues, and M. Bondarescu, *Phys. Rev. D* **77**, 024028 (2008).
 - [18] M. Broadhead and J. McDonald, *Phys. Rev. D* **72**, 043519 (2005).
 - [19] M. Gleiser, *Int. J. Mod. Phys. D* **16**, 219 (2007).
 - [20] N. Graham and N. Stamatopoulos, *Phys. Lett. B* **639**, 541 (2006).
 - [21] E. Farhi, N. Graham, A. H. Guth, N. Iqbal, R. R. Rosales, and N. Stamatopoulos, *Phys. Rev. D* **77**, 085019 (2008).
 - [22] M. A. Amin and D. Shirokoff, *Phys. Rev. D* **81**, 085045 (2010).
 - [23] M. A. Amin, *arXiv:1006.3075*.
 - [24] M. P. Hertzberg, *arXiv:1003.3459* [*Phys. Rev. D* (to be published)].
 - [25] M. Gleiser, N. Graham, and N. Stamatopoulos, *Phys. Rev. D* **82**, 043517 (2010).
 - [26] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation* (Cambridge University Press, Cambridge, UK, 2009).
 - [27] S. Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008).
 - [28] V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge (Cambridge University Press, Cambridge, UK, 2005).
 - [29] A. D. Linde, *Phys. Lett. B* **259**, 38 (1991).
 - [30] A. D. Linde, *Phys. Rev. D* **49**, 748 (1994).
 - [31] D. J. Kaup, *Phys. Rev.* **172**, 1331 (1968).
 - [32] R. Ruffini and S. Bonazzola, *Phys. Rev.* **187**, 1767 (1969).
 - [33] P. Jetzer, *Phys. Rep.* **220**, 163 (1992).
 - [34] F. E. Schunk and E. W. Mielke, *Classical Quantum Gravity* **20**, R301 (2003).
 - [35] G. Fodor, P. Forgács, and M. Mezei, *Phys. Rev. D* **81**, 064029 (2010).
 - [36] R. Friedberg, T. D. Lee, and Y. Pang, *Phys. Rev. D* **35**, 3640 (1987).
 - [37] S. Kichenassamy, *Commun. Pure Appl. Math.* **44**, 789 (1991).
 - [38] G. Fodor, P. Forgács, Z. Horváth, and Á. Lukács, *Phys. Rev. D* **78**, 025003 (2008).
 - [39] G. Fodor, P. Forgács, Z. Horváth, and M. Mezei, *J. High Energy Phys.* **08** (2009) 106.
 - [40] L. Diósi, *Phys. Lett.* **105A**, 199 (1984).
 - [41] R. Penrose, *Phil. Trans. R. Soc. A* **356**, 1927 (1998).

- [42] I. M. Moroz, R. Penrose, and P. Tod, *Classical Quantum Gravity* **15**, 2733 (1998).
- [43] P. Tod and I. M. Moroz, *Nonlinearity* **12**, 201 (1999).
- [44] R. Friedberg, T. D. Lee, and Y. Pang, *Phys. Rev. D* **35**, 3640 (1987).
- [45] R. Ferrell and M. Gleiser, *Phys. Rev. D* **40**, 2524 (1989).
- [46] J. Eiesland, *Trans. Am. Math. Soc.* **27**, 213 (1925).
- [47] K. Schleich and D. M. Witt, [arXiv:0908.4110](https://arxiv.org/abs/0908.4110).
- [48] J. Morrow-Jones and D. M. Witt, *Phys. Rev. D* **48**, 2516 (1993).
- [49] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), 3rd ed..
- [50] D. J. Griffiths, *Introduction to Quantum Mechanics* (Prentice Hall, New Jersey, 1995).
- [51] C. W. Misner and D. H. Sharp, *Phys. Rev.* **136**, B571 (1964).
- [52] K. Nakao, [arXiv:gr-qc/9507022](https://arxiv.org/abs/gr-qc/9507022).
- [53] L. Abbott and S. Deser, *Nucl. Phys.* **B195**, 76 (1982).
- [54] K. Nakao, T. Shiromizu, and K. Maeda, *Classical Quantum Gravity* **11**, 2059 (1994).
- [55] H. Kodama, *Prog. Theor. Phys.* **63**, 1217 (1980).
- [56] S. A. Hayward, *Phys. Rev. D* **53**, 1938 (1996).
- [57] E. Komatsu *et al.*, [arXiv:1001.4538](https://arxiv.org/abs/1001.4538).