Applications of Stochastic Inventory Control in Market-Making and Robust Supply Chains

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Abstract

This dissertation extends the classical inventory control model to address stochastic inventory control problems raised in market-making and robust supply chains.

In the financial market, market-makers assume the role of a counterpart so that investors can trade any fixed amounts of assets at quoted bid or ask prices at any time. Market-makers benefit from the spread between the bid and ask prices, but they have to carry inventories of assets which expose them to potential losses when the market price moves in an undesirable direction. One approach to reduce the risk associated with price uncertainty is to actively trade with other market-makers at the price of losing potential spread gain.

We propose a dynamic programming model to determine the optimal active trading quantity, which maximizes the market-maker's expected utility. For a single-asset model, we show that a threshold inventory control policy is optimal with respect to both an exponential utility criterion and a mean-variance tradeoff objective. Special properties such as symmetry and monotonicity of the threshold levels are also investigated. For a multiple-asset model, the mean-variance analysis suggests that there exists a connected no-trade region such that the market-maker does not need to actively trade with other market-makers if the inventory falls in the no-trade region. Outside the no-trade region, the optimal way to adjust inventory levels can be obtained from the boundaries of the no-trade region. These properties of the optimal policy lead to practically efficient algorithms to solve the problem.

The dissertation also considers the stochastic inventory control model in robust supply chain systems. Traditional approaches in inventory control first estimate the demand distribution among a predefined family of distributions based on data fitting of historical demand observations, and then optimize the inventory control policy using the estimated distributions, which often leads to fragile solutions in case the preselected family of distributions was inadequate. In this work, we propose a minimax robust model that integrates data fitting and inventory optimization for the single item multi-period periodic review stochastic lot-sizing problem. Unlike the classical stochastic inventory models, where demand distribution is known, we assume that histograms are part of the input. The robust model generalizes Bayesian model, and it can be interpreted as minimizing history dependent risk measures. We prove that the optimal inventory control policies of the robust model share the same structure as the traditional stochastic dynamic programming counterpart. In particular, we analyze the robust models based on the chi-square goodness-of-fit test. If demand samples are obtained from a known distribution, the robust model converges to the stochastic model with true distribution under general conditions.

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Chapter 1

Introduction

In supply chain management, inventory refers to raw materials, work-in-process goods and finished products which are held available in stock and will be used to satisfy production needs or customer demands in the future. Inventory enables businesses to cover the needs for stock occurred during the manufacturing or delivery lead times. It also provides a buffer to protect against fluctuations in customer demands as well as uncertainties in the supply process. These two aspects imply that inventory ensures satisfying demand from customers or downstream production processes. In addition, it enables taking advantage of economies of scale in purchasing, production, transportation and storage by ordering or producing more than immediate demands and storing the rest as inventory. That is, inventory management strategies can also contribute to reducing total supply chain costs. Indeed, according to the U.S. Census Bureau [52], manufacturers' and trade inventories were estimated at $1.35 trillion in April 2010, and this figure is 1.23 times the sales in that month.

Unfortunately, it is usually costly to hold inventory. This cost includes the opportunity cost associated with the capital invested in inventory, the cost of capital to finance inventory, warehousing cost, handling cost, costs associated with obsolescence and shrinkage, and insurance and taxation. Atkinson [5] pointed out that the annual inventory holding cost is approximately 15~35% of the goods' actual value.

Therefore, the key in inventory management is to achieve a tradeoff between the benefits and the costs of holding inventory, which implies the need to have the right amount of inventory at the right place and at the right time so as to balance system-wide cost and the service level. The vast amount of money invested in inventory – inventory holding cost is a significant percentage of the inventory value – is critical to the success of any business. Indeed, the success stories of giants such as Wal-Mart, Dell or Amazon demonstrate the importance of effective inventory management strategies.

The literature on inventory theory can be dated back to the beginning of last century. It is widely believed that the first inventory model is the economic order quantity (EOQ) model attributed to Harris [22]. The EOQ model assumes a constant and deterministic demand, and identifies a closed form solution which corresponds to the optimal tradeoff between an inventory holding cost and a fixed ordering cost representing economies of scale in ordering.

Wagner and Whitin [54] introduced the dynamic economic lot-size (DEL) model which considers an inventory system with deterministic time-varying demands over a discrete finite planning horizon. The cost structure is similar to that in the EOQ model and a shortest-path algorithm is developed to solve the problem.

The seminal papers of Arrow et al. [3] and Dvoretzky et al. [16] explicitly model stochastic demands using discrete-time dynamic programming formulations. Most stochastic single-commodity single-location inventory models including those proposed in this dissertation can be regarded as extensions of this fundamental model. In particular, Scarf [47] studies a stochastic counterpart of the DEL model in Wagner and Whitin [54]. He shows that an (s, S) policy is optimal by introducing and applying the notion of K-convexity. In such a policy, whenever the inventory level drops below s, an order is placed to raise the inventory level to S. Otherwise, no action is required.

Other influential research, especially those studying multiple-location inventory models, include but not limited to Roundy [39] which develops a 98% optimal strategy for the single warehouse multiple-retailer inventory system under assumptions similar to the EOQ model, and Clark and Scarf [15] which establish the optimal policy for a serial system with stochastic demands. Detailed surveys in inventory theory are provided in Porteus [42], Simchi-Levi et al. [49] and Zipkin [55].

In this dissertation, we extend the line of research in stochastic inventory control started by Arrow et al. [3] and Dvoretzky et al. [16] to investigate inventory problems associated with market-making in finance and robust systems in supply chains. Similar to the early works, we adopt the discrete-time dynamic programming framework to formulate these problems. Our objective is to identify the structures of optimal control policy in each case, and investigate properties of these policies so that efficient algorithms can be developed.

The thesis is organized as follows. Inventory management problems in market-making are analyzed in Chapter 2, where we focus on a single asset with an exponential utility objective function, Chapter 3, where we consider a single asset with a mean-variance objective function, and Chapter 4, where the focus is on the mean-variance analysis for multiple assets with correlated price movements. Chapter 5 considers the applications of inventory model in robust supply chains where the complete information about demand distributions, i.e., the cumulative distribution functions of the demands, is unknown. We conclude the dissertation in Chapter 6 with a discussion of possible directions for future research.

In the rest of this chapter, we introduce the background and motivation for this research as well as our contribution in each area: market-making and robust supply chains.

1.1 Market-Making

Investors trade foreign currencies, securities and other financial products frequently. Unfortunately, there is no guarantee that every investor who wishes to buy (or sell) a certain amount of asset will find a counterparty willing to sell (or buy) the same amount at that time. This is exactly the objective of the so-called "market-makers": to facilitate the trading process for most financial products. That is, the market-maker is ready to assume the role of a counterparty when one wishes to buy or sell financial products. For example, each stock traded on the New York Stock Exchange

(NYSE) has a market-maker called "specialist", whose sole responsibility is to serve as a market-maker for this particular stock.

Typically, market-makers quote a pair of bid/ask prices to clients and have the obligation to buy/sell at the quoted prices if their clients wish to deal at these prices. Over time, market-makers buy at bid price and sell at ask price, which is higher than the bid price at any given instant. Their objective is to profit from the "spread" between bid and ask prices, not from price movements. In that regard, they are different from ordinary investors, who seek to profit by betting on price moves.

Market-makers encounter difficulties when receiving consecutive trades in the same direction. For example, suppose a foreign currency market-maker holds no foreign currency initially and receives a series of sell orders afterwards (i.e., the clients sell to the market-maker), the market-maker's holding position becomes very large and positive.[1] This is potentially very risky because if the foreign currency depreciates, the market-maker will lose a considerable amount. For a risk-averse market-maker, this is certainly undesirable. Thus, he cannot simply wait for the arrival of a client (who wishes to buy the foreign currency from him) and sell to this client to bring his holding position back to zero.

To reduce the risk and avoid such situations, the market-maker may consider selling certain amount of the foreign currency to other market-makers to lower his position instead of waiting for sell orders. This option is available when there are multiple market-makers providing liquidity for the same asset, which is true for the foreign currency market and some stock markets, e.g., National Association of Securities Dealers Automated Quotations (NASDAQ). Of course, when the market-maker adjusts its position by selling to other market-makers, he becomes their client and has to sell at others' bid prices. As a result, he forgoes the possibility of selling to his own clients at the ask price and taking the spread. More importantly, when doing an adjustment, the market-maker will sell at the other market-makers' bid prices and

[1]Although at the beginning of this chapter we define inventory in supply chains as physical commodities held in stock, inventory also refers to the assets held or short sold by a financial institute or an individual in finance. In this particular example, the foreign currencies held or short sold by the market-maker can be regarded as inventory.

buy at the other market-makers' ask prices, thus is likely to encounter a loss. So here we have a typical trade-off between profit and risk. Our goal in this research is to apply dynamic programming techniques in order to investigate when and by how much a market-maker should sacrifice profit to reduce risk.

The observation that market-makers may carry unwanted inventories has long caught the attention of the research community, and most previous work investigates how inventories influence the market-makers' behavior when quoting bid and ask prices, in other words, it studies how to control inventory via pricing decisions. The theoretical analysis in Ho and Stoll [24] shows that risk-averse market-makers will actively induce movements toward a desirable inventory level by setting favorable bid/ask prices. Stoikov and Saglam [50] considers a market-maker in both an option and its underlying stock, and analyze the role of the derivatives of option price on the bid/ask quotes of both the option and the stock.

Empirical studies suggest that the impact of inventory levels on pricing is rather weak compared with the impact of other components such as asymmetric information (c.f. Stoll [51], Madhavan and Smidt [33], Foster and Vishwanathan [18], and Madhavan and Smidt [34]). In a later paper, Ho and Stoll [25] introduces a model that includes both the ability to change the bid/ask prices as well as opportunities to trade with other market-makers. The result suggests that trades among market-makers are necessary under certain conditions. Unfortunately, their solution is for models with only two periods.

A survey of US foreign exchange traders (c.f. Cheung and Chinn [13]) indicates that the market norm is an important determinant of the bid-ask spread and only a small proportion of bid-ask spreads differ from the conventional spread. Specifically, only 2% of the respondents in that survey reported that inventory related factors have an impact on their bid-ask spreads. This is because quoting volatile bid/ask spread may damage the market-maker's reputation and drive away potential trading opportunities. Also, many of the traders reported that they are reluctant to reveal adverse positions by quoting non-conventional spread. Thus, this empirical study implies that at least in the foreign exchange market, inventory is not managed by quoting bid/ask prices. Rather, it is controlled by trading with other market-makers. This is also supported by other evidence, for example trading volume. Indeed, trading volume is extremely high in the foreign exchange market and is believed to be a result of market-makers passing unwanted inventory from one to another (c.f. Lyons [32]). Finally, the survey of Cheung and Chinn [13] also states that more than half of respondents believe that large players dominate dollar-pound and dollar-Swiss franc markets. Therefore, many small and medium-sized players have no market power, i.e., they have no impact on future price movements when actively trading with other market-makers.

Our study is motivated by a practical problem faced by a major investment bank. Here we consider an electronic market-maker in the foreign exchange market which serves small retail orders. Since the market-maker only captures a very small fraction of the entire foreign exchange market, it quotes the conventional spread and has no market power. In this case, the primary decision the market-maker needs to make is how much to trade with other market-makers in order to limit its market exposure.

Thus, our objective is to identify effective strategies for a market-maker who does not control prices and can merely adjust inventory through active trading. In this sense, the market-making problem shares some important features with the classical inventory control problem. We need to determine the amount of assets to buy or sell during market-making process, which is analogous to the ordering quantity in inventory control. Indeed, in our case, the risk induced by inventory is analogous to the inventory holding cost, and the sacrificed spread profit due to active trading plays a similar role to the linear ordering cost. The sacrificed spread profit is the loss of spread encountered by a market-maker who sells/buys a unit of inventory to other market-makers (at their own prices) rather than holding that unit of inventory and profiting from the spread in the future. Of course, there are some important differences: in the classical inventory control model, the order quantity must be non-negative and the unit inventory holding cost is deterministic, which as we shall see, are essentially different from the market-making situation.

To present our contribution, we need to define a *threshold policy*. Such a policy is defined by two parameters, an upper limit and a lower limit. Whenever the inventory is higher (lower) than the upper (lower) limit, the market-maker will decrease (increase) the inventory to the upper (lower) limit. Otherwise, i.e., when inventory level is between the two limits, the market-maker will not change its position. We call the region where the market-maker does not adjust its inventory, the "*no-trade region*." When the inventory of the market-maker falls in the no-trade region, the market-maker will not actively trade with other market-makers, but will still accept trades from its customers. Our contributions are summarized as follows.

When the market-maker manages a single asset, we propose dynamic programming models for the market-making inventory control problem, where an exponential utility function or a mean-variance utility are used – utilities that have been applied to model risk averse decision makers. Threshold policies are proved to be optimal for both models, and the special properties of the threshold levels are also analyzed. In particular, we identify conditions under which the threshold policy is symmetric, investigate the risk neutral model, and establish various monotonicity properties of the optimal threshold levels for the mean-variance analysis.

When the market-maker manages multiple assets simultaneously, we focus on the dynamic programming formulation which optimizes the linear tradeoff of mean and variance. The optimal policy shows that there exists a simply connected no-trade region for each period and the optimal adjustment quantity is obtained directly from the no-trade region. In addition, we identify conditions under which the the no-trade region is symmetric with respect to $\mathbf{0}$.

Based on these structural properties of the optimal policy, we develop efficient algorithms to solve the corresponding dynamic program whose computational complexity is linear in the number of periods. Numerical results are also presented to illustrate properties of the optimal policies.

1.2 Robust Stochastic Lot-Sizing

The stochastic lot-sizing model has been extensively studied in the inventory literature. Most of the research has focused on models with complete information about the distribution of customer demand. However, in most real-world situations, the demand distribution is not known; only historical data is available. A common approach is to hypothesize a family of demand distributions and then to estimate the parameters specifying the distribution using the historical data. Once the probability distribution has been identified, the inventory problem is solved following this estimated distribution. This implies that the inventory policy is determined under the assumption of a perfect demand distribution.

We consider a different approach recognizing that the estimated demand distribution may not be accurate. We analyze the single-item stochastic finite-horizon periodic review lot-sizing model, under the assumption that demand is subject to an unknown distribution and only historical demand observations (given by histograms) are available. Rather than first estimating the demand distribution and then optimizing inventory decisions, as is the case in the classical approaches, we combine these two steps to minimize the worst case expected cost over a set of all possible distributions that satisfy a certain goodness-of-fit constraint. In this way, we combine distribution fitting and inventory optimization, and characterize a *robust inventory control policy* based on the historical data.

The novelty of our approach is the starting point of histograms. All practitioners in inventory control start with histograms and then they fit an underlying demand distribution (e.g., Crystal Ball from Decisioneering, Inc. allows selecting a distribution family among several listed families). Finally, based on the fitted distribution, the lot-sizing problem is solved.

The problem, of course, is that this distribution may not be the correct one. For this purpose, we develop a model that integrates both distribution fitting and lot-sizing – we refer to this model as the *robust lot-sizing model*. This novel idea of using histograms as a source of input and concurrently considering replenishment quantities and distributions leads to interesting insights. For example, as in the classical stochastic inventory setting, our results indicate that an (s, S) policy is optimal for the robust model as well. We also discuss the impact of the sample size on model performance.

The main contributions of our work are as follows

1. We develop a robust minimax model that only requires historical data, and allows correlated demand. Note that most minimax models (see, e.g., Notzon [38] and Ahmed et al. [1]) as well as Bayesian inventory models (e.g., updating the demand distributions in the way provided in Iglehart [27]) in the literature could be interpreted as special cases of our framework.

2. The optimal policy of the robust model has the same structure as the corresponding policy in the classical stochastic lot-sizing model. In particular, the optimal policy is a state-dependent base-stock policy for the multi-period inventory problem without fixed procurement costs, and a state-dependent (s, S) policy if the fixed procurement cost is considered.

3. To illustrate the general framework, we consider the special case when the set of demand distributions is directly related to the chi-square goodness-of-fit test. This set can be defined by a set of second order cone constraints.

We also prove that the robust model converges to the stochastic model with true demand distribution if samples are drawn from this distribution and sample size grows to infinity. In particular, if the demand distributions are discrete, the robust model converges to the stochastic model with the true demand distribution as the number of independent samples drawn from the true distribution for each period tends to infinity. Moreover, the rate of convergence is in the order of $1/\sqrt{k}$, where k is the number of samples. Slightly weaker results are obtained for continuous distributions.

The performance of the robust model is illustrated by means of computational experiments. We argue that the robust model outperforms the traditional approach, which optimizes the inventory decisions by using fitted distributions. We also provide insights on the performance of the robust model with different parameters and sample sizes.

Chapter 2

Single-Asset Market-Making with Exponential Utility

As we mentioned in Chapter 1, the multi-period stochastic inventory control problem has been extensively studied since 1950's, see Zipkin [55] for a detailed review of risk-neutral models. In the last two decades, a number of papers have been devoted to risk aversion in inventory management. Bouakiz and Sobel [10] focuses on minimizing the expected exponential utility of the linear ordering costs and inventory holding costs incurred during a finite or infinite planning horizon, and proves that a base stock policy is optimal. Chen et al. [11] considers risk-averse inventory (and pricing) models where the utility functions are time-separable. They show that the structure of the optimal policy is almost identical to the structure of the optimal policy in the risk-neutral counterpart, see also Simchi-Levi et al. [49].

In this chapter, we study the inventory problem in market-making introduced in Section 1.1 under the assumption that the decision maker manages a single asset and has an exponential utility function. We introduce the stochastic inputs and decision variables for the market-making inventory control problem in Section 2.1. Section 2.2 presents the dynamic program which maximizing the exponential utility throughout the planning horizon, and proves that a threshold policy is optimal for the general model. Furthermore, we discuss the special cases where the dimensions of the states determining the threshold levels can be reduced in Section 2.3, and identify sufficient conditions for the threshold policy to be symmetric in Section 2.4. Finally, Section 2.5 concludes this chapter.

2.1 Formulation

In this chapter, we consider a time horizon of one day, which reflects the observation that market-makers tend to "go home flat", i.e., market-makers prefer clearing their inventory at the end of the trading day in order to avoid significant market price movements overnight (c.f. Hasbrouck [23]). We divide the trading day into N discrete small time intervals.

The sequence of events is as follows: At the beginning of period k, we observe the current inventory level x_k. Unlike the classical inventory model, x_k can be negative as the market-maker can take a short position. Next, the bid and ask prices quoted by the dominant player, p_k^b and p_k^a are observed. After that, we adjust the inventory by the amount q_k, which is the decision variable. Note that q_k represents the amount the market-maker buys or sells (to other market-makers) at that period. We let q_k be positive if the market-maker buys q_k units of asset, and q_k is negative if the market-maker sells $|q_k|$. The market-maker, as a price follower, quote the same bid and ask prices p_k^b and p_k^a as the dominant player. Clients arrive and they sell s_k and buy d_k units of the asset to/from the market-maker. Obviously the inventory at the beginning of period $k+1$ is $x_{k+1} = x_k + q_k - d_k + s_k$.

Similar to Stoikov and Saglam [50], we consider the dynamics of the mid price $p_k = (p_k^b + p_k^a)/2$, $k = 1, ..., N+1$, which is the average of the bid and ask prices. Let the mid price at period $k+1$, be $p_{k+1} = p_k + \delta_k$. δ_k can be dependent on the mid price p_k, and we assume that δ_k conditional on p_k is independent of $\delta_{\bar{k}}$ conditional on $p_{\bar{k}}$ for any $\bar{k} \neq k$. Note that a large family of stochastic processes satisfies this assumption. For example, suppose that p_k follows a geometric random walk, i.e., $p_{k+1} = p_k \exp(\mu + \tilde{\delta}_k)$ where μ is the drift component and $\tilde{\delta}_k$ is i.i.d. distributed for any k. It is straightforward that $\delta_k = p_k \Big(\exp(\mu + \tilde{\delta}_k) - 1\Big)$ conditional on p_k is independently distributed for any k. Of course, a random walk is also a special case of the mid price model if we assume that δ_k is i.d.d. distributed for any k. Within a day, the geometric random walk is almost the same as an ordinary random walk if the price change at each stage is small ($e^x \approx 1 + x$ if $|x| \ll 1$).

We introduce another parameter ϵ_k to model the bid and ask prices. For any period k, ϵ_k is defined such that the bid and ask prices at period k for any market-maker are $p_k^b = p_k - \epsilon_k$ and $p_k^a = p_k + \epsilon_k$ respectively. Note that ϵ_k is the transaction cost the client pays when he or she trade one unit of the asset with the market-maker, or the transaction cost the market-makers pays when it trade one unit with other market-makers to control its inventory. We also refer to ϵ_k as the transaction cost in period k. For any period k, ϵ_k must be strictly positive so that the bid price is always lower than the ask price. Similar to the price movement δ_k, we also assume that ϵ_k conditional on p_k is independent for any k. For example, we can choose $\epsilon_k = \phi_k(p_k) + \varphi_k$ where $\phi_k(p_k)$ is a given function and φ_k is an independent random variable for any k. When $\varphi_k = 0$, $\epsilon_k = \phi_k(p_k)$ becomes a constant once p_k is known, e.g., ϵ_k can choose to be 0.01% of the mid price. Moreover, if all the foreign exchange market-makers quote the conventional spread, then $\epsilon_k = \varphi_k$ is a constant equal to a half of the conventional spread.

To model orders from clients, we use the random variables s_k and d_k to denote the amounts the clients buy from and sell to the market-maker in period k respectively. We also refer to s_k and d_k as the supply and demand from the clients respectively, since these amounts increase or decrease our inventory levels. Both s_k and d_k should be nonnegative, and s_k and d_k can be correlated for a given period k. For the time being, we assume that s_k and d_k conditional on p_k are independent for each period k, but we allow non-stationary distributions for s_k and d_k across different period k in order to model the intraday pattern in the trading volume, e.g., the trading volume is higher when the market opens or closes.

So far we have defined two random processes: (i) p_k and ϵ_k which jointly define the bid and ask prices of the underlying asset and (ii) s_k and d_k to model the orders from clients. These two processes can be dependent on each other in order to capture the correlations between the trading volumes and price movements, i.e., the four random variables δ_k, ϵ_k, s_k, d_k conditional on p_k can be correlated for any given k. For example, if we observe that the amount of sell orders s_k is significantly higher than the amount of buy orders d_k, we expect that the market price is more likely to go down, i.e., the probability that δ_k is negative should be higher than the case when the reverse is true. In addition, all these random variables can also depend on p_k for any k as we stated in their definition.

In any period k, the profit we obtain from the bid-ask spread by trading with our clients is $(d_k + s_k)\epsilon_k$. Note that we trade $|q_k|$ at the price quoted by other market-makers, and hence the transaction cost is $|q_k|\epsilon_k$. In addition, the market-maker's inventory is subject to the risk of price uncertainty, and hence it may incurred a profit or loss of the amount $(x_k + q_k - d_k + s_k)\delta_k$. As a result, the one-period profit at period k, $k = 1, ..., N$ is

$$\pi_k = (x_k + q_k - d_k + s_k)\delta_k + (d_k + s_k - |q_k|)\epsilon_k.$$

To simplify the notation, let $L_k = x_k + q_k$ be the inventory level after adjustment, $S_k = d_k + s_k$ and $\Delta_k = s_k - d_k$. Then

$$\pi_k = (L_k + \Delta_k)\delta_k + (S_k - |L_k - x_k|)\epsilon_k. \tag{2.1}$$

Note that here we do not consider the fee the market-maker pays to short the asset. This is because the fee is neglectable for liquid assets, e.g., foreign currency. In addition, the structure of our problem remains the same and the optimality of the threshold policy still holds even if we consider a linear short fee.

We let $\pi_{N+1} = v(x_{N+1}, \epsilon_{N+1})$ denote the profit or loss at end of the planning horizon, where $v(x_{N+1}, \epsilon_{N+1})$ is a concave function with respect to x_{N+1}. Note that π_{N+1} also depends on p_{N+1} if ϵ_{N+1} depends on p_{N+1}. If the positions at the end of the trading day can be clear at the mid price, or we mark to the mid price at the end of the day, then $\pi_{N+1} = 0$. If the inventory position at the end of the day, x_{N+1} is cleared at the price quoted by other market-makers, the market-maker incurs a salvage cost of $\epsilon_{N+1}|x_{N+1}|$, i.e., $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$.

We adopt two approaches to characterize the risk-averse attitude of the decision maker: exponential utility function and mean-variance analysis. The objective functions as well as the properties of the optimal control policies under the exponential utility criterion are presented in the remaining part of this chapter, and we discuss the corresponding results for the mean-variance analysis model in Chapter 3.

Before we end this section, we would like to point out that most of our results are not restricted by the assumptions we present here, and the generalizations are discussed in details in Section 3.7. For example, we can also allow auto-correlations in the movements of prominent bid/ask prices as well as the client orders, i.e., δ_k, ϵ_k, s_k, d_k conditional on p_k can be correlated across the period k. Furthermore, the bid/ask spread a market-maker charges its clients can be different from the spread charged by other market-makers, i.e., the bid and ask prices quote by other market-makers are $p_k^b = p_k - \epsilon_k$ and $p_k^a = p_k + \epsilon_k$ while the bid and ask prices we quote to our clients are $\tilde{p}_k^b = p_k - \tilde{\epsilon}_k^b$ and $\tilde{p}_k^a = p_k + \tilde{\epsilon}_k^a$.

2.2 Optimality of the Threshold Policy

Suppose that the market-maker has an exponential utility function $U(\pi)$, i.e.,

$$U(\pi) = -\exp(-\rho\pi), \text{ where } \rho > 0 \text{ denote the risk-aversion parameter.}$$

Note that large ρ implies higher risk aversion. Since the time horizon is one day, the market-maker does not care how much profit a particular strategy generates in the due process. Instead, he only looks at the profit at the end of the day. We should choose the amount q_k to maximize the expected utility of the total profit generated in the day, i.e., the objective function is

$$\max_{q_k} \; E\left[-\exp\left(-\rho\sum_{k=1}^{N+1}\pi_k\right)\right]. \tag{2.2}$$

Note that Bouakiz and Sobel [10] considers a similar objective function for the classical inventory model. As a result, the corresponding Bellman equation is

$$J_k(x_k, p_k, \epsilon_k) = \min_{q_k} E\left\{ e^{-\rho\pi_k} J_{k+1}\left(x_k + q_k - d_k + s_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| \, p_k, \epsilon_k \right\}$$

for any $k = 1, ..., N$, and $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1}) = \exp(-\rho\pi_{N+1})$. The state in the dynamic programming model consists of x_k, p_k and ϵ_k because we observe the inventory position x_k as well as the bid and ask prices defined by p_k and ϵ_k before we decide the adjustment quantity q_k, which is our decision variable. Note that we consider the expectation conditional on p_k and ϵ_k because the distributions of δ_k, S_k and Δ_k depends on p_k and ϵ_k.

Similar to (2.1), we define $L_k = x_k + q_k$, $S_k = d_k + s_k$ and $\Delta_k = s_k - d_k$. Inserting in (2.1), the Bellman's equation is reduced to

$$\begin{aligned} J_k(x_k, p_k, \epsilon_k) = \min_{L_k} E \Big\{ & e^{-\rho\left((L_k+\Delta_k)\delta_k + (S_k - |L_k - x_k|)\epsilon_k\right)} \\ & \times J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| \, p_k, \epsilon_k \Big\} \end{aligned} \tag{2.3}$$

for all $k = 1, ..., N$.

Under the exponential utility objective function in (2.2), we obtain the following optimal inventory control policy.

Theorem 2.1. *The optimal control policy for the dynamic programming model in* (2.2) *is as follows. For any period k, there exist threshold levels, independent of the inventory level x_k, $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ where $T_k^1(p_k, \epsilon_k) \geq T_k^2(p_k, \epsilon_k)$ for any given p_k, such that the optimal order quantity $q_k^* = T_k^1(p_k, \epsilon_k) - x_k$ if $x_k \leq T_k^1(p_k, \epsilon_k)$, $q_k^* = T_k^2(p_k, \epsilon_k) - x_k$ if $x_k > T_k^2(p_k, \epsilon_k)$, and $q_k^* = 0$ otherwise.*

In other words, given the current mid price p_k and the half of the spread ϵ_k, which specify the market bid and ask prices, the optimal policy is to keep the inventory level x_k within a certain interval $[T_k^1(p_k, \epsilon_k), T_k^2(p_k, \epsilon_k)]$. When $x_k \leq T_k^1(p_k, \epsilon_k)$, the inventory level is too low and the market-maker will lose a significant amount if the market price increases. Therefore, it is willing to pay the transaction cost and increase the inventory upto $T_k^1(p_k, \epsilon_k)$. Similarly, if the inventory level is too high, i.e., $x_k > T_k^2(p_k, \epsilon_k)$, the market-maker should decrease its inventory to $T_k^2(p_k, \epsilon_k)$ so as to protect against the case that the market price decreases drastically. Otherwise, the inventory is contained in the interval $[T_k^1(p_k, \epsilon_k), T_k^2(p_k, \epsilon_k)]$ and no action is required, i.e., the market-maker only needs to accept the orders from its clients and carry the inventory to the next period. We refer to the interval $([T_k^1(p_k, \epsilon_k), T_k^2(p_k, \epsilon_k)]$ as the *no-trade region*, where it is optimal not to actively trade with other market-makers.

In the rest of this section, we prove Theorem 2.1 by induction on the number of periods, k, and illustrate it using a numerical example.

Before we jump into the technical details of the proof of Theorem 2.1, let us first introduce the following notation, which will be used in the remaining part of Chapters 2 and 3. Since the functions we consider here may not be differentiable everywhere, e.g., the absolute value function, for any function $f(x)$, we let $f'(x)$ denote its left-hand derivative, i.e.,

$$f'(x) = \lim_{d \downarrow 0} \frac{f(x) - f(x-d)}{d},$$

which always exists if $f(x)$ is convex.

Futhermore, for a multivariate function $f(x_1, x_2, ..., x_m)$, we use $\frac{\partial f}{\partial x_i}(x_1, x_2, ..., x_m)$ to denote the left-hand derivative of $f(x_1, x_2, ..., x_m)$ with respect to the variable x_i, $i = 1, ..., m$.

Another important property for convex functions is that we can interchange the expectation and differentiation operators. To be precise, suppose that $f(x, y)$ is a convex function with respect to x, Y is a random variable, and $g(x) = E[f(x, Y)]$ is well-defined. According to the monotone convergence theorem, we have $g'(x) = E[\frac{\partial}{\partial x} f(x, Y)]$, see also Bouakiz and Sobel [10].

To prove Theorem 2.1 by induction, we start by assuming that

(A1) $J_{k+1}(x, p, \epsilon)$ is nonnegative for any x, p and ϵ,

(A2) $e^{aL} J_{k+1}(L + \Delta, p, \epsilon)$ is a convex function in L for any given a, Δ, p and ϵ.

We would like to show that (i) Theorem 2.1 is valid for period k and (ii) $J_k(x_k, p_k, \epsilon_k)$

also satisfies the induction assumptions (A1) and (A2). The proof is complete once we establish that $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1})$ has the properties (A1) and (A2).

Note that in the classical inventory problems, e.g., Scarf [47] and Bouakiz and Sobel [10], the initial inventory position x_k defines the constraint that the order upto level is greater than x_k, but it does not appear in the objective function of the Bellman equation. However, in our Bellman equation (2.3), we cannot pull x_k out of the objective function because it is included in an absolute value function.

Let us define the functions $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ as

$$\begin{aligned} f_k^1(L_k, p_k, \epsilon_k) &= E\left[e^{-\rho\left((L_k+\Delta_k)\delta_k + (S_k - L_k)\epsilon_k\right)} J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \middle| \, p_k, \epsilon_k \right] \\ f_k^2(L_k, p_k, \epsilon_k) &= E\left[e^{-\rho\left((L_k+\Delta_k)\delta_k + (S_k + L_k)\epsilon_k\right)} J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \middle| \, p_k, \epsilon_k \right]. \end{aligned} \tag{2.4}$$

It is easy to show that the Bellman equation (2.3) is equivalent to

$$\begin{aligned} J_k(x_k, p_k, \epsilon_k) &= \min\left\{ \min_{L_k \geq x_k} e^{-\rho x_k \epsilon_k} f_k^1(L_k, p_k, \epsilon_k), \min_{L_k \leq x_k} e^{\rho x_k \epsilon_k} f_k^2(L_k, p_k, \epsilon_k) \right\} \\ &= \min\left\{ e^{-\rho x_k \epsilon_k} \left(\min_{L_k \geq x_k} f_k^1(L_k, p_k, \epsilon_k) \right), e^{\rho x_k \epsilon_k} \left(\min_{L_k \leq x_k} f_k^2(L_k, p_k, \epsilon_k) \right) \right\}. \end{aligned} \tag{2.5}$$

After reformulating the Bellman equation, we decompose it into three sequential optimization problems. The problems $\min_{L_k \geq x_k} f_k^1(L_k, p_k, \epsilon_k)$ and $\min_{L_k \leq x_k} f_k^2(L_k, p_k, \epsilon_k)$ minimize a single variate function subject to a single constraint, whose structure is the same as the optimization problems in the classical inventory control models. However, we have another minimization operator which compares the optimal solution of these two problems. In this sense, the problem in (2.5) is more challenging than those in the classical inventory models.

For any given p_k and ϵ_k, let $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ be the global minimizers of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ respectively, i.e.,

$$T_k^1(p_k, \epsilon_k) = \arg\min_{L_k} f_k^1(L_k, p_k, \epsilon_k) \quad \text{and} \quad T_k^2(p_k, \epsilon_k) = \arg\min_{L_k} f_k^2(L_k, p_k, \epsilon_k).$$

We would like to establish the following properties for the functions $f_k^1(L_k, p_k, \epsilon_k)$ and

$f_k^2(L_k, p_k, \epsilon_k)$ as well as the minimizers $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$.

Lemma 2.1. *Suppose that $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ satisfies the induction assumptions (A1) and (A2). Then (i) $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ are convex functions with respect to L_k, and (ii) $T_k^1(p_k, \epsilon_k) \leq T_k^2(p_k, \epsilon_k)$ for any given p_k and ϵ_k.*

Proof. Let us consider these two functions

$$h_k^1(L_k, p_k, \epsilon_k, \Delta_k, \delta_k, \epsilon_{k+1}) = e^{-\rho(\delta_k - \epsilon_k)L_k} J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right)$$
$$h_k^2(L_k, p_k, \epsilon_k, \Delta_k, \delta_k, \epsilon_{k+1}) = e^{-\rho(\delta_k + \epsilon_k)L_k} J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right).$$

According to (A2), $h_k^1(L_k, p_k, \epsilon_k, \Delta_k, \delta_k, \epsilon_{k+1})$ and $h_k^2(L_k, p_k, \epsilon_k, \Delta_k, \delta_k, \epsilon_{k+1})$ are convex in L_k.

If we multiply both functions by $e^{-\rho(\Delta_k\delta_k + S_k\epsilon_k)}$, we have

$$\begin{aligned} g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) &= e^{-\rho(\Delta_k\delta_k + S_k\epsilon_k)} h_k^1(L_k, p_k, \epsilon_k, \Delta_k, \delta_k, \epsilon_{k+1}) \\ &= e^{-\rho\left((L_k+\Delta_k)\delta_k + (S_k - L_k)\epsilon_k\right)} J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \\ g_k^2(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) &= e^{-\rho(\Delta_k\delta_k + S_k\epsilon_k)} h_k^2(L_k, p_k, \epsilon_k, \Delta_k, \delta_k, \epsilon_{k+1}) \\ &= e^{-\rho\left((L_k+\Delta_k)\delta_k + (S_k + L_k)\epsilon_k\right)} J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right). \end{aligned}$$

Note that $e^{-\rho(\Delta_k\delta_k + S_k\epsilon_k)}$ does not depend on L_k. Hence, $g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1})$ and $g_k^2(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1})$ are convex functions with respect to L_k.

By definition, we have

$$g_k^2(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) = e^{-2\rho\epsilon_k L_k} g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}).$$

Since both functions are convex in L_k, their left-hand derivatives with respect to L_k exist. Therefore,

$$\begin{aligned} \frac{\partial g_k^2}{\partial L_k}(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) = &-2\rho\epsilon_k e^{-2\rho\epsilon_k L_k} g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) \\ &+ e^{-2\rho\epsilon_k L_k} \frac{\partial g_k^1}{\partial L_k}(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}). \end{aligned}$$

$g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1})$ is nonnegative since $J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right)$ is nonnegative by the assumption (A1). It follows directly that

$$\frac{\partial g_k^2}{\partial L_k}(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) \geq e^{-2\rho\epsilon_k L_k} \frac{\partial g_k^1}{\partial L_k}(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}). \qquad (2.6)$$

According to (2.4), it is straightforward that

$$f_k^1(L_k, p_k, \epsilon_k) = E\left[g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k \right]$$
$$f_k^2(L_k, p_k, \epsilon_k) = E\left[g_k^2(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k \right].$$

Recall that both $g_k^1(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1})$ and $g_k^2(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1})$ are convex in L_k. Therefore, $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ are also convex in L_k, which implies that

$$\begin{aligned}
\frac{\partial f_k^2}{\partial L_k}(L_k, p_k, \epsilon_k) &= E\left[\frac{\partial g_k^2}{\partial L_k}(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k \right] \\
&\geq e^{-2\rho\epsilon_k L_k} E\left[\frac{\partial g_k^1}{\partial L_k}(L_k, p_k, \epsilon_k, S_k, \Delta_k, \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k \right] \\
&= e^{-2\rho\epsilon_k L_k} \frac{\partial f_k^1}{\partial L_k}(L_k, p_k, \epsilon_k),
\end{aligned}$$

where the inequality is obtained from (2.6). Since $e^{-2\rho\epsilon_k L_k} \geq 0$, it follows directly that $\frac{\partial f_k^2}{\partial L_k}(L_k, p_k, \epsilon_k) \leq 0$ if $\frac{\partial f_k^1}{\partial L_k}(L_k, p_k, \epsilon_k) \leq 0$.

Consider any given p_k and ϵ_k. According to the convexity of $f_k^1(L_k, p_k, \epsilon_k)$ in L_k and the definition of $T_k^1(p_k, \epsilon_k)$, we know that $\frac{\partial f_k^1}{\partial L_k}(L_k, p_k, \epsilon_k) \leq 0$ for any $L_k \in (-\infty, T_k^1(p_k, \epsilon_k)]$. Consequently, $\frac{\partial f_k^2}{\partial L_k}(L_k, p_k, \epsilon_k) \leq 0$ for any $L_k \in (-\infty, T_k^1(p_k, \epsilon_k)]$, i.e., $f_k^2(L_k, p_k, \epsilon_k)$ is decreasing in $L_k \in (-\infty, T_k^1(p_k, \epsilon_k)]$. Note that $T_k^2(p_k, \epsilon_k) \leq T_k^2(p_k, \epsilon_k)$ is the global minimizer of $f_k^2(L_k, p_k, \epsilon_k)$ with given p_k and ϵ_k, we obtain that $T_k^1(p_k, \epsilon_k) \leq T_k^2(p_k, \epsilon_k)$. □

Next, we are going to show the optimal policy for period k under the assumptions (A1) and (A2).

Proposition 2.1. *Suppose that $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ satisfies the induction assumptions (A1) and (A2). Then, given p_k and ϵ_k, the optimal solution L_k^* to the problem*

(2.3) *is*

$$L_k^* = \begin{cases} T_k^1(p_k, \epsilon_k) & \text{if } x_k \le T_k^1(p_k, \epsilon_k), \\ x_k & \text{if } T_k^1(p_k, \epsilon_k) < x_k \le T_k^2(p_k, \epsilon_k), \\ T_k^2(p_k, \epsilon_k) & \text{if } x_k > T_k^2(p_k, \epsilon_k), \end{cases} \tag{2.7}$$

and the corresponding optimal value is

$$J_k(x_k, p_k, \epsilon_k) = \begin{cases} e^{-\rho x_k \epsilon_k} f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k) & \text{if } x_k \le T_k^1(p_k, \epsilon_k), \\ e^{-\rho x_k \epsilon_k} f_k^1(x_k, p_k, \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^2(x_k, p_k, \epsilon_k) & \\ & \text{if } T_k^1(p_k, \epsilon_k) < x_k \le T_k^2(p_k, \epsilon_k), \\ e^{\rho x_k \epsilon_k} f_k^2(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) & \text{if } x_k > T_k^2(p_k, \epsilon_k). \end{cases} \tag{2.8}$$

Proof. It is equivalent to show that the results defined in (2.7) and (2.8) hold for the optimization problem in (2.5). Let us consider the following three cases.

- Suppose that $x_k \le T_k^1(p_k, \epsilon_k)$.

  We proved in Lemma 2.1 that $f_k^1(L_k, p_k, \epsilon_k)$ is convex in L_k. Therefore, for the optimization problem $\min_{L_k \ge x_k} f_k^1(L_k, p_k, \epsilon_k)$, the optimal solution is $T_k^1(p_k, \epsilon_k)$ and the corresponding objective value is $f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k)$.

  Lemma 2.1 also shows that $f_k^2(L_k, p_k, \epsilon_k)$ is a convex function with respect to L_k and $T_k^1(p_k, \epsilon_k) \le T_k^2(p_k, \epsilon_k)$. As a result, for the optimization problem $\min_{L_k \le x_k} f_k^2(L_k, p_k, \epsilon_k)$, the optimal solution is x_k and the corresponding objective value is $f_k^2(x_k, p_k, \epsilon_k)$.

  The definition of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ in (2.4) suggests that

  $$e^{-\rho x_k \epsilon_k} f_k^1(x_k, p_k, \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^2(x_k, p_k, \epsilon_k).$$

  Note that $e^{-\rho x_k \epsilon_k} \ge 0$ and $f_k^1(x_k, p_k, \epsilon_k) \ge f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k)$ by the definition of $T_k^1(p_k, \epsilon_k)$. It follows directly that

  $$e^{-\rho x_k \epsilon_k} f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k) \le e^{-\rho x_k \epsilon_k} f_k^1(x_k, p_k, \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^2(x_k, p_k, \epsilon_k),$$

i.e., the optimal solution to (2.5) is $T_k^1(p_k, \epsilon_k)$ and the value of $J_k(x_k, p_k, \epsilon_k)$ is $e^{-\rho x_k \epsilon_k} f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k)$.

- Suppose that $T_k^1(p_k, \epsilon_k) < x_k \leq T_k^2(p_k, \epsilon_k)$. The results of Lemma 2.1 show that the optimal solution for both $\min_{L_k \geq x_k} f_k^1(L_k, p_k, \epsilon_k)$ and $\min_{L_k \leq x_k} f_k^2(L_k, p_k, \epsilon_k)$ is x_k. It follows immediately that the optimal solution to (2.5) is x_k and the corresponding value of $J_k(x_k, p_k, \epsilon_k)$ is $e^{-\rho x_k \epsilon_k} f_k^1(x_k, p_k, \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^2(L_k, p_k, \epsilon_k)$.

- Suppose that $x_k > T_k^2(p_k, \epsilon_k)$. The result can be established by an arguement similar to that in the case $x_k < T_k^1(p_k, \epsilon_k)$. □

Note that L_k is defined as $x_k + q_k$. Therefore, the optimal inventory control policy shown in Proposition 2.1 is exactly the same as that in Theorem 2.1. However, in order to complete the induction proof for Theorem 2.1, we need to show that the value function $J_k(x_k, p_k, \epsilon_k)$ satisfies the assumptions (A1) and (A2).

Proposition 2.2. *Suppose that $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ satisfies the induction assumptions (A1) and (A2). Then (i) $J_k(x, p, \epsilon)$ is nonnegative for any x, p and ϵ, (ii) $e^{aL} J_k(L + \Delta, p, \epsilon)$ is a convex function in L for any given a, Δ, p and ϵ.*

Proof. The part (i) of Proposition 2.2 follows from the definition of $J_k(x_k, p_k, \epsilon_k)$ in (2.8), the definition of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ in (2.4), the assumption (A1) and the fact that the exponential function is nonnegative.

For the part (ii), given a, Δ, p and ϵ, let us consider the functions $h_1(L)$, $h_2(L)$ and $h_3(L)$ such that

$$
\begin{aligned}
h_1(L) &= e^{aL - \rho(L+\Delta)\epsilon} f_k^1(T_k^1(p, \epsilon), p, \epsilon) \\
h_2(L) &= e^{aL - \rho(L+\Delta)\epsilon} f_k^1(L + \Delta, p, \epsilon) = e^{aL + \rho(L+\Delta)\epsilon} f_k^2(L + \Delta, p, \epsilon) \\
h_3(L) &= e^{aL + \rho(L+\Delta)\epsilon} f_k^2(T_k^2(p, \epsilon), p, \epsilon).
\end{aligned}
$$

Note that $f_k^1(T_k^1(p, \epsilon), p, \epsilon)$ is a constant, and the exponential function is convex. Therefore, $h_1(L)$ is convex in L. By the same argument, $h_3(L)$ is also convex in L.

Next, we would like to prove that $h_2(L)$ is convex. In this case,

$$\begin{aligned} h_2(L) &= e^{aL-\rho(L+\Delta)\epsilon} f_k^1(L+\Delta, p, \epsilon) \\ &= e^{aL-\rho(L+\Delta)\epsilon} E\left[e^{-\rho\left((L+\Delta+\Delta_k)\delta_k+(S_k-L-\Delta)\epsilon\right)} J_{k+1}\left(L+\Delta+\Delta_k, p+\delta_k, \epsilon_{k+1}\right) \Big| p, \epsilon \right] \\ &= E\left[e^{-\rho\left((\Delta+\Delta_k)\delta_k+S_k\epsilon\right)} e^{(a-\rho\delta_k)L} J_{k+1}\left(L+\Delta+\Delta_k, p+\delta_k, \epsilon_{k+1}\right) \Big| p, \epsilon \right]. \end{aligned} \tag{2.9}$$

The assumption (A2) shows that

$$e^{(a-\rho\delta_k)L} J_{k+1}\left(L+\Delta+\Delta_k, p+\delta_k, \epsilon_{k+1}\right)$$

is convex in L. Since $e^{-\rho\left((\Delta+\Delta_k)\delta_k+S_k\epsilon\right)}$ is independent of L,

$$e^{-\rho\left((\Delta+\Delta_k)\delta_k+S_k\epsilon\right)} e^{(a-\rho\delta_k)L} J_{k+1}\left(L+\Delta+\Delta_k, p+\delta_k, \epsilon_{k+1}\right)$$

is also convex in L. The definition of $h_2(L)$ in (2.9) immediately yields its convexity in L.

Let us define $h(L) = e^{aL} J_k(L+\Delta, p, \epsilon)$. According to (2.8), we have

$$h(L) = e^{aL} J_k(L+\Delta, p, \epsilon) = \begin{cases} h_1(L) & \text{if } L+\Delta \le T_k^1(p,\epsilon), \\ h_2(L) & \text{if } T_k^1(p,\epsilon) < L+\Delta \le T_k^2(p,\epsilon), \\ h_3(L) & \text{if } L+\Delta > T_k^2(p_k,\epsilon_k). \end{cases} \tag{2.10}$$

We would like to prove the convexity of $h(L)$ by showing that its left-hand derivative is non-decreasing, i.e., $h'(L) \le h'(\bar{L})$ for any $L \le \bar{L}$. It is sufficient to consider the following five cases.

- Suppose that $L+\Delta \le T_k^1(p,\epsilon)$ and $\bar{L}+\Delta \le T_k^1(p,\epsilon)$. The definition of $h(L)$ in (2.10) implies that $h'(L) = h_1'(L)$ and $h'(\bar{L}) = h_1'(\bar{L})$. We obtain $h'(L) \le h'(\bar{L})$ since $h_1'(L) \le h_1'(\bar{L})$ by the convexity of $h_1(L)$.

- Suppose that $L+\Delta \le T_k^1(p,\epsilon)$ and $T_k^1(p,\epsilon) < \bar{L}+\Delta \le T_k^2(p,\epsilon)$. We have $h'(\bar{L}) = h_2'(\bar{L})$ by the definition of $h(L)$ in (2.10). Moreover, since $T_k^1(p,\epsilon) - \Delta < \bar{L}$, the right-hand derivative of $h_2(L)$ at $T_k^1(p,\epsilon)-\Delta$ is no greater than the left-hand derivative of $h_2(L)$ at $\bar{L}$ (c.f. Artin [4]), i.e.,

$$\begin{aligned}
h'(\bar{L}) = h_2'(\bar{L}) &\geq \lim_{d\downarrow 0} \frac{h_2(T_k^1(p,\epsilon)-\Delta+d)-h_2(T_k^1(p,\epsilon)-\Delta)}{d} \\
&= (a-\rho\epsilon)e^{(a-\rho\epsilon)T_k^1(p,\epsilon)-\rho\Delta\epsilon} f_k^1(T_k^1(p,\epsilon),p,\epsilon) \\
&\quad + e^{(a-\rho\epsilon)T_k^1(p,\epsilon)-\rho\Delta\epsilon} \lim_{d\downarrow 0} \frac{f_k^1(T_k^1(p,\epsilon)+d,p,\epsilon)-f_k^1(T_k^1(p,\epsilon),p,\epsilon)}{d}.
\end{aligned}$$

Given p and ϵ, $T_k^1(p,\epsilon)$ minimizes the function $f_k^1(L_k,p,\epsilon)$. Therefore, the right-hand derivative of $f_k^1(L_k,p,\epsilon)$ with respect to L_k is nonnegative at the point $T_k^1(p,\epsilon)$, i.e.,

$$\lim_{d\downarrow 0} \frac{f_k^1(T_k^1(p,\epsilon)+d,p,\epsilon)-f_k^1(T_k^1(p,\epsilon),p,\epsilon)}{d} \geq 0.$$

Since the exponential function is nonnegative, we obtain

$$h'(\bar{L}) \geq (a-\rho\epsilon)e^{(a-\rho\epsilon)T_k^1(p,\epsilon)-\rho\Delta\epsilon} f_k^1(T_k^1(p,\epsilon),p,\epsilon).$$

The definition of $h(L)$ in (2.10) and the convexity of $h_1'(L)$ also imply that

$$h'(L) = h_1'(L) \leq h_1'(T_k^1(p,\epsilon)-\Delta) = (a-\rho\epsilon)e^{(a-\rho\epsilon)T_k^1(p,\epsilon)-\rho\Delta\epsilon} f_k^1(T_k^1(p,\epsilon),p,\epsilon).$$

As a result, we have $h'(L) \leq h'(\bar{L})$.

- Suppose $T_k^1(p,\epsilon) < L+\Delta \leq T_k^2(p,\epsilon)$ and $T_k^1(p,\epsilon) < L+\Delta \leq T_k^2(p,\epsilon)$. We can prove $h'(L) \leq h'(\bar{L})$ by the same argument as the first case.

- Suppose $T_k^1(p,\epsilon) < L+\Delta \leq T_k^2(p,\epsilon)$ and $\bar{L}+\Delta > T_k^2(p,\epsilon)$. According to the definition of $h(L)$ in (2.10) as well as the convexity of $h_2(L)$ and $h_3(L)$, we obtain

$$\begin{aligned}
h'(L) = h_2'(L) \leq h_2'(T_k^2(p,\epsilon)-\Delta) &= (a+\rho\epsilon)e^{(a+\rho\epsilon)T_k^2(p,\epsilon)-\rho\Delta\epsilon} f_k^2(T_k^2(p,\epsilon),p,\epsilon) \\
&\quad + e^{(a+\rho\epsilon)T_k^2(p,\epsilon)-\rho\Delta\epsilon} \frac{\partial f_k^2}{\partial L_k}(T_k^2(p,\epsilon),p,\epsilon)
\end{aligned}$$

and

$$h'(\bar{L}) = h_3'(L) \geq h_3'(T_k^2(p,\epsilon) - \Delta) = (a+\rho\epsilon)e^{(a+\rho\epsilon)T_k^2(p,\epsilon)-\rho\Delta\epsilon} f_k^2(T_k^2(p,\epsilon),p,\epsilon).$$

Note that $T_k^2(p,\epsilon) = \arg\min_{L_k} f_k^2(L_k,p,\epsilon)$, and hence

$$\frac{\partial f_k^2}{\partial L_k}(T_k^2(p,\epsilon),p,\epsilon) \leq 0.$$

Since the exponential function is nonnegative, we have $h_2'(T_k^2(p,\epsilon) - \Delta) \leq h_3'(T_k^2(p,\epsilon) - \Delta)$ and hence $h'(L) \leq h'(\bar{L})$.

- Suppose $L + \Delta > T_k^2(p,\epsilon)$ and $\bar{L} + \Delta > T_k^2(p,\epsilon)$. Similar to the first case, $h'(L) \leq h'(\bar{L})$ can be proved by the definition of $h(L)$ and the convexity of $h_3(L)$.

From the results of these cases, it follows directly that $h'(L)$ is increasing in L and so $h(L)$ is a convex function with respect to L, which completes the proof of part (ii). □

Lastly, we complete the proof of Theorem 2.1 by showing that the end of planning horizon value function, $J_{N+1}(x_{N+1},p_{N+1},\epsilon_{N+1})$ has the properties described in (A1) and (A2).

Proof of Theorem 2.1. Consider period $N+1$. By definition, we have

$$J_{N+1}(x_{N+1},p_{N+1},\epsilon_{N+1}) = \exp(-\rho\pi_{N+1}) = \exp(-\rho v(x_{N+1},\epsilon_{N+1})).$$

Obviously, $J_{N+1}(x_{N+1},p_{N+1},\epsilon_{N+1})$ is always nonnegative as the exponential function is nonnegative, i.e., it satisfies the induction assumption (A1).

For any given a, Δ, p, ϵ, let us define

$$h(L) = e^{aL} J_k(L+\Delta,p,\epsilon) = e^{aL}\exp(-\rho v(L+\Delta,p,\epsilon)) = \exp(aL - \rho v(L+\Delta,p,\epsilon)).$$

Let $u(L) = aL - \rho v(L+\Delta,p,\epsilon)$. Obviously, $u(L)$ is a convex function of L since $\rho > 0$

and $v(x_{N+1}, \epsilon_{N+1})$ is concave in x_{N+1}. Consider any L_1, L_2 and $\kappa \in [0, 1]$. According to the convexity of $u(L)$, we obtain

$$u(\kappa L_1 + (1-\kappa)L_2) \leq \kappa u(L_1) + (1-\kappa)u(L_2).$$

Therefore,

$$\begin{aligned} h(\kappa L_1 + (1-\kappa)L_2) = \exp(u(\kappa L_1 + (1-\kappa)L_2)) &\leq \exp(\kappa u(L_1) + (1-\kappa)u(L_2)) \\ &\leq \kappa \exp(u(L_1)) + (1-\kappa)\exp(u(L_2)) = \kappa h(L_1) + (1-\kappa)h(L_2), \end{aligned}$$

where the first and second inequalities follow from the monotonicity and convexity of the exponential function respectively, and hence $h(L)$ is a convex function of L, i.e., $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1})$ satisfies the property specified in the assumption (A2).

As a result, the properties (A1) and (A2) hold for $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1})$. Combined with the results in Propositions 2.1 and 2.2, the theorem can be proven by induction. □

Theorem 2.1 shows that we can obtain the optimal quantity to adjust the inventory level once we know $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$. We effectively reduce one dimension of the optimal solution since the values $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ are independent of x_k. For any period k, given the function $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$, we only need to (i) compute $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$, which are the global minimizers of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$, and (ii) compute $J_k(x_k, p_k, \epsilon_k)$ using the closed form (2.8). Therefore, we only need to deal with three dimensional functions in each period k. We avoid "the curse of dimensionality" in the sense that the dimension of the functions does not grow with the number of period. The computational load for each period is the same, and hence *the computational complexity is linear in the number of periods* N.

Next, we present an example which could be helpful to further understand the key tradeoff in the market-making inventory control problem.

Example 2.1. Consider a numerical example with $N = 100$ periods. Suppose that the distributions for δ_k, ϵ_k, s_k and d_k are stationary for any $k = 1, ..., N$.

We consider the case that s_k and d_k are either 0 or 1. They are independently distributed and

$$P(s_k = 1) = P(s_k = 0) = P(d_k = 1) = P(d_k = 0) = 0.5,$$

i.e., the distribution for S_k and Δ_k is

$$\begin{aligned} P(S_k = 0, \Delta_k = 0) = P(S_k = 1, \Delta_k = 1) &= P(S_k = 1, \Delta_k = -1) \\ &= P(S_k = 2, \Delta_k = 0) = 0.25. \end{aligned}$$

For the purpose of generality, we did not specify the unit of the demand and supply, e.g., s_k and d_k can be either 0 or 1 lot, i.e., 0 or $100,000$ units of base currency.

Note that the smallest commonly quoted change of exchange rates, a percentage in point (pip), is 10^{-4} for all major currencies except the Japanese yen. We restrict the mid price p_k to integral multiples of 10^{-4}. The support for δ_k is defined to be $\{\texttt{down}(p_k), 0, \texttt{up}(p_k)\}$ where

$$\texttt{up}(p_k) = -\texttt{down}(p_k) = \lfloor p_k \rfloor \times 10^{-4}, \tag{2.11}$$

i.e., $10^{-4}p_k$ rounded down to the closest integral multiple of 10^{-4}. We suppose that the price is more likely to move up if $\Delta_k = s_k - d_k$ is negative, i.e., when more clients buy from the market-maker, and vice versa. In particular, let

$$\begin{aligned} P(\delta_k = \texttt{up}(p_k) \mid \Delta_k = -1) &= 0.5 \\ P(\delta_k = 0 \mid \Delta_k = -1) = P(\delta_k = \texttt{down}(p_k) \mid \Delta_k = -1) &= 0.25, \\ P(\delta_k = 0 \mid \Delta_k = 0) &= 0.5 \\ P(\delta_k = \texttt{down}(p_k) \mid \Delta_k = 0) = P(\delta_k = \texttt{up}(p_k) \mid \Delta_k = 0) &= 0.25, \\ P(\delta_k = \texttt{down}(p_k) \mid \Delta_k = 1) &= 0.5 \\ P(\delta_k = 0 \mid \Delta_k = 1) = P(\delta_k = \texttt{up}(p_k) \mid \Delta_k = 1) &= 0.25. \end{aligned}$$

We assume that the spread is a known function of the mid price plus a random variable. In particular, let

$$\epsilon_k = \phi_k(p_k) + \varphi_k \text{ where } \phi_k(p_k) = \lfloor 0.5 p_k \rfloor \times 10^{-4}, \tag{2.12}$$

i.e., $0.5 \times 10^{-4} p_k$ rounded down to the closest integral multiple of 10^{-4}, and φ_k is a random variable independent of p_k, δ_k, s_k and d_k with the probability mass function $P(\varphi_k = 10^{-4}) = P(\varphi_k = 2 \times 10^{-4}) = 0.5$.

In addition, we set the risk aversion parameter $\rho = 100$ and consider two situations for the profit or loss at end of the planning horizon: (i) $\pi_{N+1} = v(x_{N+1}, \epsilon_{N+1}) = -\epsilon_{N+1}|x_{N+1}|$, i.e., the inventory is cleared at the bid or ask price quoted by other market-maker, (ii) $\pi_{N+1} = v(x_{N+1}, \epsilon_{N+1}) = 0$, i.e., the inventory is marked to the market mid price.

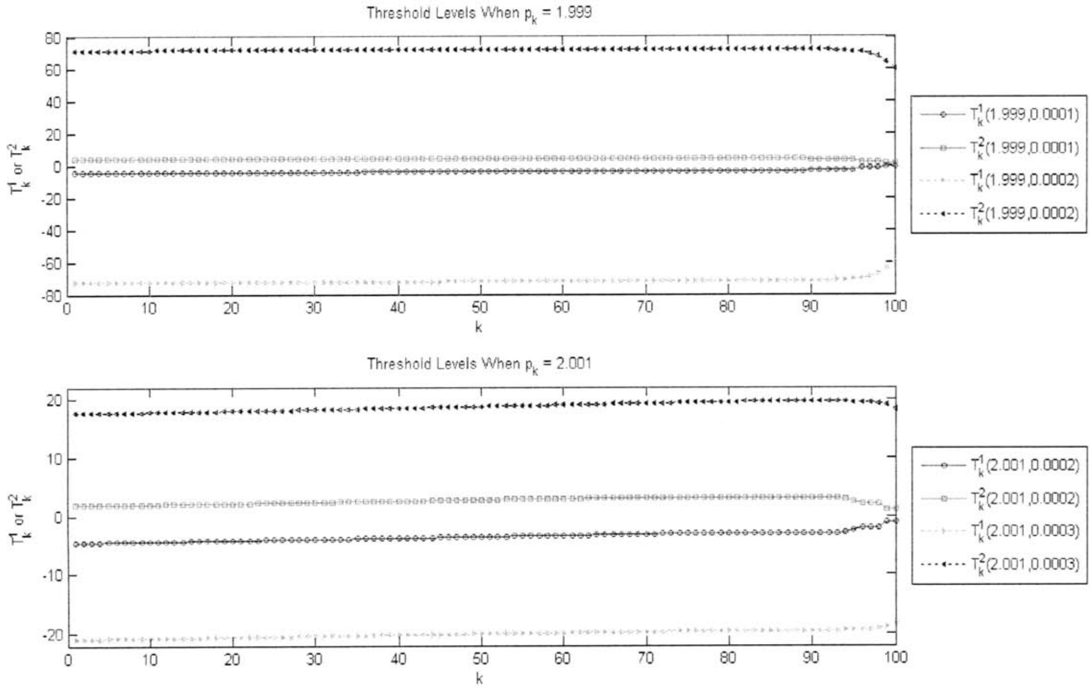
Table 2.1: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 2.1 with $\rho = 100$ and $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$

	$p_k = 1.999$				$p_k = 2.001$			
	$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0003$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−4.72	3.59	−72.85	71.34	−4.59	1.77	−20.89	17.49
50	−4.11	4.01	−72.07	71.70	−3.61	2.70	−20.01	18.64
100	−1.00	1.00	−59.42	59.42	−1.00	1.00	−18.24	18.24

The threshold levels $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ are shown in Table 2.1 and Figure 2-1. We present the results when $p_k = 1.999$ or 2.001 for any $k = 1, ..., 100$. According to (2.12), $\phi_k(1.999) = 0$ and $\phi_k(2.001) = 10^{-4}$. Hence, $\epsilon_k = 10^{-4}$ or 2×10^{-4} when $p_k = 1.999$ and $\epsilon_k = 2 \times 10^{-4}$ or 3×10^{-4} when $p_k = 2.001$.

As shown in Figure 2-1, for any given p_k and ϵ_k, the threshold levels $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ are relatively stable with respect to k when $k \leq 90$. However, when $k \geq 90$, $T_k^1(p_k, \epsilon_k)$ is increasing in k while $T_k^2(p_k, \epsilon_k)$ is decreasing in k, i.e., the no-trade region decreases as k increases. This is because x_{N+1} is cleared at the cost of ϵ_{N+1} at the end of planning horizon. For any period k, the maximum cost associated with one unit of on-hand inventory x_k is ϵ_k, because we always have the option to trade off this unit

Figure 2-1: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 2.1 with $\rho = 100$ and $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$



by paying the transaction cost ϵ_k. Note that our input parameters are stationary, and hence period $N+1$ has the highest cost associated with one unit of the inventory. Moreover, this cost decreases as k decreases, because we have more opportunities to trade off the inventory in later periods when we are further away from the end of the planning horizon. As a result, we can afford to have more inventory for earlier periods, which explains why the no-trade region deceases in k.

Also note that for the last period, $T_{100}^1(1.999, 10^{-4}) = -1$ and $T_{100}^2(1.999, 10^{-4}) = 1$. Suppose that $x_{100} = 2$. Let us consider the following two options.

- We pay the 10^{-4} transaction cost to sell one unit in period 100. After receiving the demand and supply from the clients, we hold either zero or two units to the end of planning horizon since both s_k and d_k are either zero or one, and clear these zero or two units at the transaction cost ϵ_{N+1} per unit.

- We do not actively trade with other market-makers in period 100. After receiving the demand and supply from the clients, the inventory position will be either two or three units, and we clear these two or three units at the transaction cost ϵ_{N+1} per unit.

Comparing these two options, the transaction cost for the first option is no greater than that of the second period as $\epsilon_{N+1} \geq 10^{-4}$ as $p_{100} = 1.999$. The expected return from price movements are the same since $E[\delta_k \mid p_k = 1.999] = 0$, but the first option holds less inventory which implies lower risk. As a result, a risk-averse decision maker will always choose the first option. Following this argument, we can establish that $T_{100}^1(1.999, 10^{-4}) \geq -1$ and $T_{100}^2(1.999, 10^{-4}) \leq 1$. Note that this property does not hold when $p_{100} = 1.999$ and $\epsilon_{100} = 2 \times 10^{-4}$, since $\epsilon_{N+1} \leq 2 \times 10^{-4}$ and so we have the incentive to hold the inventory in order to save the transaction cost. The case when $p_{100} = 2.001$ is the same and we omit the discussion here.

Table 2.2 and Figure 2-2 present the threshold levels when $\pi_{N+1} = 0$. Contrary to the situation with $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$, when $k \geq 90$, the no-trade region expands as k increases since $T_k^1(p_k, \epsilon_k)$ decreases in k and $T_k^2(p_k, \epsilon_k)$ increases in k. Under the situation $\pi_{N+1} = 0$, the inventory is cleared for free at the end of the planning horizon, and so we always have the incentive to hold the inventory towards the end of the planning horizon to save the transaction cost. This incentive is stronger in later periods since we bear the inventory risk caused by price uncertainty for fewer periods. As a result, we would like to hold more inventory and have a larger no-trade region as k increases. In addition, we have $T_{100}^1(p_{100}, \epsilon_{100}) = -\infty$ and $T_{100}^2(p_{100}, \epsilon_{100}) = \infty$ for both $p_{100} = 1.999$ and $p_{100} = 2.001$. It indicates that in the last period, the transaction cost to trade off the inventory cannot be compensated by the reduction in the inventory risk. Therefore, we simply hold the inventory for one more period and clear it at zero cost at the end of the planning horizon.

If we compare the threshold levels for different π_{N+1}, the threshold levels are very close for given p_k and ϵ_k when $k \leq 50$, c.f. Tables 2.1 and 2.2. The Bellman equation (2.3) indicates that the impact of π_{N+1} on $J_k(x_k, p_k, \epsilon_k)$ fades as k decreases, and hence the values of $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ are less affected by the end of the planning horizon profit or loss π_{N+1} for smaller k.

Table 2.2: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 2.1 with $\rho = 100$ and $\pi_{N+1} = 0$

	$p_k = 1.999$				$p_k = 2.001$			
	$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0003$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−4.74	3.56	−72.89	71.32	−4.65	1.71	−20.94	17.43
50	−4.40	4.19	−72.11	71.70	−3.69	2.62	−20.07	18.56
100	$-\infty$	∞	$-\infty$	∞	$-\infty$	∞	$-\infty$	∞

Both Figures 2-1 and 2-2 show that given p_k, $T_k^1(p_k, \epsilon_k)$ is higher for lower ϵ_k while $T_k^2(p_k, \epsilon_k)$ is lower for lower ϵ_k, i.e., the no-trade region is smaller for lower ϵ_k. The market-making inventory control is to find a tradeoff between the transaction cost and the inventory risk due to price movements. When the transaction cost, ϵ_k is lower, we could afford to trade with other market-makers more frequently in order to reduce the inventory risk. Consequently, we should have a smaller no-trade region.

Also note that the $T_k^1(1.999, 2 \times 10^{-4})$ is significantly low (at most -59.42) and $T_k^2(1.999, 2 \times 10^{-4})$ is significantly high (at least 59.42) for both definitions of π_{N+1}. This is mainly because there exists a 0.5 probability that the transaction cost in the next period $k+1$ will drop to 10^{-4}, therefore we can save the expected transaction cost by adopting a wider no-region in period k. If we modify the example so that $P(\varphi_k = 10^{-4}) = 0$ and $P(\varphi_k = 2 \times 10^{-4}) = 1$,[1] we cannot save the transaction cost by expecting the transaction cost to decrease, and the no-trade region for $p_k = 1.999$ and $\epsilon_k = 2 \times 10^{-4}$ will shrink significantly, e.g., $T_k^1(1.999, 2 \times 10^{-4}) = -5.96$ and $T_k^2(1.999, 2 \times 10^{-4}) = 5.00$ when we consider $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$, and $T_k^1(1.999, 2 \times 10^{-4}) = -5.94$ and $T_k^2(1.999, 2 \times 10^{-4}) = 5.05$ when we have $\pi_{N+1} = 0$.

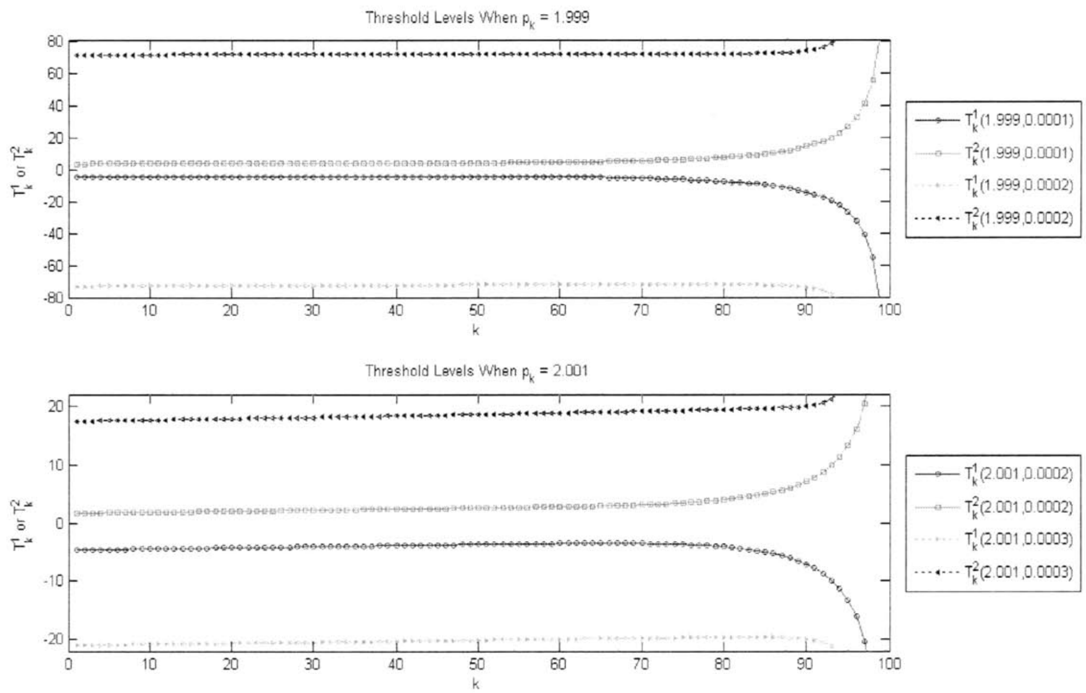
Moreover, for both $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ and $\pi_{N+1} = 0$, we have

$$T_k^1(1.999, \phi_k(1.999) + \varphi_k) \leq T_k^1(2.001, \phi_k(2.001) + \varphi_k)$$
$$T_k^2(1.999, \phi_k(1.999) + \varphi_k) \geq T_k^2(2.001, \phi_k(2.001) + \varphi_k)$$

where $\varphi_k = 10^{-4}$ or 2×10^{-4}, c.f. Tables 2.1 and 2.2 as well as Figures 2-1 and 2-2. In other words, we choose smaller no-trade region when $p_k = 2.001$ even though

[1] In this case, the threshold levels are independent of ϵ_k, c.f. Corollary 2.1.

Figure 2-2: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 2.1 with $\rho = 100$ and $\pi_{N+1} = 0$



it corresponds to higher transaction cost. According to (2.11), the support of δ_k is $\{0, \pm 2 \times 10^{-4}\}$ when $p_k = 2.001$, and it is $\{0, \pm 10^{-4}\}$ when $p_k = 1.999$. Therefore, $p_k = 2.001$ implies higher risk in price movements, and so we need to trade with other market-makers more frequently to reduce the risk, which implies a smaller no-trade region.

2.3 Reduction of the State Space

In the Bellman equation (2.3) and Theorem 2.1, the state for the dynamic programming model is x_k, p_k and ϵ_k. However, we can easily reduce the dimensions of the state space by imposing minor assumptions on the random variables. The following result can be established by the same proof as that for Theorem 2.1.

Corollary 2.1. *If δ_k, ϵ_k, S_k and Δ_k are independent of p_k for any k, then the state of the Bellman equation (2.3) is reduced to x_k and ϵ_k. Therefore, the functions $J_k(\cdot)$*

in (2.3), $f_k^1(\cdot)$ *and* $f_k^2(\cdot)$ *in* (2.4) *only depend on* x_k *and* ϵ_k, *and the threshold levels* $T_k^1(\cdot)$ *and* $T_k^2(\cdot)$ *in Theorem 2.1 only depend on* ϵ_k.

If ϵ_k *is a given function of* p_k *for any* k, *i.e.,* $\epsilon_k = \phi_k(p_k)$, *then the state of the Bellman equation* (2.3) *is reduced to* x_k *and* p_k. *Therefore, the functions* $J_k(\cdot)$ *in* (2.3), $f_k^1(\cdot)$ *and* $f_k^2(\cdot)$ *in* (2.4) *only depend on* x_k *and* p_k, *and the threshold levels* $T_k^1(\cdot)$ *and* $T_k^2(\cdot)$ *in Theorem 2.1 only depend on* p_k.

If ϵ_k *is a given constant and* δ_k, S_k *and* Δ_k *are independent of* p_k *for any* k, *then the state of the Bellman equation* (2.3) *is reduced to* x_k. *Therefore, the functions* $J_k(\cdot)$ *in* (2.3), $f_k^1(\cdot)$ *and* $f_k^2(\cdot)$ *in* (2.4) *only depend on* x_k, *and the threshold levels* $T_k^1(\cdot)$ *and* $T_k^2(\cdot)$ *in Theorem 2.1 are reduced to constants for any* k.

Let us consider an example where the threshold levels are independent of the mid-price p_k.

Example 2.2. In Example 2.1, we define the support δ_k and ϵ_k as functions of p_k presented in (2.11) and (2.12). Now let us consider the same stochastic input in Example 2.1 except that the functions $\texttt{up}(p_k)$, $\texttt{down}(p_k)$ and $\phi_k(p_k)$ in (2.11) and (2.12) are replaced by $\texttt{up}(p_k) = -\texttt{down}(p_k) = 10^{-4}$ and $\phi_k(p_k) = 0$.

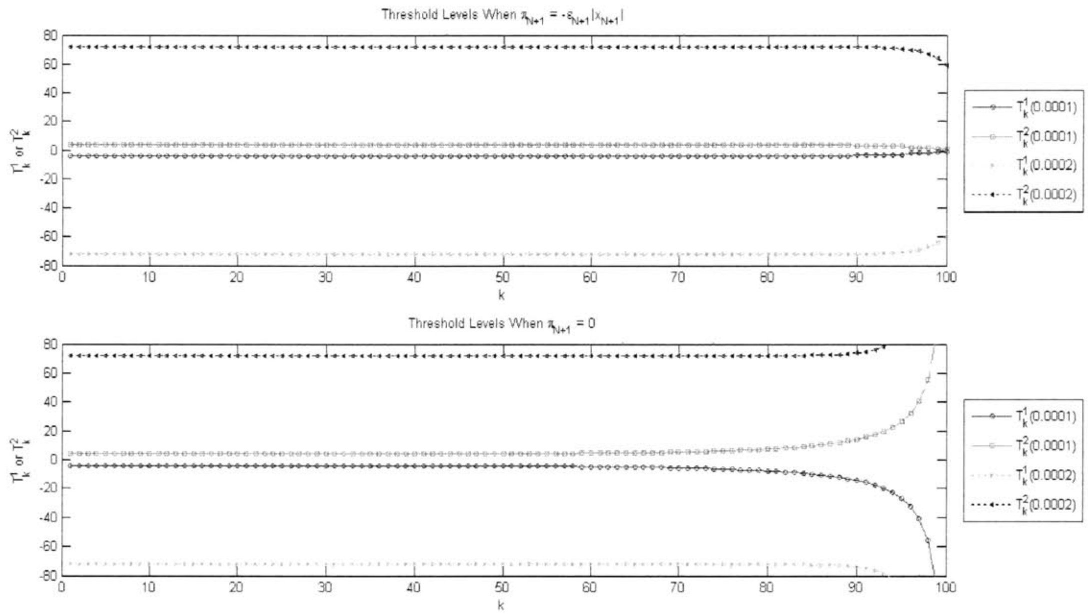
Under the assumptions, all the random variables δ_k, ϵ_k, S_k and Δ_k are independent of p_k for any k. According to Corollary 2.1 the threshold levels only depend on ϵ_k, and we denote them by $T_k^1(\epsilon_k)$ and $T_k^2(\epsilon_k)$. Again, let us consider $N = 100$ and $\rho = 100$. The corresponding threshold levels for $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ and $\pi_{N+1} = 0$ are shown in Table 2.3 and Figure 2-3.

Table 2.3: $T_k^1(\epsilon_k)$ and $T_k^2(\epsilon_k)$ for Example 2.2 with $\rho = 100$

	$\pi_{N+1} = -\epsilon_{N+1}\|x_{N+1}\|$				$\pi_{N+1} = 0$			
	$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	-4.06	4.07	-71.77	71.79	-4.08	4.09	-71.78	71.78
50	-4.00	4.01	-71.77	71.78	-4.31	4.32	-71.78	71.79
100	-1.00	1.00	-59.42	59.42	$-\infty$	∞	$-\infty$	∞

The change in the threshold levels with respect to k in Table 2.3 and Figure 2-3 is the same as that in Example 2.1. In fact, for $\epsilon_{100} = 10^{-4}$ or 2×10^{-4}, the values of

Figure 2-3: $T_k^1(\epsilon_k)$ and $T_k^2(\epsilon_k)$ for Example 2.2 with $\rho = 100$



$T_{100}^1(\epsilon_{100})$ and $T_{100}^2(\epsilon_{100})$ in Table 2.3 are the same as $T_{100}^1(p_{100}, \epsilon_{100})$ and $T_{100}^2(p_{100}, \epsilon_{100})$ with $p_{100} = 1.999$ in Example 2.1 with the corresponding π_{N+1}, because Example 2.1 specifies $\texttt{up}(p_k) = -\texttt{down}(p_k) = 10^{-4}$ and $\phi_k(p_k) = 0$ for any $p_k < 2$, and the relation between π_{N+1} and p_{N+1} solely depends on the value of ϵ_{N+1}.

Similar to Example 2.1, we also observe that the no-trade region for any period k shrinks as the spread decrease from $\epsilon_k = 2 \times 10^{-4}$ to $\epsilon_k = 10^{-4}$, since we can afford to control our inventory in a tighter region with lower transaction cost.

Also note that $T_k^1(\epsilon_k) \approx -T_k^2(\epsilon_k)$ for any k and π_{N+1} in Table 2.3. In our input, the marginal distribution of the price movements is symmetric with respect to zero, and the demand and supply from the clients have the same marginal distribution. In other words, the marginal probability for the price to increase or decrease a certain amount is the same, and the marginal probability for the clients to buy or sell a certain amount is also the same. Hence, the risk associated with holding an inventory of x_k units measured by expected exponential utility is very close to that of $-x_k$ units. As a result, the threshold levels should have the similar absolute values. We will formally prove the symmetry of the threshold levels, i.e., $T_k^1(\epsilon_k) = -T_k^2(\epsilon_k)$ later in Proposition 2.3.

Next, we introduce an example where the threshold levels are independent of the spread defined by ϵ_k.

Example 2.3. Similar to Example 2.2, we also consider a simplified case of Example 2.1. The stochastic input is the same as those in Example 2.1 except that (i) we let $\texttt{up}(p_k) = -\texttt{down}(p_k) = 10^{-4}$ instead of the definitions in (2.11), and (ii) φ_k is defined to be a constant instead of random variable. Note that the condition (ii) is sufficient for the threshold levels being indepedent of ϵ_k.

We consider two cases $\varphi_k = 10^{-4}$ and $\varphi_k = 2 \times 10^{-4}$ respectively, and for each case we let the end of the planning horizon profit or loss be either $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ and $\pi_{N+1} = 0$. The threshold levels are denoted by $T_k^1(p_k)$ and $T_k^2(p_k)$ since they are independent of ϵ_k. We display the results for $\varphi_k = 10^{-4}$ in Table 2.4 and Figure 2-4 and those for $\varphi_k = 2 \times 10^{-4}$ in Table 2.5 and Figure 2-5.

Table 2.4: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 2.3 with $\varphi_k = 10^{-4}$ and $\rho = 100$

	$\pi_{N+1} = -\epsilon_{N+1}\|x_{N+1}\|$				$\pi_{N+1} = 0$			
	$p_k = 1.999$		$p_k = 2.001$		$p_k = 1.999$		$p_k = 2.001$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−5.36	3.46	−6.16	4.82	−5.54	3.33	−6.43	4.66
50	−4.48	4.05	−5.41	5.19	−4.80	4.29	−6.96	6.61
100	−1.00	1.00	−1.00	1.00	$-\infty$	∞	$-\infty$	∞

Table 2.5: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 2.3 with $\varphi_k = 2 \times 10^{-4}$ and $\rho = 100$

	$\pi_{N+1} = -\epsilon_{N+1}\|x_{N+1}\|$				$\pi_{N+1} = 0$			
	$p_k = 1.999$		$p_k = 2.001$		$p_k = 1.999$		$p_k = 2.001$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−6.29	4.77	−6.78	5.49	−6.67	4.61	−7.23	5.47
50	−5.49	5.10	−6.00	6.00	−7.20	6.67	−9.47	9.07
100	−1.00	1.00	−1.00	1.00	$-\infty$	∞	$-\infty$	∞

The trend of threshold levels with respect to the period k is very similar to what we observed in Examples 2.1 and 2.2. Moreover, we observe that the no-trade regions

Figure 2-4: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 2.3 with $\varphi_k = 10^{-4}$ and $\rho = 100$

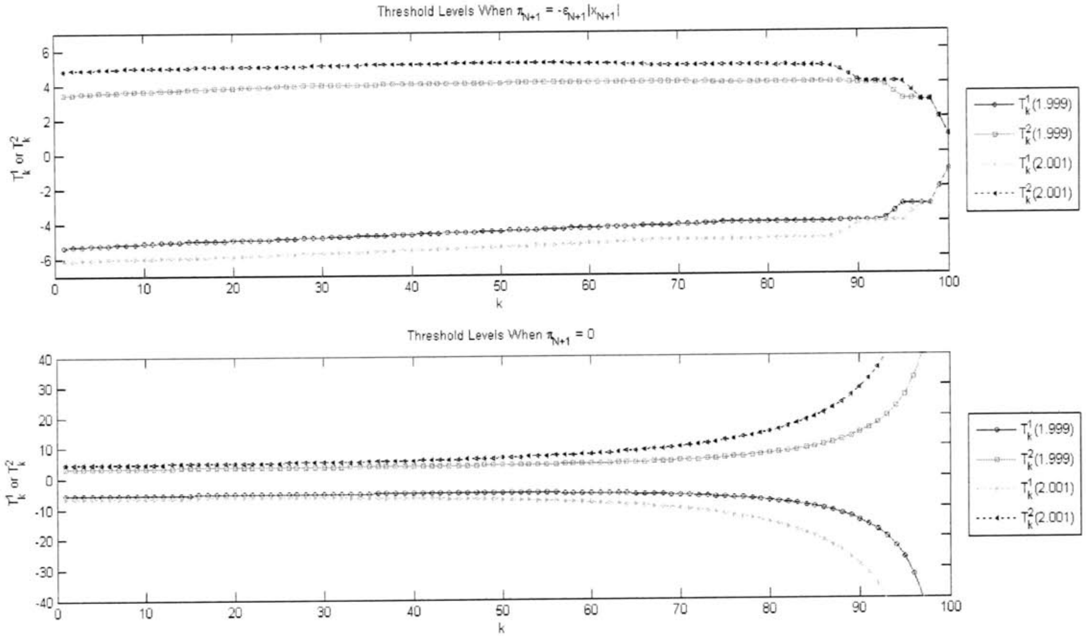
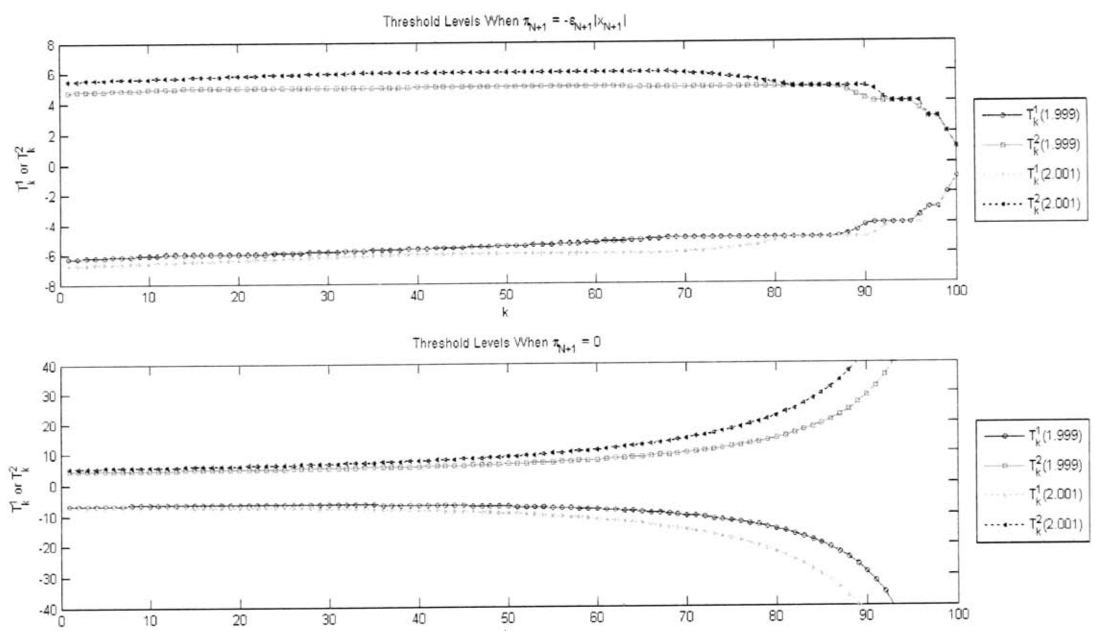


Figure 2-5: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 2.3 with $\varphi_k = 2 \times 10^{-4}$ and $\rho = 100$



are increasing in both p_k and φ_k. To be specific, the threshold levels have higher absolute values when $p_k = 2.001$ than when $p_k = 1.999$ for both values of φ_k and both definitions of π_{N+1}, which is obvious from Figures 2-4 and 2-5. In the meantime, the absolute values of threshold levels are higher when $\varphi_k = 2\times 10^{-4}$ than when $\varphi_k = 10^{-4}$ for both values of p_k and both definitions of π_{N+1} which can be observed by comparing the numbers in Tables 2.4 and 2.5. The reason behind these two observations is that higher values in p_k or φ_k are associated with higher value of ϵ_k, i.e., it is more expensive to trade with other market-makers to actively adjust the inventory. Therefore, the market-maker will tend to actively trade less frequently. It results in a larger no-trade region, which implies higher absolute values for the threshold levels for this particular example.

2.4 Symmetric Threshold Policy

Example 2.2 shows a case where the threshold levels, $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$, have very close absolute values. According to the proof of Theorem 2.1, these levels are the global minimizer of the functions $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ when given p_k and ϵ_k. Obviously, if these two functions satisfy $f_k^1(L_k, p_k, \epsilon_k) = f_k^2(-L_k, p_k, \epsilon_k)$ for any L_k, p_k and ϵ_k, then the threshold levels, $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$, have the same absolute value, i.e., $T_k^1(p_k, \epsilon_k) = -T_k^2(p_k, \epsilon_k)$. In this case, the threshold levels are symmetric with respect to zero and we refer to it as the *symmetric* threshold policy. To establish sufficient conditions for the symmetric threshold policy to be optimal, we require that (i) δ_k, ϵ_k, S_k and Δ_k are independent of the price p_k for any k, and (ii) the last period profit or loss fucntion π_{N+1} as well as the conditional distributions of Δ_k and δ_k for any k are symmetric with respect to zero. Under these assumptions, the risk associated with holding an inventory position is solely determined by the absolute value of the inventory position, and hence the no-trade region as well as the threshold levels should be symmetric with respect to zero.

Formally, we denote the cumulative distribution function of the random variable δ_k conditional on Δ_k, S_k and ϵ_k as $F_{\delta_k|\Delta_k, S_k, \epsilon_k}(\delta_k)$, and the cumulative distribution function of the random variable Δ_k conditional on δ_k, S_k and ϵ_k as $F_{\Delta_k|\delta_k,S_k,\epsilon_k}(\Delta_k)$. Let us consider the following assumptions.

(B1) δ_k, ϵ_k, S_k and Δ_k are independent of p_k for any k.

(B2) $v(x_{N+1}, \epsilon_{N+1}) = v(-x_{N+1}, \epsilon_{N+1})$ for any x_{N+1} and ϵ_{N+1}, i.e., the function $\pi_{N+1} = v(x_{N+1}, \epsilon_{N+1})$ is symmetric in x_{N+1} with respect to zero.

(B3) $F_{\delta_k|\Delta_k,S_k,\epsilon_k}(\delta_k) + F_{\delta_k|\Delta_k,S_k,\epsilon_k}(-\delta_k) = 1 + dF_{\delta_k|\Delta_k,S_k,\epsilon_k}(\delta_k)$ for any k and δ_k, i.e., the conditional distribution of δ_k is symmetric with respect to zero.

(B4) $F_{\Delta_k|\delta_k,S_k,\epsilon_k}(\Delta_k) + F_{\Delta_k|\delta_k,S_k,\epsilon_k}(-\Delta_k) = 1 + dF_{\Delta_k|\delta_k,S_k,\epsilon_k}(\Delta_k)$ for any k and Δ_k, i.e., the conditional distribution of Δ_k is symmetric with respect to zero.

The condition (B4) is equivalent to that s_k and d_k have the same distribution conditional on δ_k and ϵ_k. Let $F_{s_k,d_k|\delta_k,\epsilon_k}(s_k, d_k)$ denote the cumulative distribution function of the random variables s_k and d_k conditional on δ_k and ϵ_k.

Lemma 2.2. *$F_{\Delta_k|\delta_k,S_k,\epsilon_k}(\Delta_k) + F_{\Delta_k|\delta_k,S_k,\epsilon_k}(-\Delta_k) = 1 + dF_{\Delta_k|\delta_k,S_k,\epsilon_k}(\Delta_k)$ for any k and Δ_k if and only if $F_{s_k,d_k|\delta_k,\epsilon_k}(s_k, d_k) = F_{s_k,d_k|\delta_k,\epsilon_k}(d_k, s_k)$ for any k, s_k and d_k.*

Lemma 2.2 can be easily proved using $S_k = s_k + d_k$ and $\Delta_k = s_k - d_k$ as well as the definition of cumulative distribution function, and hence the proof is omitted here.

Now let us prove the symmetry of the threshold level under the assumptions (B1), (B2), (B3) and (B4).

Proposition 2.3. *Under the conditions (B1), (B2), (B3) and (B4), a symmetric threshold policy is optimal for the problem in* (2.2). *In particular, $J_k(x_k, \epsilon_k) = J_k(-x_k, \epsilon_k)$ and $T_k^1(\epsilon_k) = -T_k^2(\epsilon_k)$ for any k, x_k and ϵ_k.*

Proof. According to Corollary 2.1, the condition (B1) implies that the functions $J_k(\cdot)$ in (2.3), $f_k^1(\cdot)$ and $f_k^2(\cdot)$ in (2.4) only depend on x_k and ϵ_k, and the threshold levels $T_k^1(\cdot)$ and $T_k^2(\cdot)$ are functions of ϵ_k only.

From the condition (B2) and the definition of $J_{N+1}(x_{N+1}, \epsilon_{N+1})$, we obtain

$$\begin{aligned} J_{N+1}(x_{N+1}, \epsilon_{N+1}) &= \exp(-\rho v(x_{N+1}, \epsilon_{N+1})) \\ &= \exp(-\rho v(-x_{N+1}, \epsilon_{N+1})) = J_{N+1}(-x_{N+1}, \epsilon_{N+1}). \end{aligned}$$

Now let us suppose that $J_{k+1}(x_{k+1}, \epsilon_{k+1}) = J_{k+1}(-x_{k+1}, \epsilon_{k+1})$ for any x_{k+1} and ϵ_{k+1}.

The definition of $f_k^1(L_k, p_k, \epsilon_k)$ in (2.4) shows that

$$f_k^1(L_k, \epsilon_k) = E\left[e^{-\rho\left((L_k+\Delta_k)\delta_k+(S_k-L_k)\epsilon_k\right)} J_{k+1}\left(L_k + \Delta_k, \epsilon_{k+1}\right) \middle| \epsilon_k \right]$$

Consider the condition (B4) that the conditional distribution of Δ_k is symmetric, which means we can flip the signs of Δ_k inside the expectation, i.e.,

$$f_k^1(L_k, \epsilon_k) = E\left[e^{-\rho\left((L_k-\Delta_k)\delta_k+(S_k-L_k)\epsilon_k\right)} J_{k+1}\left(L_k - \Delta_k, \epsilon_{k+1}\right) \middle| \epsilon_k \right].$$

Note that $J_{k+1}(x_{k+1}, \epsilon_{k+1}) = J_{k+1}(-x_{k+1}, \epsilon_{k+1})$ by assumption. We have

$$f_k^1(L_k, \epsilon_k) = E\left[e^{-\rho\left((L_k-\Delta_k)\delta_k+(S_k-L_k)\epsilon_k\right)} J_{k+1}\left(-L_k + \Delta_k, \epsilon_{k+1}\right) \middle| \epsilon_k \right].$$

The condition (B3), the symmetry of the conditional distribution of δ_k ensures us to flip the sign of δ_k inside the expectation, and hence

$$f_k^1(L_k, \epsilon_k) = E\left[e^{-\rho\left((-L_k+\Delta_k)\delta_k+(S_k-L_k)\epsilon_k\right)} J_{k+1}\left(-L_k + \Delta_k, \epsilon_{k+1}\right) \middle| \epsilon_k \right] = f_k^2(-L_k, \epsilon_k),$$

where the second inequality follows from the definition of $f_k^2(L_k, \epsilon_k)$ in (2.4). It follows directly that $T_k^1(\epsilon_k) = -T_k^2(\epsilon_k)$ for period k.

Let us consider $J_k(x_k, \epsilon_k)$. For any $x_k \in [0, T_k^2(\epsilon_k)]$, i.e., $-x_k \in [T_k^1(\epsilon_k), 0]$,

$$J_k(x_k, \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^2(x_k, \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^1(-x_k, \epsilon_k) = J_k(-x_k, \epsilon_k),$$

where the first and last equalities follow from (2.8) and the second equality is obtained since we have shown $f_k^1(L_k, \epsilon_k) = f_k^2(-L_k, \epsilon_k)$. Similarly, for $x_k > T_k^2(\epsilon_k)$, i.e., $-x_k <$

$T_k^1(\epsilon_k)$,

$$
\begin{aligned}
J_k(x_k, \epsilon_k) &= e^{\rho x_k \epsilon_k} f_k^2(T_k^2(\epsilon_k), \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^2(-T_k^1(\epsilon_k), \epsilon_k) = e^{\rho x_k \epsilon_k} f_k^1(T_k^1(\epsilon_k), \epsilon_k) \\
&= J_k(-x_k, \epsilon_k),
\end{aligned}
$$

where the first and last equalities follow from (2.8), the second equality is obtained from $T_k^1(\epsilon_k) = -T_k^2(\epsilon_k)$, and the third equality is due to the property that $f_k^1(L_k, \epsilon_k) = f_k^2(-L_k, \epsilon_k)$. These results show that the induction assumption holds for period k, i.e. $J_k(x_k, \epsilon_k) = J_k(-x_k, \epsilon_k)$, which completes the proof. □

Next, we present an example for the symmetric threshold policy. It also illustrates how the threshold levels change with respect to the risk aversion parameter ρ.

Example 2.4. Consider an example with $N = 100$ periods. Similar to Example 2.1, we assume that the distributions for the stochastic input are stationary and independent across k. In addition, we let $\epsilon_k = 10^{-4}$ for any k and suppose that δ_k, s_k and d_k are independently distributed with the probability mass functions

$$
\begin{gathered}
P(\delta_k = -10^{-4}) = P(\delta_k = 0) = P(\delta_k = -10^{-4}) = \tfrac{1}{3}, \\
P(s_k = 1) = P(s_k = 0) = \tfrac{1}{2} \text{ and } P(d_k = 1) = P(d_k = 0) = \tfrac{1}{2}.
\end{gathered}
$$

Note that ϵ_k is a constant and δ_k, s_k and d_k are independent of p_k. It follows from Corollary 2.1 that the threshold levels are independent of both p_k and ϵ_k, and we denote them by T_k^1 and T_k^2. Moreover, Proposition 2.3 shows that $T_k^1 = -T_k^2$.

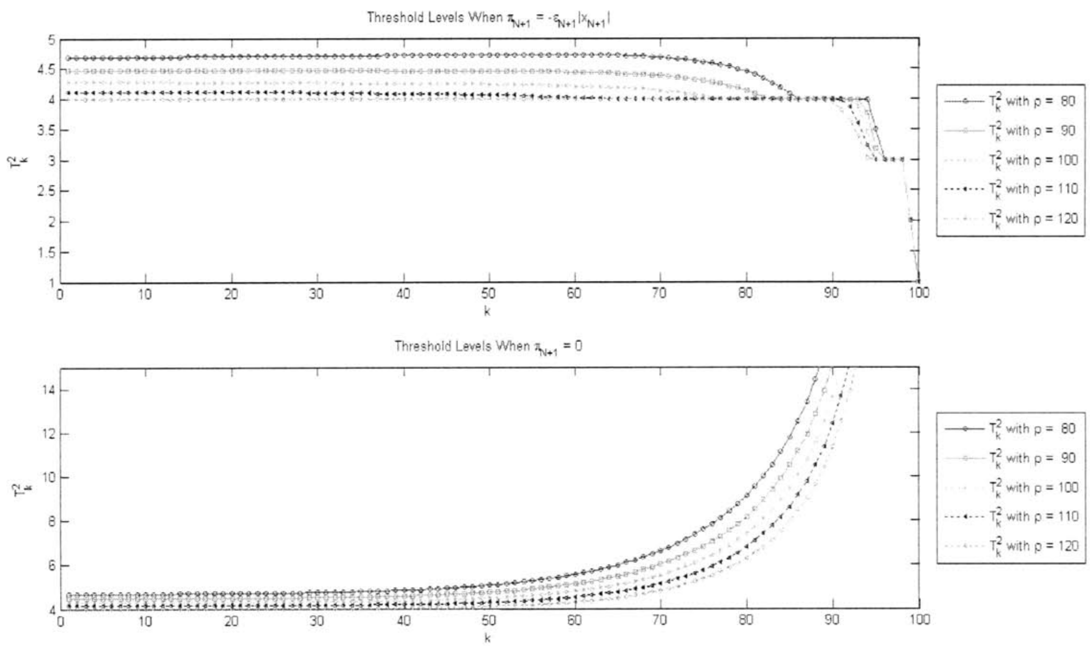
Again, we consider two situations of π_{N+1}, i.e., $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ and $\pi_{N+1} = 0$. For the risk aversion parameter $\rho = 80, 90, 100, 110, 120$, the corresponding threshold levels T_k^2 are shown in Table 2.6 and Figure 2-6. Note that the second row in Table 2.6 represent the values of ρ, and the corresponding T_k^2 are shown in the last three rows for $k = 1, 50$ and 100.

It is easy to see that the threshold levels change as k increases in the same manner as that in Examples 2.1, 2.2 and 2.3. Moreover, the threshold level T_k^2 decreases as ρ increases. Note that a more risk-averse attitude is associated with a higher value in ρ. A more risk-averse decision maker is willing to sacrifice more transaction cost in order

Table 2.6: T_k^2 for Example 2.4

	$\pi_{N+1} = -\epsilon_{N+1}\lvert x_{N+1}\rvert$					$\pi_{N+1} = 0$				
ρ	80	90	100	110	120	80	90	100	110	120
k	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2
1	4.69	4.47	4.28	4.12	4.00	4.68	4.48	4.30	4.14	4.01
50	4.73	4.46	4.24	4.07	4.00	5.07	4.75	4.49	4.29	4.11
100	1.00	1.00	1.00	1.00	1.00	∞	∞	∞	∞	∞

Figure 2-6: T_k^2 for Example 2.4



to protect against price uncertainty. Therefore, he or she will trade more frequently with other market-maker to reduce the inventory risk caused by price movements, and so the no-trade region will be smaller. Since $T_k^1 = -T_k^2$, it follows immediately that T_k^2 is increasing in the risk aversion parameter ρ.

2.5 Extensions

This chapter investigates how to control the inventory position of a single asset in the marketing-making process, where the exponential utility function is applied to model the risk-averse attitude of the decision maker. We prove that the optimal policy is a threshold policy and demonstrate that the policy can be further simplified under certain circumstances. In particular, we can reduce the dimensions of the states which determine the threshold levels, and identify conditions under which the threshold levels can be symmetric with respect to zero. The structural properties of the optimal policy lead to a computationally efficient algorithm to compute the threshold levels, which allows us to present numerical examples to illustrate the optimal policy.

As we mentioned in Section 2.1, the optimality of the threshold policy can be extended to the following settings: (i) the price dynamics and the client order processes are auto-correlated, and (ii) the spread the market-maker quotes to his or her clients are different from the spread he or she pays when actively trading with other market-makers. These extensions are discussed at the end of Chapter 3 (Section 3.7) since the results are identical and the proofs are similar for exponential utility and mean-variance models.

Chapter 3

Single-Asset Market-Making with Mean-Variance Tradeoff

Chapter 2 studies the inventory control problem in market-making with an exponential utility objective function. Another common approach to model risk-aversion is the mean-variance tradeoff. In this chapter, we adopt such an objective function for the inventory problem presented in Section 2.1 of Chapter 2. The dynamic programming formulation for the mean-variance analysis model is introduced in Section 3.1. We present the optimal threshold policy in Section 3.2, and the properties of the threshold levels are investigated in the following sections. In particular, Section 3.3 shows how to reduce the state space for the threshold levels under various condition, Section 3.4 studies the risk-neutral model which is a special case of the mean-variance model, and the symmetric and monotone properties of the threshold levels are identified in Section 3.5 and Section 3.6 respectively. Finally, Section 3.7 summarizes this chapter and presents extensions of our results.

3.1 Mean-Variance Analysis

Assume that the utility function of the market-maker is a linear trade-off between the expectation and the variance of the total profit. Adopting the notations used in

Section 2.1 of Chapter 2, the objective function is defined as

$$\max_{q_k} \quad E\left[\sum_{k=1}^{N}\left\{E\left[\pi_k|p_k,\epsilon_k\right]-\lambda\times Var\left(\pi_k|p_k,\epsilon_k\right)\right\}+\pi_{N+1}\right], \tag{3.1}$$

where the parameter $\lambda \geq 0$ represents the decision maker's risk sensitivity. Obviously, the decision maker is risk neutral when $\lambda = 0$.

We consider the expectations and the variances conditional on p_k and ϵ_k because the random variables in period k, i.e., δ_k, S_k and Δ_k are correlated with p_k and ϵ_k, and p_k and ϵ_k jointly determine the bid and ask prices in period k, which are observed before we make the decision to actively trade q_k units with other market-makers. Note that x_k is also observable before making the decision, but it is independent of other random variables defining π_k.

In addition, the profit and loss in period $N+1$, $\pi_{N+1} = v(x_{N+1}, \epsilon_{N+1})$ is defined to be a deterministic function of x_{N+1} and ϵ_{N+1}, and it follows directly that $E[\pi_{N+1}|\epsilon_{N+1}] = \pi_{N+1}$ and $Var(\pi_{N+1}|\epsilon_{N+1}) = 0$. Also note that p_{N+1} can only affect the value of π_{N+1} through the correlation with ϵ_{N+1}, and so $E[\pi_{N+1}|p_{N+1}, \epsilon_{N+1}] = \pi_{N+1}$ and $Var(\pi_{N+1}|p_{N+1}, \epsilon_{N+1}) = 0$. That is why we have the term π_{N+1} in the objective function (3.1).

It is well known that if the distribution of π_k conditional on p_k and ϵ_k is a normal distribution, the mean-variance analysis in (3.1) is equivalent to maximizing the sum of exponential utilities in each period k. Also note that this objective function is very similar to that in Stoikov and Saglam [50]. Moreover, the mean-variance type objective is also commonly adopted in the optimal order execution literature, e.g., Almgren and Chriss [2] and Engle and Ferstenberg [17].

Consider the expectation and variance of one-period profit π_k conditional on the price p_k and the spread ϵ_k. According to π_k defined in (2.1), it is straightforward to obtain

$$
\begin{aligned}
E[\pi_k|p_k, \epsilon_k] &= L_k E[\delta_k|p_k, \epsilon_k] - |L_k - x_k|\epsilon_k + E\left[\delta_k \Delta_k + S_k \epsilon_k \middle| p_k, \epsilon_k\right] \\
Var(\pi_k|p_k, \epsilon_k) &= L_k^2 Var(\delta_k|p_k, \epsilon_k) + Var\left(\delta_k \Delta_k + S_k \epsilon_k \middle| p_k, \epsilon_k\right) \\
&\quad + 2L_k \Big(E[\delta_k^2 \Delta_k|p_k, \epsilon_k] - E[\delta_k|p_k, \epsilon_k] E[\delta_k \Delta_k|p_k, \epsilon_k]\Big) \\
&\quad + 2L_k \epsilon_k \Big(E[\delta_k S_k|p_k, \epsilon_k] - E[\delta_k|p_k, \epsilon_k] E[S_k|p_k, \epsilon_k]\Big).
\end{aligned} \tag{3.2}
$$

Let $\mu_k(p_k, \epsilon_k) = E[\delta_k|p_k, \epsilon_k]$, $\sigma_k^2(p_k, \epsilon_k) = Var(\delta_k|p_k, \epsilon_k)$ and $\nu_k(p_k, \epsilon_k) = \nu_k^1(p_k, \epsilon_k) + \epsilon_k \nu_k^2(p_k, \epsilon_k)$ where

$$
\begin{aligned}
\nu_k^1(p_k, \epsilon_k) &= 2\Big(E[\delta_k^2 \Delta_k|p_k, \epsilon_k] - E[\delta_k|p_k, \epsilon_k] E[\delta_k \Delta_k|p_k, \epsilon_k]\Big) \\
\nu_k^2(p_k, \epsilon_k) &= 2\Big(E[\delta_k S_k|p_k, \epsilon_k] - E[\delta_k|p_k, \epsilon_k] E[S_k|p_k, \epsilon_k]\Big).
\end{aligned} \tag{3.3}
$$

Since the rest terms in $E[\pi_k|p_k, \epsilon_k]$ and $Var(\pi_k|p_k, \epsilon_k)$ are independent of the decision variable $L_k = x_k + q_k$, the problem in (3.1) is reduced to

$$
\min_{L_k} \; E\left[\sum_{k=1}^{N} \left\{\epsilon_k |L_k - x_k| + (\lambda \nu_k(p_k, \epsilon_k) - \mu_k(p_k, \epsilon_k)) L_k + \lambda \sigma_k^2(p_k, \epsilon_k) L_k^2\right\} - \pi_{N+1}\right].
$$

Note that the random variable S_k is no longer included in the optimization problem.

Accordingly, the Bellman equation reads

$$
\begin{aligned}
J_k(x_k, p_k, \epsilon_k) = \min_{L_k} \Big\{ &\epsilon_k |L_k - x_k| + \Big(\lambda \nu_k(p_k, \epsilon_k) - \mu_k(p_k, \epsilon_k)\Big) L_k + \lambda \sigma_k^2(p_k, \epsilon_k) L_k^2 \\
&+ E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \middle| p_k, \epsilon_k \right\} \Big\}
\end{aligned} \tag{3.4}
$$

for any $k = 1, ..., N$, and $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1}) = -\pi_{N+1} = -v(x_{N+1}, \epsilon_{N+1})$.

In the next four sections we establish the optimality of the threshold policy and analyze properties of the optimal policy. Numerical examples are also presented to illustrate the analytical results.

3.2 Optimality of the Threshold Policy

Let us define the following functions

$$f_k(L_k, p_k, \epsilon_k) = \Big(\lambda\nu_k(p_k, \epsilon_k) - \mu_k(p_k, \epsilon_k)\Big) L_k + \lambda\sigma_k^2(p_k, \epsilon_k)L_k^2 \\ + E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right)\Big| p_k, \epsilon_k \right\},$$

$$f_k^1(L_k, p_k, \epsilon_k) = f_k(L_k, p_k, \epsilon_k) + \epsilon_k L_k \quad \text{and} \quad f_k^2(L_k, p_k, \epsilon_k) = f_k(L_k, p_k, \epsilon_k) - \epsilon_k L_k. \tag{3.5}$$

Fix p_k and ϵ_k, let

$$T_k^0(p_k, \epsilon_k) = \arg\min_{L_k} f_k(L_k, p_k, \epsilon_k),$$

$$T_k^1(p_k, \epsilon_k) = \arg\min_{L_k} f_k^1(L_k, p_k, \epsilon_k),$$

$$T_k^2(p_k, \epsilon_k) = \arg\min_{L_k} f_k^2(L_k, p_k, \epsilon_k).$$

Again, the optimality of the threshold policy is proved by induction on the number of period k, and we suppose that $J_{k+1}(x_{k+1}, p_{k+1})$ is convex with respect to x_{k+1}.

Similar to the Bellman equation with an exponential utility function in (2.3), the inventory position x_k is embedded in an absolute value function and it cannot be separated from the objective function. In Chapter 2, we prove the optimal policy by first analyzing the optimal L_k in two cases, $L_k \geq x_k$ and $L_k \leq x_k$, and then comparing these two solution in order to decide whether to buy from or sell to other market-makers. In this proof, given x_k, we first determine which direction to adjust our inventory, i.e., whether the optimal L_k is greater than x_k or less than x_k.

Lemma 3.1. *Suppose that $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is convex with respect to x_{k+1}. Let L_k^* denote the optimal solution of L_k in (3.4). For any given p_k and ϵ_k, $L_k^* \leq x_k$ if $x_k \geq T_k^0(p_k, \epsilon_k)$, and $L_k^* \geq x_k$ if $x_k \leq T_k^0(p_k, \epsilon_k)$.*

Proof. Consider the function $f_k(L_k, p_k, \epsilon_k)$ defined in (3.5). Since

$$\sigma_k^2(p_k, \epsilon_k) = Var(\delta_k | p_k, \epsilon_k) \geq 0,$$

we know that $(\lambda\nu_k(p_k, \epsilon_k) - \mu_k(p_k, \epsilon_k)) L_k + \lambda\sigma_k^2(p_k, \epsilon_k)L_k^2$ is convex in L_k. Notice that $E\left\{J_{k+1}\left(L_k+\Delta_k, p_k+\delta_k, \epsilon_{k+1}\right)\right\}$ is also convex in L_k since $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is convex in x_{k+1} by assumption, and hence $f_k(L_k, p_k, \epsilon_k)$ is a convex function of L_k.

Recall that $T_k^0(p_k, \epsilon_k)$ is the global minimizer of $f_k(L_k, p_k, \epsilon_k)$ for any given L_k and p_k. Suppose that $x_k \geq T_k^0(p_k, \epsilon_k)$ and $L_k > x_k$. Consider the objective function of (3.4). The convexity of $f_k(L_k, p_k, \epsilon_k)$ implies that $f_k(L_k, p_k, \epsilon_k) \geq f_k(x_k, p_k, \epsilon_k)$. Also note that the absolute value function is nonnegative, we obtain

$$\epsilon_k|L_k - x_k| + f_k(L_k, p_k, \epsilon_k) \geq f_k(x_k, p_k, \epsilon_k),$$

and hence $L_k^* \leq x_k$ if $x_k \geq T_k^0(p_k, \epsilon_k)$.

We can prove the other part of the proposition, $L_k^* \geq x_k$ if $x_k \leq T_k^0(p_k, \epsilon_k)$, by the same argument. □

Lemma 3.1 shows that the market-maker will not increase inventory (through trading with other market-makers) when the inventory is greater than $T_k^0(p_k, \epsilon_k)$, and will not decrease inventory when the inventory is less than $T_k^0(p_k, \epsilon_k)$. With this result, we can replace the absolute value function in the Bellman equation (3.4) by a linear function, and hence we can pull the term x_k out of the objective function. The following proposition shows that the threshold policy is optimal with the threshold level $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$.

Proposition 3.1. *Suppose that $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is convex with respect to x_{k+1}. Let L_k^* denote the optimal solution of L_k in (3.4). For any p_k and ϵ_k, there exists threshold levels $T_k^1(p_k, \epsilon_k) \leq T_k^2(p_k, \epsilon_k)$ such that $L_k^* = T_k^1(p_k, \epsilon_k)$ if $x_k \leq T_k^1(p_k, \epsilon_k)$, $L_k^* = T_k^2(p_k, \epsilon_k)$ if $x_k > T_k^2(p_k, \epsilon_k)$, and $L_k^* = x_k$ otherwise.*

Proof. Suppose that $x_k \geq T_k^0(p_k)$, Lemma 3.1 shows that

$$J_k(x_k, p_k) = \min_{L_k \leq x_k} \left\{\epsilon_k(x_k - L_k) + f_k(L_k, p_k, \epsilon_k)\right\} = \epsilon_k x_k + \min_{L_k \geq x_k} f_k^2(L_k, p_k, \epsilon_k). \quad (3.6)$$

Note that $f_k^2(L_k, p_k, \epsilon_k) = f_k(L_k, p_k, \epsilon_k) - \epsilon_k L_k$ is defined in (3.5). It is a convex function of L_k because $f_k(L_k, p_k, \epsilon_k)$ is convex in L_k.

Note that we have shown that $f_k(L_k, p_k, \epsilon_k)$ is a convex function with a global minimizer $T_k^0(p_k, \epsilon_k)$ for any given p_k and ϵ_k. Since $\epsilon_k > 0$, for any fixed p_k and ϵ_k, $f_k^2(L_k, p_k, \epsilon_k)$ achieves its global minimum at $T_k^2(p_k, \epsilon_k) \geq T_k^0(p_k, \epsilon_k)$. It follows directly that $L_k^* = T_k^2(p_k, \epsilon_k)$ if $x_k > T_k^2(p_k, \epsilon_k)$, and $L_k^* = x_k$ if $T_k^0(p_k) \leq x_k \leq T_k^2(p_k)$.

Similarly, we can show that $L_k^* = T_k^1(p_k)$ if $x_k \leq T_k^1(p_k)$, and $L_k^* = x_k$ if $T_k^1(p_k) < x_k \leq T_k^0(p_k)$, where $T_k^1(p_k) \leq T_k^0(p_k)$. □

Proposition 3.1 implies that

$$J_k(x_k, p_k, \epsilon_k) = \begin{cases} -\epsilon_k x_k + f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k) & \text{if } x_k \leq T_k^1(p_k, \epsilon_k), \\ -\epsilon_k x_k + f_k^1(x_k, p_k, \epsilon_k) = \epsilon_k x_k + f_k^2(x_k, p_k, \epsilon_k) & \\ & \text{if } T_k^1(p_k, \epsilon_k) < x_k \leq T_k^2(p_k, \epsilon_k), \\ \epsilon_k x_k + f_k^2(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) & \text{if } x_k > T_k^2(p_k, \epsilon_k). \end{cases} \tag{3.7}$$

Next we complete the induction proof by showing the convexity of $J_k(x_k, p_k, \epsilon_k)$.

Proposition 3.2. *If $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is convex in x_{k+1} for any given p_{k+1}, then $J_k(x_k, p_k, \epsilon_k)$ is convex in x_k for any given p_k and ϵ_k.*

Proof. Here we show that $J_k(x_k, p_k, \epsilon_k)$ is convex and increasing in $x_k \geq T_k^0(p_k, \epsilon_k)$ for any given p_k and ϵ_k.

Let us define

$$J_k^2(x_k, p_k, \epsilon_k) = \epsilon_k x_k + f_k^2(x_k, p_k, \epsilon_k) = f_k(x_k, p_k, \epsilon_k)$$
$$J_k^3(x_k, p_k, \epsilon_k) = \epsilon_k x_k + f_k^2(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k).$$

According to (3.7), we have

$$J_k(x_k, p_k, \epsilon_k) = \begin{cases} J_k^2(x_k, p_k, \epsilon_k) & \text{if } T_k^0(p_k, \epsilon_k) \leq x_k \leq T_k^2(p_k, \epsilon_k) \\ J_k^3(x_k, p_k, \epsilon_k) & \text{if } x_k > T_k^2(p_k, \epsilon_k). \end{cases}$$

For any given p_k and ϵ_k, $J_k^2(x_k, p_k, \epsilon_k)$ is increasing in $[T_k^0(p_k, \epsilon_k), T_k^2(p_k, \epsilon_k)]$ as $f_k(L_k, p_k, \epsilon_k)$ is convex in L_k and has global minimum at $T_k^0(p_k, \epsilon_k)$. $J_k^3(x_k, p_k, \epsilon_k)$ is also increasing as $\epsilon_k > 0$ and $f_k^2(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k)$ is a constant for fixed p_k and ϵ_k. We obtain that $J_k(x_k, p_k, \epsilon_k)$ is increasing in $[T_k^0(p_k, \epsilon_k), \infty)$ since $J_k^2(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) = J_k^3(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k)$.

Note that $\frac{\partial f_k^2}{\partial L_k}(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) \leq 0$ as $T_k^2(p_k, \epsilon_k)$ minimizes $f_k^2(L_k, p_k, \epsilon_k)$ with given p_k and ϵ_k,[1] and hence

$$\frac{\partial J_k^2}{\partial x_k}(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) = \epsilon_k + \frac{\partial f_k^2}{\partial L_k}(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) \leq \epsilon_k = \frac{\partial J_k^3}{\partial x_k}(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k).$$

Besides, $J_k^2(x_k, p_k, \epsilon_k)$ is a convex function in x_k since $f_k(L_k, p_k, \epsilon_k)$ is convex in L_k, and $J_k^3(x_k, p_k, \epsilon_k)$ is a linear function with respect to x_k. It follows directly that $J_k(x_k, p_k, \epsilon_k)$ is convex for $x_k \in [T_0^k(p_k), \infty)$.

Similarly, we can show that $J_k(x_k, p_k, \epsilon_k)$ is convex and decreasing in $x_k \leq T_k^0(p_k)$, which completes the proof. □

Obviously, $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1}) = -v(x_{N+1}, \epsilon_{N+1})$ is a convex function with respect to x_{N+1}, since $v(x_{N+1}, \epsilon_{N+1})$ is concave in x_{N+1} by definition. Therefore, we establish the same optimal control policy as that in Theorem 2.1:

Theorem 3.1. *The optimal control policy for the dynamic programming model in* (3.1) *is as follows. For any period k, there exist threshold levels, independent of the inventory level x_k, $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ where $T_k^1(p_k, \epsilon) \leq T_k^2(p_k, \epsilon)$, such that the optimal order quantity $q_k^* = T_k^1(p_k, \epsilon) - x_k$ if $x_k \leq T_k^1(p_k, \epsilon)$, $q_k^* = T_k^2(p_k, \epsilon) - x_k$ if $x_k > T_k^2(p_k, \epsilon)$, and $q_k^* = 0$ otherwise.*

Observe that the proof of this theorem is significantly simpler than Theorem 2.1, the corresponding theorem for the exponential utility function. This is due to the fact that in the current case, the objective function is a summation of convex functions while in the former case, it is a multiplicative function of convex functions.

We use the stochastic input in Example 2.1 presented in Chapter 2 to illustrate the threshold policy for the mean-variance trade-off model in Theorem 3.1.

Example 3.1. Consider a probelm with $N = 100$ and $\lambda = 100$. We assume that the random variables δ_k, s_k, d_k and ϵ_k follow the stationary distribution defined in

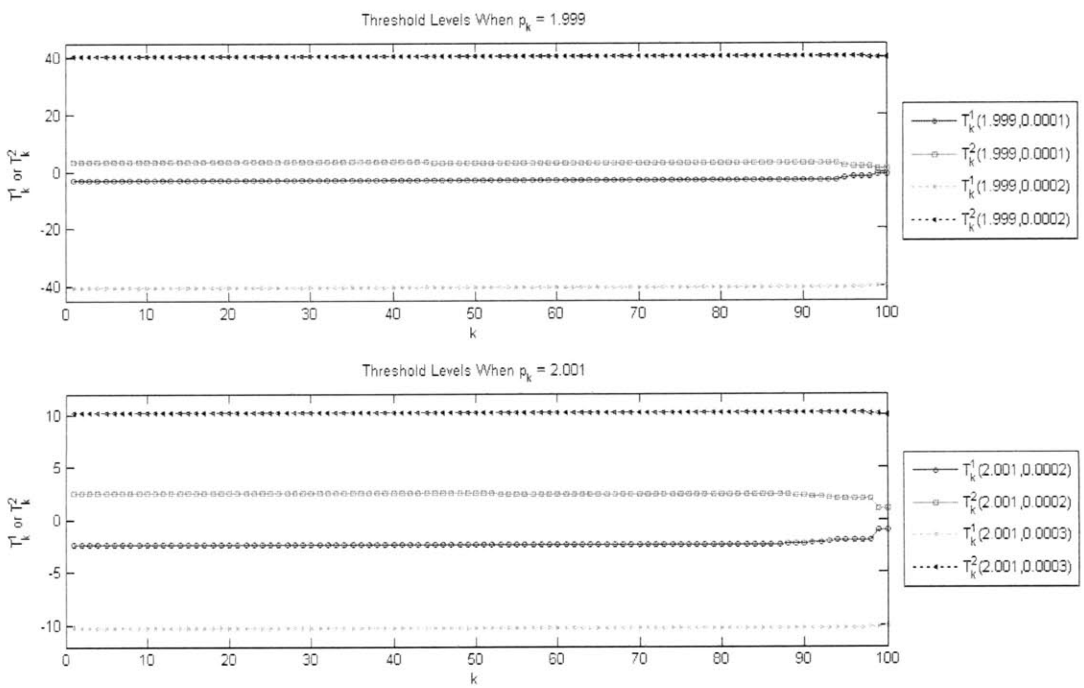
[1]Similar to Chapter 2, we use $f'(x)$ and $\frac{\partial g}{\partial x_i}(x_1, x_2, ..., x_m)$ to denote the left-hand derivatives of the functions $f(x)$ and $g(x_1, x_2, ..., x_m)$ respectively.

Example 2.1. Analogous to Tables 2.1, 2.2 and Figures 2-1, 2-2 in Example 2.1, we present the results with $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ and $\pi_{N+1} = 0$ in Tables 3.1, 3.2 and Figures 3-1, 3-2 respectively.

Table 3.1: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 3.1 with $\lambda = 100$ and $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$

	$p_k = 1.999$				$p_k = 2.001$			
	$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0003$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−3.21	3.24	−40.21	40.21	−2.41	2.47	−10.21	10.21
50	−3.19	3.21	−40.21	40.21	−2.41	2.46	−10.21	10.21
100	−1.00	1.00	−40.00	40.00	−1.00	1.00	−10.00	10.00

Figure 3-1: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 3.1 with $\lambda = 100$ and $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$

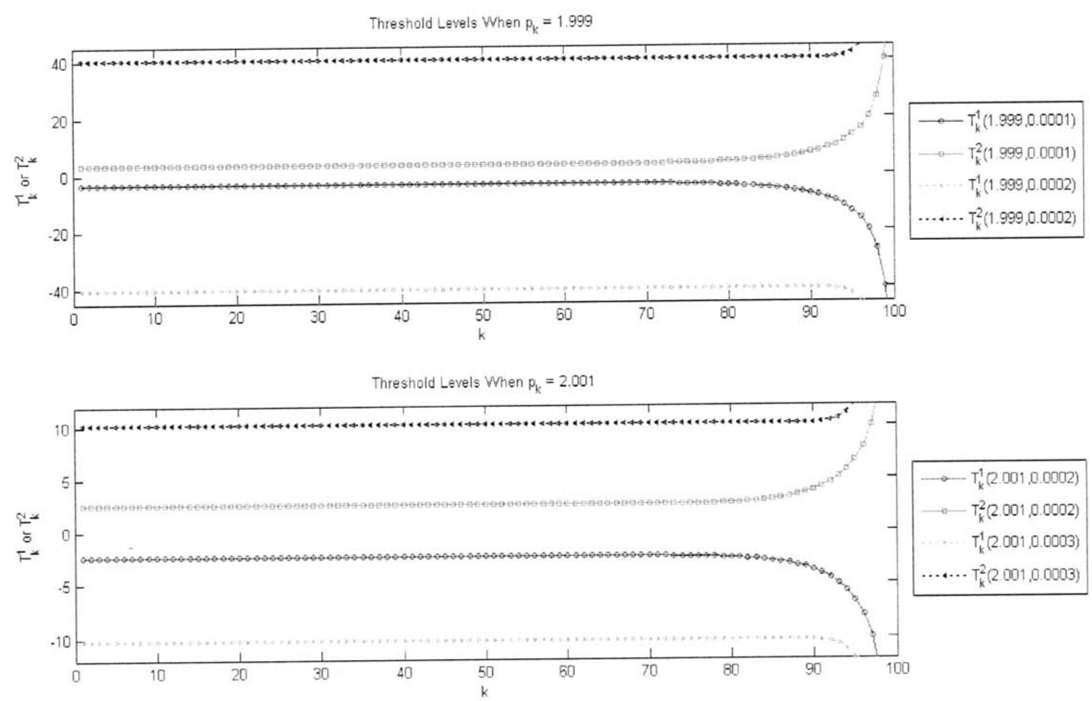


Compared with the result in Example 2.1, the no-trade regions are smaller for both definitions of π_{N+1}. In particular, the no-trade regions when $k = 100$ and $\pi_{N+1} = 0$ are finite in Table 3.2 whereas they are the entire real line in Table 2.2. This property indicates that the mean-variance model with $\lambda = 100$ is more conservative than the corresponding exponential utility model with $\rho = 100$ for the random inputs we choose. The rest observations are very similar to those in Example 2.1 and the discussions are omitted here.

Table 3.2: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 3.1 with $\lambda = 100$ and $\pi_{N+1} = 0$

	$p_k = 1.999$				$p_k = 2.001$			
	$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0003$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−3.21	3.24	−40.21	40.21	−2.41	2.47	−10.21	10.21
50	−3.24	3.26	−40.21	40.21	−2.42	2.47	−10.21	10.21
100	−80.00	80.00	−160.00	160.00	−40.00	40.00	−60.00	60.00

Figure 3-2: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 3.1 with $\lambda = 100$ and $\pi_{N+1} = 0$



3.3 Reduction of the State Space

Similar to the analysis of the exponential utility function, we can also reduce the dimensions of the state space in the dynamic program (3.1) by introducing independence conditions on the random variables. Specifically,

Corollary 3.1. *If δ_k, ϵ_k, S_k and Δ_k are independent of p_k for any k, then the state of the Bellman equation* (3.4) *is reduced to x_k and ϵ_k. Therefore, the functions $J_k(\cdot)$ in* (3.4)*, $f_k^1(\cdot)$ and $f_k^2(\cdot)$ in* (3.5) *only depend on x_k and ϵ_k, and the threshold levels $T_k^1(\cdot)$ and $T_k^2(\cdot)$ in Theorem 3.1 only depend on ϵ_k.*

If ϵ_k is a given function of p_k for any k, i.e., $\epsilon_k = \phi_k(p_k)$, then the state of the Bellman equation (3.4) *is reduced to x_k and p_k. Therefore, the functions $J_k(\cdot)$ in* (3.4)*, $f_k^1(\cdot)$ and $f_k^2(\cdot)$ in* (3.5) *only depend on x_k and p_k, and the threshold levels $T_k^1(\cdot)$ and $T_k^2(\cdot)$ in Theorem 3.1 only depend on p_k.*

If ϵ_k is a given constant and δ_k, S_k and Δ_k are independent of p_k for any k, then the state of the Bellman equation (3.4) *is reduced to x_k. Therefore, the functions $J_k(\cdot)$ in* (3.4)*, $f_k^1(\cdot)$ and $f_k^2(\cdot)$ in* (3.5) *only depend on x_k, and the threshold levels $T_k^1(\cdot)$ and $T_k^2(\cdot)$ in Theorem 3.1 are reduced to constants for any k.*

We use the stochastic inputs in Examples 2.2 and 2.3 shown in Chapter 2 to illustrate Corollary 3.1.

Example 3.2. For an example whose threshold level only depends on ϵ_k, we consider the same stochastic input as Example 2.2 with $N = 100$ and $\lambda = 100$. Table 3.3 and Figure 3-3 present the threshold levels for both $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ and $\pi_{N+1} = 0$.

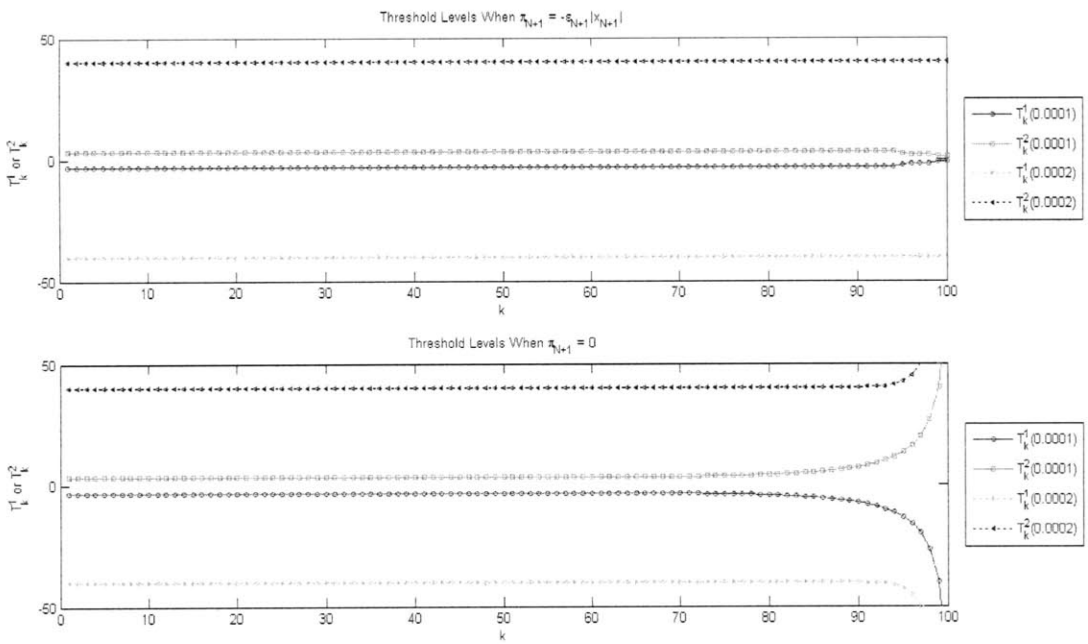
The observations from these results are very similar to those of Example 2.2, except that the no-trade regions in Table 3.3 and Figure 3-3 are smaller compared with their counterparts in Example 2.2, which agree with the observation we obtained by comparing Examples 2.1 and 3.1: the mean-variance model with $\lambda = 100$ is more conservative than the corresponding exponential utility model with $\rho = 100$.

We would like to point out that the threshold levels have very close absolute values, i.e., $T_k^1(\epsilon_k) \approx -T_k^2(\epsilon_k)$. Similar to the exponential utility model, there also exist

Table 3.3: $T_k^1(\epsilon_k)$ and $T_k^2(\epsilon_k)$ for Example 3.2 with $\lambda = 100$

	$\pi_{N+1} = -\epsilon_{N+1}\lvert x_{N+1}\rvert$				$\pi_{N+1} = 0$			
	$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$		$\epsilon_k = 0.0001$		$\epsilon_k = 0.0002$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−3.21	3.21	−40.21	40.21	−3.22	3.22	−40.21	40.21
50	−3.19	3.19	−40.21	40.21	−3.25	3.25	−40.21	40.21
100	−1.00	1.00	−40.00	40.00	−80.00	80.00	−160.00	160.00

Figure 3-3: $T_k^1(\epsilon_k)$ and $T_k^2(\epsilon_k)$ for Example 3.2 with $\lambda = 100$



certain conditions under which the mean-variance tradeoff model returns symmetric threshold levels, and we present the theoretical result in Proposition 3.3.

Example 3.3. Let us use the stochastic input defined in Example 2.3 to illustrate the situation that the threshold levels are independent of ϵ_k. Similar to Example 2.3, we consider the cases $\varphi_k = 10^{-4}$ and $\varphi_k = 2 \times 10^{-4}$ respectively, and allow π_{N+1} to be either $-\epsilon_{N+1}|x_{N+1}|$ or zero. The results for $\varphi_k = 10^{-4}$ ($\varphi_k = 2 \times 10^{-4}$, respectively) are shown in Table 3.4 and Figure 3-4 (Table 3.5 and Figure 3-5, respectively).

We observe the same trend in threshold levels when the period k changes as in the previous examples. Moreover, similar to Example 2.3, the absolute values of threshold levels for $p_k = 2.001$ are higher than those for $p_k = 1.999$, and the absolute values of threshold levels for $\varphi_k = 2 \times 10^{-4}$ are higher than those for $\varphi_k = 10^{-4}$. In general, the no-trade regions are also smaller than the counterparts in Example 2.3, which agrees with the observations from Examples 3.1 and 3.2.

Table 3.4: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 3.3 with $\varphi_k = 10^{-4}$ and $\lambda = 100$

	$\pi_{N+1} = -\epsilon_{N+1}\lvert x_{N+1}\rvert$				$\pi_{N+1} = 0$			
	$p_k = 1.999$		$p_k = 2.001$		$p_k = 1.999$		$p_k = 2.001$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−3.42	3.44	−4.43	4.47	−3.43	3.44	−4.44	4.47
50	−3.42	3.43	−4.42	4.44	−3.45	3.45	−4.70	4.70
100	−1.00	1.00	−1.00	1.00	−80.00	80.00	−160.00	160.00

Table 3.5: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 3.3 with $\varphi_k = 2 \times 10^{-4}$ and $\lambda = 100$

	$\pi_{N+1} = -\epsilon_{N+1}\lvert x_{N+1}\rvert$				$\pi_{N+1} = 0$			
	$p_k = 1.999$		$p_k = 2.001$		$p_k = 1.999$		$p_k = 2.001$	
k	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2	T_k^1	T_k^2
1	−4.44	4.46	−5.10	5.13	−4.45	4.46	−5.19	5.20
50	−4.42	4.44	−5.02	5.00	−4.70	4.70	−5.96	5.96
100	−1.00	1.00	−1.00	1.00	−160.00	160.00	−240.00	240.00

3.4 Risk Neutral Model

A very interesting case for the mean-variance model in (3.1) is when $\lambda = 0$, which gives the risk neutral model. Let us start with the special case that $\pi_{N+1} = 0$, i.e., the inventory is marked to the market mid price at the end of the planning horizon.

Example 3.4. Consider the case when $\lambda = 0$ and $\pi_{N+1} = 0$ for the model (3.1).

Suppose that the function $J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is linear in the inventory x_{k+1}, i.e.,

$$J_{k+1}(x_{k+1}, p_{k+1}, \epsilon_{k+1}) = x_{k+1}\alpha_{k+1}(p_{k+1}, \epsilon_{k+1}) + \beta_{k+1}(p_{k+1}, \epsilon_{k+1}),$$

Figure 3-4: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 3.3 with $\varphi_k = 10^{-4}$ and $\lambda = 100$

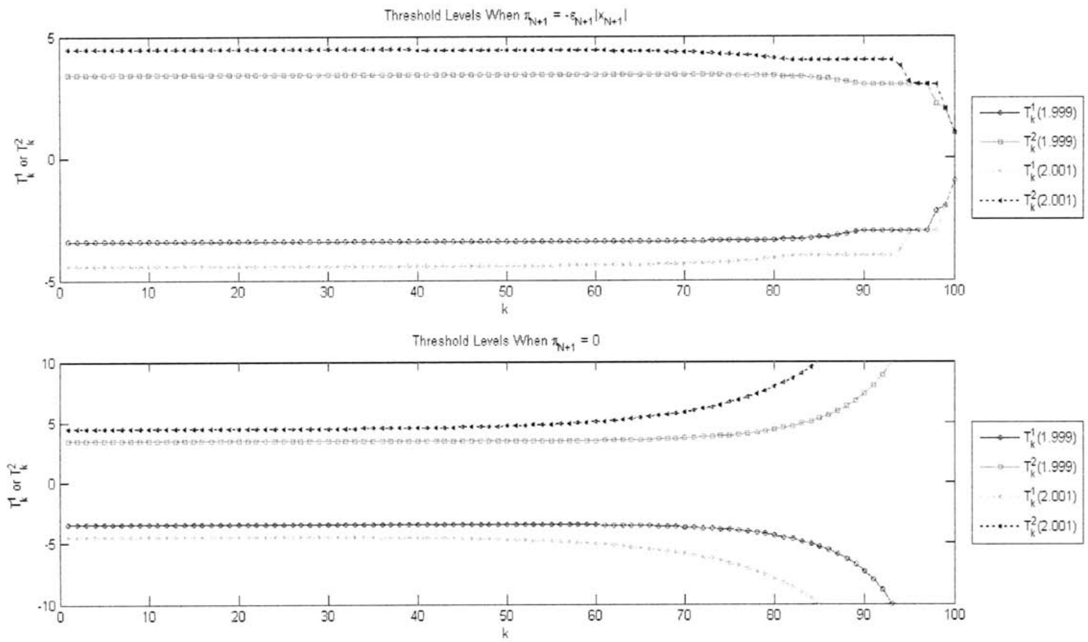
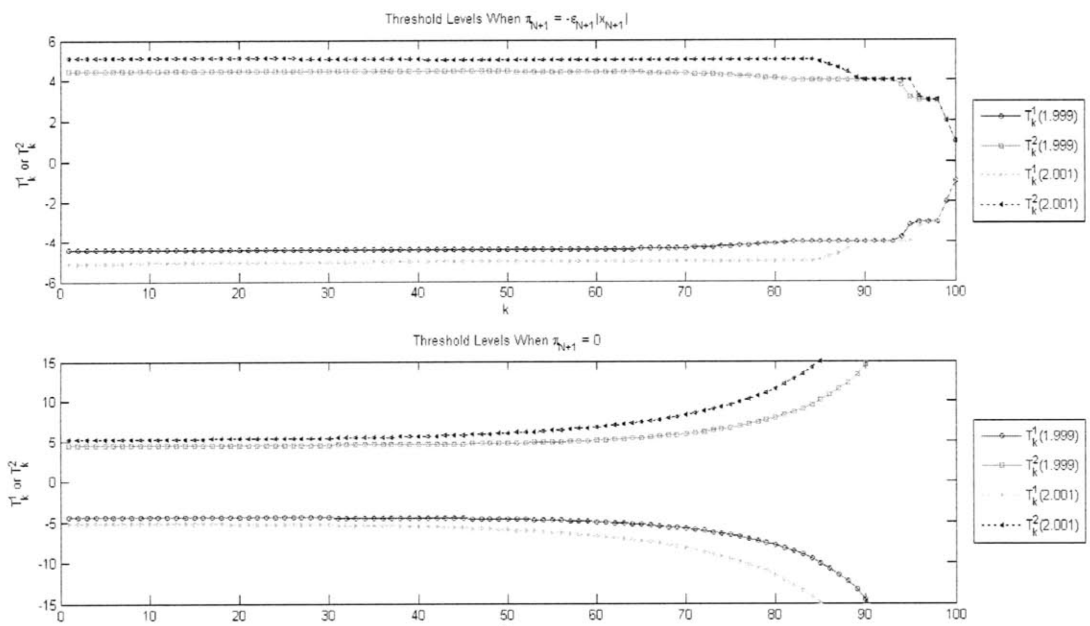


Figure 3-5: $T_k^1(p_k)$ and $T_k^2(p_k)$ for Example 3.3 with $\varphi_k = 2 \times 10^{-4}$ and $\lambda = 100$



where $\alpha_{k+1}(p_{k+1}, \epsilon_{k+1})$ and $\beta_{k+1}(p_{k+1}, \epsilon_{k+1})$ are functions of p_{k+1} and ϵ_{k+1}. Note that

$$J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1}) = -\pi_{N+1} = 0$$

obviously satisfies this assumption.

Let us move on to period k. We have

$$\begin{aligned} f_k(L_k, p_k, \epsilon_k) = & \left(-\mu_k(p_k, \epsilon_k) + E\left[\alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1})\middle| p_k, \epsilon_k\right]\right) L_k \\ & + E\left[\Delta_k \alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1}) + \beta_{k+1}(p_k + \delta_k, \epsilon_{k+1})\middle| p_k, \epsilon_k\right] \end{aligned}$$

and hence both $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ are linear functions of L_k, which implies $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ are either $-\infty$ or $+\infty$. In particular,

- If $E\left[\alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1})\middle| p_k, \epsilon_k\right] > \mu_k(p_k, \epsilon_k) + \epsilon_k$, then both $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ are linearly increasing in L_k and hence $T_k^1(p_k, \epsilon_k) = T_k^2(p_k, \epsilon_k) = -\infty$. Moreover, $J_k(L_k, p_k, \epsilon_k) = -\infty$. Note that $\mu_k(p_k, \epsilon_k) = E[\delta_k | p_k, \epsilon_k]$, which is the drift in the mid price. The intuition behind this result is that when we have very large negative drift, we always have incentive to short the asset if we are risk neutral.

- If $E\left[\alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1})\middle| p_k, \epsilon_k\right] < \mu_k(p_k, \epsilon_k) - \epsilon_k$, then both $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ are decreasing. Therefore, $T_k^1(p_k, \epsilon_k) = T_k^2(p_k, \epsilon_k) = +\infty$, and we have $J_k(L_k, p_k, \epsilon_k) = -\infty$. Similarly, this may happen when the drift in price, $\mu_k(p_k, \epsilon_k)$, is a large positive number. In this case, we would like to hold as much asset as we can to gain the expected profit.

- If $\mu_k(p_k, \epsilon_k) - \epsilon_k \leq E\left[\alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1})\middle| p_k, \epsilon_k\right] \leq \mu_k(p_k, \epsilon_k) + \epsilon_k$, we have $T_k^1(p_k, \epsilon_k) = -\infty$ and $T_k^2(p_k, \epsilon_k) = +\infty$ since $f_k^1(L_k, p_k, \epsilon_k)$ is decreasing while $f_k^2(L_k, p_k, \epsilon_k)$ is increasing. In this case, the no-trade region is the entire real line, i.e., we just receive the orders from our clients and never trade actively with other market-makers. This corresponds to the scenario that the gain in the price drift cannot compensate for the transaction cost, and a risk neutral decision maker will just keep the inventory.

At the same time, the corresponding value function is

$$J_k(x_k, p_k, \epsilon_k) = f_k(x_k, p_k, \epsilon_k) = x_k\alpha_k(p_k, \epsilon_k) + \beta_k(p_k, \epsilon_k)$$

where

$$\begin{aligned}
\alpha_k(p_k, \epsilon_k) &= -\mu_k(p_k, \epsilon_k) + E\left[\alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1})\Big| p_k, \epsilon_k\right] \\
\beta_k(p_k, \epsilon_k) &= E\left[\Delta_k\alpha_{k+1}(p_k + \delta_k, \epsilon_{k+1}) + \beta_{k+1}(p_k + \delta_k, \epsilon_{k+1})\Big| p_k, \epsilon_k\right].
\end{aligned}$$

Obviously, $J_k(x_k, p_k, \epsilon_k)$ is linear in x_k which satisfies the induction assumption. As a result, as long as $J_k(x_k, p_k, \epsilon_k)$ is well defined, there is no need to actively trade with other market-makers from period k to the end of planning horizon.

In fact, the results in Example 3.4 hold for any π_{N+1} linear in x_{N+1}. For any $\pi_{N+1} = v(x_{N+1}, \epsilon_{N+1})$ concave in x_{N+1}, $J_k(x_k, p_k, \epsilon_k)$ is well defined, i.e., $J_k(x_k, p_k, \epsilon_k) > -\infty$ for given x_k, p_k and ϵ_k, if and only if (i) $J_{k+1}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) > -\infty$ for any given x_{k+1}, p_{k+1} and ϵ_{k+1}, (ii) $f_k^1(L_k, p_k, \epsilon_k)$ is not decreasing in L_k, and (iii) $f_k^2(L_k, p_k, \epsilon_k)$ is not increasing in L_k. The conditions (ii) and (iii) are equivalent to

$$\begin{aligned}
-\mu(p_k, \epsilon_k) + \epsilon_k + \lim_{L_k \to +\infty} \frac{\partial}{\partial L_k} E\left\{J_{k+1}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1})\Big| p_k, \epsilon_k\right\} > 0 \\
-\mu(p_k, \epsilon_k) - \epsilon_k + \lim_{L_k \to -\infty} \frac{\partial}{\partial L_k} E\left\{J_{k+1}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1})\Big| p_k, \epsilon_k\right\} < 0
\end{aligned}$$

for any p_k and ϵ_k. Moreover, the no-trade region has a lower bound, i.e., $T_k^1(p_k, \epsilon_k) \in (-\infty, +\infty)$ if and only if the conditions (i), (ii), (iii) and

$$-\mu(p_k, \epsilon_k) + \epsilon_k + \lim_{L_k \to -\infty} \frac{\partial}{\partial L_k} E\left\{J_{k+1}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1})\Big| p_k, \epsilon_k\right\} < 0$$

are satisfied. Similarly, the no-trade region has an upper bound, i.e., $T_k^2(p_k, \epsilon_k) \in (-\infty, +\infty)$ if and only if the conditions (i), (ii), (iii) and

$$-\mu(p_k, \epsilon_k) - \epsilon_k + \lim_{L_k \to +\infty} \frac{\partial}{\partial L_k} E\left\{J_{k+1}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1})\Big| p_k, \epsilon_k\right\} > 0$$

are satisfied.

Next we present the results for the risk neutral model with the stochastic input used in Examples 2.1 and 3.1.

Example 3.5. Consider the problem described in Example 2.1 and suppose that the decision maker is risk-neutral. Note that $\mu_k(p_k, \epsilon_k) = E[\delta_k | p_k, \epsilon_k] = 0$. According to Example 3.4, it is straightforward that $T_k^1(p_k, \epsilon_k) = -\infty$ and $T_k^2(p_k, \epsilon_k) = +\infty$ for any p_k and ϵ_k if we let $\pi_{N+1} = 0$. We will focus on the situation that $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$.

We have $T_k^1(p_k, \epsilon_k) = -\infty$ and $T_k^2(p_k, \epsilon_k) = +\infty$ when $\varphi_k = 2 \times 10^{-4}$, i.e., when (i) $p_k = 1.999$ and $\epsilon_k = 2 \times 10^{-4}$ and (ii) $p_k = 2.001$ and $\epsilon_k = 3 \times 10^{-4}$. For these two cases, there exists a 0.5 probability that φ_{k+1} will decrease to 10^{-4}, i.e., the transaction cost in the next period may be reduced by 10^{-4}. Since the expectation of the price movement is 0, there is no expected loss associated with holding inventory. Therefore, the decision maker has no incentive to trade off the inventory in period k, and hence the no-trade region is $(-\infty, +\infty)$.

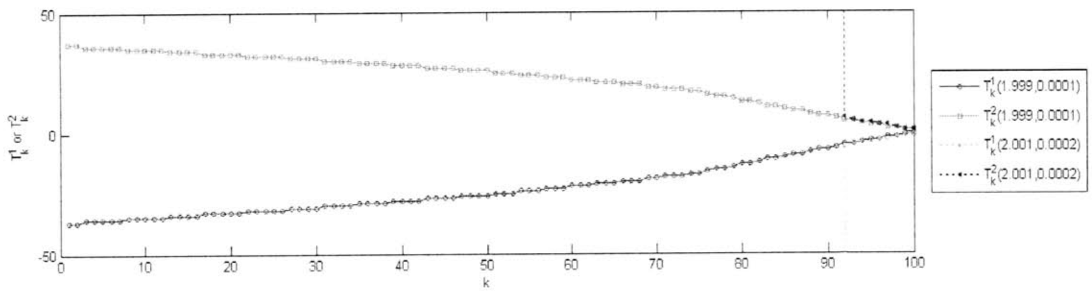
The threshold levels when (i) $p_k = 1.999$ and $\epsilon_k = 10^{-4}$ and (ii) $p_k = 2.001$ and $\epsilon_k = 2 \times 10^{-4}$ are shown in Tabel 3.6 and Figure 3-6.

Table 3.6: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 3.5

	$p_k = 1.999, \epsilon_k = 0.0001$		$p_k = 2.001, \epsilon_k = 0.0002$	
k	T_k^1	T_k^2	T_k^1	T_k^2
1	-37.00	35.00	$-\infty$	∞
50	-26.00	25.00	$-\infty$	∞
100	-1.00	1.00	-1.00	1.00

The threshold levels are always bounded for $p_k = 1.999$ and $\epsilon_k = 10^{-4}$. However, $T_k^1(2.001, 2 \times 10^{-4})$ and $T_k^2(2.001, 2 \times 10^{-4})$ are bounded only when $k \geq 92$. When $k \geq 92$, for any $\bar{k} = k+1, ..., N+1$, the probability for $p_{\bar{k}} < 2$ given $p_k = 2.001$ is very low, which, according to the definition of ϵ_k in (2.12), implies that the probability for the transaction cost to drop to 10^{-4} is low. Meanwhile, if $p_k = 2.001$ and $\epsilon_k = 2 \times 10^{-4}$, it is possible that $\epsilon_{\bar{k}}$ where $\bar{k} = k+1, ..., N+1$, the transaction cost in any future period $\bar{k}$, may increase to 3×10^{-4}. Therefore, when $k \geq 92$, $p_k = 2.001$ and $\epsilon_k = 2 \times 10^{-4}$, the decision maker has the incentive to actively trade with other market-makers in

Figure 3-6: $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ for Example 3.5



period k. On the other hand, when $k < 92$, $p_k = 2.001$ and $\epsilon_k = 2 \times 10^{-4}$, there may exist some $\bar{k} \in \{k+1, ..., N+1\}$ that $p_{\bar{k}} < 2$ and $\varphi_{\bar{k}} = 10^{-4}$, i.e., the transaction cost may decrease to 10^{-4} in some future period $\bar{k}$, and hence the decision maker would choose to hold the on-hand inventory in the hope to save the transaction cost.

In addition, we would like to point out that the no-trade region decrease as k increases, which agrees with what we observed in previous examples when $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$.

3.5 Symmetric Threshold Policy

The symmetric threshold policy is also optimal for certain special cases if we consider the mean-variance tradeoff model in (3.1). Here we establish a result analogous to Proposition 2.3 for the exponential utility model (2.2). Note that the threshold policy characterized in Proposition 2.3 is symmetric with respect to zero. For the mean-variance model, under slightly different conditions, we can generalize to the case that the threshold levels are symmetric with respect to a known constant χ, i.e., $\chi - T_k^1(p_k, \epsilon_k) = T_k^2(p_k, \epsilon_k) - \chi$ for any k, p_k and ϵ_k.

In the Bellman equation (3.4), we take the expectation of $J_{k+1}(L_{k+1}, p_{k+1}, \epsilon_{k+1})$ conditional on p_k and ϵ_k with respect to δ_k, Δ_k and ϵ_{k+1}. Since ϵ_{k+1} and Δ_k are independent, it is sufficient to consider the distribution of Δ_k conditional on δ_k, p_k and ϵ_k. Let $F_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k)$ denote the cumulative distribution function of the random variable Δ_k conditional on δ_k, p_k and ϵ_k.

For a given constant χ, we consider the following assumptions.

(C1) $v(\chi + x_{N+1}, \epsilon_{N+1}) = v(\chi - x_{N+1}, \epsilon_{N+1})$ for any x_{N+1} and ϵ_{N+1}, i.e., the function $v(x_{N+1}, \epsilon_{N+1})$ is symmetric in x_{N+1} with respect to the point χ.

(C2) $\lambda\nu_k(p_k, \epsilon_k) - \mu_k(p_k, \epsilon_k) = -2\lambda\chi\sigma_k^2(p_k, \epsilon_k)$ for any k and ϵ_k.

(C3) $F_{\Delta_k|\delta_k,p_k,\epsilon_k}(\Delta_k) + F_{\Delta_k|\delta_k,p_k,\epsilon_k}(-\Delta_k) = 1 + dF_{\Delta_k|\delta_k,p_k,\epsilon_k}(\Delta_k)$ for any k and Δ_k, i.e., the conditional distribution of Δ_k is symmetric with respect to zero.

Proposition 3.3. *Given a constant χ, under the conditions in (C1), (C2) and (C3), a symmetric threshold policy is optimal for the problem in* (3.1). *In particular, $J_k(\chi + x_k, p_k, \epsilon_k) = J_k(\chi - x_k, p_k, \epsilon_k)$ and $\chi - T_k^1(p_k, \epsilon_k) = T_k^2(p_k, \epsilon_k) - \chi$ for any k, x_k, p_k and ϵ_k.*

Proof. Consider the period $N+1$. Since $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1}) = -v(x_{N+1}, \epsilon_{N+1})$, the first condition (C1) shows that

$$J_{N+1}(\chi + x_{N+1}, p_{N+1}, \epsilon_{N+1}) = J_{N+1}(\chi - x_{N+1}, p_{N+1}, \epsilon_{N+1})$$

for any x_{N+1}, p_{N+1} and ϵ_{N+1}.

Let us assume that $J_{k+1}(\chi + x_{k+1}, p_{k+1}, \epsilon_{k+1}) = J_{k+1}(\chi - x_{k+1}, p_{k+1}, \epsilon_{k+1})$ for any x_{k+1}, p_{k+1} and ϵ_{k+1}. We can prove the proposition by induction on the number of period k.

Consider the function $f_k(L_k, p_k, \epsilon_k)$ defined in (3.5). The condition (ii) implies that

$$\begin{aligned} f_k(L_k, p_k, \epsilon_k) &= -2\lambda\chi\sigma_k^2(p_k, \epsilon_k)L_k + \lambda\sigma_k^2(p_k, \epsilon_k)L_k^2 \\ &\qquad + E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| p_k, \epsilon_k \right\} \\ &= \lambda\sigma_k^2(p_k, \epsilon_k)(L_k - \chi)^2 - \lambda\chi^2\sigma_k^2(p_k, \epsilon_k) + E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| p_k, \epsilon_k \right\}. \end{aligned}$$

Similar to the proof of Proposition 2.3, we have

$$\begin{aligned}
&f_k(\chi + L_k, p_k, \epsilon_k) \\
&= \lambda\sigma_k^2(p_k, \epsilon_k)L_k^2 - \lambda\chi^2\sigma_k^2(p_k, \epsilon_k) + E\left\{ J_{k+1}\left(\chi + L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right)\Big| \, p_k, \epsilon_k \right\} \\
&= \lambda\sigma_k^2(p_k, \epsilon_k)L_k^2 - \lambda\chi^2\sigma_k^2(p_k, \epsilon_k) + E\left\{ J_{k+1}\left(\chi - L_k - \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right)\Big| \, p_k, \epsilon_k \right\} \\
&= \lambda\sigma_k^2(p_k, \epsilon_k)L_k^2 - \lambda\chi^2\sigma_k^2(p_k, \epsilon_k) + E\left\{ J_{k+1}\left(\chi - L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right)\Big| \, p_k, \epsilon_k \right\} \\
&= f_k(\chi - L_k, p_k, \epsilon_k),
\end{aligned}$$

where the second equality follows from the assumption $J_{k+1}(\chi + x_{k+1}, p_{k+1}, \epsilon_{k+1}) = J_{k+1}(\chi - x_{k+1}, p_{k+1}, \epsilon_{k+1})$, and the third inequality can be proven by the condition (C3).

According to the definition of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ in (3.5),

$$\begin{aligned}
f_k^1(\chi + L_k, p_k, \epsilon_k) &= f_k(\chi + L_k, p_k, \epsilon_k) + \epsilon_k(\chi + L_k) \\
&= f_k(\chi - L_k, p_k, \epsilon_k) - \epsilon_k(\chi - L_k) + 2\epsilon_k\chi \\
&= f_k^2(\chi - L_k, p_k, \epsilon_k) + 2\epsilon_k\chi.
\end{aligned}$$

Given p_k and ϵ_k, $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ are global minimizers of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$, and it follows immediately that $\chi - T_k^1(p_k, \epsilon_k) = T_k^2(p_k, \epsilon_k) - \chi$.

To complete the induction proof, we still need to show that $J_k(\chi + x_k, p_k, \epsilon_k) = J_k(\chi - x_k, p_k, \epsilon_k)$. Without loss of generality, we assume that $x_k \geq 0$. Note that the proof of Lemma 3.1 shows that $f_k(L_k, p_k, \epsilon_k)$ is convex in L_k, and we have shown that $f_k(\chi + L_k, p_k, \epsilon_k) = f_k(\chi - L_k, p_k, \epsilon_k)$. Therefore, for any given p_k and ϵ_k, the global minimizer of $f_k(L_k, p_k, \epsilon_k)$ is χ, i.e., $T_k^0(p_k, \epsilon_k) = \chi$ and hence $T_k^2(p_k, \epsilon_k) \geq \chi$. As a result, we can consider the cases $x_k \in [0, T_k^2(p_k, \epsilon_k) - \chi]$ and $x_k > T_k^2(p_k, \epsilon_k)$.

For any $x_k \in [0, T_k^2(p_k, \epsilon_k) - \chi]$, the function $J_k(x_k, p_k, \epsilon_k)$ defined (3.7) shows that

$$\begin{aligned}
J_k(\chi + x_k, p_k, \epsilon_k) &= \epsilon_k(\chi + x_k) + f_k^2(\chi + x_k, p_k, \epsilon_k) \\
&= f_k(\chi + x_k, p_k, \epsilon_k) = f_k(\chi - x_k, p_k, \epsilon_k) \\
&= -\epsilon_k(\chi - x_k) + f_k^1(\chi - x_k, p_k, \epsilon_k) = J_k(\chi - x_k, p_k, \epsilon_k),
\end{aligned}$$

where the first and last equalities follow from (3.7), the second and the forth equalities are immediate results of (3.5), and the third equality is obtained by the proven fact $f_k(\chi + L_k, p_k, \epsilon_k) = f_k(\chi - L_k, p_k, \epsilon_k)$.

Similarly, for $x_k > T_k^2(p_k, \epsilon_k) - \chi$,

$$\begin{aligned} J_k(\chi + x_k, p_k, \epsilon_k) &= \epsilon_k(\chi + x_k) + f_k^2(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) \\ &= \epsilon_k(\chi + x_k - T_k^2(p_k, \epsilon_k)) + f_k(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k). \end{aligned} \tag{3.8}$$

According to the result $f_k(\chi + L_k, p_k, \epsilon_k) = f_k(\chi - L_k, p_k, \epsilon_k)$ and $\chi - T_k^1(p_k, \epsilon_k) = T_k^2(p_k, \epsilon_k) - \chi$, we can show that

$$\begin{aligned} f_k(T_k^2(p_k, \epsilon_k), p_k, \epsilon_k) &= f_k(\chi + (T_k^2(p_k, \epsilon_k) - \chi), p_k, \epsilon_k) = f_k(\chi - (T_k^2(p_k, \epsilon_k) - \chi), p_k, \epsilon_k) \\ &= f_k(\chi - (\chi - T_k^1(p_k, \epsilon_k)), p_k, \epsilon_k) = f_k(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k), \end{aligned}$$

and hence by (3.8),

$$\begin{aligned} J_k(\chi + x_k, p_k, \epsilon_k) &= \epsilon_k(\chi + x_k - T_k^2(p_k, \epsilon_k)) + f_k(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k) \\ &= \epsilon_k(-\chi + x_k + T_k^1(p_k, \epsilon_k)) + f_k(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k) \\ &= -\epsilon_k(\chi - x_k) + f_k^1(T_k^1(p_k, \epsilon_k), p_k, \epsilon_k) = J_k(\chi - x_k, p_k, \epsilon_k), \end{aligned}$$

where the second equality is obtained from the symmetry of $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$, the third equality is due to the definition of $f_k^1(L_k, p_k, \epsilon_k)$ in (3.5), and the last equality follows from (3.7). □

We use the same stochastic input as in Example 2.4 presented in Chapter 2 to illustrate the symmetric threshold policy in Proposition 3.3.

Example 3.6. Let us consider the stochastic input defined in Example 2.4. According to Corollary 3.1 and Proposition 3.3, the threshold levels T_k^1 and T_k^2 are independent of both p_k and ϵ_k, and we have $T_k^1 = -T_k^2$.

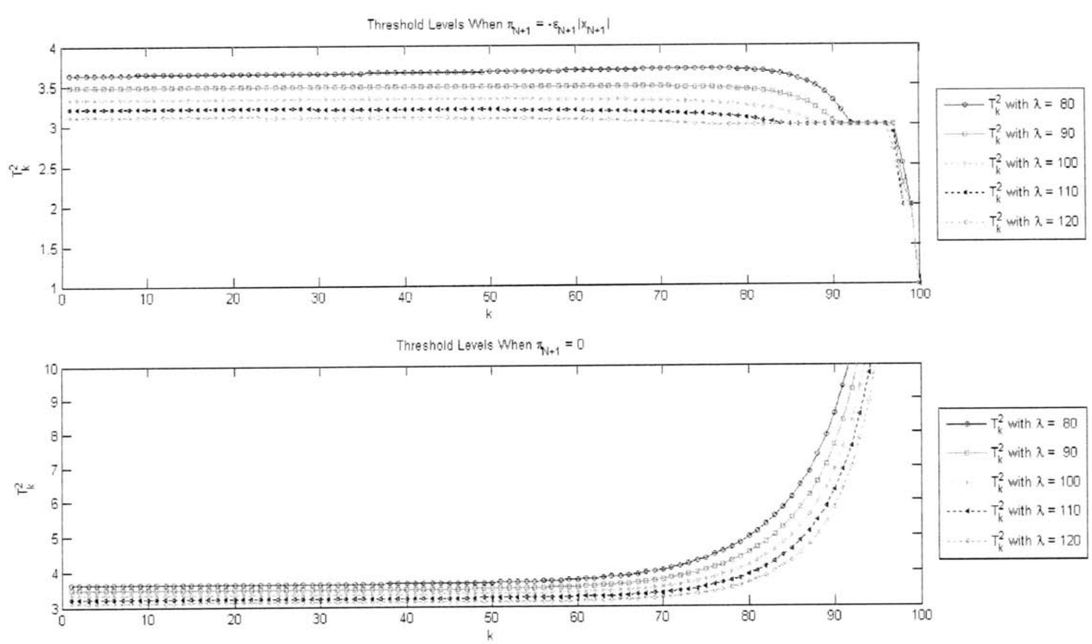
Table 3.7 and Figure 3-7 display the threshold levels T_k^2 for $\lambda = 80, 90, 100, 110, 120$ when we consider $\pi_{N+1} = -\epsilon_{N+1}|x_{N+1}|$ or $\pi_{N+1} = 0$. Let us compare the results here with those for Example 2.4 shown in Table 2.6 and Figure 2-6. We observe that

T_k^2 changes with respect to the period k in a manner similar to that of Example 2.4. Another observation analogous to Example 2.4 is that T_k^2 deceases if the decision maker tends to be more risk-averse, i.e., the value of the risk aversion parameter λ increases. Also note that the values of T_k^2 in Table 3.7 are lower than those in Table 2.6, which means that the mean variance model with $\lambda = 80, 90, 100, 110, 120$ is more conservative than the exponential model with $\rho = 80, 90, 100, 110, 120$.

Table 3.7: T_k^2 for Example 3.6

	$\pi_{N+1} = -\epsilon_{N+1}\lvert x_{N+1}\rvert$					$\pi_{N+1} = 0$				
λ	80	90	100	110	120	80	90	100	110	120
k	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2	T_k^2
1	3.64	3.49	3.34	3.22	3.11	3.64	3.49	3.34	3.22	3.12
50	3.67	3.49	3.33	3.20	3.09	3.68	3.51	3.37	3.24	3.13
100	1.00	1.00	1.00	1.00	1.00	93.75	83.33	75.00	68.18	62.50

Figure 3-7: T_k^2 for Example 3.6



3.6 Monotone Properties of the Threshold Levels

The numerical examples presented in the previous sections provide us lots of insights about the monotonicity of the threshold levels with respect to different model parameters. In this section, we identify the sufficient conditions under which the monotone properties of the threshold levels could be established analytically. In particular, we are going to investigate monotonicity with respect to the risk aversion parameter λ, the bid/ask spread determined by ϵ_k, as well as the market mid price p_k.

3.6.1 Monotonicity with Respect to the Risk Aversion Parameter

As shown in Example 3.6, more risk averse market-makers, i.e., market-markers with larger λ, should be more willing to sacrifice the transaction cost in order to reduce the inventory risk, and hence they would like to choose a smaller no-trade region. Here we identify certain sufficient conditions for this property to hold mathematically.

In particular, consider two risk aversion parameters λ_1 and λ_2 such that $0 \leq \lambda_1 \leq \lambda_2$. Let $J_{k,i}(x_k, p_k, \epsilon_k)$ denote the function $J_k(x_k, p_k, \epsilon_k)$ defined in (3.4) with the parameter λ_i, $i = 1, 2$.

Given two constant χ_1 and χ_2, we consider the following assumptions which are analogous to the assumptions (C1) and (C2).

(C1') $J_{N+1,i}(x_{N+1}, p_{N+1}, \epsilon_{N+1}) = -v_i(x_{N+1}, \epsilon_{N+1})$ where $v_i(\chi_i + x_{N+1}, \epsilon_{N+1}) = v_i(\chi_i - x_{N+1}, \epsilon_{N+1})$ for any $i = 1, 2$, x_{N+1} and ϵ_{N+1}, i.e., the function $v_i(x_{N+1}, \epsilon_{N+1})$ is symmetric in x_{N+1} with respect to the point χ_i. Moreover, we assume that

$$\frac{\partial v_1}{\partial x_{N+1}}(\chi_1 + x, \epsilon_{N+1}) \geq \frac{\partial v_2}{\partial x_{N+1}}(\chi_2 + x, \epsilon_{N+1}) \tag{3.9}$$

for any $x \geq 0$.

(C2') $\lambda_i \nu_k(p_k, \epsilon_k) - \mu_k(p_k, \epsilon_k) = -2\lambda_i \chi_i \sigma_k^2(p_k, \epsilon_k)$ for any k and ϵ_k.

We also replace the assumption (C3) by a stronger assumption.

(C3') For any k, suppose that Δ_k conditional on δ_k, p_k and ϵ_k is a continous random variable with the probability density function $f_{\Delta_k|\delta_k,p_k,\epsilon_k}(\Delta_k)$. Moreover, $f_{\Delta_k|\delta_k,p_k,\epsilon_k}(\Delta_k) = f_{\Delta_k|\delta_k,p_k,\epsilon_k}(-\Delta_k)$ and $f_{\Delta_k|\delta_k,p_k,\epsilon_k}(\Delta_k) \geq f_{\Delta_k|\delta_k,p_k,\epsilon_k}(\bar{\Delta}_k)$ for any k and $0 \leq \Delta_k \leq \bar{\Delta}_k$, i.e., the conditional density distribution $f_{\Delta_k|\delta_k,p_k,\epsilon_k}(\Delta_k)$ is unimodular and symmetric with respect to zero, e.g., Δ_k conditional on δ_k, p_k and ϵ_k is subject to a uniform or normal distribution with zero expectation.

Proposition 3.4. *For any $0 \leq \lambda_1 \leq \lambda_2$, let $T^1_{k,i}(p_k, \epsilon_k)$ and $T^2_{k,i}(p_k, \epsilon_k)$ denote the threshold levels in Theorem 3.1 corresponding to the risk aversion parameter λ_i, $i = 1, 2$. Under the assumptions (C1'), (C2') and (C3'), $T^2_{k,1}(p_k, \epsilon_k) - T^1_{k,1}(p_k, \epsilon_k) \geq T^2_{k,2}(p_k, \epsilon_k) - T^1_{k,2}(p_k, \epsilon_k)$ for any k, p_k and ϵ_k.*

Proof. Consider the induction assumption

$$\frac{\partial J_{k+1,1}}{\partial x_{k+1}}(\chi_1 + x, p_{k+1}, \epsilon_{k+1}) \leq \frac{\partial J_{k+1,2}}{\partial x_{k+1}}(\chi_2 + x, p_{k+1}, \epsilon_{k+1}) \text{ for any } x \geq 0. \tag{3.10}$$

To initiate an induction proof, we consider the period $N + 1$ and show that the function $J_{N+1,i}(x_{N+1}, \epsilon_{N+1})$ satisfies the induction assumption. The definition of $J_{N+1,i}(x_{N+1}, p_{N+1}, \epsilon_{N+1})$ in condition (C1') implies that

$$\frac{\partial J_{N+1,i}}{\partial x_{N+1}}(\chi_i + x, p_{N+1}, \epsilon_{N+1}) = -\frac{\partial v_i}{\partial x_{N+1}}(\chi_i + x, \epsilon_{N+1}),$$

and (3.9) in the condition (C1') immediately yields

$$\frac{\partial J_{N+1,1}}{\partial x_{N+1}}(\chi_1 + x, p_{N+1}, \epsilon_{N+1}) \leq \frac{\partial J_{N+1,2}}{\partial x_{N+1}}(\chi_2 + x, p_{N+1}, \epsilon_{N+1}) \text{ for any } x \geq 0.$$

Suppose that the induction assumption holds for period $k + 1$, we are going to show that $T^2_{k,1}(p_k, \epsilon_k) - T^1_{k,1}(p_k, \epsilon_k) \geq T^2_{k,2}(p_k, \epsilon_k) - T^1_{k,2}(p_k, \epsilon_k)$ for period k.

Let $f^2_{k,i}(L_k, p_k, \epsilon_k)$ denote the function $f^2_k(L_k, p_k, \epsilon_k)$ in (3.5) with the risk aversion parameter λ_i, $i = 1, 2$, and hence

$$\frac{\partial f_{k,i}^2}{\partial x_k}(\chi_i + x, p_k, \epsilon_k) = E\left\{ \frac{\partial J_{k+1,i}}{\partial x_{k+1}}(\chi_i + x + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k \right\} \\ + 2\lambda_i \sigma_k^2(p_k, \epsilon_k) x - \epsilon_k. \tag{3.11}$$

In order to simplify the notation, let us define

$$u_k(x, p_{k+1}, \epsilon_{k+1}) = \frac{\partial J_{k+1,2}}{\partial x_{k+1}}(\chi_2 + x, p_{k+1}, \epsilon_{k+1}) - \frac{\partial J_{k+1,1}}{\partial x_{k+1}}(\chi_1 + x, p_{k+1}, \epsilon_{k+1}).$$

According to Proposition 3.3 and the assumptions (C1'), (C2') and (C3'), it is staight-forward that $J_{k+1,i}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is symmetric in x_{k+1} with resepct to the point χ_i for $i = 1, 2$, and hence

$$\frac{\partial J_{k+1,i}}{\partial x_{k+1}}(\chi_i + x, p_{k+1}, \epsilon_{k+1}) = -\frac{\partial J_{k+1,i}}{\partial x_{k+1}}(\chi_i - x, p_{k+1}, \epsilon_{k+1})$$

if $J_{k+1,i}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is differentiable with respect to x_{k+1} at the point $x_{k+1} = \chi_i + x$, for $i = 1, 2$. Given p_{k+1} and ϵ_{k+1}, $J_{k+1,i}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is convex in x_{k+1}, which implies the countability of the set that $J_{k+1,i}(x_{k+1}, p_{k+1}, \epsilon_{k+1})$ is not differentiable in x_{k+1} (c.f. Roberts and Varberg [43]). Therefore, we have $u_k(x, p_{k+1}, \epsilon_{k+1}) = -u_k(-x, p_{k+1}, \epsilon_{k+1})$ for any x and p_{k+1} at any x except for a countable set.

Let us consider the following function

$$\tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) = E\left[u_k(x + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) \middle| \delta_k, \epsilon_{k+1}, p_k, \epsilon_k \right].$$

Note that Δ_k contional on p_k is independent of ϵ_{k+1} conditional on $p_{k+1} = p_k + \delta_k$. The condition (C3') implies that

$$\begin{aligned} \tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) &= \int_{\Delta_k} u_k(x + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k) d\Delta_k \\ &= \int_{\tilde{\Delta}_k} u_k(\tilde{\Delta}_k, p_k + \delta_k, \epsilon_{k+1}) f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\tilde{\Delta}_k - x) d\tilde{\Delta}_k, \end{aligned}$$

where the second equality is obtained by replacing $x+\Delta_k$ by $\tilde{\Delta}_k$. Obviously, we have

$$\begin{aligned}\tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) = & \int_{\tilde{\Delta}_k \leq 0} u_k(\tilde{\Delta}_k, p_k+\delta_k, \epsilon_{k+1}) f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\tilde{\Delta}_k - x) d\tilde{\Delta}_k \\ & + \int_{\tilde{\Delta}_k \geq 0} u_k(\tilde{\Delta}_k, p_k+\delta_k, \epsilon_{k+1}) f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\tilde{\Delta}_k - x) d\tilde{\Delta}_k.\end{aligned}$$

If we replace $\tilde{\Delta}_k$ by $-\Delta_k$ in the first integral and replace $\tilde{\Delta}_k$ by Δ_k in the second integral, it follows that

$$\begin{aligned}\tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) = & \int_{\Delta_k \geq 0} u_k(-\Delta_k, p_k+\delta_k, \epsilon_{k+1}) f_{\Delta_k|\delta_k, p_k, \epsilon_k}(-\Delta_k - x) d\Delta_k \\ & + \int_{\Delta_k \geq 0} u_k(\Delta_k, p_k+\delta_k, \epsilon_{k+1}) f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k - x) d\Delta_k.\end{aligned}$$

Since $u_k(x, p_k, \epsilon_k, \delta_k) = -u_k(-x, p_k, \epsilon_k, \delta_k)$ at any x except for a countable set, we obtain

$$\begin{aligned}\tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) = \int_{\Delta_k \geq 0} & \Big(f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k - x) - f_{\Delta_k|\delta_k, p_k, \epsilon_k}(-\Delta_k - x) \Big) \\ & u_k(\Delta_k, p_k+\delta_k, \epsilon_{k+1}) d\Delta_k.\end{aligned}$$

According to the condition (C3'), the conditional distribution of Δ_k is symmetric, which implies $f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k - x) = f_{\Delta_k|\delta_k, p_k, \epsilon_k}(|\Delta_k - x|)$ and $f_{\Delta_k|\delta_k, p_k, \epsilon_k}(-\Delta_k - x) = f_{\Delta_k|\delta_k, p_k, \epsilon_k}(|\Delta_k + x|)$. Note that $|\Delta_k - x| \leq |\Delta_k + x|$ for any $x \geq 0$ and $\Delta_k \geq 0$, and hence the unimodularity of $f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k)$ stated in the assumption (C3') shows that $f_{\Delta_k|\delta_k, p_k, \epsilon_k}(\Delta_k - x) - f_{\Delta_k|\delta_k, p_k, \epsilon_k}(-\Delta_k - x) \geq 0$. Moreover, the induction assumption in (3.10) implies that $u_k(\Delta_k, p_k+\delta_k, \epsilon_{k+1}) \geq 0$ for any $\Delta_k \geq 0$. Therefore, we obtain $\tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) \geq 0$ for any $x \geq 0$.

The first derivative of $f_{k,i}^2(x_k, p_k, \epsilon_k)$ in (3.11) suggests that

$$\begin{aligned}& \frac{\partial f_{k,2}^2}{\partial L_k}(\chi_2 + x, p_k, \epsilon_k) - \frac{\partial f_{k,1}^2}{\partial L_k}(\chi_1 + x, p_k, \epsilon_k) \\ & \quad = 2(\lambda_2 - \lambda_1)\sigma_k^2(p_k, \epsilon_k)x + E\left\{ \tilde{u}_k(x, p_k, \epsilon_k, \delta_k, \epsilon_{k+1}) \Big| p_k, \epsilon_k \right\}.\end{aligned}$$

Note that $\lambda_1 \leq \lambda_2$ and $\sigma_k^2(p_k, \epsilon_k) = Var(\delta_k | p_k, \epsilon_k) \geq 0$. It follows immediately that

$$\frac{\partial f_{k,2}^2}{\partial L_k}(\chi_2 + x, p_k, \epsilon_k) - \frac{\partial f_{k,1}^2}{\partial L_k}(\chi_1 + x, p_k, \epsilon_k) \geq 0 \text{ for any } x \geq 0. \tag{3.12}$$

For contradiction, let us suppose that $T_{k,1}^2(p_k, \epsilon_k) - \chi_1 < T_{k,2}^2(p_k, \epsilon_k) - \chi_2$. Then there exists some x such that $0 \leq T_{k,1}^2(p_k, \epsilon_k) - \chi_1 < x < T_{k,2}^2(p_k, \epsilon_k) - \chi_2$. For $i = 1, 2$, Proposition 3.3 shows that the threshold level $T_{k,i}^2(p_k, \epsilon_k) \geq \chi_i$ corresponds to the global minimizer of the convex function $f_{k,i}^2(x_k, p_k, \epsilon_k)$ for any given p_k and ϵ_k. Therefore, $\chi_1 + x > T_{k,1}^2(p_k, \epsilon_k)$ implies $\frac{\partial f_{k,1}^2}{\partial L_k}(\chi_1 + x, p_k, \epsilon_k) > 0$, and $\chi_2 + x < T_{k,2}^2(p_k, \epsilon_k)$ implies $\frac{\partial f_{k,2}^2}{\partial L_k}(\chi_2 + x, p_k, \epsilon_k) < 0$, which contradicts (3.12). As a result, we proved that $T_{k,1}^2(p_k, \epsilon_k) - \chi_1 \geq T_{k,2}^2(p_k, \epsilon_k) - \chi_2$, which yields $T_{k,1}^2(p_k, \epsilon_k) - T_{k,1}^1(p_k, \epsilon_k) \geq T_{k,2}^2(p_k, \epsilon_k) - T_{k,2}^1(p_k, \epsilon_k)$ by Proposition 3.3.

Next we complete the proof by showing that the induction assumption in (3.10) also holds for period k. According to (3.7), we have

$$\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon_k) = \begin{cases} -\epsilon_k & \text{if } x_k \leq T_k^1(p_k, \epsilon_k), \\ \frac{\partial f_k}{\partial L_k}(x_k, p_k, \epsilon_k) = \frac{\partial f_k^2}{\partial L_k}(x_k, p_k, \epsilon_k) - \epsilon_k & \\ & \text{if } T_k^1(p_k, \epsilon_k) < x_k \leq T_k^2(p_k, \epsilon_k), \\ \epsilon_k & \text{if } x_k > T_k^2(p_k, \epsilon_k). \end{cases} \tag{3.13}$$

If $0 \leq x \leq T_{k,2}^2(p_k, \epsilon_k) - \chi_2 \leq T_{k,1}^2(p_k, \epsilon_k) - \chi_1$, we have

$$\begin{aligned} &\frac{\partial J_{k,1}}{\partial x_k}(\chi_1 + x, p_k, \epsilon_k) - \frac{\partial J_{k,2}}{\partial x_k}(\chi_2 + x, p_k, \epsilon_k) \\ &\quad = \frac{\partial f_{k,1}^2}{\partial L_k}(\chi_1 + x, p_k, \epsilon_k) - \frac{\partial f_{k,2}^2}{\partial L_k}(\chi_2 + x, p_k, \epsilon_k) \leq 0, \end{aligned}$$

where the inequality is yielded by (3.12).

Suppose that $T_{k,2}^2(p_k, \epsilon_k) - \chi_2 < x \leq T_{k,1}^2(p_k, \epsilon_k) - \chi_1$. Since $J_k(x_k, p_k, \epsilon_k)$ is convex in x_k, $\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon_k)$ is non-decreasing in x_k. According to (3.13), we know

$$\frac{\partial J_{k,1}}{\partial x_k}(\chi_1 + x, p_k, \epsilon_k) \leq \epsilon_k = \frac{\partial J_{k,2}}{\partial x_k}(\chi_2 + x, p_k, \epsilon_k),$$

where the equality follows from $x + \chi_2 > T^2_{k,2}(p_k, \epsilon_k)$ and (3.13).

If $x > T^2_{k,1}(p_k, \epsilon_k) - \chi_1 \geq T^2_{k,2}(p_k, \epsilon_k) - \chi_2$, (3.13) shows that

$$\frac{\partial J_{k,1}}{\partial x_k}(\chi_1 + x, p_k, \epsilon_k) = \frac{\partial J_{k,2}}{\partial x_k}(\chi_2 + x, p_k, \epsilon_k) = \epsilon_k.$$

Summerizing the three cases, we have $\frac{\partial J_{k,1}}{\partial x_k}(\chi_1 + x, p_k, \epsilon_k) \leq \frac{\partial J_{k,2}}{\partial x_k}(\chi_2 + x, p_k, \epsilon_k)$ for any $x \geq 0$, which completes the induction proof. □

Suppose that $E[\delta_k | p_k, \epsilon_k] = E[\Delta_k | p_k, \epsilon_k] = 0$ and the distribution of δ_k conditional on p_k and ϵ_k is independent of the conditional distribution of S_k and Δ_k, which, together with the assumption (C3'), are the sufficient conditions for $\mu_k(p_k, \epsilon_k) = 0$ and $\nu_k(p_k, \epsilon_k) = 0$ for any k and ϵ_k. In this case, the increase in the risk aversion parameter λ is equivalent to increasing the conditional variance of δ_k, i.e., $\sigma^2_k(p_k, \epsilon_k) = Var(\delta_k | p_k, \epsilon_k)$, while keeping the value of the risk aversion parameter. Following the proof of Proposition 3.4, we can establish the following monotonicity property with respect to the conditional variance of δ_k.

Corollary 3.2. *Consider $\delta_{k,1}$ and $\delta_{k,2}$ such that $Var(\delta_{k,1} | p_k, \epsilon_k) \leq Var(\delta_{k,2} | p_k, \epsilon_k)$, $E[\delta_{k,i} | p_k, \epsilon_k] = 0$, and the distribution of $\delta_{k,i}$ conditional on p_k and ϵ_k is independent of the conditional distribution of S_k and Δ_k for any k and $i = 1, 2$. Let $T^1_{k,i}(p_k, \epsilon_k)$ and $T^2_{k,i}(p_k, \epsilon_k)$ denote the threshold levels in Theorem 3.1 corresponding to the price movements $\delta_{k,i}$ for any k and $i = 1, 2$. Under the assumptions (C1), (C2) and (C3'), $T^1_{k,1}(p_k, \epsilon_k) = -T^2_{k,1}(p_k, \epsilon_k) \leq T^1_{k,2}(p_k, \epsilon_k) = -T^2_{k,2}(p_k, \epsilon_k)$ for any k, p_k and ϵ_k.*

The more volatile the underlying asset price, the more risky to hold the inventory. Therefore, with the same risk aversion parameter λ, the decision maker is more likely to actively trade with other market-maker to control the inventory risk, i.e., the no-trade region shrinks as the variance of the price movement δ_k increases, which agrees with Corollary 3.2.

3.6.2 Monotonicity with Respect to the Spread

Examples 3.1 and 3.2 show that for any given period k and mid price p_k, the threshold level $T_k^1(p_k, \epsilon_k)$ is lower while $T_k^2(p_k, \epsilon_k)$ is higher for greater ϵ_k. The intuition behind this property is clear. If the spread is high, the cost of adjusting inventory is high and hence the market-maker will try to avoid trading by widening no-trade region. Formally, we can prove the following monotonicity property with respect to the spread defined by ϵ_k.

Proposition 3.5. *If ϵ_k is independent of δ_k, S_k and Δ_k in period k, then the threshold level $T_k^1(p_k, \epsilon_k)$ in Theorem 3.1 is non-increasing in ϵ_k if $\nu_k^2(p_k, \epsilon_k) \geq 0$, while $T_k^2(p_k, \epsilon_k)$ is non-decreasing in ϵ_k if $\nu_k^2(p_k, \epsilon_k) \leq 0$, where $\nu_k^2(p_k, \epsilon_k)$ is defined in* (3.3).

Proof. If ϵ_k is independent of δ_k, S_k and Δ_k, then $\mu_k(p_k, \epsilon_k)$, $\sigma_k^2(p_k, \epsilon_k)$, $\nu_k^1(p_k, \epsilon_k)$ and $\nu_k^2(p_k, \epsilon_k)$ are also independent of ϵ_k. To simplify the notation, we denote them using $\mu_k(p_k)$, $\sigma_k(p_k)$, $\nu_k^1(p_k)$ and $\nu_k^2(p_k)$ respectively, and so $\nu_k(p_k, \epsilon_k) = \nu_k^1(p_k) + \epsilon_k \nu_k^2(p_k)$. When ϵ_k is independent of δ_k and Δ_k, we also have

$$E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| \, p_k, \epsilon_k \right\} = E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| \, p_k \right\}.$$

Therefore, the function $f_k(L_k, p_k, \epsilon_k)$ can be reduced to

$$\begin{aligned} f_k(L_k, p_k, \epsilon_k) = & \Big(\lambda(\nu_k^1(p_k) + \epsilon_k \nu_k^2(p_k)) - \mu_k(p_k)\Big) L_k \\ & + \lambda \sigma_k^2 L_k^2 + E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| \, p_k \right\}, \end{aligned}$$

and hence

$$\begin{aligned} \frac{\partial f_k}{\partial L_k}(L_k, p_k, \epsilon_k) = & \lambda(\nu_k^1(p_k) + \epsilon_k \nu_k^2(p_k)) - \mu_k(p_k) + 2\lambda \sigma_k L_k \\ & + \frac{\partial}{\partial L_k} E\left\{ J_{k+1}\left(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}\right) \Big| \, p_k \right\}. \end{aligned}$$

The definition of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ in (3.5) immediately shows that

$$\frac{\partial f_k^1}{\partial L_k}(L_k, p_k, \epsilon_k) = \frac{\partial f_k}{\partial L_k}(L_k, p_k) + \epsilon_k \quad \text{and} \quad \frac{\partial f_k^2}{\partial L_k}(L_k, p_k, \epsilon_k) = \frac{\partial f_k}{\partial L_k}(L_k, p_k) - \epsilon_k,$$

and it follows that

$$\frac{\partial^2 f_k^1}{\partial L_k \partial \epsilon_k}(L_k, p_k, \epsilon_k) = \lambda \nu_k^2(p_k) + 1 \quad \text{and} \quad \frac{\partial^2 f_k^2}{\partial L_k \partial \epsilon_k}(L_k, p_k, \epsilon_k) = \lambda \nu_k^2(p_k) - 1.$$

Since $\lambda \geq 0$, $\frac{\partial f_k^1}{\partial L_k}(L_k, p_k, \epsilon_k)$ is increasing in ϵ_k if $\nu_k^2(p_k) \geq 0$, and $\frac{\partial f_k^2}{\partial L_k}(L_k, p_k, \epsilon_k)$ is decreasing in ϵ_k if $\nu_k^2(p_k) \leq 0$. Note that $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ correspond to global minimizers of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$. Following the argument in the proof of Proposition 3.4, we can prove that $T_k^1(p_k, \epsilon_k)$ is non-increasing in ϵ_k if $\nu_k^2(p_k) \geq 0$, and $T_k^2(p_k, \epsilon_k)$ is non-decreasing in ϵ_k if $\nu_k^2(p_k) \leq 0$. □

Proposition 3.5 deals with the case that we increase the spread in period k while keeping the spreads in the rest periods. Next we would like to investigate the case when we shift the spreads upwards for all periods. For instance, in Example 3.3, we consider both $\varphi_k = 10^{-4}$ and $\varphi_k = 2 \times 10^{-4}$. which correspond to a parallel shift in the spread. The computational results indicate that the lower limits $T_k^1(p_k)$ (the upper limits $T_k^2(p_k)$, repectively) have greater (smaller, respectively) values when $\varphi_k = 10^{-4}$ than when $\varphi_k = 2 \times 10^{-4}$, which is analogous to the monotonicity property in Proposition 3.5.

Formally, let us consider the following assumptions.

(C4) The end of the planning horizon profit or loss function $v(x_{N+1}, \epsilon_{N+1})$ satisfies

$$-e_{N+1} \leq \frac{\partial v}{\partial x_{N+1}}(x_{N+1}, p_{N+1}, \epsilon) - \frac{\partial v}{\partial x_{N+1}}(x_{N+1}, p_{N+1}, \epsilon + e_{k+1}) \leq e_{N+1}$$

for any x_{N+1}, p_{N+1}, ϵ, and $e_{k+1} \geq 0$.

(C5) $\nu_k^2(p_k, \epsilon_k) = 2\Big(E[\delta_k S_k | p_k, \epsilon_k] - E[\delta_k | p_k, \epsilon_k] E[S_k | p_k, \epsilon_k]\Big) = 0$ for any k, p_k, and ϵ_k.

Proposition 3.6. *For any period k, consider a random variable $\bar{\epsilon}_k = \epsilon_k + e_k$ where e_k is a given constant and $e_k \geq e_{k+1} \geq 0$. Let $\bar{T}_k^1(p_k, \bar{\epsilon}_k)$ and $\bar{T}_k^2(p_k, \bar{\epsilon}_k)$ denote the threshold levels in Theorem 2 where the spread in period k is defined by the random* *variable* $\bar{\epsilon}_k$ *for any* k. *Under the conditions (C4) and (C5),* $T_k^1(p_k, \epsilon) \geq \bar{T}_k^1(p_k, \epsilon + e_k)$ *and* $T_k^2(p_k, \epsilon) \leq \bar{T}_k^2(p_k, \epsilon + e_k)$ *for any* k, p_k *and* ϵ.

Proof. Similar to Proposition 3.4, let $\bar{J}_k(x_k, p_k, \bar{\epsilon}_k)$ denote the function $J_k(x_k, p_k, \epsilon_k)$ in (3.4) with the spread defined by the random variable $\bar{\epsilon}_k$ for period k. For the induction proof on the number of period k, we consider the following assumption

$$-e_{k+1} \leq \frac{\partial J_{k+1}}{\partial x_{k+1}}(x_{k+1}, p_{k+1}, \epsilon) - \frac{\partial \bar{J}_{k+1}}{\partial x_{k+1}}(x_{k+1}, p_{k+1}, \epsilon + e_{k+1}) \leq e_{k+1} \tag{3.14}$$

for any x_{k+1}, p_{k+1} and ϵ.

Note that the definition of $J_{N+1}(x_{N+1}, p_{N+1}, \epsilon_{N+1})$ and the assumption (C4) implies

$$-e_{N+1} \leq \frac{\partial J_{N+1}}{\partial x_{N+1}}(x_{N+1}, p_{N+1}, \epsilon) - \frac{\partial \bar{J}_{N+1}}{\partial x_{N+1}}(x_{N+1}, p_{N+1}, \epsilon + e_{N+1}) \leq e_{N+1},$$

i.e., the induction assumption (3.14) holds for period $N+1$.

Let us consider period k. Similarly, $\bar{f}_k(L_k, p_k, \bar{\epsilon}_k)$, $\bar{f}_k^1(L_k, p_k, \bar{\epsilon}_k)$ and $\bar{f}_k^2(L_k, p_k, \bar{\epsilon}_k)$ are used to denote the functions $f_k(L_k, p_k, \epsilon_k)$, $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ in (3.5) with the spread defined by the random variable $\bar{\epsilon}_k$.

Since $\bar{\epsilon}_k = \epsilon_k + e_k$, we have

$$\mu_k(p_k, \epsilon) = E[\delta_k \mid p_k, \epsilon_k = \epsilon] = E[\delta_k \mid p_k, \bar{\epsilon}_k = \epsilon + e_k]$$
$$\sigma_k^2(p_k, \epsilon) = Var(\delta_k \mid p_k, \epsilon_k = \epsilon) = Var(\delta_k \mid p_k, \bar{\epsilon}_k = \epsilon + e_k)$$

and

$$\nu_k(p_k, \epsilon) = \nu_k^1(p_k, \epsilon) = 2\left(E[\delta_k^2 \Delta_k \mid p_k, \epsilon_k = \epsilon] - E[\delta_k \mid p_k, \epsilon_k = \epsilon] E[\delta_k \Delta_k \mid \epsilon_k = \epsilon]\right)$$
$$= 2\left(E[\delta_k^2 \Delta_k \mid p_k, \bar{\epsilon}_k = \epsilon + e_k] - E[\delta_k \mid p_k, \bar{\epsilon}_k = \epsilon + e_k] E[\delta_k \Delta_k \mid p_k, \bar{\epsilon}_k = \epsilon + e_k]\right).$$

It follows immediately that

$$\begin{aligned}\frac{\partial \bar{f}_k}{\partial L_k}(L_k, p_k, \epsilon + e_k) &= \lambda \nu_k(p_k, \epsilon) - \mu_k(p_k, \epsilon) + 2\lambda \sigma_k^2(p_k, \epsilon) L_k \\ &\quad + E\left\{ \frac{\partial \bar{J}_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) \middle| p_k, \bar{\epsilon}_k = \epsilon + e_k \right\} \\ &= \lambda \nu_k(p_k, \epsilon) - \mu_k(p_k, \epsilon) + 2\lambda \sigma_k^2(p_k, \epsilon) L_k \\ &\quad + E\left\{ \frac{\partial \bar{J}_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k = \epsilon \right\},\end{aligned}$$

where the second equality is also obtained by $\bar{\epsilon}_k = \epsilon_k + e_k$. Note that

$$\begin{aligned}\frac{\partial f_k}{\partial L_k}(L_k, p_k, \epsilon) &= \lambda \nu_k(p_k, \epsilon) - \mu_k(p_k, \epsilon) + 2\lambda \sigma_k^2(p_k, \epsilon) L_k \\ &\quad + E\left\{ \frac{\partial J_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k = \epsilon \right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\partial f_k}{\partial L_k}(L_k, p_k, \epsilon) - \frac{\partial \bar{f}_k}{\partial L_k}(L_k, p_k, \epsilon + e_k) \\ &= E\left\{ \frac{\partial J_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) - \frac{\partial \bar{J}_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1}) \middle| p_k, \epsilon_k = \epsilon \right\}.\end{aligned}$$

According to the induction assumption (3.14), it follows immediately that

$$-e_{k+1} \le \frac{\partial f_k}{\partial L_k}(L_k, p_k, \epsilon) - \frac{\partial \bar{f}_k}{\partial L_k}(L_k, p_k, \epsilon + e_k) \le e_{k+1} \text{ for any } L_k,\ p_k \text{ and } \epsilon. \quad (3.15)$$

As a result, the definition of $f_k^1(L_k, p_k, \epsilon_k)$ and $f_k^2(L_k, p_k, \epsilon_k)$ in (3.5) and the assumption that $e_k \ge e_{k+1}$ implies that

$$\begin{aligned}\frac{\partial f_k^1}{\partial L_k}(L_k, p_k, \epsilon) - \frac{\partial \bar{f}_k^1}{\partial L_k}(L_k, p_k, \epsilon + e_k) &= \frac{\partial f_k}{\partial L_k}(L_k, p_k, \epsilon) - \frac{\partial \bar{f}_k}{\partial L_k}(L_k, p_k, \epsilon + e_k) - e_k \\ &\le e_{k+1} - e_k \le 0 \\ \frac{\partial f_k^2}{\partial L_k}(L_k, p_k, \epsilon) - \frac{\partial \bar{f}_k^2}{\partial L_k}(L_k, p_k, \epsilon + e_k) &= \frac{\partial f_k}{\partial L_k}(L_k, p_k, \epsilon) - \frac{\partial \bar{f}_k}{\partial L_k}(L_k, p_k, \epsilon + e_k) + e_k \\ &\ge e_k - e_{k+1} \ge 0.\end{aligned}$$

Following the arguement in the proof of Proposition 3.4, we can show that $T_k^1(p_k, \epsilon) \geq \bar{T}_k^1(p_k, \epsilon + e_k)$ and $T_k^2(p_k, \epsilon) \leq \bar{T}_k^2(p_k, \epsilon + e_k)$ for any k, p_k and ϵ.

Now let us consider

$$\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k. \epsilon + e_k).$$

For any given p_k and ϵ, we have $\bar{T}_k^1(p_k, \epsilon + e_k) \leq T_k^1(p_k, \epsilon) \leq T_k^2(p_k, \epsilon) \leq \bar{T}_k^2(p_k, \epsilon + e_k)$, and hence there exists five cases.

- If $x_k \leq \bar{T}_k^1(p_k, \epsilon + e_k)$, we obtain from (3.13) that $\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) = -\epsilon$ and $\frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) = -\epsilon - e_k$, i.e.,

$$\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) = e_k.$$

- Suppose that $\bar{T}_k^1(p_k, \epsilon + e_k) < x_k \leq T_k^1(p_k, \epsilon)$.

  The proof of Theorem 3.1 shows that $\bar{J}_k(x_k, p_k, \epsilon + e_k)$ is convex in x_k, and so its first left-hand derivative with repect to x_k should be non-decreasing in x_k, which implies $\frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) \geq -\epsilon - e_k$ by (3.13). (3.13) also shows that $\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) = -\epsilon$. As a result, we obtain

$$\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) \leq -\epsilon - (-\epsilon - e_k) = e_k.$$

  Note that $T_k^1(p_k, \epsilon) = \arg\min_{L_k} f_k^1(L_k, p_k, \epsilon_k)$, and hence $x_k \leq T_k^1(p_k, \epsilon)$ implies

$$\frac{\partial f_k^1}{\partial L_k}(x_k, p_k, \epsilon) = \frac{\partial f_k}{\partial L_k}(x_k, p_k, \epsilon) + \epsilon \leq 0, \text{ i.e., } \frac{\partial f_k}{\partial L_k}(x_k, p_k, \epsilon) \leq -\epsilon = \frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon).$$

  Moreover, we have $\frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) = \frac{\partial \bar{f}_k}{\partial L_k}(L_k, p_k, \epsilon + e_k)$ by (3.13). It follows directly that

$$\begin{aligned}\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) &\geq \frac{\partial f_k}{\partial L_k}(x_k, p_k. \epsilon) - \frac{\partial \bar{f}_k}{\partial L_k}(x_k, p_k, \epsilon + e_k) \\ &\geq -e_{k+1} \geq -e_k,\end{aligned}$$

where the second inequality follows from (3.15) and the third equality is yielded by $e_k \geq e_{k+1}$.

- If $T_k^1(p_k, \epsilon) < x_k \leq T_k^2(p_k, \epsilon)$, (3.13) shows that

$$\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) = \frac{\partial f_k}{\partial L_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{f}_k}{\partial L_k}(x_k, p_k, \epsilon + e_k).$$

  The result in (3.15) as well as the assumption that $e_k \geq e_{k+1}$ imply that

$$-e_k \leq -e_{k+1} \leq \frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) \leq e_{k+1} \leq e_k.$$

- If $T_k^2(p_k, \epsilon) < x_k \leq \bar{T}_k^2(p_k, \epsilon + e_k)$, we can show

$$-e_k \leq \frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) \leq e_k$$

  using an arguement similar to that of the second case when $\bar{T}_k^1(p_k, \epsilon + e_k) < x_k \leq T_k^1(p_k, \epsilon)$.

- If $x_k > \bar{T}_k^2(p_k, \epsilon + e_k)$, it follows immediately from (3.13) that

$$\frac{\partial J_k}{\partial x_k}(x_k, p_k, \epsilon) - \frac{\partial \bar{J}_k}{\partial x_k}(x_k, p_k, \epsilon + e_k) = \epsilon - (\epsilon + e_k) = -e_k.$$

As a result, we complete the proof by showing that the induction assumption (3.14) holds for period k. □

3.6.3 Monotonicity with Respect to the Mid Price

Let us consider the following assumption.

(C6) For any period k, $\epsilon_k = \phi_k(p_k) + \varphi_k$ where $\phi_k(p_k)$ is a given function of p_k and φ_k is a random variable independent of p_k and $\varphi_{\bar{k}}$ for any $\bar{k} \neq k$. In addition, δ_k, S_k and Δ_k are independent of p_k for any k.

Under certain conditions, the monotonicity of the threshold levels with respect to the spread ϵ_k can be transformed into the mononiticity with respect to the mid price p_k.

For example, Example 3.2 satisfies the condition (C6), and ϵ_k is non-decreasing in p_k by the definition of $\phi_k(p_k)$ in (2.12). Figures 3-4 and 3-5 show that $-T_k^1(p_k)$ and $T_k^2(p_k)$ are higher when $p_k = 2.001$ compared with the case when $p_k = 1.999$, because the higher transaction cost, i.e., ϵ_k caused by increase p_k prevent us from trading frequently with other market-maker to control the inventory. This property can be generalized to any $0 \leq p_{k,1} \leq p_{k,2}$ under some additional conditions.

Proposition 3.7. *Suppose that (i) $\phi_k(p_k)$ is non-decreasing in p_k for any k, (ii) $\phi_{k+1}(p_{k,2} + \delta_k) - \phi_{k+1}(p_{k,1} + \delta_k)$ is convex in δ_k for any $0 \leq p_{k,1} \leq p_{k,2}$ and k, and (iii) $\phi_k(p_{k,2}) - \phi_k(p_{k,1})$ is non-increasing in k for any $0 \leq p_{k,1} \leq p_{k,2}$. Under the conditions (C4), (C5) and (C6), if $E[\delta_k|\varphi_k] = 0$ for any φ_k, then the threshold level $T_k^2(p_k, \epsilon_k)$ in Theorem 3.1 is non-decreasing in p_k for any k and ϵ_k.*

Proof. The proof for Proposition 3.7 is very similar to that of Proposition 3.6. We adopt an induction proof under the following induction assumption

$$\begin{aligned}\frac{\partial J_{k+1}}{\partial x_{k+1}}(x_{k+1}, p_{k+1,1}, \phi_{k+1}(p_{k+1,1}) + \varphi_k) - \frac{\partial J_{k+1}}{\partial x_{k+1}}(x_{k+1}, p_{k+1,2}, \phi_{k+1}(p_{k+1,2}) + \varphi_k) \\ \geq \phi_{k+1}(p_{k+1,1}) - \phi_{k+1}(p_{k+1,2})\end{aligned} \tag{3.16}$$

for any $0 \leq p_{k+1,1} \leq p_{k+1,2}$, x_{k+1} and φ_k.

According to the condition (C4), the same argument in the proof of Proposition 3.6 shows that the induction assumption (3.16) is valid for period $N+1$, i.e., when $k = N$. Next, we are going to show that the monotonicity properties stated in Proposition 3.7 holds when the period $k+1$ satisfies the assumption (3.16). Let us consider any $p_{k,1}$ and $p_{k,2}$ such that $0 \leq p_{k,1} \leq p_{k,2}$.

The condition (C6) states that $\epsilon_k = \phi_k(p_k)+\varphi_k$, and δ_k, S_k and Δ_k are independent of p_k for any k. According to the definitions of $\mu_k(\epsilon_k)$, $\sigma_k^2(\epsilon_k)$ and $\nu_k(\epsilon_k)$ as well as the condition (C5), it is straightforward to show that

$$\begin{aligned}
\mu_k(p_k, \epsilon_k) &= E[\delta_k \mid p_k, \phi_k(p_k) + \varphi_k] = E[\delta_k|\varphi_k] \\
\sigma_k^2(p_k, \epsilon_k) &= Var(\delta_k \mid p_k, \phi_k(p_k) + \varphi_k) = Var(\delta_k|\varphi_k) \\
\nu_k(p_k, \epsilon_k) &= 2\Big(E[\delta_k^2\Delta_k \mid p_k, \phi_k(p_k) + \varphi_k] - E[\delta_k \mid p_k, \phi_k(p_k) + \varphi_k] \\
&\qquad \times E[\delta_k\Delta_k \mid p_k, \phi_k(p_k) + \varphi_k]\Big) \\
&= 2\Big(E[\delta_k^2\Delta_k|\varphi_k] - E[\delta_k|\varphi_k]E[\delta_k\Delta_k|\varphi_k]\Big)
\end{aligned}$$

for any p_k and $\epsilon_k = \phi_k(p_k) + \varphi_k$. Moreover, we have

$$E\left\{\frac{\partial J_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1})\middle| p_k, \epsilon_k\right\} = E\left\{\frac{\partial J_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_k + \delta_k, \epsilon_{k+1})\middle| \varphi_k\right\}.$$

for any L_k, p_k and $\epsilon_k = \phi_k(p_k) + \varphi_k$.

Therefore, the definition of $f_k(L_k, p_k, \epsilon_k)$ in (3.5) immediately shows that

$$\begin{aligned}
&\frac{\partial f_k}{\partial L_k}(L_k, p_{k,1}, \phi_k(p_{k,1}) + \varphi_k) - \frac{\partial f_k}{\partial L_k}(L_k, p_{k,2}, \phi_k(p_{k,2}) + \varphi_k) \\
&= E\left\{\frac{\partial J_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_{k,1} + \delta_k, \epsilon_{k+1}) - \frac{\partial J_{k+1}}{\partial x_{k+1}}(L_k + \Delta_k, p_{k,2} + \delta_k, \epsilon_{k+1})\middle| \varphi_k\right\} \\
&\geq E\left\{\phi_{k+1}(p_{k,1} + \delta_k) - \phi_{k+1}(p_{k,2} + \delta_k)\right| \varphi_k\} \\
&\geq \phi_{k+1}(p_{k,1} + E[\delta_k|\varphi_k]) - \phi_{k+1}(p_{k,2} + E[\delta_k|\varphi_k]) \\
&= \phi_{k+1}(p_{k,1}) - \phi_{k+1}(p_{k,2}) \geq \phi_k(p_{k,1}) - \phi_k(p_{k,2}),
\end{aligned} \tag{3.17}$$

where the first inequality follows from the induction assumption (3.16), the second inequality is yielded by the Jensen's inequality and the assumption (ii) that $\phi_{k+1}(p_{k,2}+\delta_k) - \phi_{k+1}(p_{k,1}+\delta_k)$ is convex in δ_k for any $0 \leq p_{k,1} \leq p_{k,2}$, the last equality is obtained from the assumption that $E[\delta_k|\varphi_k] = 0$ for any φ_k, and the last inequality is due to the assumption (iii) that $\phi_k(p_{k,2}) - \phi_k(p_{k,1})$ is non-increasing in k for any $0 \leq p_{k,1} \leq p_{k,2}$.

Similar to the proof of Proposition 3.6, we obtain

$$
\begin{aligned}
&\frac{\partial f_k^2}{\partial L_k}(L_k, p_{k,1}, \phi_k(p_{k,1}) + \varphi_k) - \frac{\partial f_k^2}{\partial L_k}(L_k, p_{k,2}, \phi_k(p_{k,1}) + \varphi_k) \\
= &\frac{\partial f_k}{\partial L_k}(L_k, p_{k,1}, \phi_k(p_{k,1}) + \varphi_k) - \frac{\partial f_k}{\partial L_k}(L_k, p_{k,2}, \phi_k(p_{k,2}) + \varphi_k) + \phi_k(p_{k,2}) - \phi_k(p_{k,1}) \\
\geq &\, 0,
\end{aligned}
$$

and hence $T_k^2(p_k, \epsilon_k)$ is non-decreasing in p_k for period k.

Now let us complete the proof by showing that the induction assumption (3.16) holds for period k, i.e.,

$$
\frac{\partial J_k}{\partial x_k}(x_k, p_{k,1}, \phi_k(p_{k,1}) + \varphi_k) - \frac{\partial J_k}{\partial x_k}(x_k, p_{k,2}, \phi_k(p_{k,2}) + \varphi_k) \geq \phi_k(p_{k,1}) - \phi_k(p_{k,2}),
$$

where $0 \leq p_{k,1} \leq p_{k,2}$. We can consider the two cases: $T_k^1(p_{k,1}, \epsilon_k) \leq T_k^1(p_{k,2}, \epsilon_k)$ and $T_k^1(p_{k,1}, \epsilon_k) > T_k^1(p_{k,2}, \epsilon_k)$. The proof is very similar to that in the proof of Proposition 3.6 and hence it is omitted here. □

Note that a large family of $\phi_k(p_k)$ satisfies the conditions stated in Proposition 3.7. Suppose that $\phi_{k+1}(p_{k+1})$ is differentiable and $\phi'_{k+1}(p_{k+1})$ is convex. Given $p_{k,1} \leq p_{k,2}$, let $\tilde{\phi}_k(\delta_k) = \phi_{k+1}(p_{k,2} + \delta_k) - \phi_k(p_{k,1} + \delta_k)$. For any $\delta_{k,1} \leq \delta_{k,2}$, we have

$$
\begin{aligned}
\tilde{\phi}'_k(\delta_{k,1}) - \tilde{\phi}'_k(\delta_{k,2}) = &\,\phi_{k+1}(p_{k,2} + \delta_{k,1}) - \phi_k(p_{k,1} + \delta_{k,1}) \\
&- \phi_{k+1}(p_{k,2} + \delta_{k,2}) + \phi_k(p_{k,1} + \delta_{k,2}) \leq 0
\end{aligned}
$$

since $\phi'_{k+1}(p_{k,1} + \delta_{k,2}) - \phi'_{k+1}(p_{k,1} + \delta_{k,1}) \leq \phi'_{k+1}(p_{k,2} + \delta_{k,2}) - \phi'_{k+1}(p_{k,2} + \delta_{k,1})$ by the convexity of $\phi'_{k+1}(p_{k+1})$. The increasing fist derivatie shows that $\phi_{k+1}(p_{k,2} + \delta_k) - \phi_k(p_{k,1} + \delta_k)$ is convex in δ_k for any $p_{k,1} \leq p_{k,2}$. As a result, any linear, quadratic, exponential and logarithmic function satisfies the condition (ii) in Proposition 3.7. In addition, the condition (i) and (iii) are valid if $\phi_k(p_k) = \alpha_k \phi(p_k) + \beta_k$ where $\phi(p_k)$ is a given increasing function and $\alpha_1 \geq \cdots \geq \alpha_{N+1} \geq 0$.

Furthermore, we can also achieve

$$\phi_{k+1}(p_{k,1} + E[\delta_k|\varphi_k]) - \phi_{k+1}(p_{k,2} + E[\delta_k|\varphi_k]) \geq \phi_{k+1}(p_{k,1}) - \phi_{k+1}(p_{k,2})$$

in (3.17) if $E[\delta_k|\varphi_k] \geq 0$ and $\phi_k(p_k)$ is convex in p_k. Therefore, the condition $E[\delta_k|\varphi_k] = 0$ in Proposition 3.7 can be replaced by the nonnegativity of $E[\delta_k|\varphi_k]$ and the convexity of $\phi_k(p_k)$ in p_k.

Symmetrically, we obtain the sufficient conditions under which the threshold level $T_k^1(p_k, \epsilon_k)$ is decreasing with respect to price.

Corollary 3.3. *Suppose that (i) $\phi_k(p_k)$ is a non-decreasing function of p_k for any k, (ii) $\phi_{k+1}(p_{k,2} + \delta_k) - \phi_{k+1}(p_{k,1} + \delta_k)$ is concave in δ_k for any $0 \leq p_{k,1} \leq p_{k,2}$ and k, and (iii) $\phi_k(p_{k,2}) - \phi_k(p_{k,1})$ is non-increasing in k for any $0 \leq p_{k,1} \leq p_{k,2}$. Under the conditions (C4), (C5) and (C6), if $E[\delta_k|\varphi_k] = 0$ for any φ_k, then the threshold level $T_k^1(p_k, \epsilon_k)$ in Theorem 3.1 is non-increasing in p_k for any k and ϵ_k.*

If we combine the result in Proposition 3.3 with those in Proposition 3.7 and Corollary 3.3, it immediately yields the following corollary.

Corollary 3.4. *Consider the conditions in (C1∼6). Suppose that (i) $\phi_k(p_k)$ is non-decreasing in p_k for any k, (ii) $\phi_{k+1}(p_{k,2} + \delta_k) - \phi_{k+1}(p_{k,1} + \delta_k)$ is either convex in δ_k for any k or concave in δ_k for any k, and (iii) $\phi_k(p_{k,2}) - \phi_k(p_{k,1})$ is non-increasing in k for any $0 \leq p_{k,1} \leq p_{k,2}$. If $E[\delta_k|\varphi_k] = 0$ for any φ_k, then the threshold level $T_k^1(p_k, \epsilon_k)$ in Theorem 3.1 is non-increasing in p_k for any k and ϵ_k, while $T_k^2(p_k, \epsilon_k)$ is non-decreasing in p_k for any k and ϵ_k.*

On the other hand, the monotonicity properties also holds when $\phi_k(p_k)$ is non-increasing in p_k. In particular,

Corollary 3.5. *Consider the conditions in (C4), (C5) and (C6), and suppose that (i) $\phi_k(p_k)$ is non-increasing in p_k for any k, (ii) $\phi_k(p_{k,2}) - \phi_k(p_{k,1})$ is non-decreasing in k for any $0 \leq p_{k,1} \leq p_{k,2}$, and (iii) $E[\delta_k|\varphi_k] = 0$ for any k and φ_k.*

With the additional condition (iv) $\phi_{k+1}(p_{k,2} + \delta_k) - \phi_{k+1}(p_{k,1} + \delta_k)$ is concave in

δ_k for any $0 \le p_{k,1} \le p_{k,2}$ and k, then the threshold level $T_k^2(p_k, \epsilon_k)$ in Theorem 3.1 is non-increasing in p_k for any k and ϵ_k.

With the additional condition (v) $\phi_{k+1}(p_{k,2} + \delta_k) - \phi_{k+1}(p_{k,1} + \delta_k)$ is convex in δ_k for any $0 \le p_{k,1} \le p_{k,2}$ and k, then the threshold level $T_k^1(p_k, \epsilon_k)$ in Theorem 3.1 is non-decreasing in p_k for any k and ϵ_k.

With the additional conditions (C1), (C2), (C3) and either (iv) or (v), then the threshold level $T_k^1(p_k, \epsilon_k)$ in Theorem 3.1 is non-decreasing in p_k for any k and ϵ_k, while $T_k^2(p_k, \epsilon_k)$ is non-increasing in p_k for any k and ϵ_k.

Now consider the case that the spread ϵ_k is independent of p_k for any k, i.e., $\phi_k(p_k) = 0$ for any k and p_k. Note $\phi_k(p_k) = 0$ for any k and p_k satisfies all the conditions regariding to $\phi_k(p_k)$ in Proposition 3.7 as well as Corollaries 3.3 and 3.5. Therefore, the threshold levels must be both non-decreasing and non-increasing in p_k, i.e., they are independent of p_k, which agrees with the result in Corollary 3.1.

3.7 Extentions

We analyze the optimal inventory control policy in single-asset market-making for a mean-variance analysis model, which identifies the best trade-off between the inventory risk associated with the price uncertainty and the potential loss of spread corresponding to a change in market position because of unwanted inventory level. The optimality of a threshold policy is established, where the threshold levels can be computed using an algorithm linear in the number of periods. The symmetry and monotonicity of the threshold levels are also investigated.

Although our analysis is based on the assumption that the stochastic inputs are independent across different periods, the optimality of a threshold policy can be extended to the case when the random variables are auto-correlated for both the mean-variance tradeoff discussed in this chapter and the exponential utility model analyzed in Chapter 2. Suppose that the price movements δ_k, the orders from the clients s_k and d_k as well as the spread ϵ_k are correlated across k. Let us define the vector $\mathbf{h}_k$ representing all realized information before we make our decision at period

k, i.e,

$$\mathbf{h}_k = \{p_1, ..., p_k, \epsilon_1, ..., \epsilon_k, d_1, ..., d_{k-1}, s_1, ..., s_{k-1}\}.$$

Following the proofs in Sections 2.2 and 3.1, it is straightforward to establish that a history-dependent threshold policy is optimal, which means that the threshold levels $T_k^1(\mathbf{h}_k)$ and $T_k^2(\mathbf{h}_k)$ are functions of the history vector $\mathbf{h}_k$.

In particular, suppose that there is a stochastic process I_k, $k = 1, ..., N$, measuring the market state, e.g., we may consider four states of market: (i) low volume low volatility, (ii) low volume high volatility, (iii) high volume low volatility and (iv) high volume high volatility. The process I_k is auto-correlated and the autocorrelation of the stochastic inputs is solely determined by I_k, i.e., δ_k, s_k, d_k and ϵ_k conditional on I_k are independent in k. In this case, the threshold levels in each period k are functions of p_k, ϵ_k and I_k, which can be denoted by $T_k^1(p_k, \epsilon_k, I_k)$ and $T_k^2(p_k, \epsilon_k, I_k)$ respectively.

We also mentioned in the end of Section 2.1 that we can allow the decision maker to quote bid and ask prices different from the market leader, i.e., the decision maker trades actively with other market-makers based on the bid and ask prices $p_k^b = p_k - \epsilon_k$ and $p_k^a = p_k + \epsilon_k$, while the bid and ask prices the decision maker quotes to the clients are $\tilde{p}_k^b = p_k - \tilde{\epsilon}_k^b$ and $\tilde{p}_k^a = p_k + \tilde{\epsilon}_k^a$, where ϵ_k, $\tilde{\epsilon}_k^b$ and $\tilde{\epsilon}_k^a$ are positive random variables. Note that this modification preserves the convexity of the objective function in the Bellman equation and hence the optimality of the threshold policy still holds for both an exponential utility function and a mean-variance analysis model. The threshold levels are $T_k^1(p_k, \epsilon_k)$ and $T_k^2(p_k, \epsilon_k)$ if the prices quoted to the clients $\tilde{p}_k^b$ and $\tilde{p}_k^a$ are observed after the decision maker actively trades with other market-makers. Otherwise, $\tilde{p}_k^b$ and $\tilde{p}_k^a$ are observed before the active trading decision is made, and the threshold levels are functions of p_k, ϵ_k, $\tilde{\epsilon}_k^b$ and $\tilde{\epsilon}_k^a$.

Chapter 4

Multiple-Asset Market-Making with Mean-Variance Tradeoff

Chapter 3 considers the case that the market-maker only manages a single asset. In practice, market-makers may deal with multiple assets whose prices are correlated, which requires a multiple-asset inventory model hedging the price movements of different assets. In this chapter, we propose a mean-variance model to address this issue.

This chapter is organized as follows. First, we present a brief review of the related literatures in Section 4.1. The dynamic programming formulation is introduced in Section 4.2. In order to present the optimal policy for the multiple-asset model, we start with the simplified single-period model in Section 4.3 where the planning horizon only contains one period, and then in Section 4.4 we move onto the general model with multiple periods in the planning horizon. Finally, some extensions of the multiple-asset model are presented in Section 4.5.

4.1 Literature Review

One category of multiple-item inventory models in supply chain management is the economic warehouse lot scheduling problem (EWLSP), where the orders of different items are scheduled to minimize the cost while satisfying the warehouse capacity constraint. The strategic version of the EWLSP considers the warehouse capacity as a decision variable and minimizes an objective function including a cost component related to the warehouse capacity, e.g., the cost to lease the warehouse. A massive literature has been accumulated ever since Churchman et al. [14] introduced this problem. Simchi-Levi et al. [49] provide a detailed review.

Another line of researches in multiple-item inventory models studies jointly replenishment inventory models which explores the economies of scale to jointly replenish several items, i.e., it is possible to share the fixed ordering cost if a number of items are replenished simultaneously. The joint replenishment inventory models with deterministic demands usually adopt the EOQ assumptions for each item, and consider a fixed ordering cost for each replenishment, which is independent of the number of items ordered. Although the optimal solutions to these problems are very complex and difficult to compute, various heuristics have been developed in the literature. As for the joint replenishment inventory models with stochastic demands, a large number of works focus on the (S, c, s) policy, i.e., a replenishment is triggered when the inventory position of item i drops below s_i to raise the inventory level of item i upto S_i, and any item j whose inventory position is below c_j is also ordered upto S_j. Goyal and Satir [21] review the related literature for both the deterministic and stochastic models.

Recently, inventory models with substitutable products, especially those in the EOQ or newsvendor settings, have attracted considerable attention. For example, McGillivray and Silver [36] investigated the effects of substitutability for two products in the EOQ context, where substitution occurs when one product is out of stock. Parlar and Goyal [40] considered a single-period model with two substitutable products, where the substitution occurs with a constant probability if one product is in shortage, and the revenue is not affected by the substitution. Bassok et al. [6] studied a newsvendor problem with N substitutable products under a full downward substitution rule, i.e. excess demand for product i can be satisfied using product j for any $j \geq i$.

Note that the issues studied in multiple-item inventory models in supply chain management, i.e., shard warehouse capacity or fixed ordering cost as well as demand substitutions, are not applicable in the inventory problem in multiple-asset market-making. However, we still cannot decompose the multiple-asset market-making inventory control problem into several single-asset problems because the price movements of different assets are usually correlated. For example, if the price movements of two asset i and j are negatively correlated, positive inventory positions in both assets can hedge the price movements of these two assets to certain extent. The pioneering work by Markowitz [35] proposes the mean-variance model for portfolio selection which determines which assets and how much of each asset to hold in the portfolio in order to achieve a tradeoff between expected return and risk in price uncertainty under a budget constraint. The inventory control problem in multiple-asset market-making also determines how much inventory to hold for each asset, and hence it can be considered as an extension to the portfolio selection model in Markowitz [35].

4.2 Formulation

As in the single-asset model presented in Chapters 2 and 3, we consider a time horizon of one day and divide it into N discrete small time intervals. Suppose that the market-maker manages M assets. For any asset i, $i = 1, ..., M$, the sequence of events is the same as that in the single-asset model, and we use the following notations:

- $x_{k,i}$ the inventory position of asset i at the beginning of period k
- $p_{k,i}$ the market mid price of asset i in period k
- $q_{k,i}$ the amount of asset i the market-maker actively trades with other market-makers for inventory control purpose
- $s_{k,i}$ the amount of asset i the clients sell to the market-maker in period k
- $d_{k,i}$ the amount of asset i the clients buy from the market-maker in period k.

To simplify the notation, we let $\mathbf{x}_k$, $\mathbf{p}_k$, $\mathbf{q}_k$, $\mathbf{s}_k$ and $\mathbf{d}_k$ denote the vectors consisting of $x_{k,i}$, $p_{k,i}$, $q_{k,i}$, $s_{k,i}$ and $d_{k,i}$ respectively.

We introduce the random vector $\delta_k = [\delta_{k,1}, ..., \delta_{k,M}]$ to model the evolution of the market mid price, and hence the mid price at period $k+1$ is $\mathbf{p}_{k+1} = \mathbf{p}_k + \delta_k$. δ_k are assumed to be identical and independent random vectors in k, but the components in δ_k can be correlated. In addition, the market price process can be correlated with the client orders, i.e., $\mathbf{s}_k$, $\mathbf{d}_k$ and δ_k can be correlated. However, unlike the dynamics of the mid price considered in Chapters 2 and 3, here we assume that δ_k and $\mathbf{p}_k$ are independent for any k, i.e., the asset mid prices follow a random walk. The purpose of this assumption is mainly to simplify the notation in the analysis – we will discuss in Section 4.5 extensions to models where this assumption can be easily relaxed without changing the structure of the optimal control policy. Also note that Madhavan and Smidt [34] assumes the changes in price are independent and identical normal random variables with mean 0, which is a special case of our price model.

Similar to the single-asset model in Chapters 2 and 3, the bid and ask prices at period k for any market-maker are assumed to be $\mathbf{p}_k + \epsilon_k$ and $\mathbf{p}_k - \epsilon_k$ respectively. Here ϵ_k is a nonnegative vector, and each of its component $\epsilon_{k,i}$ denotes half of the bid/ask spread for asset i. Again, we restrict ϵ_k to be either a vector of constants or a known funciton of the mid price $\mathbf{p}_k$, i.e., $\epsilon_k = \phi_k(\mathbf{p}_k)$, in order to simplify the notation, and the relaxation of this assumption is discussion in Section 4.5.

As for the client orders $\mathbf{s}_k$ and $\mathbf{d}_k$, we allow correlation within the vectors $\mathbf{s}_k$ and $\mathbf{d}_k$, and the two vectors $\mathbf{s}_k$ and $\mathbf{d}_k$ can also be correlated. That is, the amount of asset i that the clients sell to / buy from the market maker at period k can be correlated with the client orders of asset j, where $j = 1, ..., M$ and it is possible for $j = i$. The vector $[\mathbf{s}_k, \mathbf{d}_k]$ is supposed to be independent across the time period k, and we also assume $[\mathbf{s}_k, \mathbf{d}_k]$ independent of $\mathbf{p}_k$ for any k. Note that the second assumption, independence between $[\mathbf{s}_k, \mathbf{d}_k]$ and $\mathbf{p}_k$, is for the purpose to simplify the notation and it can be easily relaxed (c.f. Section 4.5).

For any period k, the profit we obtain from the bid-ask spread is $(\mathbf{d}_k + \mathbf{s}_k)^T \epsilon_k$. Note that we trade $|\mathbf{q}_k|$ at the price quoted by other market-makers, and hence the transaction cost is $|\mathbf{q}_k|^T \epsilon_k$. In addition, the market-maker's inventory is subject to the risk of price uncertainty, and hence he may incurred a profit or loss of the amount

$(\mathbf{x}_k + \mathbf{q}_k - \mathbf{d}_k + \mathbf{s}_k)^T \delta_k$. As a result, the one-period profit in period k is

$$\pi_k = (\mathbf{x}_k + \mathbf{q}_k - \mathbf{d}_k + \mathbf{s}_k)^T \delta_k + (\mathbf{d}_k + \mathbf{s}_k - |\mathbf{q}_k|)^T \epsilon_k. \tag{4.1}$$

Similar to the single-asset model in Chapters 2 and 3, π_{N+1} denote our profit or loss at end of the planning horizon. We assume that

$$\pi_{N+1} = \sum_{i=1}^{M} v_i(x_{N+1,i}, \epsilon_{N+1,i}) \tag{4.2}$$

where $v_i(x_{N+1,i}, \epsilon_{N+1,i})$ is a concave function with respect to $x_{N+1,i}$ for any $i = 1, ..., M$. Note that here we consider ϵ_{N+1} to be a contant or a known function of $\mathbf{p}_{N+1}$. Therefore, π_{N+1} is well defined once $\mathbf{x}_{N+1}$ and $\mathbf{p}_{N+1}$ are given.

In this chapter, we focus on mean-variance analysis to cater for the risk-aversion in market-making, i.e., a term of the profit variance is subtracted from the risk-neutral objective function, and the resulted objective function is

$$\max_{q_k} \; E\left[\sum_{k=1}^{N} \left\{E\left[\pi_k|\mathbf{p}_k\right] - \lambda \times Var\left(\pi_k|\mathbf{p}_k\right)\right\} + \pi_{N+1}\right], \tag{4.3}$$

which is an immediate extension of the mean-variance objective function for the single-asset model defined in (3.1). It is straight forward that the state variables are the inventory position $\mathbf{x}_k$ and the market mid price $\mathbf{p}_k$, and the decision variable is the quantity to adjust the inventory by active trading $\mathbf{q}_k$.

Let us consider the expectation and variance of the one period profit conditional on the market mid price $\mathbf{p}_k$ for any $k = 1, ..., N$. To simplify the notation, we define $\mathbf{S}_k = \mathbf{s}_k + \mathbf{d}_k$, $\Delta_k = \mathbf{s}_k - \mathbf{d}_k$ and Σ_k to be the variance-covariance matrix of δ_k. Without loss of generality, we let the diagonal components of Σ_k be all ones, which can be easily obtained by rescaling the units of the assets. According to (4.1), it is easy to establish that

$$
\begin{aligned}
E[\pi_k \mid \mathbf{p}_k] &= (\mathbf{x}_k + \mathbf{q}_k)^T E[\delta_k] - |\mathbf{q}_k|^T \epsilon_k + E[\Delta_k^T \delta_k] + E[\mathbf{S}_k]^T \epsilon_k \\
Var(\pi_k \mid \mathbf{x}_k, \mathbf{p}_k) &= (\mathbf{x}_k + \mathbf{q}_k)^T \Sigma_k (\mathbf{x}_k + \mathbf{q}_k) + Var(\Delta_k^T \delta_k) + \epsilon^T Var(\mathbf{S}_k)\epsilon_k \\
&\quad + 2(\mathbf{x}_k + \mathbf{q}_k)^T E[(\delta_k - E[\delta_k])(\Delta_k^T \delta_k - E[\Delta_k^T \delta_k])] \\
&\quad + 2(\mathbf{x}_k + \mathbf{q}_k)^T E[(\delta_k - E[\delta_k])(\mathbf{S}_k - E[\mathbf{S}_k])^T \epsilon_k] \\
&\quad + 2E[(\Delta_k^T \delta_k - E[\Delta_k^T \delta_k])(\mathbf{S}_k - E[\mathbf{S}_k])^T \epsilon_k].
\end{aligned}
$$

Note that the terms $E[\Delta_k^T \delta_k] + E[\mathbf{S}_k]^T \epsilon_k$ in $E[\pi_k \mid \mathbf{x}_k, \mathbf{q}_k]$ and

$$
Var(\Delta_k^T \delta_k) + \epsilon^T Var(\mathbf{S}_k)\epsilon_k + 2E[(\Delta_k^T \delta_k - E[\Delta_k^T \delta_k])(\mathbf{S}_k - E[\mathbf{S}_k])^T \epsilon_k]
$$

in $Var(\pi_k \mid \mathbf{x}_k, \mathbf{q}_k)$ are independent of the state and decision variables. Therefore, defining $\mathbf{L}_k = \mathbf{x}_k + \mathbf{q}_k$ and

$$
\gamma_k = -E[\delta_k] + 2\lambda E[(\delta_k - E[\delta_k])(\Delta_k^T \delta_k - E[\Delta_k^T \delta_k])] + 2\lambda E[(\delta_k - E[\delta_k])(\mathbf{S}_k - E[\mathbf{S}_k])^T \epsilon_k],
$$

our objective function in (4.3) is equivalent to

$$
\min_{\mathbf{L}_k} \; E\left[\sum_{k=1}^{N} \left\{\lambda \mathbf{L}_k^T \Sigma_k \mathbf{L}_k + \gamma_k^T \mathbf{L}_k + |\mathbf{L}_k - \mathbf{x}_k|^T \epsilon_k\right\} - \pi_{N+1}\right], \tag{4.4}
$$

In the remaining part of this chapter, we first analyze properties of the optimal solutions to (4.4) for the single-period model, i.e., when $N = 1$, and then extend the results to the multiple-period model.

4.3 Single-Period Multiple-Asset Model

When $N = 1$ and $\pi_{N+1} = 0$, the objective function in (4.4) is reduced to

$$
\min_{\mathbf{L}} \; E\left[\lambda \mathbf{L}^T \Sigma \mathbf{L} + \gamma^T \mathbf{L} + |\mathbf{L} - \mathbf{x}|^T \epsilon\right].
$$

Since the absolute value functions are convex, it is equivalent to

$$\min_{\mathbf{L},\mathbf{z}} \left\{ E\left[\lambda \mathbf{L}^T \Sigma \mathbf{L} + \gamma^T \mathbf{L} + \mathbf{z}^T \epsilon\right] : \mathbf{z} \geq \mathbf{L} - \mathbf{x}, \mathbf{z} \geq \mathbf{x} - \mathbf{L} \right\}. \tag{4.5}$$

Note that Σ is a variance-covariance matrix, and hence it is positive semi-definite. Therefore, (4.5) is a convex quadratic optimization problem subject to linear constraint. It follows directly that the KKT condition is the sufficient and necessary condition for optimal solutions. Let α and β denote the Lagrangian multipliers of the constraints (4.5). The KKT condition reads

$$\begin{aligned} 2\lambda\Sigma\mathbf{L} + \gamma + \alpha - \beta &= 0 \\ \epsilon - \alpha - \beta &= 0 \\ \mathbf{z} &\geq \mathbf{L} - \mathbf{x} \\ \mathbf{z} &\geq \mathbf{x} - \mathbf{L} \\ \alpha_i(L_i - z_i - x_i) &= 0 \qquad \forall\, i = 1, ..., M \\ \beta_i(L_i + z_i - x_i) &= 0 \qquad \forall\, i = 1, ..., M \\ \alpha, \beta &\geq 0. \end{aligned} \tag{4.6}$$

Also note that $\mathbf{z} = |\mathbf{L} - \mathbf{x}|$ in any optimal solution to (4.5), and therefore the KKT condition in (4.6) is equivalent to

$$2\lambda\Sigma\mathbf{L} + \gamma = \epsilon - 2\alpha \text{ where } \begin{cases} \alpha_i = 0 & \text{if } L_i < x_i \\ \alpha_i = \epsilon_i & \text{if } L_i > x_i \\ 0 \leq \alpha_i \leq \epsilon_i & \text{if } L_i = x_i \end{cases} \text{ for } i = 1, ..., M.$$

Here ϵ_i is the ith component in the vector ϵ. Let Σ_i denote the ith row of the variance-covariance matrix Σ and γ_i be the ith component in γ. It follows directly that the optimal inventory control policy is as the follows.

Theorem 4.1. *Under single-period mean-variance analysis, there exists a parallelotropic no-trade region defined by* $R = \{\mathbf{x} : -\epsilon \leq 2\lambda\Sigma\mathbf{x} + \gamma \leq \epsilon\}$, *where* ϵ *can be a given vector or a given function with respect to the market price* $\mathbf{p}$.

Outside the no-trade region, the inventory is adjusted to the boundary of the no-trade region. Let $\mathbf{L}^*$ *denote the optimal adjusted inventory level. For any asset* i, $i = 1, ..., M$, *the optimal adjusted inventory level* L_i^* *satisfies the following conditions:* $x_i > L_i^*$ *iff* $2\lambda\Sigma_i\mathbf{L}^* + \gamma_i = \epsilon_i$, $x_i < L_i^*$ *iff* $2\lambda\Sigma_i\mathbf{L}^* + \gamma_i = -\epsilon_i$.

For each asset i, Theorem 4.1 indicates that the no-trade region is defined by an upper limit $2\lambda\Sigma_i\mathbf{x} + \gamma_i = \epsilon_i$ and an lower limit $2\lambda\Sigma_i\mathbf{x} + \gamma_i = -\epsilon_i$. These two limits are parallel straight lines. Consider any two assets i, j where $i \neq j$. We can fix the components in $\mathbf{x}$ except for x_i and x_j, and focus on the (x_i, x_j) plane. Let ρ_{ij} denote the correlation coefficient between δ_i and δ_j. Since we choose the units of assets so that the variances of δ_i and δ_j are both ones, the limits defining the no-trade region on the (x_i, x_j) plane is parrelel to the line

$$2\lambda x_i + 2\lambda\rho x_j = 0, \text{ which is equivalent to } x_i = -\rho x_j.$$

If we interpret the limits of x_i on the (x_i, x_j) plane as funcitons in x_j, the slope of the limits is alway flatter than the 45° line as $|\rho| \leq 1$.

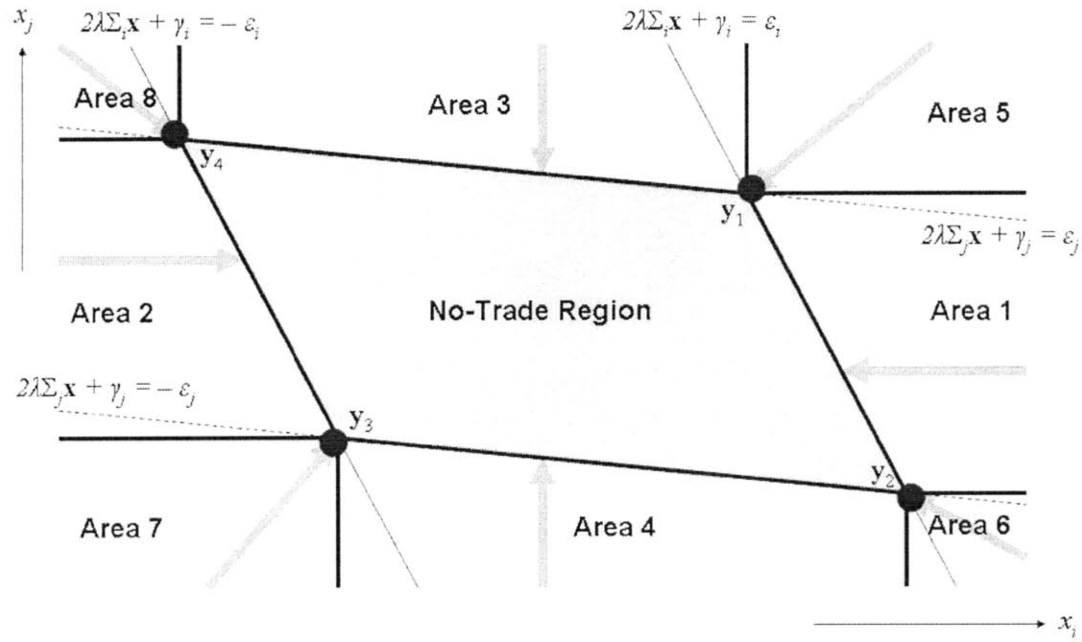
In addition, when the inventory position is not contained in the no-trade region, Theorem 4.1 implies that the optimal solution only allows two types of inventory adjustment: for any asset i, $i = 1, ..., M$,

- increase the inventory of asset i to hit the lower limit of asset i on the boundary of no-trade region

- decrease the inventory of asset i to hit the upper limit of asset i on the boundary of no-trade region.

Obviously, the optimal policy is reduced to a threshold policy if we only have one asset in the portfolio. Theorem 4.1 with two assets is illustrated in Figure 4-1. The thin solid lines and the dash lines correspond to the limits for asset i and j respectively. Hence, the no-trade region, which corresponds to the intersection of the area between asset i's limits and asset j's limits, is the parallelogram defined by the four vertices $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and $\mathbf{y}_4$. The area outside the no-trade region is divided into 8

Figure 4-1: Illustration of Optimal Solution for Single-Period Model



subareas by the bold lines. The adjustment of inventory is indicated by the arrows. More specifically,

- if the inventory is in Area 1 (Area 2, resp.), the inventory of asset i is decreased (increased, resp.) and the inventory of asset j is unchanged so that the inventory after adjustment lies on the straight line between $\mathbf{y}_1$ and $\mathbf{y}_2$ ($\mathbf{y}_3$ and $\mathbf{y}_4$, resp.)

- if the inventory is in Area 3 (Area 4, resp.), the inventory of asset j is decreased (increased, resp.) and the inventory of asset i is unchanged so that the inventory after adjustment lies on the straight line between $\mathbf{y}_1$ and $\mathbf{y}_4$ ($\mathbf{y}_2$ and $\mathbf{y}_3$, resp.)

- if the inventory is in Area 5 (Area 6, 7, 8 resp.), the inventory of both assets are modified so that the inventory after adjustment is $\mathbf{y}_1$ ($\mathbf{y}_2$, $\mathbf{y}_3$, $\mathbf{y}_4$, resp.).

4.4 Multiple-Period Multiple-Asset Model

When $N > 1$, we define $J_k(\mathbf{x}_k, \mathbf{p}_k)$ to be the optimal mean-variance value when the initial inventory at period k is $\mathbf{x}_k$, the price at period k is $\mathbf{p}_k$ and we act optimally from period k onwards, i.e.,

$$J_k(\mathbf{x}_k, \mathbf{p}_k) = \min_{\mathbf{L}_l} \left\{ E\left[\sum_{l=k}^{N} \left\{ \lambda \mathbf{L}_l^T \Sigma_l \mathbf{L}_l + \gamma_l^T \mathbf{L}_l + |\mathbf{L}_l - \mathbf{x}_l|^T \epsilon_l \right\} \right] \right\}.$$

It follows directly that the dynamic programming model in (4.4) has the following Bellman's equation

$$J_k(\mathbf{x}_k, \mathbf{p}_k) = \min_{\mathbf{L}_k} \left\{ \lambda \mathbf{L}_k^T \Sigma_k \mathbf{L}_k + \gamma_k^T \mathbf{L}_k + |\mathbf{L}_k - \mathbf{x}_k|^T \epsilon_k + E[J_{k+1}(\mathbf{L}_k + \Delta_k, \mathbf{p}_k + \delta_k)] \right\} \tag{4.7}$$

where $k = 1, ...N$, and $J_{N+1}(\mathbf{x}_{N+1}, \mathbf{p}_{N+1}) = -\pi_{N+1}$. Note that we assume that $J_{N+1}(\mathbf{x}_{N+1}, \mathbf{p}_{N+1}) = -\pi_{N+1} = 0$ in this section. However, all the results can be established for π_{N+1} defined in (4.2) where $v_i(x_{N+1,i}, \epsilon_{N+1,i})$ is concave in $x_{N+1,i}$ for any $i = 1, ..., M$.

In this part, we establish the structural property of optimal solutions by induction. The induction assumptions and proofs are presented after we discuss the optimal policy.

First, let us consider the notations used to characterize the optimal inventory control policy. We use $x_{k,i}$, $\delta_{k,i}$, $\epsilon_{k,i}$ and $\gamma_{k,i}$ to denote the elements corresponding to asset i in the vectors $\mathbf{x}_k$, δ_k, ϵ_k and γ_k. The row in Σ_k corresponding to asset i is denoted by $\Sigma_{k,i}$. Let $\rho_{k,ij}$ denote the correlation coefficient between $\delta_{k,i}$ and $\delta_{k,j}$ for any two assets i, j where $i \neq j$. Moreover, we let $\nabla_i J_k(\mathbf{x}, \mathbf{p}) = \partial J_k(\mathbf{x}, \mathbf{p}) / \partial x_{k,i}$ and $\nabla J_k(\mathbf{x}, \mathbf{p})$ be the vector consisting of $\nabla_i J_k(\mathbf{x}, \mathbf{p})$.

The following theorem presents the optimal policy for the multiple-period model.

Theorem 4.2. *Under mean-variance analysis, for any $k = 1, ..., N$, there exists a* *connected no-trade region without holes defined by the following inequality*

$$R_k = \{\mathbf{x} : -\epsilon_k \leq 2\lambda\Sigma_k\mathbf{x}_k + \gamma_k + E[\nabla J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)] \leq \epsilon_k\},$$

where ϵ_k can be a given vector or a given function with respect to the market price $\mathbf{p}_k$.

Outside the no-trade region, the inventory is adjusted to the boundary of the no-trade region. Let $\mathbf{L}^*$ denote the optimal adjusted inventory level. For any asset i, $i = 1, ..., M$, the optimal adjusted inventory level L_i^* satisfies the following conditions:

- $x_{k,i} > L_{k,i}^*$ *iff* $2\lambda\Sigma_{k,i}\mathbf{L}_k^* + \gamma_{k,i} + E[\nabla_i J_{k+1}(\mathbf{L}_k^* + \Delta_k, \mathbf{p}_k + \delta_k)] = \epsilon_{k,i}$,
- $x_{k,i} < L_{k,i}^*$ *iff* $2\lambda\Sigma_{k,i}\mathbf{L}_k^* + \gamma_{k,i} + E[\nabla_i J_{k+1}(\mathbf{L}_k^* + \Delta_k, \mathbf{p}_k + \delta_k)] = -\epsilon_{k,i}$.

Similar to the single-period model, again we can consider the no-trade region as imposing an upper limit

$$\max\left\{x_{k,i} : 2\lambda\Sigma_{k,i}\mathbf{x}_k + \gamma_{k,i} + E[\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)] = \epsilon_{k,i} \text{ given } x_{k,j}\ \forall j \neq i\right\}$$

and an lower limit

$$\min\left\{x_{k,i} : 2\lambda\Sigma_{k,i}\mathbf{x}_k + \gamma_{k,i} + E[\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)] = -\epsilon_{k,i} \text{ given } x_{k,j}\ \forall j \neq i\right\}$$

for the inventory of asset i. Outside the no-trade region, we trade actively to adjust our inventory. The optimal adjustment is the same as the single-period model: for any asset i, $i = 1, ..., M$,

- increase the inventory of asset i to hit the lower limit of asset i on the boundary of no-trade region
- decrease the inventory of asset i to hit the upper limit of asset i on the boundary of no-trade region.

Note that for the single-period model, the upper and lower limits of each asset are hyperplanes, which does not hold for the multiple-period model. To characterize the properties of the limits in the multi-period model, we focus on the $(x_{k,i}, x_{k,j})$ plane for any assets i and j by fixing the inventory level of the rest assets in the portfolio.

Proposition 4.1. *Consider any pair of asset i and j, $i \neq j$. For any inventory level of the assets other than asset i and j, the upper and lower limits of asset i on the $(x_{k,i}, x_{k,j})$ plane are functions in $x_{k,j}$ denoted by $u(x_{k,j})$ and $l(x_{k,j})$.*

If $\rho_{k,ij} \geq 0$, $u(x_{k,j})$ and $l(x_{k,j})$ are continuous non-increasing functions. For any $c > 0$, $u(x_{k,j}) + c \geq u(x_{k,j} - c)$ and $l(x_{k,j}) + c \geq l(x_{k,j} - c)$.

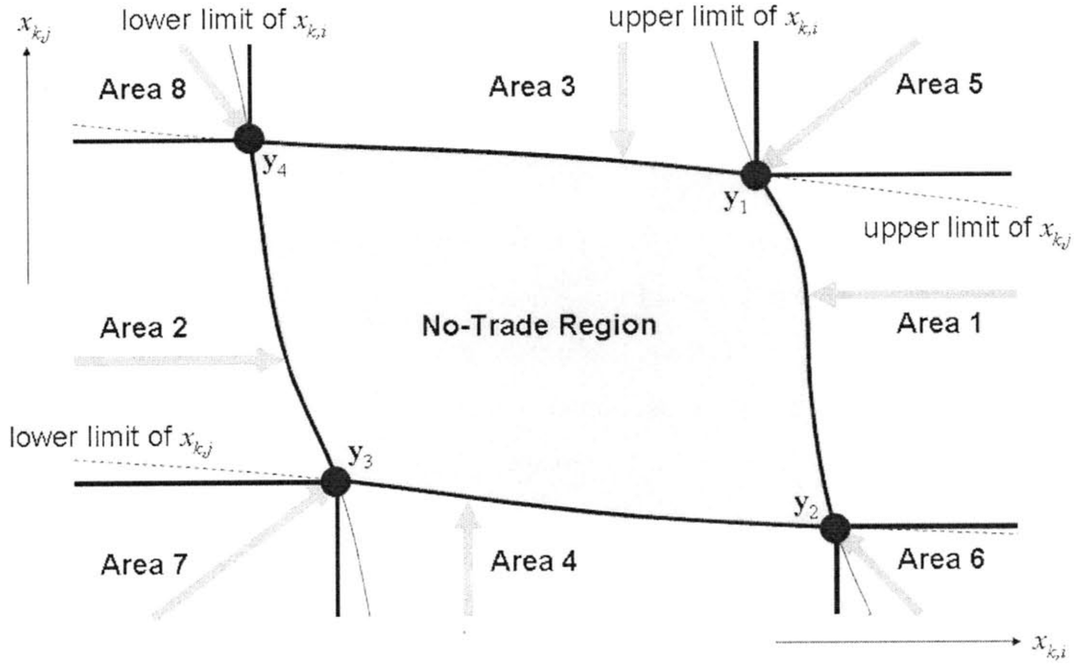
If $\rho_{k,ij} \leq 0$, $u(x_{k,j})$ and $l(x_{k,j})$ are continuous non-decreasing functions. For any $c > 0$, $u(x_{k,j}) + c \geq u(x_{k,j} + c)$ and $l(x_{k,j}) + c \geq u(x_{k,j} + c)$.

Note that the relationship between $u(x_{k,j}), l(x_{k,j})$ and $u(x_{k,j} + c), l(x_{k,j} + c)$ essentially says that $u(x_{k,j})$ and $l(x_{k,j})$ are flatter than the 45° line. Also, when $\rho_{k,ij} = 0$, the limits are both monotone non-increasing and non-decreasing, which implies that they are constants, i.e., the no-trade region on the $(x_{k,i}, x_{k,j})$ plane (if exists) is a rectangle. This property agrees with the single-asset result, which implies that the no-trade region is a hyperrectangle if the price movements of all assets in the portfolio are independent.

Again, Theorem 4.2 implies that the optimal policy is reduced to a threshold policy when the market-maker only manages a single asset since a connected region in one demention is an interval. Figure 4-2 illustrates Theorem 4.2 and Proposition 4.1 when the market price movements of these two assets are positively correlated. Similar to Figure 4-1, the thin solid lines and the dash lines correspond to the limits for asset i and j respectively, and the no-trade region is the shaded area with corner points $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and $\mathbf{y}_4$. The area outside the no-trade region is divided into 8 subareas by the bold lines. The adjustment of inventory is indicated by the arrows. More specifically,

- if the inventory is in Area 1 (Area 2, resp.), the inventory of asset i is decreased (increased, resp.) and the inventory of asset j is unchanged so that the inventory after adjustment lies on the section of the upper limit of asset i between $\mathbf{y}_1$ and $\mathbf{y}_2$ (the lower limit of asset i between $\mathbf{y}_3$ and $\mathbf{y}_4$, resp.)

Figure 4-2: Illustration of Optimal Solution for Multiple-Period Model



- if the inventory is in Area 3 (Area 4, resp.), the inventory of asset j is decreased (increased, resp.) and the inventory of asset i is unchanged so that the inventory after adjustment lies on the upper limit of asset j between $\mathbf{y}_1$ and $\mathbf{y}_4$ (the lower limit of asset j between $\mathbf{y}_2$ and $\mathbf{y}_3$, resp.)

- if the inventory is in Area 5 (Area 6, 7, 8 resp.), the inventory of both assets are modified so that the inventory after ajustment is $\mathbf{y}_1$ ($\mathbf{y}_2$, $\mathbf{y}_3$, $\mathbf{y}_4$, resp.).

In order to prove Theorem 4.2 and Proposition 4.1, we consider the following induction assumptions.

Assumption 4.1. The function $J_k(\mathbf{x}_k, \mathbf{p}_k)$ is convex and continuously differentiable in $\mathbf{x}_k$.

Assumption 4.2. For any asset i, $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k)$ is monotone non-decreasing in $x_{k,j}$ for any asset j such that $\rho_{k,ij} \geq 0$, and it is monotone non-increasing in $x_{k,j}$ for any asset j such that $\rho_{k,ij} \leq 0$.

Assumption 4.3. Consider any asset $i = 1, ..., M$ and $c > 0$.

For any asset j such that $\rho_{k,ij} \geq 0$, $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ where $x_{k,i}^c = x_{k,i} + c$, $x_{k,j}^c = x_{k,j} - c$ and the rest components in $\mathbf{x}_k^c$ are the same as those in $\mathbf{x}_k$.

For any asset j such that $\rho_{k,ij} \leq 0$, $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ where $x_{k,i}^c = x_{k,i} + c$, $x_{k,j}^c = x_{k,j} + c$ and the rest components in $\mathbf{x}_k^c$ are the same as those in $\mathbf{x}_k$.

Note that $J_{N+1}(\mathbf{x}_{N+1}, \mathbf{p}_{N+1}) = 0$ satisfies all induction assumptions. It is sufficient to i) prove that Theorem 4.2 and Proposition 4.1 for period k when they are true for period $k+1$ and the induction assumptions hold for period $k+1$, and ii) prove $J_k(\mathbf{x}_k, \mathbf{p}_k)$ satisfies the induction assumptions.

First, let us prove Theorem 4.2 for period k assuming that Assumption 4.1 and 4.2 hold for period $k+1$.

Proof of Theorem 4.2. Assumption 4.1 implies that $E[J_{k+1}(\mathbf{x}_k+\Delta_k, \mathbf{p}_k+\delta_k)]$ is convex in $\mathbf{x}_k$. According to the monotone convergence theorem, we have

$$\frac{\partial}{\partial x_i} E[J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)] = E[\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)]$$

for any $i = 1, ..., M$. Therefore, $E[J_{k+1}(\mathbf{x}_k+\Delta_k, \mathbf{p}_k+\delta_k)]$ is continuously differentiable in $\mathbf{x}_k$. Applying the same argument as in the single-period model to the Bellman's equation in (4.7), we obtain the inequalities defining the no-trade region R_k as well as the optimal rules to adjust the inventory outside the no-trade region.

To simplify the notation, let us define

$$f_i(\mathbf{x}_k) = 2\lambda\Sigma_{k,i}\mathbf{x}_k + \gamma_{k,i} + E[\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)].$$

Assumption 4.1 implies that $\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)$ is non-decreasing in $x_{k,i}$. Assumption 4.2 assumes that $\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)$ is monotone in any $x_{k,j}$ where $j \neq i$. Note that δ_k are iid random vectors and hence $\rho_{k,ij} = \rho_{k+1,ij}$ for any i, j. Therefore, $\Sigma_{k,i}\mathbf{x}_k$ has the same monotonicity in $x_{k,j}$ as $\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)$, where $j = 1, ..., M$. It follows directly that $f_i(\mathbf{x}_k)$ is also monotone in $x_{k,j}$ for any $j = 1, ..., M$.

Also note that $R_k = \{\mathbf{x}_k : -\epsilon_{k,i} \leq f_i(\mathbf{x}_k) \leq \epsilon_{k,i} \ \forall i = 1, ..., M\}$. Since $f_i(\mathbf{x}_k)$ is monotone in $x_{k,j}$ for any $i, j = 1, ..., M$, R_k is connected without holes. $\square$

Next, we prove that Proposition 4.1 holds for period k under Assumption 4.1, 4.2 and 4.3.

Proof of Proposition 4.1. Here we assume that $\rho_{k,ij} \geq 0$ and consider the upper limit. The results for the lower limit when $\rho_{k,ij} \geq 0$ as well as the case when $\rho_{k,ij} \leq 0$ follow from the same argument.

Let $\mathbf{x}_{k,-ij}$ denote the inventory level of the assets other than asset i and j. The upper limit can be defined as

$$u(x_{k,j}) = \max\left\{x_{k,i} : f_i(\mathbf{x}_k) = \epsilon_{k,i} \text{ given } x_{k,j} \text{ and } \mathbf{x}_{k,-ij}\right\}.$$

It follows directly that $u(x_{k,j})$ is continuous as $f_i(\mathbf{x}_k)$ is continuous by Assumption 4.1.

To prove the monotonicity of $u(x_{k,j})$, let us consider $x_j^1 < x_j^2$. Let $\mathbf{x}^1$ be the vector such that the ith component is $u(x_j^1)$, the jth component is x_j^1 and the rest components are set to $\mathbf{x}_{k,-ij}$. According to the definition of $u(x_{k,j})$, we know that $f_i(\mathbf{x}^1) = \epsilon_{k,i}$. Let $\mathbf{x}^2$ denote the vector such that the jth component is x_j^2 and the rest components are equal to that of $\mathbf{x}^1$. The proof of Theorem 4.2 indicates that $f_i(\mathbf{x}_k)$ is non-decreasing in $x_{k,j}$. Therefore, we have $f_i(\mathbf{x}^2) \geq f_i(\mathbf{x}^1) = \epsilon_{k,i}$. Note that we also prove that $f_i(\mathbf{x}_k)$ is non-decreasing in $x_{k,i}$ in the proof of Theorem 4.2. It follows directly that $u(x_j^1) \geq u(x_j^2)$, i.e., $u(x_{k,j})$ is non-increasing in $x_{k,j}$.

Let $\mathbf{x}_k$ be the vector such that the ith component is $u(x_{k,j})$, the jth component is $x_{k,j}$ and the rest components are equal to $\mathbf{x}_{k,-ij}$. For any $c > 0$, define $\mathbf{x}_k^c$ such that the ith component is $u(x_{k,j}) + c$, the jth component is $x_{k,j} - c$ and the rest elements are $\mathbf{x}_{k,-ij}$. Assumption 4.3 implies that

$$E[\nabla_i J_{k+1}(\mathbf{x}_k + \Delta_k, \mathbf{p}_k + \delta_k)] \leq E[\nabla_i J_{k+1}(\mathbf{x}_k^c + \Delta_k, \mathbf{p}_k + \delta_k)].$$

Also note that

$$\Sigma_{k,i}\mathbf{x}_k - \Sigma_{k,i}\mathbf{x}_k^c = u(x_{k,j}) + \rho_{k,ij}x_{k,j} - (u(x_{k,j}) + c) - \rho_{k,ij}(x_{k,j} - c) = (\rho_{k,ij} - 1)c \leq 0.$$

It follows directly that $f_i(\mathbf{x}_k) \leq f_i(\mathbf{x}_k^c)$. Note that $f_i(\mathbf{x}_k) = \epsilon_{k,i}$ by definition. Since $f_i(\mathbf{x}_k)$ is non-decreasing in $x_{k,i}$, we have $u(x_{k,j}) + c \geq u(x_{k,j} - c)$. □

To complete the induction proof, it remains to show that the function $J_k(\mathbf{x}_k, \mathbf{p}_k)$ satisfies all the induction assumptions.

Proposition 4.2. *Suppose that $J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$ has all the properties specified in Assumption 4.1, 4.2 and 4.3, then $J_k(\mathbf{x}_k, \mathbf{p}_k)$ also satisfies Assumption 4.1, 4.2 and 4.3.*

Proof. To prove the convexity of $J_k(\mathbf{x}_k, \mathbf{p}_k)$, let us consider any $\mathbf{x}^1$, $\mathbf{x}^2$ and $\kappa \in (0, 1)$. Let $\mathbf{x} = \kappa\mathbf{x}^1 + (1-\kappa)\mathbf{x}^2$. Given the market price $\mathbf{p}_k$, suppose that $\mathbf{L}^1$ and $\mathbf{L}^2$ are optimal solutions to the optimization problem in (4.7) corresponding to $\mathbf{x}^1$ and $\mathbf{x}^2$ respectively, i.e.,

$$\begin{aligned} J_k(\mathbf{x}^1, \mathbf{p}_k) &= \lambda\mathbf{L}^{1T}\Sigma_k\mathbf{L}^1 + \gamma_k\mathbf{L}^1 + |\mathbf{L}^1 - \mathbf{x}^1|^T\epsilon_k + E[J_{k+1}\left(\mathbf{L}^1 + \Delta_k, \mathbf{p}_k + \delta_k\right)] \\ J_k(\mathbf{x}^2, \mathbf{p}_k) &= \lambda\mathbf{L}^{2T}\Sigma_k\mathbf{L}^2 + \gamma_k\mathbf{L}^2 + |\mathbf{L}^2 - \mathbf{x}^2|^T\epsilon_k + E[J_{k+1}\left(\mathbf{L}^2 + \Delta_k, \mathbf{p}_k + \delta_k\right)]. \end{aligned} \tag{4.8}$$

Let $\mathbf{L} = \kappa\mathbf{L}^1 + (1-\kappa)\mathbf{L}^2$. It follows directly from the definition of $J_k(\mathbf{x}, \mathbf{p}_k)$ that

$$J_k(\mathbf{x}, \mathbf{p}_k) \leq \lambda\mathbf{L}^T\Sigma_k\mathbf{L} + \gamma_k\mathbf{L} + |\mathbf{L} - \mathbf{x}|^T\epsilon_k + E[J_{k+1}\left(\mathbf{L} + \Delta_k, \mathbf{p}_k + \delta_k\right)]. \tag{4.9}$$

Note that for each asset i, we have

$$|L_i - x_i| = |(\kappa L_i^1 + (1-\kappa)L_i^2) - (\kappa x_i^1 + (1-\kappa)x_i^2)| \leq \kappa|L_i^1 - x_i^1| + (1-\kappa)|L_i^2 - x_i^2|,$$

and hence

$$|\mathbf{L} - \mathbf{x}|^T\epsilon_k \leq \kappa|\mathbf{L}^1 - \mathbf{x}^1|^T\epsilon_k + (1-\kappa)|\mathbf{L}^2 - \mathbf{x}^2|^T\epsilon_k.$$

Also note that Σ_k is positive semi-definite and $J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$ is convex in $\mathbf{x}_k$. According to (4.8) and (4.9), we have

$$J_k(\mathbf{x}, \mathbf{p}_k) \le \kappa J_k(\mathbf{x}^1, \mathbf{p}_k) + (1-\kappa) J_k(\mathbf{x}^2, \mathbf{p}_k),$$

and hence $J_k(\mathbf{x}_k, \mathbf{p}_k)$ is convex in $\mathbf{x}_k$.

To prove the differentiability of $J_k(\mathbf{x}_k, \mathbf{p}_k)$, let us focus on the two-dimensional plane $(x_{k,i}, x_{k,j})$ by fixing the inventory of the assets other than asset i and j. According to the definition of the first derivative, it is straightforward to obtain $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k)$ in Figure 4-2.

- $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = -c_{k,i}$ if $\mathbf{x}_k$ is in Area 2, 7 and 8.
  $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = \epsilon_{k,i}$ if $\mathbf{x}_k$ is in Area 1, 5 and 6.
  $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = f_i(\mathbf{x}_k)$ if $\mathbf{x}_k$ is in the no-trade region.

- For any $\mathbf{x}_k$ in Area 3, define $\mathbf{x}_k^c$ such that $x_{k,j}^c = x_{k,j} + c$ and $x_{k,m}^c = x_{k,m}$ for any $m \neq j$. If $\mathbf{x}_k^c$ is also in Area 3, then $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$. In other words, for any two points in Area 3, if they only differ in the component corresponding to asset j, they have the same first derivative of $J_k(\mathbf{x}_k, \mathbf{p}_k)$ with respect to the inventory of asset i.

- Similarly, for any $\mathbf{x}_k$ in Area 4, define $\mathbf{x}_k^c$ such that $x_{k,j}^c = x_{k,j} + c$ and $x_{k,m}^c = x_{k,m}$ for any $m \neq j$. If $\mathbf{x}_k^c$ is also in Area 4, then $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$.

Since $f_i(\mathbf{x}_k)$ is continuous, it follows directly that $J_k(\mathbf{x}_k, \mathbf{p}_k)$ is continuously differentiable with respect to $x_{k,i}$ on the plane $(x_{k,i}, x_{k,j})$. Similar results can be obtained for the case when the plane $(x_{k,i}, x_{k,j})$ does not intersect with the no-trade region. Because this property holds for any asset i and j, we conclude that $J_k(\mathbf{x}_k, \mathbf{p}_k)$ is continuously differentiable in $\mathbf{x}_k$.

Note that we establish the monotonicity of $f_i(\mathbf{x}_k)$ in the proof of Theorem 4.2. Assumption 4.2 can be verified using the properties of $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k)$ in the previous part to prove continuous differentiability as well as the inequalities to define the no-trade region.

Now let us prove the validity of Assumption 4.3. We consider $\rho_{k,ij} \geq 0$ and restrict to the case illustrated in Figure 4-2. For any $\mathbf{x}_k$ and c, we define $\mathbf{x}_k^c$ such that $x_{k,i}^c = x_{k,i} + c$, $x_{k,j}^c = x_{k,j} - c$ and the rest elements in $\mathbf{x}_k^c$ are the same as those in $\mathbf{x}_k$. Assumption 4.3 holds as long as we can prove that $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ for any $c > 0$.

- If $\mathbf{x}_k$ is in Area 2, 7 and 8,$\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = -\epsilon_{k,i}$ which is the minimum value of $J_k(\mathbf{x}_k, \mathbf{p}_k)$, and hence $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ for any $c > 0$.

- If $\mathbf{x}_k$ is in Area 1, 5 and 6,Proposition 4.1 implies that $\mathbf{x}_k^c$ is also in Area 1, 5, and 6 for any $c > 0$. Therefore, $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k) = \epsilon_{k,i}$ for any $c > 0$.

- If $\mathbf{x}_k$ is in Area 4, Proposition 4.1 shows $\mathbf{x}_k^c$ must be in Area 4 or Area 6. It is sufficient to consider the case that $\mathbf{x}_k^c$ is also in Area 4 as $\nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ achieves its maximum in Area 6. Consider $\mathbf{x}'_k$ such that $x'_{k,j} = x_{k,j} - c$ and $x'_{k,m} = x_{k,m}$ for any $m \neq j$. We know that $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = \nabla_i J_k(\mathbf{x}'_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ where the equality follows from the results when proving the differentiability of $J_k(\mathbf{x}_k, \mathbf{p}_k)$, and the inequality is obtained from the convexity of $J_k(\mathbf{x}_k, \mathbf{p}_k)$.

- If $\mathbf{x}_k$ is in the no-trade region, we know that $\mathbf{x}_k^c$ must be in Area 1, 4, 6 or the no-trade region by Proposition 4.1. Again, it is sufficient to consider that $\mathbf{x}_k^c$ is also in the no-trade region. In this case, $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) = f_i(\mathbf{x}_k)$ and $\nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k) = f_i(\mathbf{x}_k^c)$. Similar to the arguement in the proof of Proposition 4.1, we can show that $f_i(\mathbf{x}_k) \leq f_i(\mathbf{x}_k^c)$, and hence $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$.

- If $\mathbf{x}_k$ is in Area 3, similarly, it is sufficient to show $\nabla_i J_k(\mathbf{x}_k, \mathbf{p}_k) \leq \nabla_i J_k(\mathbf{x}_k^c, \mathbf{p}_k)$ if $\mathbf{x}_k^c$ is also in Area 3, which can be proved by an argument similar to that in Area 4.

The other situations, i.e., when $\rho_{k,ij} \leq 0$ or the $(x_{k,i}, x_{k,j})$ plane does not intersect with the no-trade region, can be proved following the same argument, and this completes the proof. □

4.4.1 Symmetric Optimal Control Policy

In the single-asset model, we identified the conditions under which the threshold levels are symmetric. For the multiple-asset model, we can also show that the no-trade region is symmetric with respect to $\mathbf{0}$, i.e., $\mathbf{x}$ is in the no-trade region if and only if $-\mathbf{x}$ is in the no-trade region, under the following condition:

(D1) The market price movements are independent of the order arrivals, i.e., $[\mathbf{s}_k, \mathbf{d}_k]$ and δ_k are independent for any $k = 1, ..., N$.

(D2) The market price process is a symmetric random walk, i.e., $P(\delta_{k,i} \leq v_i \ \forall i = 1, ..., M) = P(\delta_{k,i} \geq -v_i \ \forall i = 1, ..., M)$ for any $[v_1, ..., v_M] \in \mathbb{R}^M$ and $k = 1, ..., N$.

(D3) The buy and sell orders from the clients are subject to the same distribution, i.e., $P(s_{k,i} \leq v_i \ \forall i = 1, ..., M) = P(d_{k,i} \leq v_i \ \forall i = 1, ..., M)$ for any $[v_1, ..., v_M] \in \mathbb{R}^M$ and $k = 1, ..., N$.

Proposition 4.3. *Under assumptions (D1), (D2) and (D3), for both single-period and multiple-period models, the no-trade region is symmetric with respect to* $\mathbf{0}$ *and so is the optimal control policy.*

Proof. Note that $\gamma_k = \mathbf{0}$ for any $k = 1, ..., N$ under assumptions (D1), (D2) and (D3). It follows immediately that the single-period model has a symmetric no-trade region and a symmetric optimal policy.

For the multiple-period model, we prove the proposition under the induction assumption that $J_k(\mathbf{x}_k, \mathbf{p}_k)$ is symmetric in $\mathbf{x}_k$ with respect to $\mathbf{0}$, i.e., $J_k(\mathbf{x}_k, \mathbf{p}_k) = J_k(-\mathbf{x}_k, \mathbf{p}_k)$. $J_{N+1}(\mathbf{x}_{N+1}, \mathbf{p}_{N+1}) = 0$ obviously satisfies the induction assumption.

Suppose that $J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1}) = J_{k+1}(-\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$ for any $\mathbf{x}_{k+1}$ and $\mathbf{p}_{k+1}$. The assumptions (D1), (D2) and (D3) specify that Δ_k and δ_k are independent random vectors and their distributions are symmetric with respect to $\mathbf{0}$. According to Theorem 4.2, the no-trade region R_k is symmetric with respect to $\mathbf{0}$. The symmetry of the optimal control policy as well as the function $J_k(\mathbf{x}_k, \mathbf{p}_k)$ can be established straightforwardly, which completes the induction proof. □

4.4.2 Numerical Results

Theorem 4.2 also suggests an efficient algorithm to compute the optimal solutions. Suppose that we are given $J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$ and $\nabla J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$. Theorem 4.2 shows that the no-trade region of period k can be obtained from $\nabla J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$. Outside the no-trade region of period k, the optimal adjusted inventory can also be obtained directly, and hence we can compute the value function of period k from the value function of period $k+1$, i.e., $J_k(\mathbf{x}_k, \mathbf{p}_k)$ from $J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$. To calculate $\nabla J_k(\mathbf{x}_k, \mathbf{p}_k)$, besides the numerical methods, we can also utilize the property of $\nabla J_k(\mathbf{x}_k, \mathbf{p}_k)$ discussed in the proof of Proposition 4.2. It is straightforward that the computational complexity of this approach is linear in the number of periods N. In fact, according to Theorem 4.2 and the proof of Proposition 4.2, we only need the values of $J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$ and $\nabla J_{k+1}(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$ within the no-trade region of period $k+1$ to compute $J_k(\mathbf{x}_k, \mathbf{p}_k)$ and $\nabla J_k(\mathbf{x}_k, \mathbf{p}_k)$.

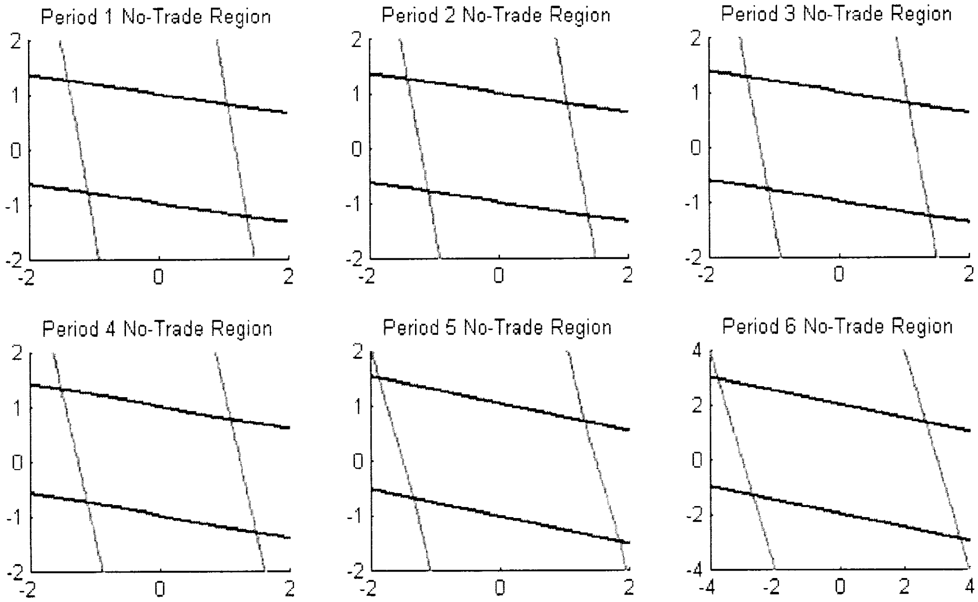
In the remaining part of this section, the algorithm is implemented for a 6-period problem with two assets, which helps us to further understand the properties of the no-trade region.

We assume that the independence and symmetry assumptions hold for this example and the input parameters are stationary throughout the whole planning horizon. The difference between sell and buy orders from the clients, Δ. has a uniform distribution among the vertices of the $\{-1, 1\}$ square, i.e.,

$$P\left(\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = P\left(\Delta = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = P\left(\Delta = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$
$$= P\left(\Delta = \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \frac{1}{4}.$$

The correlation coefficient between the market price movements of these two assets

Figure 4-3: Illustration of Optimal Solution for Multiple-Period Model



is set to 1/4, and hence the variance-covariance matrix of the price movements is

$$\Sigma = \begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}.$$

The vector ϵ determining the bid/ask spread is independent of the market price, and we let $\epsilon_1 = 3$ and $\epsilon_2 = 2$, which correspond to half of the spread for asset 1 and 2 respectively. Moreover, the risk aversion parameter is set to $\lambda = 1/2$.

Suppose that we can clear our inventory at the end of the planning horizon without additional cost, i.e., clear the position at the market price. It implies that $J_{N+1}(\mathbf{x}_{N+1}, \mathbf{p}_{N+1}) = 0$. The no-trade regions for period 1 to 6 are shown in Figure 4-3. The no-trade region in period k is the area defined by the intersections of 4 lines in the corresponding sub-figure. There are three important observations here.

- The no-trade region for period 6 is significantly larger than those of the previous periods, which is analogous to the observations in the examples of Chapters 2 and 3 when we mark to the market mid price at the end of the planning horizon. Note that the axes are from -4 to 4 in the sub-figure corresponding to period

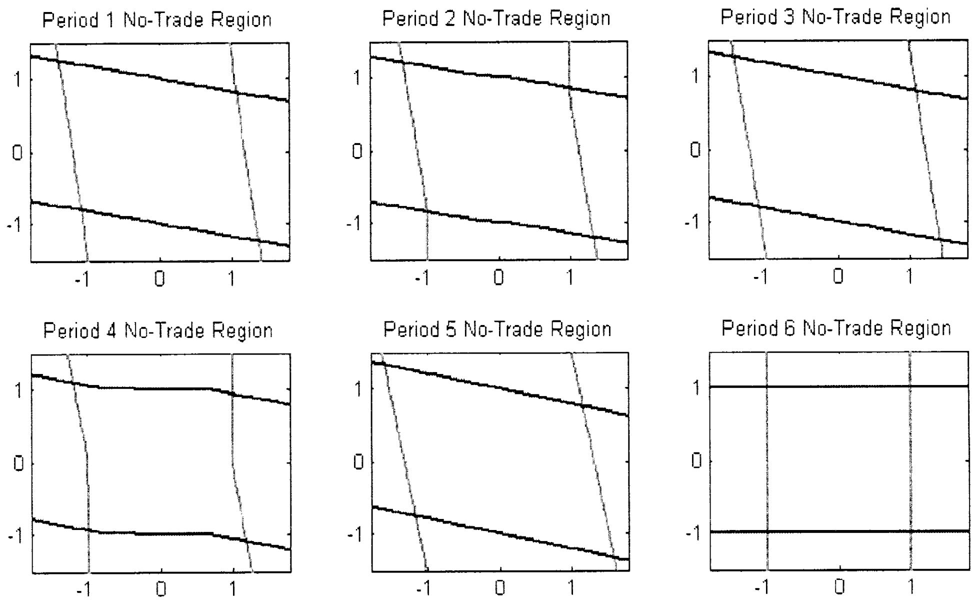
6 while those for the rest periods are from -2 to 2. Here we do not have transaction cost at the end of planning horizon. In period 6, we only need to hold the inventory for one more period and clear it without incurring any spread loss. Therefore, the optimal policy would actively trade less frequently and result in a larger no-trade region.

- In general, the no-trade region is longer along the horizontal axis corresponding to asset 1 and narrower along the vertical axis corresponding to asset 2. Note that the spread of asset 1 is higher than that of asset 2, i.e., the transaction cost is higher to actively adjust the inventory of asset 1. Therefore, we adjust the inventory of asset 1 less frequently and hence the no-trade region is slightly wider along that direction.

- The no-trade region has greater area in the 2nd and 4th quadrants than the 1st and 3rd quadrants. The 1st and 3rd quadrants correspond to the portfolios with two assets having positions with the same sign while the 2nd and 4th quadrants correspond to the portfolios with positions having opposite signs. Therefore, this property implies that we trade the portfolios with positions in the same sign more frequently. Note that the price movements of the two assets are positively correlated in this example. As a result, a portfolio with positions in the same sign implies higher risk compared with a portfolio with the same absolute positions but in opposite signs, which explains why we actively adjust portfolios having positions with the same sign more frequently.

Similar to the examples in Chapters 2 and 3, we also consider the case that the inventory at the end of the planning horizon is cleared at the bid/ask price quoted by other market-makers, i.e., $J_{N+1}(\mathbf{x}_{N+1}, \mathbf{p}_{N+1}) = \epsilon_{N+1}^T |\mathbf{x}_{N+1}|$. The corresponding no-trade regions are shown in Figure 4-4. Note that the last two observations for Figure 4-3 are also applicable to the no-trade regions for period 1 to 5. The no-trade region for period 6 is the $\{-1, 1\}$ square corresponding to the support of Δ. To understand this property, let us investigate a simple example. Suppose that we have 2 units of asset 1 at the beginning of period 6. Consider the following two options.

Figure 4-4: Illustration of Optimal Solution for Multiple-Period Model



Option 1. We actively sell 1 unit in period 6. After receiving the orders from the clients, our inventory is 0 or 2 units. We hold 0 or 2 units for 1 period and clear the inventory at the end of the planning horizon.

Option 2. We do not adjust inventory actively in period 6. After observing the orders from the clients, our inventory is 1 or 3 units, which is held for 1 period. We clear 1 or 3 units at the end of the planning horizon.

These two options have the same transaction cost, but option 1 holds less inventory. Therefore, a risk-averse decision maker would always go for option 1. The same argument also applies to asset 2, and hence the no-trade region should be contained in the $\{-1, 1\}$ square. Moreover, our risk aversion parameter $\lambda = 0.5$ is not very conservative, so the no-trade region for this example is the $\{-1, 1\}$ square. Notice that this argument is very similar to how we explain $T^1_{100}(1.999, 10^{-4}) = -1$ and $T^2_{100}(1.999, 10^{-4}) = 1$ in Example 2.1 of Chapter 2.

4.5 Extensions

In this chapter, we analyzed the multiple-asset inventory model in market-making under the assumption that the market-maker is risk averse. The optimal policy is fully characterized and can be computed efficiently. We show that the no-trade region R_k depends on the market mid price $\mathbf{p}_k$ if the spread determined by ϵ_k is a function of $\mathbf{p}_k$. Otherwise, i.e., when ϵ_k is a vector of constants, the no-trade region in period k is the same for any realized market mid price $\mathbf{p}_k$.

As we pointed out in Section 4.2, the optimal policy characterized by Theorem 4.2 and Proposition 4.1 is valid under more general assumptions.

- The random walk assumption of the market mid price can be relaxed. We can assume that the price movement δ_k depends on the market mid price $\mathbf{p}_k$, and δ_k conditional on $\mathbf{p}_k$ is independent in k. As we mentioned in Section 2.1, the geometric random walk is a special case of this price dynamics. Under these assumptions, Theorem 4.2 characterizes the optimal control policy where the no-trade region R_k relies on the realized market mid price $\mathbf{p}_k$ for any period k. In addition, Proposition 4.1 also holds if the correlation coefficient of $\delta_{k,i}|\mathbf{p}_k$ and $\delta_{k,j}|\mathbf{p}_k$ has the same sign for any pair of assets i and j, for any period k and any market price $\mathbf{p}_k$.

- We can allow the spread in any period k to be a random vector correlated with other stochastic inputs, i.e., ϵ_k is a random vector correlated with $\mathbf{p}_k$, δ_k, $\mathbf{s}_k$ and $\mathbf{d}_k$. As long as ϵ_k conditional on $\mathbf{p}_k$ is independent across the period k, e.g., $\epsilon_k = \phi_k(\mathbf{p}_k) + \varphi_k$ where $\phi_k(\mathbf{p}_k)$ is a givn function and φ_k is a random vector independent in k, the optimal policy described in Theorem 4.2 and Proposition 4.1 still holds except that the no-trade region R_k is determined by the realization of both the mid price and the spread in period k, i.e., $\mathbf{p}_k$ and ϵ_k.

- Suppose that the orders from the clients are correlated with the market mid price, i.e., $[\mathbf{s}_k, \mathbf{d}_k]$ and $\mathbf{p}_k$ are correlated. If $[\mathbf{s}_k, \mathbf{d}_k]$ conditional on $\mathbf{p}_k$ are independent in k, we have the same optimal policy as that in Theorem 4.2 and

Proposition 4.1. Notice that under the correlations between $[\mathbf{s}_k, \mathbf{d}_k]$ and $\mathbf{p}_k$, the no-trade region depends on $\mathbf{p}_k$ even if ϵ_k is a constant vector for any k.

Similar to the single-asset model, if we consider auto-correlated stochastic random inputs, then Theorem 4.2 holds with the no-trade region R_k depending on all realized information before we make our decision in period k, i.e., the vector

$$\mathbf{h}_k = \{\mathbf{p}_1, ..., \mathbf{p}_k, \epsilon_1, ..., \epsilon_k, \mathbf{d}_1, ..., \mathbf{d}_{k-1}, \mathbf{s}_1, ..., \mathbf{s}_{k-1}\}.$$

In order to obtain Proposition 4.1, we also need the assumption that the correlation coefficients of $\delta_{k,i}|\mathbf{h}_k$ and $\delta_{k,j}|\mathbf{h}_k$ have the same sign for any pair of assets i and j, for any period k and any history $\mathbf{h}_k$.

Finally, the optimal policy is the same if the decision maker quotes bid and ask prices different from the prices at which he or she trades with other market-makers.

Chapter 5

Robust Stochastic Lot-Sizing by Means of Histograms

Recall the discussion in Section 1.2 that most inventory model relies on the complete distribution functions of demands, which are usually not available in practice. Therefore, in this chapter, we investigate how to find a robust solution to the classical inventory model of Scarf [47] when only historical data is available and the demand distribution functions are not explicitly given.

This chapter is organized as follows. The literature related to robust inventory control is briefly reviewed in Section 5.1. In Section 5.2 we describe our robust model, which incorporates historical data and present the optimality equation in a compact form. The structure of the optimal policies is characterized in Section 5.3. Section 5.4 considers a special case with robustness defined by the chi-square goodness-of-fit test. We also discuss selected convergence results for the chi-square test based models in the same section. The computational results are presented in Section 5.5. Finally, additional extensions are presented in Section 5.6.

5.1 Literature Review

The notion of robust inventory control is not new in the literature. The earliest work in minimax inventory control is attributed to Scarf [46], where minimization of the maximum expected cost of the newsvendor model over all distributions with a given mean and variance is considered. Gallego and Moon [19] present another proof of Scarf's result and consider various extension of the model. The recent work by Natarajan et al. [37] extends the result of Scarf [46] by considering the set of distributions with a given mean, variance and semivariance information. Perakis and Roels [41] minimize the maximum regret of the newsvendor model over a convex set of distributions with certain moments and shape. Notzon [38] considers a multiple period model where the demand in each period is assumed to be independent. The demand distribution function is ambiguous, but it is within a specified class of distribution functions. The minimax control policy minimizes the maximum expected cost. The optimality of (s, S) policy is proved. In addition, Gallego et al. [20] propose the minimax finite-horizon inventory models where the set of distributions is defined by linear constraints, and solve the optimization problems by a sequence of linear programs.

Bertsimas and Thiele [7] analyze distribution-free inventory problems, in which demand in each period is assumed to be a random variable that takes values in a given range. The demand is assumed to be a random variable controlled only by two values: the lower and upper estimators. To capture the trade-off between robustness and optimality, a parameter is defined to control the budgets of uncertainty at every time period. They show that for a variety of problems, the structures of the optimal policy remain the same as in the associated model with complete information about the distribution of customer demand. A related model from the base stock perspective is analyzed in Bienstock and Ozbay [8].

See and Sim [48] consider a factor-based demand model with given mean, support, and deviation measures. To obtain tractable replenishment policies, the worst case expected cost among all distributions satisfying the demand model is minimized by solving a second order cone optimization problem.

Ahmed et al. [1] propose an inventory control model which minimizes a coherent risk measure instead of the overall cost function. They show that risk aversion treated in the form of coherence risk measures is equivalent to the minimax formulations, and it is proved that the optimal policies conserve the properties of the stochastic dynamic programming counterparts. They do not consider demand dependent evolutions.

Liyanage and Shanthikumar [30] first provide concrete examples in a single period (newsvendor) setting, which illustrate that separating distribution estimation and inventory optimization, as done in the classical approach, may lead to suboptimal solutions. They propose the use of operational statistics where it is assumed that the demand distribution function belongs to a specific (predetermined) family and estimate the (single) parameter of the family within an inventory optimization model.

In addition, selected recent papers also consider lost-sale inventory problems with censored demand data, i.e., the observed historical demand data excludes the lost-sale information as the lost sales are not observable. Huh and Rusmevichientong [26] propose nonparametric adaptive policies to solve this problem and provide a bound for the asymptotic performance, which interestingly is the same as the converenge rate of our model under discrete distributions.

The models by Notzon [38] and Ahmed et al. [1] do not take historical data into account, and they predefine the class of distribution functions. The robust optimization approaches from Bertsimas and Thiele [7] as well as See and Sim [48] do not use any historical data except to determine the support, expectation and deviation measures. On the other hand, Liyanage and Shanthikumar [30] use historical data but predetermine the family of distributions. In fact, they consider only distributions characterized by a single unknown parameter. This is the only work besides the one proposed in this chapter that concurrently optimizes the expected cost and the distribution or the parameter to determining the demand distribution. Our research combines both strategies by integrating distribution fitting with robust optimization. Specifically, we consider the set of demand distributions that satisfy a certain data fitting criterion with respect to historical data and characterize an optimal policy that minimizes the maximum expected cost.

5.2 Formulation of Robust Stochastic Lot-Sizing

The classical multi-period inventory problem considers a finite planning horizon of T periods. For each period $t = 1, ..., T$, let $\tilde{D}_t$ be a random variable representing demand in that period. The sequence of events is as follows. At the beginning of each period t, the decision maker reviews the inventory level x_t, and places an order for q_t (possibly zero) units. Since lead time is assumed to be zero, this order arrives immediately and hence increases the inventory level up to y_t, where $y_t = x_t + q_t$. After observing demand $\tilde{D}_t$, the net inventory at the beginning of period $t+1$ is reduced to $x_{t+1} = y_t - \tilde{D}_t$.

The procurement cost in each period $t = 1, ..., T-1$ includes two components, a fixed procurement cost K if $q_t > 0$, and a unit procurement cost c_t for each unit ordered. Inventory holding cost is charged at a rate of h_t for any unit of excess inventory at the end of period t, and a unit back-order cost b_t is incurred for any unit of unsatisfied demand. We assume that all shortages are backlogged. Thus, the total cost for period t given the inventory levels before and after ordering (x_t and y_t respectively) as well as demand $\tilde{D}_t$ in that period is

$$C_t\left(x_t, y_t, \tilde{D}_t\right) = K\mathbb{I}(y_t - x_t) + c_t(y_t - x_t) + h_t\left(y_t - \tilde{D}_t\right)^+ + b_t\left(y_t - \tilde{D}_t\right)^- \quad (5.1)$$

for any $t = 1, ..., T$, where $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $\mathbb{I}(x) = 1$ if $x > 0$ and $\mathbb{I}(x) = 0$ otherwise.

In the standard dynamic programming formulation, we consider $\tilde{V}_t(x_t)$ for any $t = 1, ..., T$, which denotes the optimal expected cost over horizon $[t, T]$, given that the inventory level at the beginning of period t is x_t and an optimal policy is adopted over horizon $[t, T]$. We assume $\tilde{V}_{T+1}(x_{T+1}) = 0$. Let $\theta \in [0, 1]$ be the discount rate. The optimality equation reads

$$\tilde{V}_t(x_t) = \min_{y_t \geq x_t}\left\{E\left[C_t\left(x_t, y_t, \tilde{D}_t\right)\right] + \theta E\left[\tilde{V}_{t+1}\left(y_t - \tilde{D}_t\right)\right]\right\} \qquad t = 1, ..., T. \quad (5.2)$$

Note that the distribution of $\tilde{D}_t$, $t = 1, ..., T$ is required to solve this dynamic programming formulation.

In practice, the demand distribution is not known. Rather, an inventory manager has at her disposal only historical data. Depending on the realized past demand in the planning horizon, the manager may choose different aggregations of historical data to forecast the demand distribution. For example,

- the demand data of the last n observations are considered, which is analogous to the moving average forecast, or
- the realized demand in periods 1 to $t-1$ is accounted for when forecasting the demand in period t.

Historical observations are often aggregated. Let $[D_{t,i}, D_{t,i+1})$ denote the ith possible range of the demand in period t (all observations within a given range are indistinguishable), and let the vector $\mathbf{d}_t = [d_1, ..., d_{t-1}]$ denote the realized demand in periods 1 to $t-1$, where d_τ, $\tau = 1, ..., t-1$ corresponds to the realized demand in period τ. The number of observations that fall within $[D_{t,i}, D_{t,i+1})$ is a function of the realized demand $\mathbf{d}_t$ and is denoted by $N_{t,i}(\mathbf{d}_t)$. Finally, we define $n_t(\mathbf{d}_t) = \sum_i N_{t,i}(\mathbf{d}_t)$, which corresponds to the total number of available observations under the realized demand $\mathbf{d}_t$.

Hypothetically we can think of $N_{t,i}(\mathbf{d}_t)$ as forming a histogram with respect to unknown distribution $\tilde{D}_t$. The bins are $[D_{t,i}, D_{t,i+1})$ and the number of values falling within the ith bin is $N_{t,i}(\mathbf{d}_t)$. In practice, the decision maker observes only these histograms, i.e., the historical samples.

We assume that $D_{t,1} = 0$ and $D_{t,M_t+1} = +\infty$, where M_t corresponds to the number of bins in the histogram for time period t. Let $P_{t,i} = P\left(\tilde{D}_t \in [D_{t,i}, D_{t,i+1})\right)$ be the probability that demand in period t falls in the interval $[D_{t,i}, D_{t,i+1})$ under the fitted distribution. Clearly, $n_t(\mathbf{d}_t)P_{t,i}$ is the expected number of observations that fall in this interval according to the fitted distribution.

The classical approach to identify the best distribution representing the observed data is to use a goodness-of-fit test. The objective is to fit a distribution that "closely" follows the observed data. Under this criterion, there should be a set of distributions depending on $\mathbf{d}_t$, which satisfy the given goodness-of-fit test. We denote this set by $\mathcal{P}_t(\mathbf{d}_t)$. Throughout this paper, we assume that $\mathcal{P}_t(\mathbf{d}_t)$ is compact for any t and $\mathbf{d}_t$.

As defined in the dynamic programming field, a decision rule μ_t at time t is a function of inventory x_t, which decides the ordering quantity at time t given x_t, i.e., $y_t = \mu_t(x_t)$. We formally state our problem in the context of a two-player game, which is also presented in Iyengar [28]. The first player chooses the decision rule μ_t at time t and pays the cost. The second player chooses a distribution of $\tilde{D}_t$ in $\mathcal{P}_t(\mathbf{d}_t)$ after observing the order quantity, and receives a reward equal to the cost paid by the first player. Therefore, the second player may select a different distribution for different x_t and μ_t. Given decision rule μ_t, the set of all distributions player two could choose is

$$\mathcal{Q}^{\mu_t} = \left\{\mathbf{P}(x_t, \mu_t(x_t)) \in \mathcal{P}_t(\mathbf{d}_t) \text{ over all } x_t, \mathbf{d}_t\right\}.$$

In $\mathcal{Q}^{\mu_t}$ we merely express that for each x_t, μ_t, $\mathbf{d}_t$, we might have a different distribution. Moreover, a policy π is defined as the decision rule to be used at every period, i.e., $\pi = (\mu_1, ..., \mu_T)$. A policy π also yields a set of distributions $\mathcal{Q}^\pi$ which can be used by the second player or adversary, where

$$\mathcal{Q}^\pi = \mathcal{Q}^{\mu_1} \times \mathcal{Q}^{\mu_2} \times \cdots \times \mathcal{Q}^{\mu_T}. \tag{5.3}$$

As the second player will maximize her reward, given policy π, inventory x_t, and realized demand $\mathbf{d}_t$, the cost paid by player one from period t to T is

$$V_t^\pi(x_t, \mathbf{d}_t) = \max_{\mathbf{Q} \in \mathcal{Q}^\pi} E^{\mathbf{Q}, \tilde{\mathbf{D}}} \left[\sum_{\tau=t}^{T} \theta^{\tau-t} C_\tau \left(x_\tau, \mu_\tau(x_\tau), \tilde{D}_\tau \right) + \theta^{T+1-t} V_{T+1}(x_{T+1}, \mathbf{d}_{T+1}) \right],$$

where $C_\tau\left(x_\tau, \mu_\tau(x_\tau), \tilde{D}_\tau\right)$ denotes the total cost incurred in period τ defined in (5.1), and $V_{T+1}(x_{T+1}, \mathbf{d}_{T+1})$ is the terminal cost. Also note that $\mathbf{Q}$ defines the distributions $\tilde{D}_\tau$, $\tau = t, ..., T$. Unless stated otherwise, we assume that $V_{T+1}(\cdot) = 0$. We also have

$$x_{\tau+1} = \mu_\tau(x_\tau) - \tilde{D}_\tau \quad \text{and} \quad \mathbf{d}_{\tau+1} = \left[\mathbf{d}_\tau, \tilde{D}_\tau\right].$$

Since the first player will choose a policy that minimizes the cost, the optimal cost from period t to T given inventory x_t at time t, and the realized demand $\mathbf{d}_t$ from period 1 to $t-1$, is

$$V_t(x_t, \mathbf{d}_t) = \min_{\pi} \max_{\mathbf{Q} \in \mathcal{Q}^{\pi}} E^{\mathbf{Q}, \tilde{\mathbf{D}}} \left[\sum_{\tau=t}^{T} \theta^{\tau-t} C_{\tau} \left(x_{\tau}, \mu_{\tau}(x_{\tau}), \tilde{D}_{\tau} \right) + \theta^{T+1-t} V_{T+1}(x_{T+1}, \mathbf{d}_{T+1}) \right], \tag{5.4}$$

for $t = 1, ..., T$. Note that the model minimizes the maximum expected cost arising from any distribution in the set $\mathcal{P}_t(\mathbf{d}_t)$, which is known as the minimax robust approach. We next state an optimality equation, which is essential to establish the optimal control policies.

Proposition 5.1. *The optimality equation of the robust model is*

$$V_t(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} \max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \left\{ \sum_i P_{t,i} \left(C_t(x_t, y_t, D_{t,i}) + \theta V_{t+1} \left(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}] \right) \right) \right\} \tag{5.5}$$

for $t = 1, ..., T$, *where* $\mathcal{P}_t(\mathbf{d}_t)$ *is the set of distributions satisfying the goodness-of-fit condition at period* t, *and* $C_t(x_t, y_t, D_{t,i})$ *is defined by* (5.1).

Proof. It follows from Theorem 2.1 in Iyengar [28] when $\mathcal{P}_t(\mathbf{d}_t)$ is arbitrary. If $\mathcal{P}_t(\mathbf{d}_t)$ is convex, the proposition can also be proved by the Von Neumann's minimax theorem (see, e.g., Von Neumann [53]). □

An immediate observation from Proposition 5.1 is that we minimize the worst case expected cost over a set of distributions. Therefore, our robust stochastic model may not be as conservative as the classical minimax models, where the worst case is defined by the realized demand instead of distribution, e.g., the minimax model discussed in Section 2.4 of Notzon [38].

Note that the Bayesian inventory models assume a prior demand distribution, and the posterior distribution at time t is obtained by updating the prior distribution using $\mathbf{d}_t$, e.g., Iglehart [27] updates the demand distribution belonging to the exponential and range families after observing realized demand information. Our model only requires the set of distributions $\mathcal{P}(\mathbf{d}_t)$ to be a function of the realized demand $\mathbf{d}_t$.

Therefore, we can define it as a singleton updated by a Bayesian rule. In this case, the robust minmax model is reduced to a Bayesian inventory model, which indicates that the Bayesian models are special cases of our minimax model.

Proposition 5.1 also gives us an interpretation of the robust model from a risk measure perspective when set $\mathcal{P}_t(\mathbf{d}_t)$ is convex. Ahmed et al. [1] establish the correspondence between coherent risk measures and minimax models over convex sets of distributions. From this perspective, our minimax robust model essentially minimizes a coherent risk measure with respect to the total cost. If we consider $\mathbf{P}_t(\mathbf{d}_t) \equiv \mathbf{P}_t$ for any $\mathbf{d}_t$ and t, then our minimax robust model (5.5) minimizes a coherent risk measure in any period t and it reduces to the model considered in Ahmed et al. [1]. In the case when the set of distributions in our minimax model depends on demand realization $\mathbf{d}_t$, our robust model is to minimize a coherent risk measure in every period t. The risk measure we consider in period t is updated by the realized demand in previous periods. Intuitively, if the decision maker lost a significant amount in the previous period, he or she would tend to be more risk-averse in subsequent periods. Therefore, it is reasonable to adjust the risk measure based on the realized demand information $\mathbf{d}_t$.

5.3 Properties of Optimal Policies

In this section we study optimal policies of the general robust stochastic model (5.5). Notzon [38] and Ahmed et al. [1] show the optimality of (s, S) policy when the set of distributions in the minimax model is independent of the realized demand $\mathbf{d}_t$ (Ahmed et al. [1] also assumes the set of distributions is convex). Here we extend the optimality of (s, S) policy to the more general model in (5.5).

We assume that the reader is familiar with standard concepts in inventory theory such as K-convexity and (s, S) policies (see, e.g., Zipkin [55] and Porteus [42]).

Let us define

$$U_i(y, \mathbf{d}) = h_t\left(y_t - D_{t,i}\right)^+ + b_t\left(y_t - D_{t,i}\right)^- + \theta V_{t+1}\left(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]\right), \qquad (5.6)$$

which corresponds to the expected cost incurred from period t to T if the inventory level after receiving the order in period t is y_t and the demand in period t is $D_{t,i}$. Consider the function

$$f(y, \mathbf{d}) = \max_{\mathbf{P} \in \mathcal{P}(\mathbf{d})} \sum_i U_i(y, \mathbf{d}) P_i.$$ [1]

Since optimality of the (s, S) policy follows directly from K-convexity, first we are going to establish that the function $f(y, \mathbf{d})$ is K-convex in y.

Lemma 5.1. *If $U_i(y, \mathbf{d})$ is K-convex in y for any given $\mathbf{d}$, then $f(y, \mathbf{d})$ is a K-convex function in y for any given $\mathbf{d}$.*

Proof. Consider the function

$$g(\mathbf{U}, \mathbf{d}) = \max_{\mathbf{P} \in \mathcal{P}(\mathbf{d})} \mathbf{U}^T \mathbf{P}$$

Note that $f(y, \mathbf{d}) = g(\mathbf{U}(y, \mathbf{d}), \mathbf{d})$.

We first show that $g(\mathbf{U}, \mathbf{d})$ is an increasing function of $\mathbf{U}$ for any given $\mathbf{d}$. Suppose that $\mathbf{U}_1 \leq \mathbf{U}_2$ and $g(\mathbf{U}_1, \mathbf{d}) = \mathbf{U}^T \mathbf{P}_1^*$. Since $\mathbf{P}_1^* \geq 0$,

$$g(\mathbf{U}_1, \mathbf{d}) = \mathbf{U}_1^T \mathbf{P}_1^* \leq \mathbf{U}_2^T \mathbf{P}_1^* \leq g(\mathbf{U}_2, \mathbf{d}).$$

Consider now the value of $g(\mathbf{U} + K\mathbf{e})$, where $\mathbf{e}$ is the vector with all entries of 1. Let $\mathbf{P}^*$ denote the maximizer of $g(\mathbf{U} + K\mathbf{e}, \mathbf{d})$. We have

$$g(\mathbf{U} + K\mathbf{e}, \mathbf{d}) = (\mathbf{U} + K\mathbf{e})^T \mathbf{P}^* = \mathbf{U}^T \mathbf{P}^* + K\mathbf{e}^T \mathbf{P}^* \leq g(\mathbf{U}, \mathbf{d}) + K, \tag{5.7}$$

where the last inequality follows from $g(\mathbf{U}, \mathbf{d}) \geq \mathbf{U}^T \mathbf{P}^*$ and $\mathbf{e}^T \mathbf{P}^* = \sum_i P_i^* = 1$ as $\mathbf{P}^*$ defines a distribution.

For any $y_1 \leq y_2$ and $\lambda \in [0, 1]$, since $U_i(y, \mathbf{d})$ is K-convex in y for any given $\mathbf{d}$, we have

$$U_i((1 - \lambda) y_1 + \lambda y_2, \mathbf{d}) \leq (1 - \lambda) U_i(y_1, \mathbf{d}) + \lambda U_i(y_2, \mathbf{d}) + \lambda K.$$

[1] Note that here we drop subscript t in order to simplify the notation.

As $g(\mathbf{U}, \mathbf{d})$ is increasing in $\mathbf{U}$,

$$g(\mathbf{U}((1-\lambda)y_1 + \lambda y_2, \mathbf{d}), \mathbf{d}) \leq g((1-\lambda)\mathbf{U}(y_1, \mathbf{d}) + \lambda \mathbf{U}(y_2, \mathbf{d}) + \lambda K\mathbf{e}, \mathbf{d}).$$

It is straightforward to show that $g(\mathbf{U}, \mathbf{d})$ is a convex function of $\mathbf{U}$, as it is the maximum of linear functions of $\mathbf{U}$. Therefore,

$$g((1-\lambda)\mathbf{U}(y_1, \mathbf{d}) + \lambda \mathbf{U}(y_2, \mathbf{d}) + \lambda K\mathbf{e}, \mathbf{d}) \leq (1-\lambda) g(\mathbf{U}(y_1, \mathbf{d}), \mathbf{d}) + \lambda g(\mathbf{U}(y_2, \mathbf{d}) + K\mathbf{e}, \mathbf{d}).$$

According to (5.7) we have

$$g(\mathbf{U}(y_2, \mathbf{d}) + K\mathbf{e}, \mathbf{d}) \leq g(\mathbf{U}(y_2, \mathbf{d}), \mathbf{d}) + K.$$

As a result, it follows

$$g(\mathbf{U}((1-\lambda)y_1 + \lambda y_2, \mathbf{d}), \mathbf{d}) \leq (1-\lambda) g(\mathbf{U}(y_1, \mathbf{d}), \mathbf{d}) + \lambda g(\mathbf{U}(y_2, \mathbf{d}), \mathbf{d}) + \lambda K.$$

and therefore $f(y, \mathbf{d}) = g(\mathbf{U}(y, \mathbf{d}), \mathbf{d})$ is a K-convex function in y. □

Base on this property, we show the K-convexity of the cost-to-go functions.

Proposition 5.2. *If $V_{t+1}(x_{t+1}, \mathbf{d}_{t+1})$ is a K-convex function in x_{t+1} for any fixed $\mathbf{d}_{t+1}$, the cost-to-go function $V_t(x_t, \mathbf{d}_t)$ is a K-convex function in x_t for any fixed $\mathbf{d}_t$, and for any $t = 1, ..., T$.*

Proof. The proposition is trivially true for $t = T + 1$. Suppose that the proposition holds for period $t + 1$, and consider period t.

To simplify the notation, let us define

$$\begin{aligned} f_t(y_t, \mathbf{d}_t) = c_t y_t + \max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \sum_i P_{t,i} \left[h_t (y_t - D_{t,i})^+ + b_t (y_t - D_{t,i})^- \right. \\ \left. + \theta V_{t+1} (y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) \right]. \end{aligned} \tag{5.8}$$

Therefore, the optimality equation in (5.5) is equal to

$$V_t(x_t, \mathbf{d}_t) = -c_t x_t + \min_{y_t \geq x_t} \{K\mathbb{I}(y_t - x_t) + f_t(y_t, \mathbf{d}_t)\}.$$

According to Lemma 5.1, if $V_{t+1}(x_{t+1}, \mathbf{d}_{t+1})$ is K-convex in x_t, $f_t(y_t, \mathbf{d}_t)$ is K-convex in y_t. Let $S_t(\mathbf{d}_t)$ be a global minimizer of $f_t(y_t, \mathbf{d}_t)$ for any given $\mathbf{d}_t$. Moreover, let $s_t(\mathbf{d}_t)$ be the smallest element of the set $\{s_t(\mathbf{d}_t) \mid s_t(\mathbf{d}_t) \leq S_t(\mathbf{d}_t),\ f_t(s_t, \mathbf{d}_t) = f_t(S_t, \mathbf{d}_t) + K\}$. According to the properties of K-convex functions (see, e.g., Zipkin [55] and Porteus [42]), we have

$$V_t(x_t, \mathbf{d}_t) = \begin{cases} K - c_t x_t + f_t(S_t(\mathbf{d}_t), \mathbf{d}_t) & \text{if } x_t \leq s_t(\mathbf{d}_t), \\ -c_t x_t + f_t(x_t, \mathbf{d}_t) & \text{otherwise.} \end{cases}$$

K-convexity of $V_t(x_t, \mathbf{d}_t)$ follows from K-convexity of $f_t(y_t, \mathbf{d}_t)$. □

From the structure of $V_{t+1}(\cdot)$, we can derive an optimal policy.

Theorem 5.1. *A state dependent (s, S) policy is optimal for the robust stochastic model. More precisely, for any t and $\mathbf{d}_t$, there exists $S_t(\mathbf{d}_t)$ and $s_t(\mathbf{d}_t)$ such that $S_t(\mathbf{d}_t) - x_t$ units are ordered in period t if $x_t \leq s_t(\mathbf{d}_t)$ and no order is placed otherwise.*

Proof. The structure of the policy follows directly from the proof of Proposition 5.2 and general theory of K-convexity (see, e.g., Zipkin [55] and Porteus [42]). □

If there is no fixed cost, then $V_t(x_t, \mathbf{d}_t)$ is convex in x_t for any t. Therefore, a state dependent base-stock policy is optimal, and the base-stock level given the realized demand $\mathbf{d}_t$ is $S_t(\mathbf{d}_t)$.

A drawback from the practical point of view is the fact that s_t and S_t depend on $\mathbf{d}_t$. We next characterize a special case when this is circumvented. Suppose that $\mathbf{d}_t$ and $\mathbf{d}'_t$ denote two different demand realizations from period 1 to $t-1$. Let us assume that if demand realizations in periods 1 to $t-1$ are $\mathbf{d}_t$ or $\mathbf{d}'_t$, then the same demand realization in period t to T generates the same histogram in any period $t, ..., T$. Then vectors $\mathbf{d}_t$ and $\mathbf{d}'_t$ correspond to the same (s, S) levels. To formalize this property, let $s_t(\mathbf{d}_t)$ and $S_t(\mathbf{d}_t)$ (respectively $s_t(\mathbf{d}'_t)$ and $S_t(\mathbf{d}'_t)$) denote the (s, S) levels corresponding to history $\mathbf{d}_t$ (respectively $\mathbf{d}'_t$). For any $\tau \geq t$, let the vector $[\mathbf{d}_t, \tilde{d}_t, \tilde{d}_{t+1}, ..., \tilde{d}_{\tau-1}]$ denote the realized demand up to period $\tau - 1$ where the demands from periods 1 to $t-1$ are aggregated in vector $\mathbf{d}_t$, and the realized demand in periods t to $\tau - 1$ is labeled by $\tilde{d}_t, \tilde{d}_{t+1}, ..., \tilde{d}_{\tau-1}$, respectively.

Proposition 5.3. *Let $V_{T+1}(x_{T+1}, \mathbf{d}_{T+1}) = V_{T+1}(x_{T+1}, \mathbf{d}'_{T+1})$ for any x_{T+1}, $\mathbf{d}_{T+1}$, $\mathbf{d}'_{T+1}$, and consider any $\tau = t, ..., T$. Suppose that realizations $\mathbf{d}_t$ and $\mathbf{d}'_t$ give the same number of samples in interval $[D_{\tau,i}, D_{\tau,i+1})$ for any i as long as the realized demand in periods t to $\tau - 1$ is the same, i.e.,*

$$N_{\tau,i}([\mathbf{d}_t, \tilde{d}_t, \tilde{d}_{t+1}, ..., \tilde{d}_{\tau-1}]) = N_{\tau,i}([\mathbf{d}'_t, \tilde{d}_t, \tilde{d}_{t+1}, ..., \tilde{d}_{\tau-1}])$$

for any i and any realization $[\tilde{d}_t, \tilde{d}_{t+1}, ..., \tilde{d}_{\tau-1}]$ of $[\tilde{D}_t, \tilde{D}_{t+1}, ..., \tilde{D}_{\tau-1}]$. Then we have $s_t(\mathbf{d}_t) = s_t(\mathbf{d}'_t)$, $S_t(\mathbf{d}_t) = S_t(\mathbf{d}'_t)$, and $V_t(x_t, \mathbf{d}_t) = V_t(x_t, \mathbf{d}'_t)$ for any x_t.

Proof. Consider period T. According to the assumption stated, $N_{T,i}(\mathbf{d}_T) = N_{T,i}(\mathbf{d}'_T)$ for any i, and hence we have $n_T(\mathbf{d}_T) = n_T(\mathbf{d}'_T)$ and $\mathcal{P}_T(\mathbf{d}_T) = \mathcal{P}_T(\mathbf{d}'_T)$. By assumption on $V_{T+1}(\cdot)$, we obtain $s_T(\mathbf{d}_T) = s_T(\mathbf{d}'_T)$ and $S_T(\mathbf{d}_T) = S_T(\mathbf{d}'_T)$ from Theorem 5.2. Moreover, the result $V_T(x_T, \mathbf{d}_T) = V_T(x_T, \mathbf{d}'_T)$ follows from (5.5).

Suppose that the proposition is true for any period $\tau > t$, which implies that $V_{t+1}(x_{t+1}, [\mathbf{d}_t, D_{t,i}]) = V_{t+1}(x_{t+1}, [\mathbf{d}'_t, D_{t,i}])$ for any x_{t+1} and i. Moreover, we have $N_{t,i}(\mathbf{d}_t) = N_{t,i}(\mathbf{d}'_t)$ for any i, which implies $n_t(\mathbf{d}_t) = n_t(\mathbf{d}'_t)$ and $\mathcal{P}_t(\mathbf{d}_t) = \mathcal{P}_t(\mathbf{d}'_t)$. According to Theorem 5.2 and (5.5), the results hold for period t. □

Suppose that we use the same bin intervals $[D_{t,i}, D_{t,i+1})$ for any period t in the planning horizon. Furthermore, let us assume that we update the histogram in time period t only based on the realized demand in periods 1 to $t-1$, or, for example, given a fixed n, we update the histogram in time period t only based on realized demand in time periods $t-n$ through $t-1$. Observe that these two scenarios do not allow any forecasting based on the just realized demand. From Proposition 5.3, it now follows that the number of different (s, S) levels at time t cannot exceed the number of bins to the power of t. This observation substantially reduces the computational burden.

5.4 Robust Models Based on Chi-Square Test

The most widely used goodness-of-fit test is the chi-square test (see, e.g., Chernoff and Lehmann [12]) with the statistical test

$$\sum_i \frac{(N_{t,i}(\mathbf{d}_t) - n_t(\mathbf{d}_t)P_{t,i})^2}{n_t(\mathbf{d}_t)P_{t,i}} \le \chi_t^2 \qquad t = 1, ..., T,$$

where parameter χ_t^2 controls how close the observed sample data is to the estimated expected number of observations according to the fitted distribution $(P_{t,i})_{i=1,...,M_t}$.

More specifically, suppose that k is the number of bins, c is the number of estimated parameters for the fitted distribution (e.g., $c = 2$ for normal distributions due to the mean and variance), and consider the null hypothesis H_0 that the observations are independent random samples drawn from the fitted distribution. Chernoff and Lehmann [12] show that if H_0 is true, the test statistic converges to a distribution function that lies between the distribution functions of chi-square distributions with $k-1$ and $k-c-1$ degrees of freedom. Let α denote the significance level, and consider $\chi^2_{k-1,1-\alpha}$ such that $F(\chi^2_{k-1,1-\alpha}) = 1-\alpha$, where $F(x)$ is the distribution function of the chi-square distribution with $k-1$ degrees of freedom. It is often recommended that we reject the null hypothesis at the significance level α if the test statistic is greater than $\chi^2_{k-1,1-\alpha}$ (see, e.g., Law and Kelton [29]). In our context, $k = M_t$ and α, whose interpretation is as above, is given by the decision maker.

Since $P_{t,i}$ should define a probability distribution, we have $\sum_i P_{t,i} = 1$ and $P_{t,i} \ge 0$. Let $\mathbf{P}_t$ denote the vector of $(P_{t,i})_i$. The set of distributions that satisfy the chi-square test is

$$\mathcal{P}_t(\mathbf{d}_t) = \left\{ \mathbf{P}_t \,\middle|\, \mathbf{A}_t\mathbf{P}_t = \mathbf{b}_t,\ \sum_i \frac{(N_{t,i}(\mathbf{d}_t) - n_t(\mathbf{d}_t)P_{t,i})^2}{n_t(\mathbf{d}_t)P_{t,i}} \le \chi_t^2,\ \mathbf{P}_t \ge \mathbf{0} \right\} \qquad (5.9)$$

for any $t = 1, ..., T$. The linear constraints $\mathbf{A}_t\mathbf{P}_t = \mathbf{b}_t$ capture at least the fact that $\sum_i P_{t,i} = 1$. They can also be used to model more complicated properties of the distribution set, such as constraints on the expected value, any moment or desired percentiles of the distributions. It is straightforward to establish the compactness of

$\mathcal{P}_t(\mathbf{d}_t)$.

We next give an alternative optimality equation that exploits the structure of (5.9). We first provide an alternative characterization of $\mathcal{P}_t(\mathbf{d}_t)$. We assume that every norm is the Euclidean norm.

Lemma 5.2. *The set of demand distributions $\mathcal{P}_t(\mathbf{d}_t)$ defined in (5.9) is equivalent to the projection of the set*

$$\left\{(\mathbf{P}_t, \mathbf{Q}_t) \left| \mathbf{A}_t\mathbf{P}_t = \mathbf{b}_t, \sum_i N_{t,i}(\mathbf{d}_t)^2 Q_{t,i} - n_t(\mathbf{d}_t)^2 \le n_t(\mathbf{d}_t)\chi_t^2, \right.\right.$$
$$\left.\left\| \begin{bmatrix} P_{t,i} - Q_{t,i} \\ 2 \end{bmatrix} \right\| \le P_{t,i} + Q_{t,i} \right\}$$

on the space of $\mathbf{P}_t$.

Proof. Since $\sum_i P_{t,i} = 1$ and $\sum_i N_{t,i}(\mathbf{d}_t) = n_t(\mathbf{d}_t)$, we have

$$\begin{aligned}\sum_i \frac{(N_{t,i}(\mathbf{d}_t) - n_t(\mathbf{d}_t)P_{t,i})^2}{n_t(\mathbf{d}_t)P_{t,i}} &= \sum_i \frac{N_{t,i}(\mathbf{d}_t)^2}{n_t(\mathbf{d}_t)P_{t,i}} - \sum_i 2N_{t,i}(\mathbf{d}_t) + \sum_i n_t(\mathbf{d}_t)P_{t,i} \\ &= \sum_i \frac{N_{t,i}(\mathbf{d}_t)^2}{n_t(\mathbf{d}_t)P_{t,i}} - n_t(\mathbf{d}_t).\end{aligned}$$

As χ_t^2 and $n_t(\mathbf{d}_t)$ are finite, we have $P_{t,i} > 0$ for any i. Therefore,

$$\sum_i \frac{N_{t,i}(\mathbf{d}_t)^2}{n_t(\mathbf{d}_t)P_{t,i}} - n_t(\mathbf{d}_t) \le \chi_t^2$$

is equivalent to

$$\sum_i N_{t,i}(\mathbf{d}_t)^2 Q_{t,i} - n_t(\mathbf{d}_t)^2 \le n_t(\mathbf{d}_t)\chi_t^2, \ \frac{1}{P_{t,i}} \le Q_{t,i}, \ P_{t,i}, Q_{t,i} > 0.$$

Obviously, the constraints $\frac{1}{P_{t,i}} \le Q_{t,i}$ and $P_{t,i}, Q_{t,i} > 0$ are equivalent to

$$P_{t,i}Q_{t,i} \ge 1, \ P_{t,i}, Q_{t,i} \ge 0, \quad \Longleftrightarrow \quad \begin{bmatrix} P_{t,i} & 1 \\ 1 & Q_{t,i} \end{bmatrix} \succeq 0.$$

Note that the eigenvalues of the matrix $\begin{bmatrix} P_{t,i} & 1 \\ 1 & Q_{t,i} \end{bmatrix}$ are

$$\frac{P_{t,i} + Q_{t,i} \pm \sqrt{(P_{t,i} - Q_{t,i})^2 + 4}}{2},$$

therefore the positive semidefinite constraint is equivalent to

$$\frac{P_{t,i} + Q_{t,i} - \sqrt{(P_{t,i} - Q_{t,i})^2 + 4}}{2} \geq 0 \iff \left\| \begin{bmatrix} P_{t,i} - Q_{t,i} \\ 2 \end{bmatrix} \right\| \leq P_{t,i} + Q_{t,i},$$

which proves the proposition. □

Lemma 5.2 shows that the set $\mathcal{P}_t(\mathbf{d}_t)$ can be defined by a set of linear and second order cone constraints (see, e.g., Lobo et al. [31]). Note that the second order cone constraints are a special class of positive semidefinite constraints and they have better computational properties than general positive semidefinite constraints. This alternative definition of the set $\mathcal{P}_t(\mathbf{d}_t)$ also suggests a compact optimality equation.

Proposition 5.4. *The optimality equation of the robust stochastic model* (5.5) *is equivalent to*

$$\begin{aligned} V_t(x_t, \mathbf{d}_t) \quad = \quad \min_{y_t, \mathbf{U}_t, \mathbf{p}_t, \mathbf{u}_t, \lambda_t} \quad & K\mathbb{I}(y_t - x_t) + c_t(y_t - x_t) + \mathbf{p}_t^T \mathbf{b}_t - 2\sum_i u_{t,i} N_{t,i}(\mathbf{d}_t) \\ & +\lambda_t \left(n_t(\mathbf{d}_t)^2 + n_t(\mathbf{d}_t)\chi_t^2\right) \\ s.t. \quad & \left\| \begin{bmatrix} \mathbf{p}_t^T - U_{t,i}\mathbf{A}_{t,i} - \lambda_t \\ 2u_{t,i} \end{bmatrix} \right\| \leq \mathbf{p}_t^T - U_{t,i}\mathbf{A}_{t,i} + \lambda_t \qquad \forall i \\ & U_{t,i} \geq h_t\left(y_t - D_{t,i}\right) + \theta V_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) \qquad \forall i \\ & U_{t,i} \geq b_t\left(y_t - D_{t,i}\right) + \theta V_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) \qquad \forall i \\ & y_t \geq x_t, \end{aligned} \tag{5.10}$$

for any $t = 1, ..., T$.

Note that this is not the standard optimality equation since $V_{t+1}(\cdot)$ is present in constraints and not the objective function. We use it later to obtain computationally tractable control policies.

Proof. The optimality equation defined in (5.5) is equivalent to

$$
\begin{aligned}
V_t(x_t, \mathbf{d}_t) \quad = \quad & \min_{y_t \geq x_t, \mathbf{U}_t} \quad K\mathbb{I}(y_t - x_t) + c_t(y_t - x_t) + \max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \sum_i P_{t,i} U_{t,i} \\
& \text{s.t.} \quad U_{t,i} \geq h_t\,(y_t - D_{t,i}) + \theta V_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) \quad \text{for every } i \\
& \qquad\quad U_{t,i} \geq b_t\,(y_t - D_{t,i}) + \theta V_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) \quad \text{for every } i.
\end{aligned}
\tag{5.11}
$$

According to Lemma 5.2, the maximization problem $\max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \sum_i P_{t,i} U_{t,i}$ is the second order cone problem and hence it is equivalent to its Lagrangian dual

$$
\begin{aligned}
\min_{\mathbf{p}_t, \mathbf{u}_t, \lambda_t} \quad & \mathbf{p}_t^T \mathbf{b}_t - 2 \sum_i u_{t,i} N_{t,i}(\mathbf{d}_t) + \lambda_t \left(n_t(\mathbf{d}_t)^2 + n_t(\mathbf{d}_t) \chi_t^2 \right) \\
\text{s.t.} \quad & \left\| \begin{bmatrix} \mathbf{p}_t^T - U_{t,i} \mathbf{A}_{t,i} - \lambda_t \\ 2u_{t,i} \end{bmatrix} \right\| \leq \mathbf{p}_t^T - U_{t,i} \mathbf{A}_{t,i} + \lambda_t \qquad \text{for every } i,
\end{aligned}
\tag{5.12}
$$

where $\mathbf{A}_{t,i}$ denotes the ith row of matrix $\mathbf{A}_t$ (see, e.g., Lobo et al. [31]).

Note that (5.10) is obtained by replacing the maximization problem in (5.11) by (5.12). Therefore, the proposition is equivalent to proving that problem (5.10) is equivalent to problem (5.11). Let z_1^* and z_2^* denote the optimal values of problems (5.11) and (5.10), respectively.

We first show that $z_1^* \geq z_2^*$. Let y_t^*, $\mathbf{U}_t^*$ and $\mathbf{P}_t^*$ denote an optimal solution for problem (5.11). Problem $\max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \sum_i P_{t,i} U_{t,i}$ has a finite optimal value if we set $U_{t,i}$ to $U_{t,i}^*$. Therefore, there exists an optimal solution $\mathbf{p}_t^*$, $\mathbf{u}_t^*$, and λ_t^* for its dual, problem (5.12), and the corresponding optimal value is $z_1^* - K\mathbb{I}(y_t^* - x_t) - c_t(y_t^* - x_t)$. Obviously y_t^*, $\mathbf{U}_t^*$, $\mathbf{p}_t^*$, $\mathbf{u}_t^*$, and λ_t^* is a feasible solution to (5.10) with the objective value z_1^*, and therefore we have $z_1^* \geq z_2^*$.

It remains to show $z_1^* \leq z_2^*$. Let y^*, $\mathbf{U}^*$, $\mathbf{p}^*$, $\mathbf{u}^*$ and λ^* be an optimal solution for problem (5.10). Problem (5.12) with $U_{t,i} = U_{t,i}^*$ has a finite optimal value, and therefore the problem $\max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \sum_i P_{t,i} U_{t,i}^*$ has an optimal solution $\mathbf{P}^*$ with the optimal cost $z_2^* - K\mathbb{I}(y_t^* - x_t) - c_t(y_t^* - x_t)$. Since y^*, $\mathbf{U}^*$, and $\mathbf{P}^*$ give a feasible solution to problem (5.11) and the corresponding objective value is z_2^*, we have $z_1^* \leq z_2^*$. □

5.4.1 Computation of $(s,\ S)$ Levels

Next we give a computational approach to compute $s_t(\mathbf{d}_t)$ and $S_t(\mathbf{d}_t)$.

Theorem 5.2. *Let $S_t(\mathbf{d}_t)$ be an optimal solution to the minimization problem*

$$\begin{array}{lll} \min\limits_{y_t, \mathbf{U}_t, \mathbf{p}_t, \mathbf{u}_t, \lambda_t} & c_t y_t + \mathbf{p}_t^T \mathbf{b}_t - 2\sum\limits_i u_{t,i} N_{t,i}(\mathbf{d}_t) + \lambda_t \left(n_t(\mathbf{d}_t)^2 + n_t(\mathbf{d}_t)\chi_t^2 \right) & \\ \text{s.t.} & \left\| \begin{bmatrix} \mathbf{p}_t^T \mathbf{A}_{t,i} - U_{t,i} - \lambda_t \\ 2u_{t,i} \end{bmatrix} \right\| \leq \mathbf{p}_t^T \mathbf{A}_{t,i} - U_{t,i} + \lambda_t & \text{for every } i \\ & U_{t,i} \geq h_t \left(y_t - D_{t,i} \right) + \theta V_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) & \text{for every } i \\ & U_{t,i} \geq b_t \left(D_{t,i} - y_t \right) + \theta V_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) & \text{for every } i, \end{array}$$

and let $s_t(\mathbf{d}_t)$ be the smallest element of the set

$$\{s_t(\mathbf{d}_t) \mid s_t(\mathbf{d}_t) \leq S_t(\mathbf{d}_t),\ f_t(s_t, \mathbf{d}_t) = f_t(S_t, \mathbf{d}_t) + K\},$$

where $f_t(y_t, \mathbf{d}_t)$ is defined by (5.8).

A state dependent (s, S) policy is optimal for the robust stochastic model (5.5) *with $\mathcal{P}_t(\mathbf{d}_t)$ defined by* (5.9)*, and the (s, S) levels are given by $s_t(\mathbf{d}_t)$ and $S_t(\mathbf{d}_t)$ respectively. If there is no fixed cost, a state dependent base-stock policy is optimal, and the base-stock level given the realized demand $\mathbf{d}_t$ is $S_t(\mathbf{d}_t)$.*

Proof. The minimization problem to calculate $S_t(\mathbf{d}_t)$ follows from the alternative optimality equation (5.10). □

Consider the models where the historical data used for period t is independent of the realized demand from periods 1 to $t-1$, i.e., the number of observations $N_{t,i}$ in the ith bin and the total number of available observations n_t are constant for any realized demand $\mathbf{d}_t$. Therefore, the set of distributions that satisfy the chi-square test is defined by

$$\mathcal{P}_t = \left\{ \mathbf{P}_t \middle| \mathbf{A}_t\mathbf{P}_t = \mathbf{b}_t, \ \sum_i \frac{(N_{t,i} - n_t P_{t,i})^2}{n_t P_{t,i}} \leq \chi_t^2, \ \mathbf{P}_t \geq \mathbf{0} \right\} \qquad t = 1, ..., T. \quad (5.13)$$

In this case, the optimality equation of the robust model is reduced to

$$V_t(x_t) = \min_{y_t \geq x_t} \max_{\mathbf{P}_t \in \mathcal{P}_t} \left\{ \sum_i P_{t,i} \left(C_t(x_t, y_t, D_{t,i}) + \theta V_{t+1}(y_t - D_{t,i}) \right) \right\} \qquad t = 1, ..., T, \quad (5.14)$$

where $\mathcal{P}_t$ and $C_t(x_t, y_t, D_{t,i})$ are defined by (5.13) and (5.1) respectively.

Alternatively, it can be written as

$$\begin{aligned} V_t(x_t) \ = \ \min_{y_t, \mathbf{U}_t, \mathbf{p}_t, \mathbf{u}_t, \lambda_t} \quad & K\mathbb{I}(y_t - x_t) + c_t(y_t - x_t) + \mathbf{p}_t^T \mathbf{b}_t - 2\sum_i u_{t,i} N_{t,i} \\ & + \lambda_t \left(n_t^2 + n_t \chi_t^2\right) \\ \text{s.t.} \quad & \left\| \begin{bmatrix} \mathbf{p}_t^T - U_{t,i}\mathbf{A}_{t,i} - \lambda_t \\ 2u_{t,i} \end{bmatrix} \right\| \leq \mathbf{p}_t^T - U_{t,i}\mathbf{A}_{t,i} + \lambda_t \quad \text{for every } i \\ & U_{t,i} \geq h_t(y_t - D_{t,i}) + \theta V_{t+1}(y_t - D_{t,i}) \qquad \text{for every } i \\ & U_{t,i} \geq b_t(D_{t,i} - y_t) + \theta V_{t+1}(y_t - D_{t,i}) \qquad \text{for every } i \\ & y_t \geq x_t, \end{aligned} \quad (5.15)$$

for any $t = 1, ..., T$.

The corresponding optimal (s, S) policy levels are also independent of the realized demand $\mathbf{d}_t$.

Theorem 5.3. *The (s, S) policy is optimal for the robust stochastic model* (5.14)*. In particular, let S_t be the optimal solution to the minimization problem*

$$\begin{aligned} \min_{y_t, \mathbf{U}_t, \mathbf{p}_t, \mathbf{u}_t, \lambda_t} \quad & c_t y_t + \mathbf{p}_t^T \mathbf{b}_t - 2\sum_i u_{t,i} N_{t,i} + \lambda_t \left(n_t^2 + n_t \chi_t^2\right) \\ \text{s.t.} \quad & \left\| \begin{bmatrix} \mathbf{p}_t^T \mathbf{A}_{t,i} - U_{t,i} - \lambda_t \\ 2u_{t,i} \end{bmatrix} \right\| \leq \mathbf{p}_t^T \mathbf{A}_{t,i} - U_{t,i} + \lambda_t \quad \text{for every } i \\ & U_{t,i} \geq h_t(y_t - D_{t,i}) + \theta V_{t+1}(y_t - D_{t,i}) \qquad \text{for every } i \\ & U_{t,i} \geq b_t(D_{t,i} - y_t) + \theta V_{t+1}(y_t - D_{t,i}) \qquad \text{for every } i, \end{aligned}$$

and let s_t be the smallest element of the set

$$\{s_t \mid s_t \leq S_t,\ f_t(s_t) = f_t(S_t) + K\},$$

where

$$f_t(y_t) = c_t y_t + \max_{\mathbf{P}_t \in \mathcal{P}_t} \sum_i P_{t,i} \left[h_t \left(y_t - D_{t,i}\right)^+ + b_t \left(y_t - D_{t,i}\right)^- + \theta V_{t+1}(y_t - D_{t,i}) \right].$$

The policy is to order $S_t - x_t$ units in period t if $x_t \leq s_t$, and no order is placed otherwise.

Without fixed procurement cost, a basestock policy is optimal, that is, $S_t - x_t$ units are ordered in period t if $x_t \leq S_t$, and no order is placed otherwise.

5.4.2 Convergence of Robust Models Based on Chi-Square Test

Up to this point we assumed that the bins are given. In this part we shed light on the robust models based on the chi-square test with varying number of bins and their sizes. In particular, we explore the case when the number of samples increases and accordingly the bin sizes tend to 0. The main results concern with such a case when χ_t^2 also converges to 0. We show that the cost-to-go function of the robust model converges to the cost-to-go function of the nominal distribution under mild technical assumptions. In a special case we are able to establish a rate of convergence result. The convergence study does not only provide the asymptotic performance of the robust model when the sample size approaches infinitely, but also indicates to select small bins and χ^2 values in the presence of a significant number of samples.

In this part our starting point is that the demand random variables $\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_T$ are subject to some multivariate distribution. Although the distribution may not be known, we assume that histograms pertaining to the robust model are obtained from samples from these distributions. We study the behavior of cost-to-go functions as the number of samples increases.

Let $\tilde{\mathbf{d}}_t = \left[\tilde{D}_1, ..., \tilde{D}_{t-1}\right]$, and let $F_t\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ denote the conditional cumulative distribution function of demand $\tilde{D}_t$ given realized demand $\mathbf{d}_t$ from periods 1 to $t-1$. Assuming that the conditional distribution function is known for each t and $\mathbf{d}_t$, we can solve the corresponding dynamic programming problem, and obtain $\bar{V}(x_t, \mathbf{d}_t)$ for each period t,

$$\bar{V}_t(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} \left\{ \int_{D_t} \left(C_t(x_t, y_t, D_t) + \theta \bar{V}_{t+1}\left(y_t - D_t, [\mathbf{d}_t, D_t]\right)\right) dF_t\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right) \right\},$$

where $t = 1, ..., T$.

We investigate how accurately the value functions $V_t(x_t, \mathbf{d}_t)$ of our robust model approximate the true cost-to-go function $\bar{V}(x_t, \mathbf{d}_t)$ if histograms are based on samples.

We start by analyzing the convergence of the robust model as χ_t^2 converges to 0. Let $\hat{V}_t(x_t, \mathbf{d}_t)$ denote the cost-to-go function of the stochastic model with the distribution defined by

$$P\left(\tilde{D}_\tau = D_{\tau,i} \middle| \tilde{\mathbf{d}}_\tau = \mathbf{d}_\tau\right) = \frac{N_{\tau,i}(\mathbf{d}_t)}{n_\tau(\mathbf{d}_t)}, \qquad \tau = t, ..., T. \tag{5.16}$$

Formally,

$$\hat{V}_t(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} \left\{ \sum_i \frac{N_{t,i}(\mathbf{d}_t)}{n_t(\mathbf{d}_t)} \left(C_t(x_t, y_t, D_{t,i}) + \theta \hat{V}_{t+1}\left(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]\right)\right) \right\},$$

for any $t = 1, ..., T$.

Proposition 5.5. *If $V_{T+1}(\cdot) = \hat{V}_{T+1}(\cdot)$, then for any x_t, $\mathbf{d}_t$, and t, we have*

$$\lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} V_t(x_t, \mathbf{d}_t) = \hat{V}_t(x_t, \mathbf{d}_t).$$

Proof. The proposition clearly holds for $t = T + 1$. Suppose that it holds for any τ such that $\tau > t$. To simplify notation, let

$$U_{t,i}(x_t, y_t) = C_t(x_t, y_t, D_{t,i}) + \theta V_{t+1}\left((y_t - D_{t,i})^+, [\mathbf{d}_t, D_{t,i}]\right) \tag{5.17}$$

and therefore

$$V_t(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} \max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \left\{ \sum_i P_{t,i} U_{t,i}(x_t, y_t) \right\} \qquad t = 1, ..., T.$$

According to the definition of $\mathcal{P}_t(\mathbf{d}_t)$ in (5.9), we have

$$\mathcal{P}_t(\mathbf{d}_t) \subset \left\{ \mathbf{P}_t \left| \frac{(N_{t,i}(\mathbf{d}_t) - n_t(\mathbf{d}_t)P_{t,i})^2}{n_t(\mathbf{d}_t)P_{t,i}} \leq \chi_t^2 \text{ for every } i \right. \right\}.$$

Let $\underline{P}_{t,i}$ and $\overline{P}_{t,i}$ correspond to the solutions of $\frac{(N_{t,i} - n_t P_{t,i})^2}{n_t P_{t,i}} = \chi_t^2$, i.e.,

$$\underline{P}_{t,i}(\mathbf{d}_t) = \frac{N_{t,i}(\mathbf{d}_t)}{n_t(\mathbf{d}_t)} + \frac{\chi_t^2}{2n_t(\mathbf{d}_t)} - \frac{\sqrt{4N_{t,i}(\mathbf{d}_t)\chi_t^2 + (\chi_t^2)^2}}{2n_t(\mathbf{d}_t)},$$
$$\overline{P}_{t,i}(\mathbf{d}_t) = \frac{N_{t,i}(\mathbf{d}_t)}{n_t(\mathbf{d}_t)} + \frac{\chi_t^2}{2n_t(\mathbf{d}_t)} + \frac{\sqrt{4N_{t,i}(\mathbf{d}_t)\chi_t^2 + (\chi_t^2)^2}}{2n_t(\mathbf{d}_t)}.$$

It follows directly that

$$\mathcal{P}_t(\mathbf{d}_t) \subset \overline{\mathcal{P}}_t(\mathbf{d}_t) = \left\{ \mathbf{P}_t(\mathbf{d}_t) \left| \underline{P}_{t,i}(\mathbf{d}_t) \leq P_{t,i} \leq \overline{P}_{t,i}(\mathbf{d}_t) \text{ for every } i \right. \right\}.$$

Therefore we obtain

$$\max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \left\{ \sum_i P_{t,i} U_{t,i}(x_t, y_t) \right\} \leq \max_{\mathbf{P}_t \in \overline{\mathcal{P}}_t(\mathbf{d}_t)} \left\{ \sum_i P_{t,i} U_{t,i}(x_t, y_t) \right\}$$
$$= \sum_{i: U_{t,i}(x_t, y_t) \leq 0} \underline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t) + \sum_{i: U_{t,i}(x_t, y_t) > 0} \overline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t).$$

Minimizing both sides over $\{x_t | y_t \geq x_t\}$ yields

$$V_t(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} \max_{\mathbf{P}_t \in \mathcal{P}_t(\mathbf{d}_t)} \left\{ \sum_i P_{t,i} U_{t,i}(x_t, y_t) \right\}$$
$$\leq \min_{y_t \geq x_t} \left\{ \sum_{i: U_{t,i}(x_t, y_t) \leq 0} \underline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t) + \sum_{i: U_{t,i}(x_t, y_t) > 0} \overline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t) \right\}.$$

Let $y_t^* \geq x_t$ be a minimizer of $\hat{V}_t(x_t, \mathbf{d}_t)$. Then

$$V_t(x_t, \mathbf{d}_t) \leq \sum_{i: U_{t,i}(x_t, y_t^*) \leq 0} \underline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t^*) + \sum_{i: U_{t,i}(x_t, y_t^*) > 0} \overline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t^*). \quad (5.18)$$

Taking the limit on both sides yields

$$\lim_{\substack{\chi_\tau^2 \to 0, \\ \forall \tau \geq t}} V_t(x_t, \mathbf{d}_t) \leq \lim_{\substack{\chi_\tau^2 \to 0, \\ \forall \tau \geq t}} \left\{ \sum_{i: U_{t,i}(x_t, y_t^*) \leq 0} \underline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t^*) + \sum_{i: U_{t,i}(x_t, y_t^*) > 0} \overline{P}_{t,i}(\mathbf{d}_t) U_{t,i}(x_t, y_t^*) \right\}.$$

Note that

$$\lim_{\chi_t^2 \to 0} \underline{P}_{t,i}(\mathbf{d}_t) = \lim_{\chi_t^2 \to 0} \overline{P}_{t,i}(\mathbf{d}_t) = \frac{N_{t,i}(\mathbf{d}_t)}{n_t(\mathbf{d}_t)},$$

and by the induction assumption

$$\lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} U_{t,i}(x_t, y_t^*) = C_t(x_t, y_t^*, D_{t,i}) + \theta \hat{V}_{t+1}\left((y_t^* - D_{t,i})^+, [\mathbf{d}_t, D_{t,i}]\right).$$

By the definition of y_t^*,

$$\lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} V_t(x_t, \mathbf{d}_t) \leq \hat{V}_t(x_t, \mathbf{d}_t).$$

Since the distribution defined by (5.16) is in $\mathcal{P}_t(\mathbf{d}_t)$, it is easy to verify that $\hat{V}_t(x_t, \mathbf{d}_t) \leq V_t(x_t, \mathbf{d}_t)$. Therefore, we have

$$\lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} V_t(x_t, \mathbf{d}_t) = \hat{V}_t(x_t, \mathbf{d}_t). \qquad \square$$

Now suppose that for each period t, we have a sequence of samples. For any $k = 1, 2, \ldots$ we have the set of m_t^k available samples for period t,

$$\mathsf{d}_t^k = \left\{ \mathsf{d}_{t,1}^k, \ldots, \mathsf{d}_{t,m_t^k}^k \right\}.$$

The samples are drawn from the distribution $\tilde{D}_t$ conditioned on realized demand $\mathbf{d}_t$. Therefore, given realized demand $\mathbf{d}_t$ from periods 1 to $t-1$, and the kth sample set d_t^k for period t, we can construct a histogram such that the total number of samples selected in the histogram is $n_t^k(\mathbf{d}_t)$, the boundary of bins are $\left\{D_{t,1}^k, ..., D_{t,M_t^k}^k\right\}$, and the number of samples falling in the ith bin $[D_{t,i}^k, D_{t,i+1}^k)$ is denoted by $N_{t,i}^k(\mathbf{d}_t)$. This histogram naturally defines an empirical distribution with the conditional cumulative distribution function $F_t^k\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ defined by

$$F_t^k\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right) = \frac{1}{n_t^k(\mathbf{d}_t)} \sum_{i=1}^{i^*(D_t)} N_{t,i}^k(\mathbf{d}_t),$$

where $i^* = i^*(D_t)$ is such that $D_{t,i^*}^k \leq D_t < D_{t,i^*+1}^k$.

Note that the kth set of samples d_t^k, $t = 1, ..., T$, also defines a robust model $V_t^k(x_t, \mathbf{d}_t)$ based on the just described parameters $n_t^k(\mathbf{d}_t)$ and $N_{t,i}^k(\mathbf{d}_t)$. In the remainder of this section we analyze under what conditions $V_t^k(x_t, \mathbf{d}_t)$ converge to $\bar{V}_t(x_t, \mathbf{d}_t)$, which denotes the cost-to-go function of the stochastic model with respect to true distributions. We always assume that the distribution of $\tilde{D}_t$ has finite support $[0, D_t^{\max}]$ for any t, and $V_{T+1}(\cdot) = \hat{V}_{T+1}(\cdot) = \bar{V}_{T+1}(\cdot)$. We first study the case with general distributions, and we derive stronger results when the distributions are discrete.

General Distributions

We first show convergence under general distributions. We only need the distribution functions of samples to converge pointwise to the distribution function of the true distribution and $\chi_t^2 \to 0$.

Proposition 5.6. *Suppose that for any $\mathbf{d}_t$ and t we have*

$$\lim_{k\to\infty} F_t^k\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right) = F_t\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$$

for every D_t. If there is no fixed procurement cost, then

$$\lim_{k\to\infty} \lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} V_t^k(x_t, \mathbf{d}_t) = \bar{V}_t(x_t, \mathbf{d}_t).$$

Furthermore, the convergence is uniform with respect to k.

In the proof, we need some concepts from measure theory and a known result. A sequence of measures μ_k converge to a measure μ *weakly* if $\int f d\mu_k \overset{k\to\infty}{\longrightarrow} \int f d\mu$ for every continuous bounded function f. A sequence of measures μ_k converge to a measure μ *setwise* if $\mu_k(B) \overset{k\to\infty}{\longrightarrow} \mu(B)$ for every measurable set B.

It is well known that convergence in distribution does not imply setwise convergence of the underlying probability measures. Indeed, convergence in distribution is equivalent to weak convergence. It is not difficult to see that setwise convergence implies weak convergence.

The following result can be found in Royden [45], page 232.

Proposition 5.7. *Let μ_k be a sequence of measures converging setwise to a measure μ. Let $\{f_k\}_k$, $\{g_k\}_k$ be two sequences of measurable functions converging pointwise to f and g respectively. Furthermore, let $|f_k| \le g_k$ for every k and $\lim_k \int g_k d\mu_k = \int g d\mu < \infty$. Then*

$$\lim_k \int f_k d\mu_k = \int f d\mu.$$

Next, we will give the proof for Proposition 5.6.

Proof of Proposition 5.6. Let $\hat{V}_t^k(x_t, \mathbf{d}_t)$ denote the cost-to-go function of the stochastic model with respect to the empirical distribution $F_\tau^k\left(D_\tau \middle| \tilde{\mathbf{d}}_\tau = \mathbf{d}_\tau\right)$ for any $\tau \ge t$. As shown in Proposition 5.5 we have

$$\begin{aligned}&\lim_{\chi_\tau^2 \to 0, \forall \tau \ge t} V_t^k(x_t, \mathbf{d}_t) = \hat{V}_t^k(x_t, \mathbf{d}_t) \\ &= \min_{y_t \ge x_t}\left\{\int_0^{D_t^{\max}} \left(C_t(x_t, y_t, D_t) + \theta \hat{V}_{t+1}^k(y_t - D_t, [\mathbf{d}_t, D_t])\right) dF_t^k\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)\right\}.\end{aligned}$$

Therefore, it is sufficient to show that

$$\lim_{k\to\infty} \hat{V}_t^k(x_t, \mathbf{d}_t) = \bar{V}_t(x_t, \mathbf{d}_t).$$

Let us fix x_0. Then under an optimal policy the inventory is always within

$$\left[x_0 - \sum_{\tau=1}^{T} D_\tau^{\max}, x_0 + \sum_{\tau=1}^{T} D_\tau^{\max}\right]. \tag{5.19}$$

Therefore we can assume that y_t and $x_t = y_t - \tilde{D}_t$ are always within this range for any t. It is easy to show by induction that $\left|\hat{V}_t^k(x_t, \mathbf{d}_t)\right| \leq M(x_0) < \infty$, where $M(x_0)$ is a constant depending only on x_0.

Note that as $V_{T+1}(\cdot) = \hat{V}_{T+1}(\cdot) = \bar{V}_{T+1}(\cdot)$, the proposition holds for period $T+1$. Suppose that for any $\tau > t$, $\hat{V}_\tau^k(x_\tau, \mathbf{d}_\tau) \to \bar{V}_\tau(x_\tau, \mathbf{d}_\tau)$ pointwise.

Let

$$\begin{aligned} f_t^k(x_t, y_t, \mathbf{d}_t) &= \int_0^{D_t^{\max}} \left(C_t(x_t, y_t, D_t) + \theta \hat{V}_{t+1}^k(y_t - D_t, [\mathbf{d}_t, D_t])\right) dF_t^k\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right), \\ f_t(x_t, y_t, \mathbf{d}_t) &= \int_0^{D_t^{\max}} \left(C_t(x_t, y_t, D_t) + \theta \bar{V}_{t+1}(y_t - D_t, [\mathbf{d}_t, D_t])\right) dF_t\left(D_t \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right). \end{aligned} \tag{5.20}$$

Note that $\hat{V}_t^k(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} f_t^k(x_t, y_t, \mathbf{d}_t)$ and $\bar{V}_t(x_t, \mathbf{d}_t) = \min_{y_t \geq x_t} f_t(x_t, y_t, \mathbf{d}_t)$.

Let $\mu_{t,k}^{\mathbf{d}_t}$ be the Lebesgue-Stieltjes measure based on $F_t^k\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$,[2] and we define similarly $\mu_t^{\mathbf{d}_t}$ with respect to $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$. By assumption, F_t^k converge pointwise to F_t at any point. It is now easy to see that as a result $\mu_{t,k}^{\mathbf{d}_t}$ converge setwise to $\mu_t^{\mathbf{d}_t}$.

Let now $g_{t,k}(D_t) = M(x_0)$, i.e., a sequence of constant functions, and

$$f_{t,k}^{\mathbf{d}_t, y_t}(D_t) = \hat{V}_{t+1}^k(y_t - D_t, [\mathbf{d}_t, D_t]).$$

By definition we have $\left|f_{t,k}^{\mathbf{d}_t, y_t}(D_t)\right| \leq g_{t,k}(D_t)$. Let also $g_t(D_t) = M(x_0)$. Clearly $g_{t,k}$ converge pointwise to g_t and by the induction assumption $f_{t,k}^{\mathbf{d}_t, y_t}(D_t)$ converge pointwise to $f_t^{\mathbf{d}_t, y_t}(D_t)$ defined by $f_t^{\mathbf{d}_t, y_t}(D_t) = \bar{V}_{t+1}(y_t - D_t, [\mathbf{d}_t, D_t])$.

Furthermore, $\int g_{t,k} d\mu_{t,k}^{\mathbf{d}_t} = M(x_0) = \int g_t d\mu_t^{\mathbf{d}_t}$. Thus we can apply Proposition 5.7,

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[2] $\mu_{t,k}^{\mathbf{d}_t}([a, b]) = F_t^k\left(b \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right) - F_t^k\left(a \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$, and then $\mu_{t,k}^{\mathbf{d}_t}$ is extended by the Riesz representation theorem.

which implies

$$\lim_{k\to\infty}\int_0^{D_t^{\max}} \hat{V}_{t+1}^k(y_t - D_t, [\mathbf{d}_t, D_t])dF_t^k\left(D_t\,\middle|\,\tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$$
$$= \int_0^{D_t^{\max}} \bar{V}_{t+1}(y_t - D_t, [\mathbf{d}_t, D_t])dF_t\left(D_t\,\middle|\,\tilde{\mathbf{d}}_t = \mathbf{d}_t\right).$$

Note that this holds at every y_t and $\mathbf{d}_t$.

Since setwise convergence implies weak convergence and since $C_t(x_t, y_t, D_t)$ is continuous and bounded, by weak convergence we obtain

$$\lim_{k\to\infty}\int_0^{D_t^{\max}} C_t(x_t, y_t, D_t)dF_t^k\left(D_t\,\middle|\,\tilde{\mathbf{d}}_t = \mathbf{d}_t\right) = \int_0^{D_t^{\max}} C_t(x_t, y_t, D_t)dF_t\left(D_t\,\middle|\,\tilde{\mathbf{d}}_t = \mathbf{d}_t\right).$$

Therefore, we have

$$\lim_{k\to\infty} f_t^k(x_t, y_t, \mathbf{d}_t) = f_t(x_t, y_t, \mathbf{d}_t).$$

Note that if finite convex functions $f_k(\mathbf{x}) \to f(\mathbf{x})$ pointwise, then $f_k(\mathbf{x}) \to f(\mathbf{x})$ uniformly on each compact subset of the domain (see, e.g., Rockafellar [44]). Since there are no fixed costs, both $f_t^k(x_t, y_t, \mathbf{d}_t)$ and $f_t(x_t, y_t, \mathbf{d}_t)$ are convex in both x_t and y_t. Therefore, given $\mathbf{d}_t$, $f_t^k(x_t, y_t, \mathbf{d}_t) \to f_t(x_t, y_t, \mathbf{d}_t)$ pointwise implies that $f_t^k(x_t, y_t, \mathbf{d}_t) \to f_t(x_t, y_t, \mathbf{d}_t)$ uniformly.

Let $y_t^k(x_t, \mathbf{d}_t)$ and $y_t^*(x_t, \mathbf{d}_t)$ denote the minimizers of $\hat{V}_t^k(x_t, \mathbf{d}_t)$ and $\bar{V}_t(x_t, \mathbf{d}_t)$, respectively. Clearly,

$$\hat{V}_t^k(x_t, \mathbf{d}_t) = f_t^k(x_t, y_t^k(x_t, \mathbf{d}_t), \mathbf{d}_t) \quad \text{and} \quad \bar{V}_t(x_t, \mathbf{d}_t) = f_t(x_t, y_t^*(x_t, \mathbf{d}_t), \mathbf{d}_t).$$

According to uniform convergence, for any $\epsilon > 0$, there exists a positive integer K such that

$$\left|f_t^k(x_t, y_t, \mathbf{d}_t) - f_t(x_t, y_t, \mathbf{d}_t)\right| < \epsilon$$

for any x_t, y_t, and $k > K$. Therefore,

$$f_t(x_t, y_t^k(x_t, \mathbf{d}_t), \mathbf{d}_t) - \epsilon < f_t^k(x_t, y_t^k(x_t, \mathbf{d}_t), \mathbf{d}_t) = \hat{V}_t^k(x_t, \mathbf{d}_t).$$

Note that $f_t(x_t, y_t^k(x_t, \mathbf{d}_t), \mathbf{d}_t) \geq f_t(x_t, y_t^*(x_t, \mathbf{d}_t), \mathbf{d}_t) = \bar{V}_t(x_t, \mathbf{d}_t)$ and therefore we have

$$\bar{V}_t(x_t, \mathbf{d}_t) - \epsilon = f_t(x_t, y_t^k(x_t, \mathbf{d}_t), \mathbf{d}_t) - \epsilon < \hat{V}_t^k(x_t, \mathbf{d}_t).$$

Also note that

$$f_t^k(x_t, y_t^*(x_t, \mathbf{d}_t), \mathbf{d}_t) \geq f_t^k(x_t, y_t^k(x_t, \mathbf{d}_t), \mathbf{d}_t) = \hat{V}_t^k(x_t, \mathbf{d}_t)$$

and

$$f_t^k(x_t, y_t^*(x_t, \mathbf{d}_t), \mathbf{d}_t) < f_t(x_t, y_t^*(x_t, \mathbf{d}_t), \mathbf{d}_t) + \epsilon = \bar{V}_t(x_t, \mathbf{d}_t) + \epsilon.$$

Thus

$$\hat{V}_t^k(x_t, \mathbf{d}_t) < \bar{V}_t(x_t, \mathbf{d}_t) + \epsilon.$$

As a result, for any x_t and $k > K$, we have

$$\left| \hat{V}_t^k(x_t, \mathbf{d}_t) - \bar{V}_t(x_t, \mathbf{d}_t) \right| < \epsilon,$$

i.e., $\hat{V}_t^k(x_t, \mathbf{d}_t) \to \bar{V}_t(x_t, \mathbf{d}_t)$ uniformly for any given $\mathbf{d}_t$, which completes the induction step. □

If F_t is continuous, then the following result is obtained.

Corollary 5.1. *If* $F_t^k\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ *converge in distribution to* $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ *for any* $\mathbf{d}_t$ *and* t*, and* $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ *is continuous for any* $\mathbf{d}_t$ *and* t*, then*

$$\lim_{k\to\infty} \lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} V_t^k(x_t, \mathbf{d}_t) = \bar{V}_t(x_t, \mathbf{d}_t).$$

Proof. The definition of convergence in distribution implies that $F_t^k\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ converge to $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ at any point D_t where $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ is continuous. Since by assumption $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ is continuous, it follows that $F_t^k\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ converge pointwise to $F_t\left(\cdot \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right)$ and thus we can apply Proposition 5.6. □

Now suppose that the demand distributions for each time period are independent, and let $F_t(D_t)$ denote the cumulative distribution function of $\tilde{D}_t$. Let $\{\mathtt{d}_{t,1}, \mathtt{d}_{t,2}, \ldots\}$

denote a sequence of random samples drawn from the true distribution $\tilde{D}_t$. We can define the kth sample set for period t as $\mathtt{d}_t^k = \{\mathtt{d}_{t,1}, ..., \mathtt{d}_{t,k}\}$. Consider the robust model independent of realized demand. The histogram for time period t is based on $\mathtt{d}_t^k$ with the bins' boundaries being all distinct elements in this set. The corresponding empirical distribution is defined by

$$F_t^k(D_t) = \frac{1}{k} \times |\{\mathtt{d}_{t,j} : \mathtt{d}_{t,j} \leq D_t, j = 1, ..., k\}|.$$

Let $V_t^k(x_t)$ denote the cost-to-function of the robust model defined by the histogram based on the kth sample set, and let $\bar{V}_t(x_t)$ denote the cost-to-go function corresponding to the stochastic model given distribution functions $F_t(D_t)$.

Corollary 5.2. *If F_t is continuous and there is no fixed procurement cost, then*

$$\lim_{k \to \infty} \lim_{\chi_\tau^2 \to 0, \forall \tau \geq t} V_t^k(x_t) = \bar{V}_t(x_t) \qquad a.s.$$

Proof. As $k \to \infty$, the Glivenko-Cantelli theorem (see, e.g., Billingsley [9]) shows that $F_t^k(D_t)$ converges to $F_t(D_t)$ uniformly a.s. at every point D_t where $F_t(D_t)$ is continuous. The result follows immediately from Corollary 5.1. □

Discrete Distributions

Under the setting of Proposition 5.6, consider the case of $\tilde{D}_t$ being subject to a discrete distribution with finite support $\{D_{t,1}, ..., D_{t,M_t}\} \subset [0, D_t^{\max}]$, and let

$$P\left(\tilde{D}_t = D_{t,i} \middle| \tilde{\mathbf{d}}_t = \mathbf{d}_t\right) = p_{t,i}(\mathbf{d}_t).$$

Without loss of generality, we let this finite support be the boundaries of the bins for all the histograms associated with time period t. A result similar to Proposition 5.6 is next proved for the robust stochastic model with both fixed and variable procurement cost.

Proposition 5.8. *Suppose that for any $\mathbf{d}_t$, i and t, $N_{t,i}^k(\mathbf{d}_t)/n_t^k(\mathbf{d}_t)$ converge to*

$p_{t,i}(\mathbf{d}_t)$. *Then with fixed and variable procurement cost, we have*

$$\lim_{k\to\infty}\lim_{\chi_\tau^2\to 0,\forall\tau\geq t} V_t^k(x_t,\mathbf{d}_t)=\bar{V}_t(x_t,\mathbf{d}_t).$$

Proof. Consider $f_t^k(x_t,y_t,\mathbf{d}_t)$ and $f_t(x_t,y_t,\mathbf{d}_t)$ defined in (5.20). Following the proof of Proposition 5.6, it is sufficient to show that $f_t^k(x_t,y_t,\mathbf{d}_t)\to f_t(x_t,y_t,\mathbf{d}_t)$ uniformly for any fixed $\mathbf{d}_t$, under the induction assumption that $\hat{V}_\tau^k(x_\tau,\mathbf{d}_\tau)\to\bar{V}_\tau(x_\tau,\mathbf{d}_\tau)$ uniformly for any given $\mathbf{d}_\tau$ and $\tau>t$.

As $k\to\infty$, $N_{t,i}^k(\mathbf{d}_t)/n_t^k(\mathbf{d}_t)\to p_{t,i}(\mathbf{d}_t)$ uniformly with respect to i (note that there are only finitely many i's). That is, for any $\epsilon>0$, there exists a positive integer K_1 such that

$$\left|\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)}-p_{t,i}(\mathbf{d}_t)\right|<\epsilon$$

for any i and $k>K_1$.

The induction assumption implies that for any $\epsilon>0$, there exists a positive integer K_2 such that

$$\left|\hat{V}_{t+1}^k(x_{t+1},\mathbf{d}_{t+1})-\bar{V}_{t+1}(x_{t+1},\mathbf{d}_{t+1})\right|<\epsilon$$

for any x_{t+1} and $k>K_2$.

Consider $k>\max\{K_1,K_2\}$. Given $\mathbf{d}_t$, for any x_t and y_t we have

$$\begin{aligned}
&\left|f_t^k(x_t,y_t,\mathbf{d}_t)-f_t(x_t,y_t,\mathbf{d}_t)\right|\\
=&\left|\sum_{i=1}^{M_t}\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)}\Big(h_t(y_t-D_{t,i})^+ + b_t(y_t-D_{t,i})^- + \theta\hat{V}_{t+1}^k(y_t-D_{t,i},[\mathbf{d}_t,D_{t,i}])\Big)\right.\\
&\left.-\sum_{i=1}^{M_t}p_{t,i}(\mathbf{d}_t)\left(h_t(y_t-D_{t,i})^+ + b_t(y_t-D_{t,i})^- + \theta\bar{V}_{t+1}(y_t-D_{t,i},[\mathbf{d}_t,D_{t,i}])\right)\right|\\
\leq&\left|\sum_{i=1}^{M_t}\Big(h_t(y_t-D_{t,i})^+ + b_t(y_t-D_{t,i})^-\Big)\left(\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)}-p_{t,i}(\mathbf{d}_t)\right)\right|\\
&+\theta\left|\sum_{i=1}^{M_t}\left(\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)}\hat{V}_{t+1}^k(y_t-D_{t,i},[\mathbf{d}_t,D_{t,i}])-p_{t,i}(\mathbf{d}_t)\bar{V}_{t+1}(y_t-D_{t,i},[\mathbf{d}_t,D_{t,i}])\right)\right|.
\end{aligned}$$

According to (5.19), $|h_t(y_t-D_{t,i})^+ + b_t(y_t-D_{t,i})^-|\leq M'(x_0)<\infty$ where $M'(x_0)$ is a constant depending on the initial inventory x_0. Also note that

$$\left|N_{t,i}^k(\mathbf{d}_t)/n_t^k(\mathbf{d}_t) - p_{t,i}(\mathbf{d}_t)\right| < \epsilon.$$

and hence

$$\begin{aligned}\left|\sum_{i=1}^{M_t}\left(h_t(y_t - D_{t,i})^+ + b_t(y_t - D_{t,i})^-\right)\left(\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)} - p_{t,i}(\mathbf{d}_t)\right)\right| &< \sum_{i=1}^{M_t} M'(x_0)\epsilon \\ &= M_t M'(x_0)\epsilon.\end{aligned}$$

Since y_t and $D_{t,i}$ are bounded again by (5.19), $\left|\hat{V}_{t+1}^k(x_{t+1}, \mathbf{d}_{t+1})\right| \leq M(x_0) < \infty$. Also note that

$$\sum_{i=1}^{M_t} p_{t,i}(\mathbf{d}_t) = 1,$$

$$\left|\hat{V}_{t+1}^k(x_{t+1}, \mathbf{d}_{t+1}) - \bar{V}_{t+1}(x_{t+1}, \mathbf{d}_{t+1})\right| < \epsilon.$$

$$\left|N_{t,i}^k(\mathbf{d}_t)/n_t^k(\mathbf{d}_t) - p_{t,i}(\mathbf{d}_t)\right| < \epsilon.$$

We obtain

$$\begin{aligned}&\left|\sum_{i=1}^{M_t}\left(\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)}\hat{V}_{t+1}^k(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) - p_{t,i}(\mathbf{d}_t)\bar{V}_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}])\right)\right| \\ \leq &\sum_{i=1}^{M_t}\left|\hat{V}_{t+1}^k(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}])\right|\left|\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)} - p_{t,i}(\mathbf{d}_t)\right| \\ &+ \sum_{i=1}^{M_t} p_{t,i}(\mathbf{d}_t)\left|\hat{V}_{t+1}^k(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}]) - \bar{V}_{t+1}(y_t - D_{t,i}, [\mathbf{d}_t, D_{t,i}])\right| \\ < &\sum_{i=1}^{M_t} M(x_0)\epsilon + \epsilon = (M_t M(x_0) + 1)\epsilon.\end{aligned}$$

As a result,

$$\left|f_t^k(x_t, y_t, \mathbf{d}_t) - f_t(x_t, y_t, \mathbf{d}_t)\right| < M_t M'(x_0)\epsilon + \theta(M_t M(x_0) + 1)\epsilon,$$

and hence $f_t^k(x_t, y_t, \mathbf{d}_t)$ converge uniformly to $f_t(x_t, y_t, \mathbf{d}_t)$. □

So far we assumed that χ_t^2 converges to zero. Next we establish a convergence result for any fixed χ_t^2. We must require that the number of samples goes to infinity.

Proposition 5.9. *Suppose that for any $\mathbf{d}_t$, i and t, $N_{t,i}^k(\mathbf{d}_t)/n_t^k(\mathbf{d}_t)$ converge to $p_{t,i}(\mathbf{d}_t)$, and that the number of samples $n_t^k(\mathbf{d}_t)$ converges to infinity. Then for any fixed χ_t^2 and with fixed and variable procurement cost we have*

$$\lim_{k\to\infty} V_t^k(x_t, \mathbf{d}_t) = \bar{V}_t(x_t, \mathbf{d}_t).$$

Proof. Since $V_{T+1}^k(\cdot) = \bar{V}_{T+1}(\cdot)$, the proposition holds for period $T+1$. Suppose that for any $\tau > t$, $\hat{V}^k(x_\tau, \mathbf{d}_\tau) \to \bar{V}_\tau(x_\tau, \mathbf{d}_\tau)$ uniformly for any fixed $\mathbf{d}_\tau$. Consider period t.

According to the definition of the distribution set $\mathcal{P}_t^k(\mathbf{d}_t)$ for $V_t^k(x_t, \mathbf{d}_t)$, similarly to the proof of Proposition 5.5, we have

$$\mathcal{P}_t^k(\mathbf{d}_t) \subset \overline{\mathcal{P}}_t^k(\mathbf{d}_t) = \left\{\mathbf{P}_t \,\middle|\, \underline{P}_{t,i}^k(\mathbf{d}_t) \le P_{t,i} \le \overline{P}_{t,i}^k(\mathbf{d}_t) \text{ for every } i\right\}$$

where

$$\begin{aligned}\underline{P}_{t,i}^k(\mathbf{d}_t) &= \frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)} + \frac{\chi_t^2}{2n_t^k(\mathbf{d}_t)} - \frac{\sqrt{4N_{t,i}^k(\mathbf{d}_t)\chi_t^2 + (\chi_t^2)^2}}{2n_t^k(\mathbf{d}_t)},\\ \overline{P}_{t,i}^k(\mathbf{d}_t) &= \frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)} + \frac{\chi_t^2}{2n_t^k(\mathbf{d}_t)} + \frac{\sqrt{4N_{t,i}^k(\mathbf{d}_t)\chi_t^2 + (\chi_t^2)^2}}{2n_t^k(\mathbf{d}_t)}.\end{aligned} \tag{5.21}$$

Consider $\hat{V}_t^k(x_t, \mathbf{d}_t)$ as defined in Propositions 5.6 and 5.8, which denotes the cost-to-go function of the stochastic model under the empirical distribution. Note that $\hat{V}_t^k(x_t, \mathbf{d}_t) \le V_t^k(x_t, \mathbf{d}_t)$, and hence

$$\lim_{k\to\infty} \hat{V}_t^k(x_t, \mathbf{d}_t) \le \lim_{k\to\infty} V_t^k(x_t, \mathbf{d}_t).$$

Proposition 5.8 shows that $\lim_{k\to\infty} \hat{V}_t^k(x_t, \mathbf{d}_t) = \bar{V}_t(x_t, \mathbf{d}_t)$ whenever $\frac{N_{t,i}^k(\mathbf{d}_t)}{n_t^k(\mathbf{d}_t)} \to p_{t,i}(\mathbf{d}_t)$, and we obtain

$$\bar{V}_t(x_t, \mathbf{d}_t) \le \lim_{k\to\infty} V_t^k(x_t, \mathbf{d}_t).$$

Also note that $N_{t,i}^k(\mathbf{d}_t)/n_t^k(\mathbf{d}_t) \to p_{t,i}(\mathbf{d}_t)$ and $\frac{\chi_t^2}{2n_t^k(\mathbf{d}_t)} \pm \frac{\sqrt{4N_{t,i}^k(\mathbf{d}_t)\chi_t^2+(\chi_t^2)^2}}{2n_t^k(\mathbf{d}_t)} \to 0$ as

$k \to \infty$. Therefore,

$$\lim_{k\to\infty} \underline{P}^k_{t,i}(\mathbf{d}_t) = \lim_{k\to\infty} \overline{P}^k_{t,i}(\mathbf{d}_t) = p_{t,i}(\mathbf{d}_t).$$

Following the same argument as in the proof of Proposition (5.5), it is easy to see that

$$\lim_{k\to\infty} V^k_t(x_t, \mathbf{d}_t) \leq \bar{V}_t(x_t, \mathbf{d}_t),$$

which completes the proof. □

Now consider the setting of Corollary 5.2, where the demand distributions are assumed to be independent, and the kth sample set is defined to be the first k elements in a sequence of independent random samples drawn from the true distribution. Using Proposition 5.8, we can obtain a result analogous to Corollary 5.2. We also establish the rate of convergence.

Corollary 5.3. *With fixed and variable procurement cost, we have* $\lim_{k\to\infty} V^k_t(x_t) = \bar{V}_t(x_t)$ *a.s., and the rate of convergence is* $O(1/\sqrt{k})$.

Proof. The convergence follows from the Glivenko-Cantelli theorem (c.f. Billingsley [9]) and Proposition 5.9.

Since $V^k_{T+1}(\cdot) = \bar{V}_{T+1}(\cdot)$, the rate of convergence holds for time period $T+1$. Suppose that it holds for any time period $\tau > t$.

Consider the set of distributions $\mathcal{P}^k_t$ defined for the robust model $V^k_t(x_t)$. The definition in (5.9) shows that

$$\mathcal{P}^k_t \supset \left\{ \mathbf{P}_t \,\middle|\, \mathbf{A}_t\mathbf{P}_t = \mathbf{b}_t,\ \frac{(N^k_{t,i} - kP_{t,i})^2}{kP_{t,i}} \leq \frac{\chi^2_t}{M_t}\ \forall i,\ \mathbf{P}_t \geq \mathbf{0} \right\}.$$

The inequality $\frac{(N_{t,i}^k - kP_{t,i})^2}{kP_{t,i}} \leq \frac{\chi_t^2}{M_t}$ is equivalent to

$$P_{t,i} \in \left[\frac{N_{t,i}^k}{k} + \frac{\chi_t^2/M_t}{2k} - \frac{\sqrt{4N_{t,i}^k \chi_t^2/M_t + (\chi_t^2/M_t)^2}}{2k}, \right.$$
$$\left. \frac{N_{t,i}^k}{k} + \frac{\chi_t^2/M_t}{2k} + \frac{\sqrt{4N_{t,i}^k \chi_t^2/M_t + (\chi_t^2/M_t)^2}}{2k} \right].$$

Therefore, the rate at which $\mathcal{P}_t^k$ shrinks to the single point

$$\mathbf{P}_t = \left\{ P_{t,1} = \frac{N_{t,1}^k}{k}, ..., P_{t,M_t} = \frac{N_{t,M_t}^k}{k} \right\}$$

is $O(1/\sqrt{k})$. According to the law of large numbers, $N_{t,i}^k/k$ converge to $p_{t,i} = P\left(\tilde{D}_t = D_{t,i}\right)$ exponentially (see, e.g., Billingsley [9]). As a result, for sufficiently large k, $\mathcal{P}_t^k$ contains vector $\mathbf{p}_t$ a.s., and hence $\bar{V}_t(x_t) \leq V_t^k(x_t)$ a.s.

As shown in (5.18),

$$\bar{V}_t(x_t) \leq V_t^k(x_t) \leq \sum_{i:U_{t,i}^k(x_t,y_t^*)\leq 0} \underline{P}_{t,i}^k U_{t,i}^k(x_t, y_t^*) + \sum_{i:U_{t,i}^k(x_t,y_t^*)>0} \overline{P}_{t,i}^k U_{t,i}^k(x_t, y_t^*) \qquad \text{a.s.},$$

where $U_{t,i}^k(x_t, y_t)$ is defined in the same way as (5.17), and y_t^* denotes an optimal solution to $\bar{V}_t(x_t)$.

Note that

$$\lim_{k\to\infty} U_{t,i}^k(x_t, y_t^*) = C_t(x_t, y_t^*, D_{t,i}) + \theta \bar{V}_{t+1}\left((y_t^* - D_{t,i})^+\right).$$

The rate of convergence of $U_{t,i}^k(x_t, y_t^*)$ is determined by the convergence rate of $\bar{V}_{t+1}^k(\cdot)$, and hence it is in the order of $O(1/\sqrt{k})$.

According to the definition of $\underline{P}_{t,i}^k$ and $\overline{P}_{t,i}^k$ in (5.21), both $\underline{P}_{t,i}^k$ and $\overline{P}_{t,i}^k$ converge to $p_{t,i}$ at the convergence rate of $O(1/\sqrt{k})$, since $N_{t,i}^k/k$ converge to $p_{t,i}$ exponentially.

Finally, note that

$$\bar{V}_t(x_t) = \lim_{k\to\infty} \left\{ \sum_{i:U^k_{t,i}(x_t,y^*_t)\le 0} \underline{P}^k_{t,i} U^k_{t,i}(x_t, y^*_t) + \sum_{i:U^k_{t,i}(x_t,y^*_t)>0} \overline{P}^k_{t,i} U^k_{t,i}(x_t, y^*_t) \right\}.$$

We conclude that the rate of convergence of $V^k_t(x_t)$ is in the order of $O(1/\sqrt{k})$. $\square$

5.5 Computational Results

In this section, we describe computational experiments and present numerical results to support the effectiveness of the minimax robust model based on the chi-square test. As we have mentioned in the previous sections, the traditional approach is to fit the historical data with a distribution and then apply stochastic inventory optimization using the fitted distribution. The main objective of our experiments is to compare performances of this separated approach and the studied minimax robust model with respect to optimality and robustness. At the same time, we would like to assess sensitivity of the robust model to the choices of the bin sizes and χ^2 parameters, and provide an empirical approach to choose these values.

We consider inventory control problems without fixed ordering costs. Following the notation in the previous sections, we let T denote the planning horizon and c_t, h_t, b_t denote the variable order cost, unit inventory holding cost, and backorder cost for any period t, $t = 1, ..., T$, respectively. The demand distributions for any period t are assumed to be i.i.d. In the robust model, we restrict ourselve to the case of equal bin sizes and these, together with χ^2, are the same for every period in the planning horizon. To simplify the notation, pair $\langle \epsilon, \chi^2 \rangle$ denotes the choice of the bin size and χ^2 in the robust model, where the first parameter ϵ denotes the bin size.

The procedure of the computational experiments is as follows.

Step 1. Suppose that the underlying demand distribution has support $\{0, 1, ..., \bar{D}\}$.

We randomly generate a distribution among all distributions whose support is a subset of $\{0, 1, ..., \bar{D}\}$. In particular, we pick distribution

$$p_i = P(\tilde{D}_t = i) = \frac{U_i}{\sum_{i=0}^{\bar{D}} U_i},$$

for any $i = 0, 1, ..., \bar{D}$, where U_i for all i are i.i.d. random variables uniformly distributed in the interval $[0, 1]$. We refer to the distribution $\mathbf{p} = \{p_i\}_i$ as the true distribution.

Step 2. Generate n random samples according to the true distribution selected in Step 1.

Step 3. Fit the samples obtained in Step 2 using Crystal Ball and then choose the l best fitted distributions according to the χ^2 goodness-of-fit statistic.

Step 4. Solve the standard stochastic inventory control problem with distributions generated in Steps 1 and 3.

Step 5. Solve the robust inventory control model using a set of bin-size and χ^2 combinations.

Step 6. Evaluate the total expected cost with respect to the true distribution $\mathbf{p}$ corresponding to the policies of the stochastic models and robust models computed in Steps 4 and 5. We use this step to investigate the optimality of the robust models.

Step 7. The n samples generated in Step 2 define the empirical distribution $\hat{\mathbf{p}}$ such that

$$\hat{p}_i = \frac{\text{the number of times value } i \text{ appears in the } n \text{ samples}}{n}$$

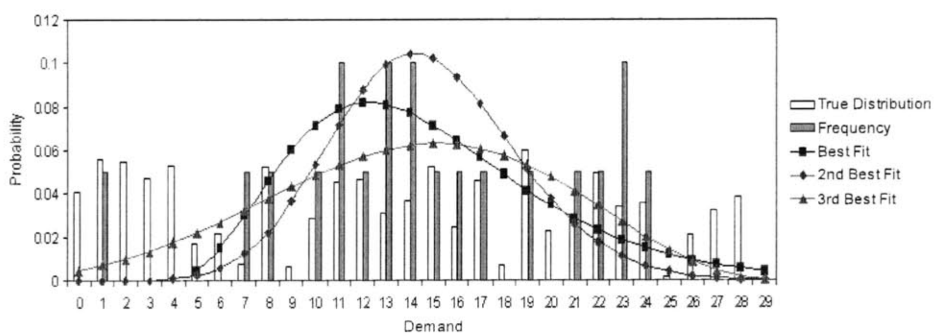
for any $i = 0, 1, ..., \bar{D}$. Let $\delta = \mathbf{p} - \hat{\mathbf{p}}$. We generate m random permutations of vector δ and denote the jth permutation of the coordinates by δ^j. Vector $\hat{\mathbf{p}}^j = \hat{\mathbf{p}} + \delta^j$ also defines a distribution.[3] Note that $\hat{\mathbf{p}}^j$ is equal to $\mathbf{p}$ if $\delta^j = \delta$, i.e., when δ^j is not permuted.

[3]If $\hat{\mathbf{p}}^j$ contains any negative component, we set $\hat{\mathbf{p}}^j$ to be the positive part of $\hat{\mathbf{p}}^j$ plus a random permutation of its negative part, and we repeat this process until $\hat{\mathbf{p}}^j \geq \mathbf{0}$.

For each distribution defined by vector $\hat{\mathbf{p}}^j$, we can evaluate the corresponding cost for each policy computed in Steps 4 and 5. Therefore, we obtain m costs for each policy and we report the conditional value-at-risk[4] (CVaR) at the 5% level of the m costs for each policy. The purpose of this step is to understand the robustness of different approaches.

Let us consider a 10-period problem. The support for the demand distribution is assumed to be the set $\{0, 1, ..., 29\}$, i.e., $\bar{D} = 29$. The cost parameters c_t, h_t and b_t are generated independently according to the uniform distributions within the intervals $[12, 15]$, $[2, 5]$ and $[22, 25]$, respectively. Following the computational procedure, we first draw $n = 20$ samples from the selected true distribution. Fitting the samples using Crystal Ball, the three best fitted distributions according to the chi-square values are negative binomial, Poisson, and beta. The true distribution $\mathbf{p}$, sample frequency $\hat{\mathbf{p}}$ and the three distributions are displayed in Figure 5-1.

Figure 5-1: True Distribution, Frequency and Fitted Distributions with 20 Samples



In Steps 4 and 5 of our procedure, we compute the base stock levels corresponding to different models: the stochastic model using the true distribution, the stochastic model using the three best fitted distributions, and robust models with different bin-size and χ^2 value combinations. In particular, the following set of bin-size and χ^2 value combinations are considered: $\langle 3, 1\rangle$, $\langle 3, 3\rangle$, $\langle 3, 5\rangle$, $\langle 5, 1\rangle$, $\langle 5, 3\rangle$, $\langle 5, 5\rangle$.

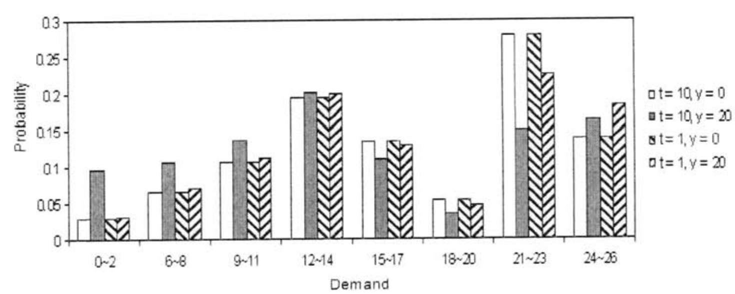
As stated in our analysis, the robust model picks the demand distribution based on the on-hand inventory after the order is received, i.e., the order-up-to level y_t.

[4]Given random variable X, the conditional value-at-risk at a quantile-level q is defined as $E[X|X < \mu]$ where μ is defined by $P(X < \mu) = q$.

Although we use the same histogram in each period, the demand distribution returned by the robust model depends also on t. We use the robust model with the bin-size/χ^2 value $\langle 3, 3 \rangle$ to illustrate these properties.

In Figures 5-2 and 5-3, and Table 5.1, we use a simple representative sample of cost parameters. Figure 5-2 shows the robust distributions for the last period $t = 10$ and the first period $t = 1$ when the inventory levels after receiving the order y_t are 0 and 20 respectively. For both periods, the distributions returned by the robust model for $y_t = 20$ have lower probabilities in the region 15 to 26 than those for $y_t = 0$. The intuition behind this observation is that the robust model picks a demand distribution maximizing the expected cost. For any possible value of the demand, we incur a certain cost corresponding to $U_{t,i}(y_t, \mathbf{d}_t)$ defined in (5.6). Therefore, the robust model chooses a lower probability for demand values with lower costs. Value $y_t = 20$ is very close to the demand when the demand falls in the region 15 to 26. The amount we over- or under-order is low and hence the corresponding over- or under-order cost is also low.[5] Therefore, the corresponding costs associated with the demand values are lower than the costs corresponding to other demand values. As a result, the robust model assigns lower probabilities in these regions compared with the case when $y_t = 0$.

Figure 5-2: Demand Distributions Returned by the Robust Model with Bin Size = 3 and $\chi^2 = 3$



If we compare the robust distributions when $y_t = 20$ for period 10 and period 1, we observe that the probability for period 10 is higher for small demand values. This

[5]In this section, the over-order (under-order, respectively) cost includes not only the inventory holding cost h_t (backorder cost b_t, respectively) incurred in period t, but also the impact of over-order (under-order, respectively) in period t based on the cost-to-go function $V_{t+1}(\cdot)$.

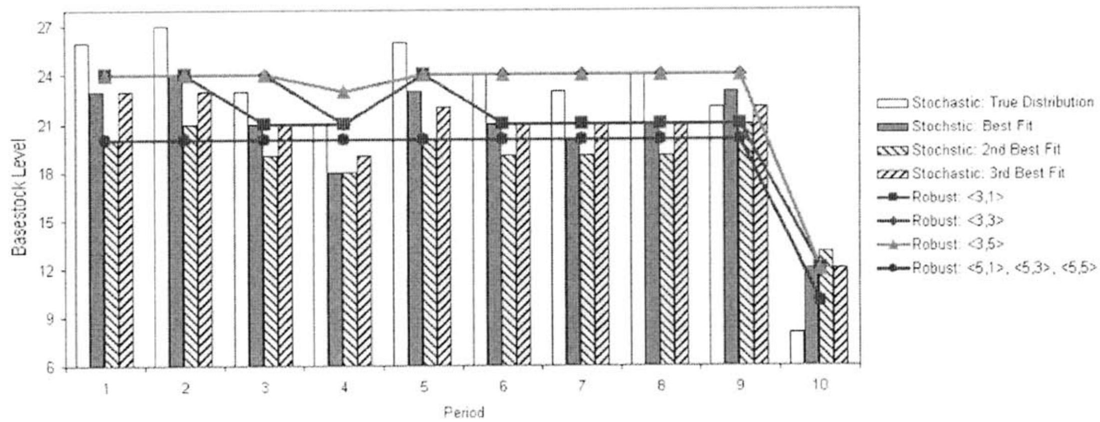
can also be explained by the tradeoff between the over- and under-order costs. In the last period, the over-order cost is $c_{10} + h_{10}$ and the under-order cost is b_{10} since we set $V_{T+1}(\cdot) = 0$. For any earlier period $t < 10$, the over-order costs are significantly lower as we can carry the inventory to the next period and save the order cost c_t, but the under-order cost is $b_t + c_{t+1}$ since we not only pay the backorder cost but also procure the product in period $t + 1$ to satisfy the unmet demand in period t. When $y_t = 20$, we pay the over-order costs when the demands are low (e.g., in the region 0 to 11), and the under-order costs are incurred when the demand are high (e.g., in the region 21 to 26). As the over-order costs are higher and the under-order costs are lower in the last period, it implies that the ratio between the costs for low demands and the costs for high demands is greater in period 10 than period 1. This is the reason why the robust model assigns higher probabilities for low demands in period 10.

On the other hand, the robust distributions when $y_t = 0$ are almost the same for the two periods with $t = 10$ and $t = 1$. In this case, we only have the under-order cost no matter if the demand is high or low. Although the under-order cost is higher in period 1 than period 10, the ratios between the costs for low and high demands are almost the same for period 1 and 10. Therefore, the worst case distributions are similar for these two periods.

The basestock levels computed in Steps 4 and 5 are displayed in Figure 5-3. For any of the stochastic or robust models, the basestock level for period 10 is significantly lower than the remaining periods. As explained before, this is caused by the fact that the overorder cost is much higher while the underorder cost is lower in period 10 because of $V_{T=1}(\cdot) = 0$, and thus we should order less in that period. In addition, the basestock level for period 4 is slightly lower for most of the models since period 4 has the highest order and inventory holding cost while its backorder cost is relatively low.

For the three robust models with the bin-size 3, the basestock levels are nondecreasing with respect to the χ^2 value, since the sets of distributions are inclusion-wise increasing in the χ^2 value. In our instances, the backorder cost is much higher than the inventory holding cost. Intuitively, the worst case distribution should assign higher

Figure 5-3: Basestock Levels Computed Using Different Models



probabilities for high demand values. Therefore, the larger the χ^2 value is, the higher the probabilities for high demand values in the worst case distribution, and hence we should order more to minimize the worst case expected cost. As a result, the basestock levels are higher for the robust models with greater χ^2 values. However, if we set the bin-size to 5 for the robust models, the basestock levels are the same when the χ^2 values are equal to 1, 3 and 5. This observation indicates that the basestock levels are less sensitive to the χ^2 values when we have larger bins.

We use Steps 6 and 7 to understand the performance of different models. The results are summarized in Table 5.1. The first four columns correspond to the results for the stochastic models using true distribution $\mathbf{p}$ and the three best fitted distributions, respectively. The next four columns show the results for the robust models. Note that the last column corresponds to the robust models with bin-size 5 and χ^2 values 1, 3 and 5. These three robust models have the same performance for this example as they have the same basestock levels. We show the expected cost for different models with respect to the true distribution in the first line, which corresponds to the output of Step 6. In the second line, we report the output of Step 7, i.e., the CVaR at 5% level for the costs of $m = 1000$ distributions generated by $\hat{\mathbf{p}}$ plus random permutations of $\mathbf{p} - \hat{\mathbf{p}}$. For the purpose of comparison, the numbers in Table 5.1 are calculated by subtracting the cycle stock order cost, i.e., $\left(\sum_{t=1}^{T} c_t\right)\left(\sum_{i=1}^{\bar{D}} ip_i\right)$, from the original cost or CVaR, and normalizing with respect to that of the stochastic model using true distribution.

Table 5.1: Performance of Different Models for the Instance in Figure 5-1

	Stochastic Models				Robust Models			
	True Dist	Best Fit	2nd Best Fit	3rd Best Fit	$\langle 3,1\rangle$	$\langle 3,3\rangle$	$\langle 3,5\rangle$	$\langle 5,1$ or 3 or $5\rangle$
Cost	1	1.0595	1.1834	1.0582	1.0415	1.0211	1.0249	1.1511
CVaR	1	1.0486	1.1802	1.0511	1.0356	0.9774	0.9739	1.1662

Obviously, the stochastic model using the true distribution gives the lowest expected cost. The output of Step 7, CVaR, also indicates that this model is robust with respect to perturbations in the input distribution as it has the third lowest CVaR, which is only 2.61% higher than the lowest CVaR.

For the three stochastic models using fitted distributions, the models using the 1st and 3rd best fitted distributions have a very similar performance. The best-fit case has the best performance among the fitted stochastic models as its CVaR is 0.25% better than the 3rd best-fit stochastic model and the cost is only 0.13% higher than that. The performance of the model using the 2nd best distribution is much worse compared with the other two. Its cost and CVaR values are at lest 12% higher than the remaining two models.

The three robust models with bin-size 3 outperform all of the stochastic models using fitted distributions in terms of both optimality (cost) and robustness (CVaR). The robust models with bin-size 5 also have better values of the cost and CVaR than the stochastic model using the 2nd best fitted distribution. In particular, the robust models with bin-size/χ^2 value combinations of $\langle 3,3\rangle$ and $\langle 3,5\rangle$ are significantly better than the stochastic models using fitted distributions. They reduce the cost by more than 3% and CVaR by more than 7% when comparing with the fitted stochastic models. Among the robust models we prefer the model with bin-size/χ^2 value combination $\langle 3,3\rangle$, since it improves the cost by 0.38% at the price of a 0.35% increase in CVaR.

Next we repeated the experiment from Step 1 to Step 7 for 10 times, i.e., each time with a different true distribution, demand data and cost parameters. Table 5.2 shows the average and standard deviation of the cost and CVaR values for the 10 data samples for the stochastic model using the true distribution, the stochastic model using the best fitted distribution as well as the 6 robust models already considered. All robust models have lower average and standard deviation of cost and CVaR compared with the stochastic model using the best fitted distribution. In terms of both optimality (cost) and robustness (CVaR), the performance of our robust models is better on average (smaller average) and more stable (smaller standard deviation) than the stochastic model using the best fitted distribution.

Table 5.2: Performance of Different Models in 10 Instances

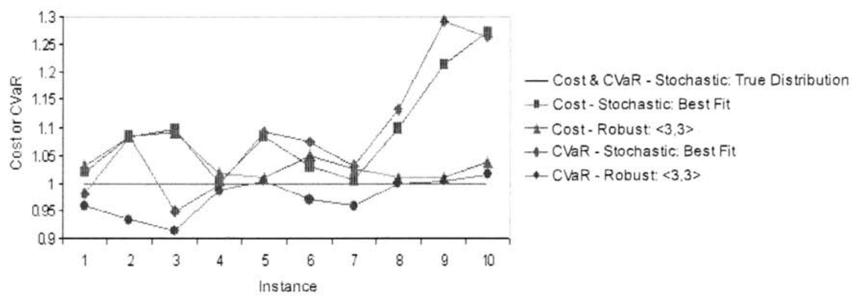
	Stochastic Models		Robust Models					
	True Dist	Best Fit	$\langle 3,1\rangle$	$\langle 3,3\rangle$	$\langle 3,5\rangle$	$\langle 5,1\rangle$	$\langle 5,3\rangle$	$\langle 5,5\rangle$
Cost Ave.	1	1.0902	1.0412	1.0361	1.0499	1.0725	1.0597	1.0570
Cost Stdev.	0	0.0893	0.0210	0.0302	0.0330	0.0736	0.0511	0.0519
CVaR Ave.	1	1.0894	1.0014	0.9745	0.9746	1.0648	1.0281	1.0212
CVaR Stdev.	0	0.1142	0.0507	0.0327	0.0355	0.0917	0.0639	0.0618

The robust models with bin-size 3 have lower values of average and standard deviation of both measures than the robust models with bin-size 5. Moreover, the robust models with higher χ^2 values, e.g., when χ^2 is set to 3 or 5, have lower CVaR than those with χ^2 values set to 1. This observation agrees with our understanding that increasing χ^2 values can improve the robustness of the models. However, it may also affect the cost of the models, e.g., the average cost for the $\langle 3,5\rangle$ robust model is 0.8% higher than that of the $\langle 3,1\rangle$ robust model.

The robust model with bin-size 3 and χ^2 value 3 has the lowest average cost, lowest CVaR, and lowest standard deviation of CVaR among all robust models, and its standard deviation of the cost is the second lowest. This agrees with our suggestion drawn from Figure 5-1: the robust model with bin-size/χ^2 value combination $\langle 3,3\rangle$ should be the best among the robust models.

Figure 5-4 shows the cost and CVaR values for the stochastic model using the best fitted distribution and the $\langle 3, 3 \rangle$ robust model in each of the 10 instances. The cost values of the $\langle 3, 3 \rangle$ robust model are at least 7.5% lower than the stochastic model with the best fitted distribution for instances 5, 8, 9 and 10. The improvement in instances 9 and 10 even exceeds 20%. The cost values of instances 2 and 3 are almost the same for both models. Instance 7 is the only case where the cost of the robust model is more than 2% (2.04% to be exact) higher than the cost of the stochastic model.

Figure 5-4: The Stochastic Model Using Best Fitted Distribution vs. the Robust Model with Parameters $\langle 3, 3 \rangle$ for 10 Instances



The values of CVaR for the robust model are less than one for 7 out of the 10 instances, they are very close to one (at most 0.04% higher than one) for the other 2 instances, and the largest value is 1.0150. On the other hand, the values of CVaR for the stochastic model with the best fitted distribution is less than one only for 3 instances and the largest value is 1.2923. We conclude that the $\langle 3, 3 \rangle$ robust model is much more robust compared with the stochastic model using the true distribution.

In order to understand the sensitivity of different models with respect to the number of samples drawn from the true distribution, we ran 10 additional experiments in which we generate $n = 40$ samples from the true distribution in Step 2.

Table 5.3 summarizes the main statistics of the stochastic model using the best fitted distribution and our robust models. Similar to the result in Table 5.2 where we have 20 samples from the true distribution, all of the robust models outperform the stochastic model with the best fitted distribution in both the average and standard deviation of the two measures.

Table 5.3: Performance of Different Models for 10 Instances and 40 Samples

	Stochastic Models		Robust Models					
	True Dist	Best Fit	$\langle 3,1\rangle$	$\langle 3,3\rangle$	$\langle 3,5\rangle$	$\langle 5,1\rangle$	$\langle 5,3\rangle$	$\langle 5,5\rangle$
Cost Ave.	1	1.0749	1.0278	1.0223	1.0364	1.0457	1.0317	1.0304
Cost Stdev.	0	0.0500	0.0232	0.0262	0.0346	0.0353	0.0297	0.0297
CVaR Ave.	1	1.0820	1.0141	0.9901	0.9832	1.0539	1.0167	1.0091
CVaR Stdev.	0	0.0765	0.0406	0.0268	0.0217	0.0433	0.0167	0.0127

As expected, the average cost of all robust models and the best fit stochastic model improves when the sample size increases from 20 to 40. The robust models with bin size 5 have a slightly greater improvement than the remaining models. For the other three statistics, we also observe improvements for the stochastic model using the best fitted distribution as well as the robust models with bin-size 5 when the sample size is increased to 40. Again, the robust models with bin-size 5 show slightly better improvements in these statistics.

If we compare the robust models with different bin sizes, those with bin-size 3 still perform better than those with bin-size 5. However, compared with the case of 20 samples, the differences are slightly smaller for all statistics, which suggests that the robust models with bin-size 5 improve faster as the sample size increases. Similar to the experiments with 20 samples, the increase in χ^2 values also helps to improve the robustness of the models, which is measured by CVaR. The improvements in robustness as χ^2 values increase are more significant for 40-sample experiments than those with 20 samples. In addition, the increased χ^2 may also increase the cost, e.g., the average cost increases from 1.0278 to 1.0364 if we increase the χ^2 value from 1 to 5 for the robust models with bin-size 3.

The robust model with parameters $\langle 3,3\rangle$ has the lowest average cost, the second lowest average CVaR and the second lowest standard deviation of the cost. Besides, its standard deviation of CVaR is less than 3%. We still consider it as the most efficient model among all the robust models and the stochastic model using the best fitted distribution.

To summarize the numerical results, the computational experiments show that the robust models outperform the stochastic models using fitted distributions in terms of both optimality and robustness. The robust models with a lower bin size perform better than those with a larger bin size, but an increase in sample size may decrease the difference in performance caused by the choice of the bin size. In addition, a higher χ^2 value helps to increase the robustness but it may sacrifice the cost of the robust models.

5.6 Extensions

In this chapter, we propose a robust stochastic model for the multi-period lot sizing problem, in which the demand distribution is unknown and the only available information is historical data. The convergence results for the chi-square test based models suggest that the solutions to the robust approach are very close to the optimal stochastic programming solutions when the sample size is sufficiently large. This robust framework based on historical data can be extended to many more general finite-horizon dynamic programming problems. and the convergence properties can also be extended to more general problems.

Although we consider back-order models, most of our results can be extended to lost sales models if the historical data also reflect the amount of lost sales. In particular, for lost-sales models with only linear procurement cost and under the same technical assumptions, the optimal policy under the robust model is a state-dependent base-stock policy.

Chapter 6

Conclusions

This dissertation considers two applications of inventory control models, which are both formulated as discrete-time finite-horizon dynamic programs. We analyze the Bellman equations of the dynamic programming formulation, fully characterize the optimal control policies, and investigate various properties of these policies. The two applications naturally partition the dissertation into two parts.

In the first part, the inventory control problem in market-making is analyzed, where the decision maker, i.e., the market-maker, controls the inventory in order to limit the exposure to market price movements at the risk of losing possible gain of the bid/ask spread. We prove that a threshold policy is optimal for risk-averse inventory control in market-making when considering a single asset, and we establish sufficient conditions for the threshold levels being symmetric or monotone.

For the market-making problem with multiple assets, the optimal policy shows that there exists a simple connected no-trade region, which is proved to be symmetric under certain conditions. The boundaries of the no-trade region also determine the optimal quantity to actively trade with other market makers if the inventory position falls outside the no-trade region. The optimal policies lead to efficient algorithms to solve the dynamic programming problems, and the computational complexity is linear in the number of periods. The structural results of the optimal policy provide insights to significantly reduce the search region if the market-makers would like to identify an inventory control strategy by simulation or backtesting.

Of course, important limitations of our model exist. For example, we assume that the decision maker is a small player in the market, which means that the decision maker follows the bid and ask prices quoted by the market leader, and its actions to trade with its clients or other market-makers have no impact on the future price movements. In the case that the decision maker is a market leader, it can determine the bid and ask prices as well as the amount of active trades with other market-makers, and it needs to take into account their impact on the price movements. For these big players in the market, we need to optimize the pricing and inventory decision simultaneously. This type of results also benefits the small players as it quantifies the loss of efficiency of being the price follower. The joint pricing and inventory problem for big players are left for future research.

In the second part, we return to the single-item single-location inventory problem in supply chain management, where the assumptions are the same as the classical inventory control model except that the future demands are specified by historical data instead of cumulative distribution functions. We propose a minimax model which optimizes the worst-case expected total cost over a set of demand distributions defined by the historical data. We show that the corresponding optimal control policy is the same as the stochastic counterpart in the inventory control literature, i.e., a basestock policy is optimal if there is no fixed ordering cost and an (s, S) policy is optimal when a fixed ordering cost is considered. One way to construct the set of demand distribution using historical data is to consider the test statistics in data fitting. In particular, we present how to define the set using χ^2 test and prove that the minimax robust model converges to the stochastic model as the number of available data points goes to infinity. The computational procedure adopted in Section 5.5 also serves as an empirical approach to determine the parameters such as the bin sizes and χ^2 values in the robust model.

In Section 5.6, we mentioned that most results for back-order models can be extended to lost-sale models. Note that this is under the assumption that we have historical demand data which also includes lost sales. However, in practice, lost sales may not be observable and we only know the amount sold during a period, i.e., the available historical data is sales data instead of demand data which consists of both sales and lost sales. Therefore, future research could study the lost sales models with historical sales data.

Moreover, incomplete distribution information is not unique to the stochastic inventory control problem. Many problems in supply chain management, such as network design, production sourcing, and process flexibility, to name a few, require stochastic input data, i.e., information on the distribution of various parameters, such as demand, lead time, yield and etc. Unfortunately, much like the inventory control problem analyzed in this dissertation, in most cases, only historical data is available. Therefore, the idea to integrate inventory optimization with data fitting presented in this dissertation can be extended to other supply chain problems, where we choose the supply chain decision by optimizing the worst-case expected cost or profit over a set of distributions generated by the available data. The structure of the corresponding minimax or maximin models may vary from problem to problem and thus require carefully applying various optimization techniques besides dynamic programming.

Also note that the minimax inventory control model proposed in Chapter 5 returns not only the optimal robust inventory policy but also the demand distributions corresponding to the worst-case scenario associated with this policy. An interesting question is whether it is possible extend this approach to demand forecasting, which provides an indispensable input to supply chain models. Although most forecasting tools are based on the point-of-sale data, store shipment information is also available in the retail industry, and it is usually more accurate and reliable than the point-of-sale data. In this case, shipment data corresponds to the quantity ordered for each item in each period, which is determined by the inventory control policy. Our robust inventory model takes historical data as input and returns demand distribution and inventory policy as output, i.e., it establishes a relationship among historical data, demand distribution and inventory control policy. Therefore, this approach may help to develop forecasting models utilizing shipment data.

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