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Optimized design of a low-resistance electrical conductor for the multimegahertz range

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We propose a design for a conductive wire composed of several mutually insulated coaxial conducting shells. With the help of numerical optimization, it is possible to obtain electrical resistances significantly lower than those of a heavy-gauge copper wire or litz wire in the 2–20 MHz range. Moreover, much of the reduction in resistance can be achieved for just a few shells; in contrast, litz wire would need to contain $\sim 10^4$ strands to perform comparably in this frequency range.

In this letter, we show that a structure of concentric cylindrical conducting shells can be designed to have much lower electrical resistance for ~ 10 MHz frequencies than heavy gauge wire or available litz wires. At such frequencies, resistance is dominated by skin-depth effects, which prevent the current from being uniformly distributed over the cross section; this is typically combatted by breaking the wire into a braid of many thin insulated wires (litz wire[1]), but the $\sim 10\,\mu$m skin depth at these frequencies makes traditional litz wire impractical ($\sim 10^4$ μm-scale strands). In contrast, we show that as few as 10 coaxial shells can improve resistance by more than a factor of 3 compared to solid wire, and thin concentric shells can be fabricated by a variety of processes (such as electroplating, electrodeposition, or even a fiber-drawing process[2–4]). Good conductors at these frequencies are increasingly important, e.g. to make low-loss resonators for wireless power transfer[5,6], or for other applications (e.g. RFID) operating at ISM (Industrial, Scientific, and Medical[7]) frequencies (e.g. 6.78 and 13.56 MHz). We derive an analytical expression for the impedance matrix of both litz wire and nested cylindrical conductors starting from the quasistatic Maxwell equations; in particular, a key factor turns out to be the proximity losses[8] induced by one conductor in another conductor via magnetic fields. Using this result combined with numerical optimization, we are able to quickly optimize all of the shell thicknesses to minimize the resistance with a given frequency and number of shells.

For a cylindrically symmetrical system of nested conductors oriented along the z direction, we first see below that Maxwell's equations reduce to a Helmholtz equation in each annular layer. In the quasistatic limit of low frequency, we show that this further simplifies into a scalar Helmholtz equation for E_z alone, which can be solved in terms of Bessel functions, the coefficients of which are determined by the boundary conditions at each interface: continuity of E_z and of $H_\phi \sim \partial E_z/\partial r$. Once the solution for E_z, and thus the current density σE_z (for conductivity σ) and the magnetic fields (from Ampere's law), are obtained, the impedance matrix can be derived from energy considerations. Of course, a real wire is not perfectly cylindrical because of bending and other perturbations, but these effects can typically be neglected (e.g., if the bending radius is much larger than the wire radius).

We start by analytically solving Maxwell's equations in each medium (air, copper, and insulator):

$$\nabla^2\mathbf{E}(\mathbf{r}) + \left(k^2 + 2i\kappa^2\right)\mathbf{E}(\mathbf{r}) = 0. \quad (1)$$

Here $k = \sqrt{\epsilon_r}\omega/c$, ω is angular frequency, ϵ_r is relative permittivity, c is the speed of light in vacuum, $\kappa = 1/\delta = \sqrt{\omega\mu_0\sigma/2}$ (δ is skin depth), μ_0 is the magnetic constant, and σ is the conductivity (5.9×10^7 S/m for copper, zero otherwise). Since the wavelength ($\lambda = 2\pi/k \simeq 15$ m in vacuum at 20 MHz) is much longer than the conductor thickness or the skin depth, the k^2 term in Eq. 1 is negligible for solving within a given cross-section z. The solutions $E_z(r,\phi)$ of Eq. 1 are then linear combinations of $\cos(m\phi)$ and/or $\sin(m\phi)$ (m an integer) multiplied by Bessel functions $J_m(\eta)$ and $Y_m(\eta)$, where $\eta = \sqrt{2i}\kappa r$, or equivalently Hankel functions $H_m^{(\pm)}(\eta) = J_m(\eta) \pm iY_m(\eta)$. The magnetic field is $i\omega B_\phi = \partial E_z/\partial r$. The specific linear combination is determined by continuity of E_z and H_ϕ at interfaces, and by using Ampere's law to relate a line integral of the magnetic field around a conductor to the enclosed current. Finiteness at $r = 0$ dictates that the innermost layer must have $E_z(\eta) \sim J_m(\eta)$.

Given a set of N conductors, one can find the impedance matrix by first using the procedure above to solve for the electric and magnetic fields associated with the N distinct cases where a single conductor k ($k = 1, 2\ldots, N$) carries a net current $I_k\exp(i\omega t)$. The impedance matrix can then be derived by enforcing conservation of energy[8]. For example, one can find the complex-symmetric impedance matrix $Z_{k,l}$ by exciting the elements with currents $I_k\exp(i\omega t)$, superposing the previously computed solutions for the electric and magnetic fields, and computing the (complex) energy U of the system by integrating the net Poynting flux

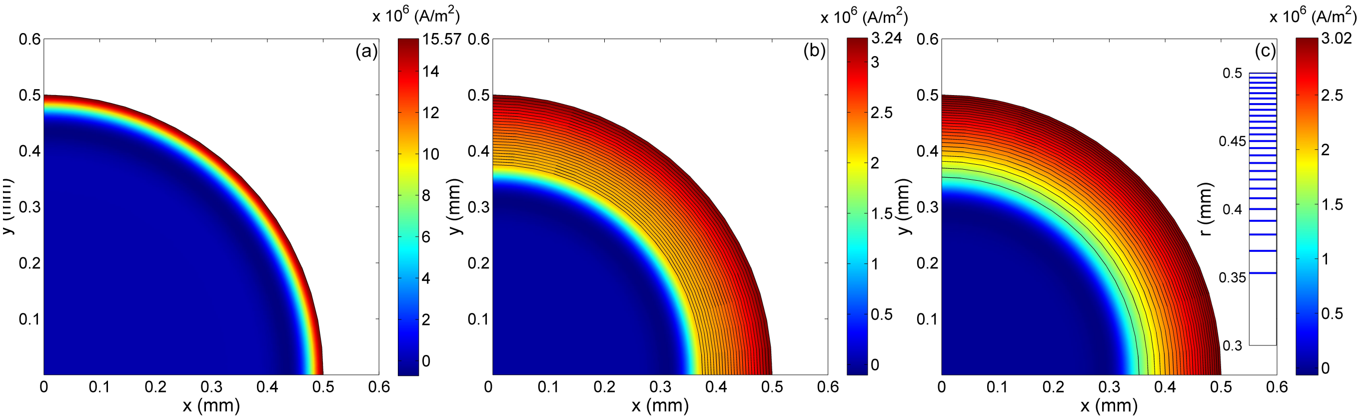

FIG. 1: Optimal current density at 10 MHz for a copper conductor with diameter 1 mm when the wire consists of: (a) one piece of copper (resistance per length of 265.9 mΩ/m), (b) 24 mutually insulated conductive concentric shells of equal thickness 5.19 μm around a solid copper conductor (55.2 mΩ/m), and (c) 25 elements whose thicknesses are optimized so as to minimize the overall resistance (51.6 mΩ/m). In (c), inset shows radial locations (blue) of interfaces between shells. The overall current is 1 A. For simplicity, the insulating gap between shells is taken to be negligibly small. These geometries were solved analytically in the text, whereas these images were generated by a finite-element method[9] as a check.

$\mathbf{S} = \mathbf{E} \times \mathbf{H}^*/2$ flowing into each shell and integrating the magnetic energy density $\mathbf{B} \cdot \mathbf{H}^*/4$ elsewhere. $Z_{k,l}$ then follows from $U = \sum_{k,l} I_k Z_{k,l} I_l^*/(2\omega)$. Once the impedance matrix is known, it is straightforward to compute the power dissipated by any currents $I_k \exp(i\omega t)$: $P_{\text{dis}} = \sum_{k,l} \text{Re}\left\{I_k Z_{k,l} I_l^*\right\}/2$. Since the total current is $\sum_k^N I_k \exp(i\omega t)$, the overall resistance is

$$R = \text{Re}\left\{\sum_{k,l}^{N} I_k Z_{k,l} I_l^*\right\} \Bigg/ \left|\sum_{k}^{N} I_k\right|^2 . \qquad (2)$$

Alternatively, if the current distribution is known beforehand (as in litz wire), one can compute the power dissipated and thus the overall resistance of the system by solving for the fields and integrating the Poynting flux into each conductor without computing Z.

We begin by reviewing traditional litz wire. It is convenient to split the problem into two steps: we first solve Eq. 1 for an isolated cylindrical strand of diameter d carrying current I, and then consider the proximity effect on a single strand from the net magnetic field of all strands. The first step is cylindrically symmetric, so $E_z \sim J_0(\eta)$, $H_\phi \sim J_1(\eta)$ and the power/length dissipated is $P_{\text{own}} = \text{Re}\left\{\eta_s J_0(\eta_s)/J_1(\eta_s)\right\} |I|^2/(\pi d^2 \sigma)$, where $\eta_s = \sqrt{2i}\kappa d/2$. If there are N identical strands carrying identical currents, uniformly arranged into a circular bundle of overall diameter D, then the interior magnetic field at a radius R is well approximated by $H_{\text{int}}(R) = 2NIR/(\pi D^2)$. If $d/D \ll 1$, H_{int} is essentially uniform over each strand, in which case the induced fields in a strand are $E_z \sim J_1(\eta)\sin(\phi)$, $H_\phi \sim [J_0(\eta) - J_2(\eta)]\sin(\phi)$ (with ϕ relative to the impinging magnetic field) and the resulting loss/length is $P_{\text{prox}} = \text{Re}\left\{\eta_s J_1(\eta_s)\left[J_2(\eta_s) - J_0(\eta_s)\right]^*/|J_0(\eta_s)|^2\right\} \pi |\mathbf{H}_{\text{int}}|^2/\sigma$. Summing all these losses, the total resistance/length (ignoring a small correction from the finite strand-winding pitch) is

$$R_{\text{litz}} = \frac{2}{N\pi d^2\sigma}\text{Re}\left\{\eta_s \frac{J_0(\eta_s)}{J_1(\eta_s)}\right\} + \frac{N}{\pi D^2 \sigma}\text{Re}\left\{\frac{\eta_s J_1(\eta_s)\left[J_2(\eta_s) - J_0(\eta_s)\right]^*}{|J_0(\eta_s)|^2}\right\}. \qquad (3)$$

In the limit $\kappa d \ll 1$, the lowest-order deviation from the DC resistance of N strands $[R_{\text{DC}} = 4/(N\pi d^2\sigma)]$ is $R_{\text{litz}} \simeq R_{\text{DC}}\left[1 + (Nd/D)^2(\kappa d)^4/128\right]$. Thus, to keep the resistance near the DC resistance of a solid conductor of diameter D $[4/(\pi D^2\sigma)]$, one would need the scalings $N \sim 1/d^2$ and $d \sim 1/\kappa^2 \sim 1/\omega$. For example, with $D = 1$ mm at 10 MHz, one would need about $N \simeq 10^4$ strands and $d < 10\,\mu$m diameters in order to have a total resistance within a factor of 3 of the DC value. Common commercially available litz wires ($\sim 10^2$ strands per mm^2) would typically perform significantly worse than a solid copper wire of the same overall diameter [Fig. 1 (a)] in this frequency range.

We now consider concentric shells and solve for the fields when shell k is excited by a current $I_k \exp(i\omega t)$. This reduces to two cases: a shell carrying a current with no external field and a shell carrying no net current but immersed in a magnetic field H_ϕ generated by a current-carrying inner shell. In either case, the solutions are determined by continuity and by the fact that the magnetic field in a non-conductive medium at a radius r is $H_\phi = I/(2\pi r)$, where I is the total current inside the radius r. For a shell with inner radius a, outer radius b,

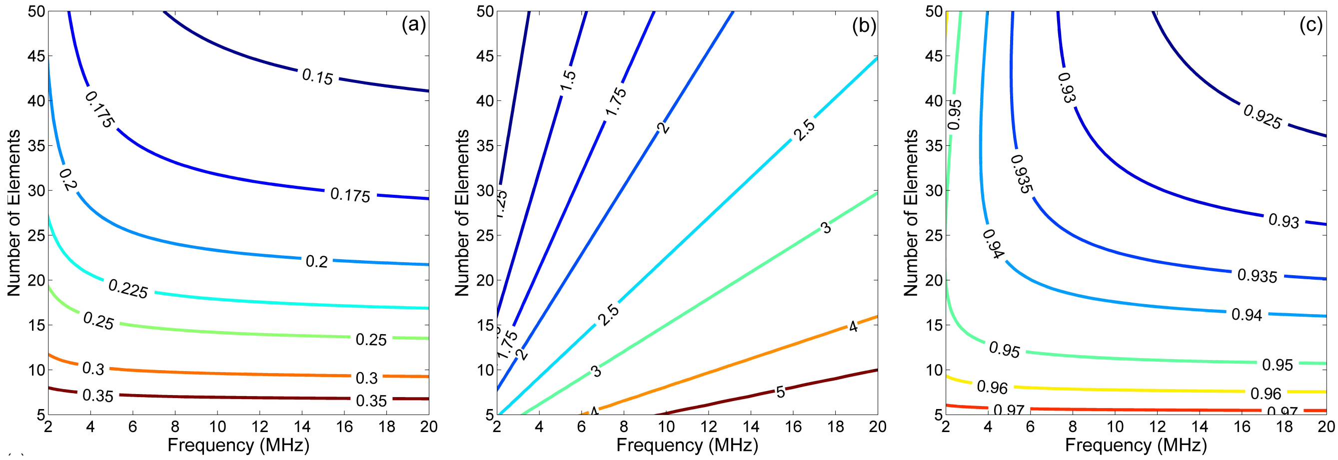

FIG. 2: Ratio of the resistance of an optimized conducting-shell structure with overall diameter 1 mm to: (a) the AC resistance of a solid conductor of the same diameter, (b) the DC resistance of the same conductor (21.6 mΩ/m), and (c) the resistance with the same number of elements, but with shells of (optimized) uniform thickness around a copper core.

and carrying current I, the fields are then:

$$E_z(\eta) = \frac{\sqrt{2i}\kappa I}{2\pi\sigma}\left[C_{a,b}^{(+)}H_0^{(+)}(\eta) + C_{a,b}^{(-)}H_0^{(-)}(\eta)\right] \quad (4)$$

$$H_\phi(\eta) = \frac{I}{2\pi}\left[C_{a,b}^{(+)}H_1^{(+)}(\eta) + C_{a,b}^{(-)}H_1^{(-)}(\eta)\right], \quad (5)$$

where the constants $C_{a,b}^{(\pm)}$ are given by

$$C_{a,b}^{(\pm)} = \frac{\mp H_1^{(\mp)}(\eta_a)/b}{H_1^{(+)}(\eta_a)H_1^{(-)}(\eta_b) - H_1^{(+)}(\eta_b)H_1^{(-)}(\eta_a)}, \quad (6)$$

with $\eta_a = \sqrt{2i}\kappa a$ and $\eta_b = \sqrt{2i}\kappa b$. Similarly, for the case of a shell with no current but enclosing a total current I, the fields are identical to Eqs. 4–5 except that $C_{a,b}^{(\pm)}$ are replaced by

$$D_{a,b}^{(\pm)} = \frac{\mp\left[H_1^{(\mp)}(\eta_a)/b - H_1^{(\mp)}(\eta_b)/a\right]}{H_1^{(+)}(\eta_a)H_1^{(-)}(\eta_b) - H_1^{(+)}(\eta_b)H_1^{(-)}(\eta_a)}. \quad (7)$$

It is now straightforward to derive the full impedance matrix, as described previously, to compute the resistance of any given current distribution via Eq. 2. A uniform current distribution over N shells of equal thickness $t = D/(2N)$, for instance, would have losses similar to those of a litz wire, although with a much smaller number of components ($N \sim \sqrt{N_{\text{strands}}}$, where N_{strands} is for litz wire with $d = t$). Given that the exact impedance matrix is known, however, elementary calculus yields the currents I_k that minimize Eq. 2 for a given structure. A more dramatic reduction in the resistance, especially when $N \lesssim \kappa D$, comes from letting the geometry of the shells vary (changing the impedance matrix) and then minimizing Eq. 2 using numerical optimization[10,11]. We have performed both a single-parameter optimization where a fixed number of shells of uniform thickness (the optimization parameter) surround a copper core (a cylindrical shell with inner radius 0) and a multi-parameter optimization where the thickness of each shell is a parameter [Figs. 1(b) and (c) show the resulting structures and current densities of, respectively, the single- and multi-parameter optimizations for 25 shells at 10 MHz]. Although we here model the innermost conductor as a solid core for simplicity, most of the current density flows within a skin-depth of this conductor (Fig. 1), and in practical applications substituting a cylindrical shell a couple of skin-depths thick or more for the solid core would result in a negligible increase in resistance. Fig. 2 compares the lowest resistance (as a function of N and the frequency) achievable by fixing the overall diameter of the conductor at $D = 1$ mm and optimizing both the individual thicknesses of the $(N-1)$ outer shells and the current distribution I_k to the resistance of: (a) a solid, heavy-gauge, conductor of the same overall D [resistance/length $\simeq \kappa/(\pi D\sigma)$ for $\kappa D \gg 1$], (b) the DC resistance per length of a solid copper wire $[4/(\pi D^2\sigma)]$ (corresponding to the $N \to \infty$ limit where proximity losses vanish and the current flows uniformly over the cross section), and (c) the optimal current distribution for $N-1$ shells of optimal *equal* thickness around a thick central conductor. Fig. 2(a) shows that an optimized cylindrical shell conductor can significantly outperform a solid conductor (the best currently available solution in this regime) of the same radius, while Fig. 2(c) shows that the conductors resulting from the single-parameter optimization (which may be significantly easier to realize experimentally) perform nearly as well as the more complex multi-parameter structures. For conductors with shells of uniform thickness, a regression analysis shows that the optimal thickness t_{opt} is well approximated by $\kappa t_{\text{opt}} = 1.104 \times N^{-0.4618}$ to within $< 1\%$ provided that $4 \lesssim N \lesssim \kappa D$.

A potential disadvantage of a concentric-shell structure compared to traditional litz wire is that the braiding of the latter automatically takes care of the impedance matching. However, one can see from Fig. 2(a) that much of the relative improvement of an optimized concentric shell conductor over a solid conductor occurs for structures with only a handful of elements, where even a brute force approach of individually matching the impedance of each shell to achieve the optimal current distribution could be implemented (similar impedance matching considerations arise in multi-layer high-T_c superconducting power cables[12,13], albeit at much lower frequencies). As shown in Fig. 2(a), an optimized concentric shell conductor with 10 elements and overall diameter 1 mm would have roughly 30% of the resistance of a solid conductor of the same diameter over the entire 2–20 MHz range. Equivalently, since the resistance of a solid conductor in the regime $\kappa D \gg 1$ scales as $1/D$, our optimized conductor with ten elements would have the same resistance per length as a solid conductor with a diameter $\sim 1/0.3 \simeq 3.33$ times greater (and $\simeq 10$ times the area).

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