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Asymptotically Unbiased Inference for a Dynamic Panel Model with Fixed Effects When Both n and T are Large

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December, 2000

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Abstract

We consider a dynamic panel AR(1) model with fixed effects when both n and T are large. Under the "T fixed n large" asymptotic approximation, the maximum likelihood estimator is known to be inconsistent due to the well-known incidental parameter problem. We consider an alternative asymptotic approximation where n and T grow at the same rate. It is shown that, although the MLE is asymptotically biased, a relatively simple fix to the MLE results in an asymptotically unbiased estimator. The bias corrected MLE is shown to be asymptotically efficient by a Hajék type convolution theorem.

Key Words: dynamic Panel, VAR, large n-large T asymptotics, bias correction, efficiency

JEL codes: C13, C23, C33

1 Introduction

In this paper, we consider estimation of the autoregressive parameter of a dynamic panel data model with fixed effects. The model has additive individual time invariant intercepts (fixed effects) along with a parameter common to every individual. The total number of parameters is therefore equal to the number of individuals plus the dimension of the common parameter, say K. When the number of individuals (n) is large relative to the time series dimension (T), a maximum likelihood estimator of all $n+K$ parameters would lead to inconsistent estimates of the common parameter of interest. This is the well-known incidental parameter problem.[1] Inconsistency of the MLE under T fixed n large asymptotics lead to a focus on instrumental variables estimation in the recent literature. Most instrumental variables estimators are at least partly based on the intuition that first differencing yields a model free of fixed effects.[2] Despite its appeal as a procedure which avoids the incidental parameter problem, the instrumental variables based procedure is problematic as a general principle to deal with potentially nonlinear panel models because of its inherent reliance on first differencing. Except for a small number of cases where conditioning on some sufficient statistic eliminates fixed effects, there does not seem to exist any general strategy even for potentially nonlinear panel models.

In this paper, we develop such a general strategy by considering an alternative asymptotic approximation where both n and T are large. We analyze properties of the MLE under this approximation. It is shown that the MLE is consistent and asymptotically normal, although it is not centered at the true value of the parameter. The noncentrality parameter under our alternative asymptotic approximation implicitly captures bias of order $O\left(T^{-1}\right)$, which can be viewed as an alternative form of the incidental parameter problem. We develop a bias-corrected estimator by examining the noncentrality parameter. Our strategy can be potentially replicated in nonlinear panel models, although analytic derivations for nonlinear models are expected to be much more involved than in linear dynamic panel models. We can in principle iterate our strategy to eliminate biases of order $O\left(T^{-2}\right)$ or $O\left(T^{-3}\right)$, although we do not pursue such a route here.

Having removed the asymptotic bias, we raise an efficiency question. Is the bias-corrected MLE asymptotically efficient among the class of all reasonable estimators? In order to assess efficiency, we derive a Hajék-type convolution theorem, and show that the asymptotic distribution of the bias-corrected MLE is equal to the minimal distribution in the convolution theorem.

Our alternative asymptotic approximation is expected to be of practical relevance if T is not too small compared to n as is the case for example in cross-country studies.[3] The properties of dynamic panel models are usually discussed under the implicit assumption that T is small and n is large relying on T fixed n large asymptotics. Such asymptotics seem quite natural when T is indeed very small compared to n. In cases where T and n are of comparable size we expect our approximation to be more accurate.

[1] See Neyman and Scott (1948) for general discussion on the incidental parameter problem, and Nickell (1981) for its implication in the particular linear dynamic panel model of interest.

[2] For discussion of various instrumental variables estimators and moment restrictions, see Holtz-Eakin, Newey, and Rosen (1988), Arellano and Bond (1992), Chamberlain (1992), Ahn and Schmidt (1995), Arellano and Bover (1995), Blundell and Bond (1995), and Hahn (1997).

[3] Inter-country comparison studies seems to be a reasonable application for such perspective. See Islam (1995) and/or Lee, Pesaran, and Smith (1998) for recent examples of inter-country comparison studies.

It should be emphasized that some of the results in Sections 3 are independently found by Alvarez and Arellano (1998). They derived basically the same result (and more) for the MLE and other IV estimators under the assumption that (i) the initial observation has a stationary distribution, and (ii) the fixed effects are normally distributed with zero mean. Although our result is derived under slightly more general assumptions in that we do not impose such conditions, this difference should be regarded as mere technicality. The more fundamental difference is that they were concerned with the comparison of various estimators for dynamic panel data models whereas we are concerned with bias correction and efficiency. Phillips and Moon (1999) recently considered a panel model where both T and n are large. They considered asymptotic properties of OLS estimators for a panel cointegrating relation when both T and n go to infinity. This paper differs from theirs with respect to the assumption that $0 < \lim n/T < \infty$ whereas they assume $\lim n/T = 0$ as $n, T \to \infty$. It is shown in Section 3 that the asymptotic bias of the MLE (OLS) is proportional to $\sqrt{n/T}$. Phillips and Moon (1999) showed that the OLS estimator is consistent and asymptotically normal with zero mean. Although their setup is different from ours in the sense that their regressor is assumed to be nonstationary, it is plausible that their asymptotic unbiasedness of OLS critically hinges on the assumption that $\lim n/T = 0$.

2 Bias Corrected MLE for Panel VAR with Fixed Effects

In this section, we consider estimation of the autoregressive parameter θ_0 in a dynamic panel model with fixed effects

$$y_{it}' = \alpha_i' + y_{it-1}'\theta_0' + \varepsilon_{it}', \quad i = 1, \ldots, n; \ t = 1, \ldots, T, \tag{1}$$

where y_{it} is an m-dimensional vector and ε_{it}' is i.i.d. normal. We establish the asymptotic distribution of the OLS estimator (MLE) for θ_0 under the alternative asymptotics, and develop an estimator free of (asymptotic) bias. We go on to argue that the bias corrected MLE is efficient using a Hajék type convolution theorem, and provide an intuitive explanation of efficiency by considering the limit of the Cramer-Rao lower bound. Finally, we point out that the asymptotic distribution of the bias corrected MLE is robust to nonnormality by presenting an asymptotic analysis for a model where ε_{it}' violates the normality assumption. We leave the efficiency analysis of models with nonnormal innovations for future research.

Model (1) may be understood as a parametric completion of the univariate dynamic panel AR(1) model with additional regressors. If we write $y_{it} = \left(Y_{it}, X_{it+1}'\right)'$, then the first component of the model (1) can be rewritten as

$$Y_{it} = c_i + \beta_0 \cdot Y_{it-1} + \gamma_0' X_{it} + e_{it}, \quad i = 1, \ldots, n; \ t = 1, \ldots, T \tag{2}$$

where c_i and $(\beta_0, \gamma_0')'$denote the first component of α_i and the first column of θ_0'. This implies that, under the special circumstances where X_{it} follows a first order VAR, we can regard model (1) as a completion of model (2). Under this interpretation, model (1) encompasses panel models with further regressors such as (2).

Even more generally, model (1) can be parametrized to be the reduced form of a dynamic simultaneous equation system in y_{it} allowing for higher order VAR dynamics as well as exogenous regressors. This requires imposing blockwise zero and identity restrictions on θ_0. It is well-known that MLE reduces to blockwise OLS as long the restrictions are block recursive. Even though we do not spell out the details of this interpretation of our model it is clear that extending our results to this more general case is straightforward.

If we assume ε_{it} is i.i.d. over t and i, and has a zero mean multivariate normal distribution, then the MLE (fixed effects estimator/OLS) takes the form

$$\widetilde{\theta}' = \left(\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)'\right)^{-1}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it}-\overline{y}_{i}\right)'\right),$$

where $\bar{y}_i \equiv \frac{1}{T}\sum_{t=1}^{T} y_{it}$, $\overline{y}_{i-} \equiv \frac{1}{T}\sum_{t=1}^{T} y_{it-1}$. We examine properties of $\widehat{\theta}$ under potential nonnormality of ε_{it} under the alternative asymptotics. If the innovations ε_{it} are not normal then the resulting estimator $\widetilde{\theta}'$ is a pseudo-MLE, and does no longer possess the efficiency properties of the exact MLE. For this reason we impose the additional assumption of normality for our discussion of asymptotic efficiency later in this section. We impose the following conditions:

Condition 1 *(i)* ε_{it} *is* i.i.d. *across* i *and strictly stationary in* t *for each* i, $E\left[\varepsilon_{it}\right] = 0$ *for all* i *and* t, $E\left[\varepsilon_{it}\varepsilon_{is}'\right] = \Omega \cdot 1\{t = s\}$; *(ii)* $0 < \lim \frac{n}{T} \equiv \rho < \infty$; *(iii)* $\lim_{n\to\infty}\theta_0^n = 0$; *and (iv)* $\frac{1}{n}\sum_{i=1}^{n}\left|y_{i0}\right|^2 = O(1)$ *and* $\frac{1}{n}\sum_{i=1}^{n}\left|\alpha_i\right|^2 = O(1)$.

The innovations ε_{it} are uncorrelated but not independent. Their higher order dependence allows for conditional heteroskedasticity. In order to be able to establish central limit theorems for our estimators and to justify covariance matrix estimation we need to impose additional restrictions on the distribution of the innovations. The dependence is limited by a fourth order cumulant summability restriction slightly stronger than in Andrews (1991). These conditions could be related to more primitive mixing conditions on the underlying ε_{it} as shown in Andrews(1991). We define $u_{it}^* \equiv \sum_{j=0}^{\infty}\theta_0^j\varepsilon_{it-j}$.

Condition 2

$$\sum_{t_1,t_2,t_3=-\infty}^{\infty}\left|\text{cum}_{j_1,\ldots,j_4}\left(u_{it_1}^*,\varepsilon_{it_2},u_{it_3}^*,\varepsilon_{i0}\right)\right| < \infty\ \forall i \text{ and } j_1,\ldots,j_k \in \{1,\ldots,m\}.$$

In the same way as Andrews (1991), we define $z_{it} \equiv \left(I \otimes u_{it-1}^*\right)\varepsilon_{it}$ and impose an additional eighth order moment restriction on ε_{it}, which takes the form of a fourth order cumulant summability condition on z_{it}.

Condition 3

$$\sum_{t_1,t_2,t_3=-\infty}^{\infty}\left|\text{cum}_{j_1,\ldots,j_4}\left(z_{it_1},z_{it_2},z_{it_3},z_{i0}\right)\right| < \infty\ \forall i \text{ and } j_1,\ldots,j_k \in \{1,\ldots,m\}.$$

Remark 1 *In Condition 1, our requirement that* $0 < \lim \frac{n}{T} \equiv \rho < \infty$ *corresponds to the choice of a particular set of asymptotic sequences. The choice of these sequences is guided by the desire to obtain asymptotic approximations that mimic certain moments of the finite sample distribution, in our case the mean of the estimator. Bekker (1994, p.661) argues that the choice of a particular sequence can be justified by its ability to "generate acceptable approximations of known distributional properties of related statistics".*

In our case it seems most appropriate to investigate the properties of the score related to the dynamic panel model. After concentrating out the fixed effects, we are lead to consider the normalized score process $S_{nT} = \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes \left(y_{it-1} - \overline{y}_{i-} \right) \right) \left(\varepsilon_{it} - \overline{\varepsilon}_{i} \right)$. *In the appendix, we show that*[4] $S_{nT} \xrightarrow{d} S$ *under the alternative asymptotics with* $n/T \rightarrow \rho$, *where* S *has a normal distribution with mean equal to* $-\sqrt{\rho}\left(I \otimes I - (I \otimes \theta_0)\right)^{-1} \operatorname{vec}(\Omega)$. *Clearly, under fixed* T *large* n *asymptotics the score process has an explosive mean leading to the inconsistency result. The exact finite sample bias for the score is given by* $E\left[S_{nT}\right] = -\sqrt{n/T}T^{-1} \sum_{t=1}^{T} \sum_{j=0}^{T} \left(I \otimes \theta_0^j \right) \operatorname{vec}(\Omega)$. *The term* $-\sqrt{\rho}T^{-1} \sum_{t=1}^{T} \sum_{j=0}^{T} \left(I \otimes \theta_0^j \right) \operatorname{vec}(\Omega)$ *converges to* $-\sqrt{\rho}\left(I \otimes I - (I \otimes \theta_0)\right)^{-1} \operatorname{vec}(\Omega) = E[S]$ *by the Toeplitz lemma as* $T \rightarrow \infty$, *and is closer to* $E[S]$ *for small values of* θ_0. *In other words our asymptotic sequence preserves the mean of the score process in the limit. The form of the approximation error also may explain simulation findings indicating that the approximation improves for larger values of* T *and deteriorates with* θ *getting closer to the unit circle.*

Our asymptotics may also be understood as an attempt to capture the bias of the score of order $O\left(T^{-1}\right)$. *We show in the appendix that the score process is well approximated by a process, say* S_{nT}^{*},[5] *such that* $E\left[S_{nT}^{*}\right] = -\sqrt{\frac{n}{T}} \frac{1}{T} \sum_{t=1}^{T} \sum_{j=0}^{t-1} \left(I \otimes \theta_0^j \right) \operatorname{vec}(\Omega)$.[6] *Because the term* $\frac{1}{T} \sum_{t=1}^{T} \sum_{j=0}^{t-1} \left(I \otimes \theta_0^j \right) \operatorname{vec}(\Omega)$ *is of order* $O(1)$, *the approximate mean of the normalized score process can be elicited only by considering the alternative approximation where* n *and* T *grow to infinity at the same rate. The mean of the score process that our asymptotics captures may also be identified as the bias of the score up to* $O\left(T^{-1}\right)$. *Because the score* $\frac{1}{\sqrt{nT}} S_{nT}$ *is approximated by* $\frac{1}{\sqrt{nT}} S_{nT}^{*}$, *and because*

$$E\left[\frac{1}{\sqrt{nT}} S_{nT}^{*}\right] = -\frac{1}{T}\frac{1}{T} \sum_{t=1}^{T} \sum_{j=0}^{t-1} \left(I \otimes \theta_0^j \right) \operatorname{vec}(\Omega) = \frac{1}{T}\left(-\left(I \otimes I - (I \otimes \theta_0)\right)^{-1} \operatorname{vec}(\Omega) + o(1) \right),$$

we may understand $-\frac{1}{T}\left(I \otimes I - (I \otimes \theta_0)\right)^{-1} \operatorname{vec}(\Omega)$ *as the bias of the score of order* $O\left(T^{-1}\right)$.

Remark 2 *Condition 2 implies* $\sum_{j=-\infty}^{\infty} \left| \operatorname{Cov}_{k_1,k_2}\left(z_{it}, z_{it-j}\right) \right| < \infty$, *because*

$$\begin{aligned} \operatorname{Cov}_{k_1,k_2}(z_{it}, z_{it-j}) = {} & \operatorname{cum}_{l_1,..,l_4}\left(u_{it-1}^{*}, \varepsilon_{it}, u_{it-1-j}^{*}, \varepsilon_{it-j}\right) + \operatorname{Cov}_{l_1,l_3}\left(u_{it-1}^{*}, u_{it-1-j}^{*}\right) \operatorname{Cov}_{l_2,l_4}\left(\varepsilon_{it}, \varepsilon_{it-j}\right) \\ & + \operatorname{Cov}_{l_1,l_4}\left(u_{it-1}^{*}, \varepsilon_{it-j}\right) \operatorname{Cov}_{l_3,l_2}\left(u_{it-1-j}^{*}, \varepsilon_{it}\right), \end{aligned}$$

where $k_1 = l_1 m + l_2 + 1$ *and* $k_2 = l_3 m + l_4 + 1$ *with* $l_1, ..., l_4 \in \{0, 1, ..., m\}$. *In this sense our Condition 2 is stronger than the first part of Assumption A in Andrews (1991). Condition 3 is identical to the second part of Assumption A in Andrews (1991).*

Remark 3 *In the special case where* ε_{it} *is iid across* i *and* t *Conditions 2 and 3 are equivalent to* $E\left[\left|\varepsilon_{it}^{(j)}\right|^8\right] < \infty$ *for all* j *where* $\varepsilon_{it}^{(j)}$ *is the* j*-th element in* ε_{it}. *See Lemma 1 in Appendix A.*

We show below that the MLE $\widehat{\theta}$ is consistent, but $\sqrt{nT} \operatorname{vec}\left(\widehat{\theta}' - \theta_0'\right)$ is not centered at zero:

[4] Lemma 6 in Appendix A.

[5] The exact definition of S_{nT}^{*} is given in (12) in Appendix A.

[6] See Lemma 3.

Theorem 1 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3, we have*

$$\sqrt{nT}\,\mathrm{vec}\left(\widetilde{\theta}' - \theta_0'\right) \to \mathcal{N}\left(-\sqrt{\rho}(I\otimes\Upsilon)^{-1}\left(I\otimes I-(I\otimes\theta_0)\right)^{-1}\mathrm{vec}\,(\Omega)\,, (I\otimes\Upsilon)^{-1}\left(\Omega\otimes\Upsilon+\mathcal{K}\right)(I\otimes\Upsilon)^{-1}\right),$$

where $\Upsilon \equiv \Omega+\theta_0\Omega\theta_0'+\theta_0^2\Omega\left(\theta_0'\right)^2+\cdots$, $\mathcal{K}\equiv\sum_{t=-\infty}^{\infty}\mathcal{K}\,(t,0)$, $\mathcal{K}\,(t_1,t_2)\equiv E\left[\left(I\otimes u_{it_1-1}^*\right)\varepsilon_{it_1}\varepsilon_{it_2}'\left(I\otimes u_{it_2-1}^{*\prime}\right)\right]$ $-\ E\left[\varepsilon_{it_1}\varepsilon_{it_2}'\right]\otimes E\left[u_{i0}^*u_{i0}^{*\prime}\right]$, *and* $u_{it}^*\equiv\sum_{j=0}^{\infty}\theta_0^j\varepsilon_{it-j}$. *If in addition all the innovations ε_{it} are independent for all i and t then*

$$\sqrt{nT}\,\mathrm{vec}\left(\widetilde{\theta}' - \theta_0'\right) \to \mathcal{N}\left(-\sqrt{\rho}(I\otimes\Upsilon)^{-1}\left(I\otimes I-(I\otimes\theta_0)\right)^{-1}\mathrm{vec}\,(\Omega)\,,\Omega\otimes\Upsilon^{-1}\right).$$

Under our alternative asymptotic sequence the MLE is therefore consistent but has a limiting distribution that is not centered at zero. The non-centrality parameter results from correlation between the averaged error terms and the regressors y_{it-1}. Because averaging takes place for each individual the estimated sample means do not converge to constants fast enough to eliminate their effect on the limiting distribution. Under our asymptotics the convergence is however fast enough to eliminate the inconsistency problem found for fixed T large n asymptotic approximations.

When the innovations are not iid then the limiting distribution is affected by higher order moments reflecting the conditional heteroskedasticity in the data. The limiting covariance matrix $\Omega\otimes\Upsilon+\mathcal{K}$ can also be expressed as $\lim T^{-1}\sum_{t_1t_2=-T}^{T}E\left[\left(I\otimes u_{it_1-1}^*\right)\varepsilon_{it_1}\varepsilon_{it_2}'\left(I\otimes u_{it_2-1}^{*\prime}\right)\right]$. Standard tools for consistent and optimal estimation of $\Omega\otimes\Upsilon+\mathcal{K}$ were discussed in Andrews (1991). Under our conditions the results of Andrews are directly applicable.

Our theorem 1 roughly predicts that

$$\mathrm{vec}\left(\widetilde{\theta}' - \theta_0'\right) \to \mathcal{N}\left(-\frac{1}{T}(I\otimes\Upsilon)^{-1}\left(I\otimes I-(I\otimes\theta_0)\right)^{-1}\mathrm{vec}\,(\Omega)\,,\frac{1}{nT}(I\otimes\Upsilon)^{-1}\left(\Omega\otimes\Upsilon+\mathcal{K}\right)(I\otimes\Upsilon)^{-1}\right).$$

Therefore, the noncentrality-parameter $-\sqrt{\rho}(I\otimes\Upsilon)^{-1}\left(I\otimes I-(I\otimes\theta_0)\right)^{-1}\mathrm{vec}\,(\Omega)$ can be viewed as a device to capture bias of order up to $O\left(T^{-1}\right)$.

Our bias corrected MLE is given by

$$\begin{aligned}\mathrm{vec}\left(\widehat{\widetilde{\theta}}\right) \equiv & \left[I\otimes\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)'\right)^{-1}\right] \\ & \cdot\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\left(y_{it-1}-\overline{y}_{i-}\right)\right)\left(y_{it}-\overline{y}_i\right)'+\frac{1}{T}\left(I\otimes I-\left(I\otimes\widehat{\theta}\right)\right)^{-1}\mathrm{vec}\left(\widehat{\Omega}\right)\right], \quad (3)\end{aligned}$$

where

$$\widehat{\Upsilon}\equiv\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)', \quad \text{and} \quad \mathrm{vec}\left(\widehat{\Omega}\right)\equiv\left(I\otimes I-\left(\widehat{\theta}\otimes\widehat{\theta}\right)\right)\mathrm{vec}\left(\widehat{\Upsilon}\right). \quad (4)$$

We show below that the bias-corrected MLE $\widehat{\widetilde{\theta}}$ is consistent, and $\sqrt{nT}\,\mathrm{vec}\left(\widehat{\widetilde{\theta}}' - \theta_0'\right)$ is centered at zero:

Theorem 2 *Let y_{it} be generated by (1). Then, under Conditions 1, 2 and 3, we have*

$$\sqrt{nT}\,\mathrm{vec}\left(\widehat{\widehat{\theta}}' - \theta_0'\right) \to \mathcal{N}\left(0, (I\otimes\Upsilon)^{-1}(\Omega\otimes\Upsilon+\mathcal{K})(I\otimes\Upsilon)^{-1}\right).$$

If in addition all the innovations ε_{it} are independent for all i and t then

$$\sqrt{nT}\,\mathrm{vec}\left(\widehat{\widehat{\theta}}' - \theta_0'\right) \to \mathcal{N}\left(0, \Omega\otimes\Upsilon^{-1}\right).$$

We now show that the bias-corrected MLE is asymptotically efficient. We do so by showing that the asymptotic distribution of the bias corrected MLE is 'minimal' in the sense of a Hajék type convolution theorem.[7] We show that the asymptotic distribution of any reasonable estimator can be written as a convolution of the 'minimal' normal distribution and some other arbitrary distribution. In this sense, the bias-corrected MLE can be understood to be asymptotically efficient.

Condition 4 *(i)* $\varepsilon_{it} \sim \mathcal{N}(0,\Omega)$ i.i.d.; *(ii)* $0 < \lim \frac{n}{T} \equiv \rho < \infty$; *(iii)* $\lim_{n\to\infty}\theta_0^n = 0$; *and (iv)* $\frac{1}{n}\sum_{i=1}^n |y_{i0}|^2 = O(1)$ *and* $\frac{1}{n}\sum_{i=1}^n |\alpha_i|^2 = O(1)$.

In order to discuss efficiency we naturally have to guarantee that $\hat{\theta}$ is the exact MLE. For this reason we impose the additional requirement of normal innovations in condition (4).

Theorem 3 *Let y_{it} be generated by (1). Suppose that Condition (4) is satisfied. Then, the asymptotic distribution of any regular estimator of* $\mathrm{vec}(\theta_0)$ *cannot be more concentrated than* $\mathcal{N}\left(0,\Omega\otimes\Upsilon^{-1}\right)$.

Proof. See Appendix C.3. ■

It should be emphasized that Theorem 3 in itself does not say anything about the attainability of the bound $\Omega\otimes\Upsilon^{-1}$. The asymptotic variance bound it provides is a lower bound of the asymptotic variances of regular estimators. On the other hand, it is not clear whether such a bound is attainable. Comparison with Theorem 2 leads us to conclude that the bound is attained by the bias corrected MLE as long as the innovations ε_{it} are iid Gaussian.

Corollary 1 *Under Condition 4, the bias corrected MLE $\widehat{\widehat{\theta}}$ is asymptotically efficient.*

3 Application to Univariate Dynamic Panel Model with Fixed Effects

In this section, we apply Theorems 1 and 2 in the previous section to the univariate stationary panel AR(1) model with fixed effects

$$y_{it} = \alpha_i + \theta_0 y_{it-1} + \varepsilon_{it}, \quad i = 1,\ldots,n;\ t = 1,\ldots,T. \tag{5}$$

We also consider estimation of fixed effects α_i in the univariate contexts. Finally, we examine how the result changes under the unit root. It turns out that the distribution of the MLE is quite sensitive to such

[7] See Appendix C.1 for the exact sense under which the asymptotic distribution of bias corrected MLE is 'minimal'.

a specification change. As such, we expect that our bias corrected estimator will not be (approximately) unbiased under a unit root.

We first apply Theorems 1 and 2 to the univariate case. Obviously, Condition 4 would now read (i) $\varepsilon_{it} \sim \mathcal{N}(0,\Omega)$ *i.i.d.*; (ii) $0 < \lim \frac{n}{T} \equiv \rho < \infty$; (iii) $|\theta_0| < 1$; and (iv) $\frac{1}{n}\sum_{i=1}^{n} y_{i0}^2 = O(1)$ and $\frac{1}{n}\sum_{i=1}^{n} \alpha_i^2 = O(1)$. Note that the MLE (OLS) is given by

$$\widehat{\theta} = \frac{\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it}-\overline{y}_i\right)\cdot\left(y_{it-1}-\overline{y}_{i-}\right)}{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)^2}.$$

Applying (3) and (4) to the univariate model, we obtain

$$\widehat{\widehat{\theta}} \equiv \left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)^2\right)^{-1}\left[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it}-\overline{y}_i\right)+\frac{1}{T}\left(1-\widehat{\theta}\right)^{-1}\widehat{\Omega}\right]$$

where

$$\widehat{\Omega} = \left(1-\widehat{\theta}^2\right)\widehat{\Upsilon} = \left(1-\widehat{\theta}^2\right)\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)^2\right).$$

Therefore, our bias corrected estimator is given by

$$\widehat{\widehat{\theta}} = \widehat{\theta} + \frac{1}{T}\frac{1}{1-\widehat{\theta}}\left(1-\widehat{\theta}^2\right) = \widehat{\theta} + \frac{1}{T}\left(1+\widehat{\theta}\right) = \frac{T+1}{T}\widehat{\theta} + \frac{1}{T}. \tag{6}$$

Because $\Upsilon = \frac{\Omega}{1-\theta_0^2}$ in the univariate case, we can conclude from Theorem 2 that

$$\sqrt{nT}\left(\widehat{\widehat{\theta}} - \theta_0\right) \to \mathcal{N}\left(0, 1-\theta_0^2\right)$$

From Theorem 3, we can also conclude that $\widehat{\widehat{\theta}}$ is efficient under the alternative asymptotics where $n, T \to \infty$ at the same rate.

Our theoretical result may be related to Kiviet's (1995) result. He derived an expression of some approximate bias of the MLE, which depends on the unknown parameter values including θ_0. He showed by simulation that the *infeasible* bias-corrected MLE, based on such knowledge of θ_0, has much more desirable finite sample properties than various instrumental variable type estimators. Because his bias correction depends on the unknown parameter value θ_0, feasible implementation appears to require a preliminary estimator of θ_0. He considered instrumental variable type estimators as preliminary estimators in his simulation study, but he failed to produce a theory for the corresponding estimator. Our bias corrected estimator, which does not require a preliminary estimator of θ_0, may be understood as an implementable version of Kiviet's estimator. Also, our convolution theorem may be understood as a formalization of his simulation result.

We now consider estimation of fixed effects α_i. Recently, Geweke and Keane (1996), Chamberlain and Hirano (1997), and Hirano (1998) examined predictive aspects of the dynamic panel model from a Bayesian perspective. From a Frequentist perspective, prediction requires estimation of individual specific intercept terms. We argue that intercept estimation is asymptotically unbiased to begin with, and is affected very little by bias corrected estimation of θ_0. It follows that estimation of θ_0 can be separately analyzed even for the purpose of prediction. Observe that the MLE of α_i is given by

$$a_i \equiv \frac{1}{T}\sum_{t=1}^{T}\left(y_{it}-\widehat{\theta}y_{it-1}\right)=\alpha_i+\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}-\left(\widehat{\theta}-\theta_0\right)\frac{1}{T}\sum_{t=1}^{T}y_{it-1}. \tag{7}$$

so that

$$\begin{aligned}\sqrt{T}\left(a_i-\alpha_i\right) &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}-\sqrt{T}\left(\widehat{\theta}-\theta_0\right)\cdot\frac{1}{T}\sum_{t=1}^{T}y_{it-1}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}-O_p\left(\frac{1}{\sqrt{n}}\right)\cdot O_p(1)\\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}+o_p(1).\end{aligned}$$

Because $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}$ converges in distribution to $N\left(0,\sigma_0^2\right)$ as $T\to\infty$, the MLE is asymptotically unbiased. Furthermore, we have

$$\sqrt{T}\left(\widehat{a}_i-\alpha_i\right)=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}+o_p(1),$$

where $\widehat{a}_i$ denotes the estimator of α_i obtained by replacing the MLE $\widehat{\theta}$ in (7) by the bias corrected estimator $\widehat{\widehat{\theta}}$. It follows that more efficient estimation of θ_0 does not affect the estimation of α_i.

We now consider the nonstationary case where $\theta_0=1$. We first consider a simple dynamic panel model with a unit root, where individual specific intercepts are all equal to zero but the econometrician does not know that. The econometrician therefore estimates fixed effects along with θ.

Theorem 4 *Suppose that (i)* $\varepsilon_{it}\sim\mathcal{N}\left(0,\sigma^2\right)$ i.i.d; *(ii)* $\alpha_i\equiv 0$; *(iii)* $\theta_0=1$; *and (iv)* $n,T\to\infty$.[8] *We then have*

$$\sqrt{nT^2}\left(\widehat{\theta}-\theta_0+\frac{3}{T+1}\right)\to\mathcal{N}\left(0,\frac{51}{5}\right).$$

Proof. See Appendix D.1. ∎

One obvious implication of Theorem 4 is that the bias correction for the stationary case is not expected to work under the unit root. In order to understand the intuition, it is useful to note that Theorem 4 roughly predicts that

$$\widehat{\theta}\approx\mathcal{N}\left(1-\frac{3}{T+1},\frac{10.2}{nT^2}\right), \tag{8}$$

which indicates that the bias only depends on T. For example, Theorem 4 roughly predicts that the MLE is centered around $\frac{1}{5}$ if $T=5$. The bias correction for the stationary model critically hinges on the fact that the rough bias in a finite sample is a function of n and T, and hence, is not expected to be robust to the unit root specification.

We now consider the case where individual specific intercepts are nonzero, and the econometrician estimates them along with θ.

Theorem 5 *Suppose that (i)* $\varepsilon_{it}\sim\mathcal{N}\left(0,\sigma^2\right)$ i.i.d; *(ii)* $\lim\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2>0$; *(iii)* $\theta_0=1$; *and (iv)* $\lim\sqrt{\frac{n}{T}}$ *exists. We then have*

$$n^{1/2}T^{3/2}\left(\widehat{\theta}-\theta_0\right)\to\mathcal{N}\left(-\frac{6\sigma^2\lim\sqrt{\frac{n}{T}}}{\lim\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2},\frac{12\sigma^2}{\lim\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2}\right).$$

[8] No particular rate on the growth of n and T is imposed.

Proof. See Appendix D.2. ■

Although Theorem 5 shares the same feature as Theorem 1 as far as the asymptotic bias being proportional to $\lim \sqrt{\frac{n}{T}}$, it is quite clear that the bias correction for the stationary case does not work because the asymptotic bias under the unit root depends on $\lim \frac{1}{n}\sum_{i=1}^{n}\alpha_i^2$.

4 Monte Carlo

We conduct a small Monte Carlo experiment to evaluate the accuracy of our asymptotic approximations to the small sample distribution of the MLE and bias corrected MLE. We generate samples from the model

$$y_{it} = \alpha_i + \theta_0 y_{it-1} + \varepsilon_{it}$$

where $y_{it} \in \mathbb{R}$, $\theta_0 \in \{0, .3, .6, .9\}$, $\alpha_i \sim \mathcal{N}(0,1)$ independent across i, and $\varepsilon_{it} \sim \mathcal{N}(0,1)$ independent across i and t. We generate α_i and ε_{it} such that they are independent of each other. We chose $y_{i0}|\,\alpha_i \sim \mathcal{N}\left(\frac{\alpha_i}{1-\theta_0}, \frac{\mathrm{Var}(\varepsilon_{it})}{1-\theta_0^2}\right)$. The effective sample sizes we consider are $n = \{100, 200\}$ and $T \in \{5, 10, 20\}$.[9] For each sample of size n and T we compute the bias corrected MLE $\widehat{\widehat{\theta}}$ based on the formulation (6). We also compute the usual GMM estimator $\widehat{\theta}_{GMM}$ based on the first differences

$$y_{it} - y_{it-1} = \theta_0 \left(y_{it-1} - y_{it-2}\right) + \varepsilon_{it}$$

using past levels $(y_{i0}, \cdots, y_{it-2})$ as instruments. In order to avoid the complexity of weight matrix estimation, we considered Arellano and Bover's (1995) modification.[10]

Finite sample properties of both estimators obtained by 5000 Monte Carlo runs are summarized in Table 1. We can see that both estimators have some bias problems. Unfortunately, our bias corrected estimator does not completely remove the bias. This suggests that an even more careful small sample analysis based on higher order expansions of the distribution might be needed to account for the entire bias. On the other hand, the efficiency of $\widehat{\widehat{\theta}}$ measured by the root mean squared error (RMSE) often dominates that of the GMM estimator, suggesting that our crude higher order asymptotics and the related convolution theorem provided a reasonable prediction about the efficiency of the bias-corrected MLE.

5 Summary

In this paper, we considered a dynamic panel model with fixed effects where n and T are of the same order of magnitude. We developed a method to remove the asymptotic bias of MLE, and showed that such a bias corrected MLE is asymptotically efficient in the sense that its asymptotic variance equals that of the limit of the Cramer-Rao lower bound. Our simulation results compare our efficient bias corrected MLE to more conventional GMM estimators. It turns out that our estimator has comparable bias properties and often dominates the GMM estimator in terms of mean squared error loss for the sample sizes that we think our procedure is most appropriate for.

[9] A more extensive Monte Carlo results are available from authors upon request.

[10] Arellano and Bover (1995) proposed to use the moment restrictions obtained by Helmert's transformation of the same set of information. Strictly speaking, therefore, their estimator is not based on the first differences with past level instruments.

One advantage of using bias corrected MLE as the guiding principle for constructing estimators is that it can more naturally be extended to nonlinear models. GMM estimators on the other hand ultimately rely on transformations such as first differencing or similar averaging techniques to remove the individual fixed effects. Such transformations are inherently linear in nature and therefore not suited for generalizations to a nonlinear context.

Our technique can in principle be generalized to remove bias of higher order than T^{-1} by repeating the alternative asymptotic approximation scheme for an appropriately rescaled version of the bias corrected estimator. We are planning to pursue this avenue in future research.

Our bias corrected MLE is not expected to be asymptotically unbiased under a unit root. We leave development of a bias corrected estimator robust to nonstationarity to future research.

Appendix

A Proof of Theorem 1

Theorem 1 is established by combining Lemmas 6 and 7 below. Note that

$$y_{it} = \theta_0^t y_{i0} + (I - \theta_0)^{-1}\left(1 - \theta_0^t\right)\alpha_i + \theta_0^{t-1}\varepsilon_{i1} + \theta_0^{t-2}\varepsilon_{i2} + \cdots + \varepsilon_{it}. \tag{9}$$

In the stationary case where $\lim_n \theta_0^n = 0$, we work with the stationary approximation to y_{it} which is given by

$$u_{it}^* \equiv \sum_{j=0}^{\infty} \theta_0^j \varepsilon_{it-j}, \quad t \geq 1 \tag{10}$$

$$y_{it}^* \equiv (I - \theta_0)^{-1}\alpha_i + u_{it}^*, \quad t \geq 0. \tag{11}$$

The vectorized representation of the OLS estimator for θ_0' is given by

$$\operatorname{vec}\left(\widehat{\theta}' - \theta_0'\right) = \left[I \otimes \left(\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1} - \overline{y}_{i-}\right)\left(y_{it-1} - \overline{y}_{i-}\right)'\right)^{-1}\right]\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I \otimes \left(y_{it-1} - \overline{y}_{i-}\right)\right)\left(\varepsilon_{it} - \overline{\varepsilon}_i\right)\right]$$

where $\overline{\varepsilon}_i \equiv \frac{1}{T}\sum_t^T \varepsilon_{it}$. We define the joint cumulant next. Let $\xi = (\xi_1, ..., \xi_k) \in \mathbb{R}^k$ and $\varepsilon = \left(\varepsilon_{t_1}^{(j_1)}, ..., \varepsilon_{t_k}^{(j_k)}\right)$ where $\varepsilon_t^{(j)}$ is the j-th element of ε_t, then $\phi_{j_1,...j_k,t_1,...,t_k}(\xi) \equiv E\left[e^{i\xi'\varepsilon}\right]$ is the joint characteristic function with corresponding cumulant generating function $\ln \phi_{j_1,...j_k,t_1,...,t_k}(\xi)$. The joint v-th order cross-cumulant function is

$$\operatorname{cum}_{j_1,...j_k}\left(\varepsilon_{t_1}, ..., \varepsilon_{t_k}\right) \equiv \left.\frac{\partial^{v_1+...+v_k}}{\partial\xi_1^{v_1}\cdots\partial\xi_k^{v_k}} \ln \phi_{j_1,...j_k,t_1,...,t_k}(\xi)\right|_{\xi=0}$$

where v_i are nonnegative integers $v_1 + ... + v_k = v$.

Lemma 1 *Let ε_{it} be iid across i and t and $E\left[\left|\varepsilon_{it}^{(j)}\right|^8\right] < \infty$. Assume Conditions 4 (ii) - (iv) hold. Then Conditions 2 and 3 hold.*

Proof. Available upon request from the authors. ■

Lemma 2 *Let y_{it} be generated by (1). Also, let*

$$S_{nT}^* \equiv \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I \otimes u_{it}^*\right)\left(\varepsilon_{it} - \overline{\varepsilon}_i\right)$$

Then under Conditions 1, 2 and 3

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I \otimes \left(y_{it-1} - \overline{y}_{i-}\right)\right)\left(\varepsilon_{it} - \overline{\varepsilon}_i\right) = S_{nT}^* + o_p(1) \tag{12}$$

Proof. Because $\sum_{t=1}^{T}\left(I\otimes\overline{y}_{i-}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)=0$, and

$$\sum_{t=1}^{T}\left(I\otimes y_{it-1}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)=\sum_{t=1}^{T}\left(I\otimes\left(\left(I-\theta_0\right)^{-1}\alpha_i+u_{it}^{*}\right)\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)=\sum_{t=1}^{T}\left(I\otimes u_{it}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right),$$

it suffices to prove that

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes y_{it-1}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes y_{it-1}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)+o_p(1).$$

From (9), (10), and (11), we obtain $y_{it}=y_{it}^{*}+\theta_0^{t}\left(y_{i0}-u_{i0}^{*}\right)-\left(I-\theta_0\right)^{-1}\theta_0^{t}\alpha_i$. We may therefore write

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes y_{it-1}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)-\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes y_{it-1}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)\\
&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}y_{i0}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right) \qquad (13)\\
&-\left(I-\theta_0\right)^{-1}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}\alpha_i\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right) \qquad (14)\\
&-\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}u_{i0}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right). \qquad (15)
\end{aligned}$$

We analyze (13) first. By assumption, its expectation is zero. Because of independence of ε_{it} and y_{i0}, we have

$$\operatorname{Var}\left(\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}y_{i0}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)\right)=\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}y_{i0}\right)\Omega\left(I\otimes y_{i0}'\theta_0'^{t}\right)-\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(I\otimes\theta_0^{t}y_{i0}\right)\Omega\left(I\otimes y_{i0}'\theta_0'^{s}\right),$$

from which it follows that

$$\begin{aligned}
&\operatorname{vec}\left[\operatorname{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}y_{i0}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)\right)\right]\\
=\;&T^{-1}\left(I\otimes I\otimes I\otimes I-\left(I\otimes\theta_0\right)\otimes\left(I\otimes\theta_0\right)\right)^{-1}\left(\left(I\otimes\theta_0\right)\otimes\left(I\otimes\theta_0\right)-\left(\left(I\otimes\theta_0\right)\otimes\left(I\otimes\theta_0\right)\right)^{T+1}\right)\\
&\times\operatorname{vec}\left(n^{-1}\sum_{i=1}^{n}\left(I\otimes y_{i0}\right)\Omega\left(I\otimes y_{i0}'\right)\right)\\
&-T^{-2}\left(I\otimes\left(I-\theta_0\right)^{-1}\left(\theta_0-\theta_0^{T+1}\right)\right)\otimes\left(I\otimes\left(I-\theta_0\right)^{-1}\left(\theta_0-\theta_0^{T+1}\right)\right)\operatorname{vec}\left(n^{-1}\sum_{i=1}^{n}\left(I\otimes y_{i0}\right)\Omega\left(I\otimes y_{i0}'\right)\right)\\
=\;&o(1).
\end{aligned}$$

It therefore follows that (13) is $o_p(1)$. In the same way it follows that (14) is $o_p(1)$. We turn to the analysis of (15). Because of independence of ε_{it} and u_{i0}^{*}, we can see that it has a mean equal to zero, and

$$\begin{aligned}
\operatorname{vec}\left[\operatorname{Var}\left(\sum_{t=1}^{T}\left(I\otimes\theta_0^{t}u_{i0}^{*}\right)\varepsilon_{it}\right)\right]&=\sum_{t_1,t_2=1}^{T}\left(\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right)\operatorname{vec}\left(E\left[\left(I\otimes u_{i0}^{*}\right)\varepsilon_{it_1}\varepsilon_{it_2}'\left(I\otimes u_{i0}^{*\prime}\right)\right]\right)\\
&=\sum_{t_1,t_2=1}^{T}\left(\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right)\left(E\left[\varepsilon_{it_1}\varepsilon_{it_2}'\right]\otimes E\left[u_{i0}^{*}u_{i0}^{*\prime}\right]+\mathcal{K}_0\left(t_1,t_2\right)\right),
\end{aligned}$$

where the matrix $\mathcal{K}_0(t_1, t_2)$ contains elements of the form $\mathrm{cum}_{j_1,\dots,j_4}(u_{i0}^*, u_{i0}^*, \varepsilon_{it_1}, \varepsilon_{it_2})$. The sum over the first term then can be expressed as

$$\sum_{t=1}^{T}\left((I\otimes\theta_0^t)\otimes(I\otimes\theta_0^t)\right)\mathrm{vec}\left[\Omega\otimes E(u_{i0}^* u_{i0}^{*\prime})\right] = \left(I\otimes I\otimes I\otimes I-\left((I\otimes\theta_0)\otimes(I\otimes\theta_0)\right)\right)^{-1}$$
$$\times\left(\left((I\otimes\theta_0)\otimes(I\otimes\theta_0)\right)-\left((I\otimes\theta_0)\otimes(I\otimes\theta_0)\right)^{T+1}\right)\mathrm{vec}\left[\Omega\otimes E\left[u_{i0}^* u_{i0}^{*\prime}\right]\right].$$

The second term is bounded by

$$\sum_{t_1,t_2=1}^{T}\left\|\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right\|\left\|\mathrm{vec}\,\mathcal{K}_0(t_1,t_2)\right\| \le \sup_{t_1,t_2}\left\|\mathrm{vec}\,\mathcal{K}_0(t_1,t_2)\right\|\sum_{t_1,t_2=1}^{T}\left\|\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right\|<\infty$$

Together these results imply that $\mathrm{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^t u_{i0}^*\right)\varepsilon_{it}\right)=o(1)$. Next consider

$$\begin{aligned}
\left\|\mathrm{Var}\left(\sum_{t=1}^{T}\left(I\otimes\theta_0^t u_{i0}^*\right)\bar{\varepsilon}_i\right)\right\| &= \left\|T^{-2}\sum_{t_1,\dots,t_4=1}^{T}\left(\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right)\mathrm{vec}\,E\left[\left(I\otimes u_{i0}^*\right)\varepsilon_{it_3}\varepsilon_{it_4}'\left(I\otimes u_{i0}^{*\prime}\right)\right]\right\| \\
&\le\left\|\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\otimes\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\right\| \\
&\quad\times\left\|T^{-1}\mathrm{vec}\left[\Omega\otimes E\left[u_{i0}^* u_{i0}^{*\prime}\right]\right]+T^{-2}\sum_{t_3,t_4=1}^{T}\mathrm{vec}\,\mathcal{K}_0(t_3,t_4)\right\| \\
&\le\left\|\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\otimes\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\right\| \\
&\quad\times\left(\left\|T^{-1}\mathrm{vec}\left[\Omega\otimes E\left[u_{i0}^* u_{i0}^{*\prime}\right]\right]\right\|+T^{-2}\sum_{t_3,t_4=1}^{T}\left\|\mathrm{vec}\,\mathcal{K}_0(t_3,t_4)\right\|\right) \\
&=O\left(T^{-1}\right),
\end{aligned}$$

which shows that $\mathrm{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^t u_{i0}^*\right)\bar{\varepsilon}_i\right)=o(1)$. It therefore follows that (15) is $o_p(1)$. ■

Lemma 3 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3,*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it}^*\right)\bar{\varepsilon}_i=\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\mathrm{vec}(\Omega)+o_p(1).$$

Proof. We have

$$\begin{aligned}
E\left[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it}^*\right)\bar{\varepsilon}_i\right] &= \frac{n}{\sqrt{nT}}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[\left(I\otimes u_{it}^*\right)\varepsilon_{is}\right] \\
&= \sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{t}E\left[\left(I\otimes u_{it}^*\right)\varepsilon_{is}\right]=\sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}\left(I\otimes\theta_0^j\right)\mathrm{vec}(\Omega)
\end{aligned} \quad (16)$$

By the usual result on Cesàro averages, we have $\frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}(I\otimes\theta_0^j)=\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}+o(1)$. Therefore,

$$E\left[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it}^*\right)\bar{\varepsilon}_i\right]=\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\mathrm{vec}(\Omega)+o(1). \quad (17)$$

We also have

$$
\begin{aligned}
&\operatorname{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(I\otimes u_{it}^{*})\bar{\varepsilon}_i\right)\\
&= E\left[\frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\left(\left(I\otimes u_{it_1}^{*}\right)\varepsilon_{it_2}-E\left[\left(I\otimes u_{it_1}^{*}\right)\varepsilon_{it_2}\right]\right)\left(\left(I\otimes u_{it_3}^{*}\right)\varepsilon_{it_4}-E\left[\left(I\otimes u_{it_3}^{*}\right)\varepsilon_{it_4}\right]\right)'\right]\\
&= \frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(I\otimes\theta_0^{j_1}\right)\left(\operatorname{vec}\left(\operatorname{Cov}\left(\varepsilon_{it_1-j_1},\varepsilon_{it_4}\right)\right)\operatorname{vec}\left(\operatorname{Cov}\left(\varepsilon_{it_3-j_2},\varepsilon_{it_2}\right)\right)'\right)\left(I\otimes\theta_0^{'j_2}\right)\\
&+\frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(I\otimes\theta_0^{j_1}\right)\left[\operatorname{Cov}\left(\varepsilon_{it_2},\varepsilon_{it_4}\right)\otimes\operatorname{Cov}\left(\varepsilon_{it_1-j_1},\varepsilon_{it_3-j_2}\right)\right]\left(I\otimes\theta_0^{'j_2}\right)\\
&+\frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\left(I\otimes\theta_0^{j_1}\right)\mathcal{K}\left(t_1,t_2,t_3,t_4\right)\left(I\otimes\theta_0^{'j_2}\right)
\end{aligned}
$$

where $\mathcal{K}(t_1,t_2,t_3,t_4)$ is a matrix containing elements of the form $\operatorname{cum}_{j_1,\ldots,j_4}\left(u_{it_1}^{*},\varepsilon_{it_2},u_{it_3}^{*},\varepsilon_{it_4}\right)$. This leads to

$$
\begin{aligned}
\operatorname{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(I\otimes u_{it}^{*})\bar{\varepsilon}_i\right) &= \frac{1}{T^3}\sum_{t_1,t_3=1}^{T}\sum_{t_2=1}^{t_3}\sum_{t_4=1}^{t_1}\left(I\otimes\theta_0^{t_1-t_4}\right)\operatorname{vec}(\Omega)\operatorname{vec}(\Omega)'\left(I\otimes\theta_0^{t_3-t_2}\right)\\
&+\frac{1}{T^2}\sum_{t_3=1}^{T}\sum_{j_1=0}^{\infty}\sum_{t_1=1}^{\min(T,t_3-j_1)}\left(I\otimes\theta_0^{j_1}\right)\left(\Omega\otimes\Omega'\right)\left(I\otimes\theta_0^{t_3+j_1-t_1}\right)\\
&+\frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\left(I\otimes\theta_0^{t_1}\right)\mathcal{K}\left(t_1,t_2,t_3,t_4\right)\left(I\otimes\theta_0^{t_4}\right)'.
\end{aligned}
$$

The first term on the right is $O\left(T^{-1}\right)$ because

$$
\sum_{t_1=1}^{T}\sum_{t_4=1}^{t_1}\left(I\otimes\theta_0^{t_1-t_4}\right)=\sum_{t_1=1}^{T}\left(I\otimes I-I\otimes\theta_0\right)^{-1}\left(I\otimes I-I\otimes\theta_0^{t_1}\right)=O(T).
$$

The second term is $O\left(T^{-1}\right)$ because

$$
\begin{aligned}
&\left\|\operatorname{vec}\left[\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\min(T,t_3-j_1+1)}\left(I\otimes\theta_0^{j_1}\right)\left(\Omega\otimes\Omega'\right)\left(I\otimes\theta_0^{j_2}\right)\right]\right\|\\
&\qquad\le\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left\|\left(I\otimes\theta_0^{j_1}\right)\otimes(I\otimes I)\right\|\left\|(I\otimes I)\otimes\left(I\otimes\theta_0^{j_2}\right)\right\|\left\|\operatorname{vec}(\Omega\otimes\Omega')\right\|=O(1)
\end{aligned}
$$

Finally, the third term is $O\left(T^{-2}\right)$ because $\frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\operatorname{cum}_{j_1,\ldots,j_4}\left(u_{it_1}^{*},\varepsilon_{it_2},u_{it_3}^{*},\varepsilon_{it_4}\right)=O\left(T^{-2}\right)$ by the cumulant summability assumption. ■

Lemma 4 *Assume ε_t is a sequence of independent, identically distributed random vectors with $E[\varepsilon_t]=0$ for all t. Then $\operatorname{cum}_{j_1,\ldots j_k}\left(\varepsilon_{t_1},\ldots,\varepsilon_{t_k}\right)=0$ unless $t_1=t_2=\cdots=t_k$. In this case we define $\operatorname{cum}(j_1,\ldots,j_k)\equiv\operatorname{cum}_{j_1,\ldots j_k}\left(\varepsilon_t,\ldots,\varepsilon_t\right)$.*

Lemma 5 *Let Conditions 1, 2 and 3 be satisfied. Then*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\rightarrow\mathcal{N}\left(0,\Omega\otimes\Upsilon+\mathcal{K}\right)$$

where $\mathcal{K}=\sum_{t=-\infty}^{\infty}\mathcal{K}\left(t,0\right)$ *and* $\mathcal{K}\left(t_1,t_2\right)\equiv E\left[\left(I\otimes u_{it_1-1}^{*}\right)\varepsilon_{it_1}\varepsilon_{it_2}^{\prime}\left(I\otimes u_{it_2-1}^{*\prime}\right)\right]-E\left[\varepsilon_{it_1}\varepsilon_{it_2}^{\prime}\right]\otimes E\left[u_{i0}^{*}u_{i0}^{*\prime}\right]$. *If in addition all the innovations* ε_{it} *are independent for all* i *and* t *then*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\rightarrow\mathcal{N}\left(0,\Omega\otimes\Upsilon\right).$$

Proof. We need to check the generalized Lindeberg Feller condition for joint asymptotic normality as in Theorem 2 of Phillips and Moon (1999). A sufficient condition for the theorem to hold is that for all $\ell\in\mathbb{R}^{m^2}$ such that $\ell'\ell=1$ it follows $E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\ell'(I\otimes u_{it-1}^{*})\varepsilon_{it}\right)^4\right]<\infty$ uniformly in i and T. Letting $\mathbf{z}_{it}=\ell'\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}$ and noting that $E\left[\mathbf{z}_{it}\right]=0$ we consider

$$\begin{aligned}\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}E\left[\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}\right]&=\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}\left[\operatorname{Cov}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_2}\right)\operatorname{Cov}\left(\mathbf{z}_{it_3},\mathbf{z}_{it_4}\right)+\operatorname{Cov}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_3}\right)\operatorname{Cov}\left(\mathbf{z}_{it_2},\mathbf{z}_{it_4}\right)\right]\\&+\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}\left[\operatorname{Cov}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_4}\right)\operatorname{Cov}\left(\mathbf{z}_{it_2},\mathbf{z}_{it_3}\right)+\operatorname{cum}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_2},\mathbf{z}_{it_3},\mathbf{z}_{it_4}\right)\right],\end{aligned}$$

where

$$\begin{aligned}\operatorname{Cov}\left(\mathbf{z}_{it},\mathbf{z}_{is}\right)&=\ell'E\left[\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\varepsilon_{is}^{\prime}\left(I\otimes u_{is-1}^{\prime *}\right)\right]\ell\\&=\ell'\operatorname{vec}\left(E\left[u_{it-1}^{*}\varepsilon_{is}^{\prime}\right]\right)\operatorname{vec}\left(E\left[u_{is-1}^{*}\varepsilon_{it}^{\prime}\right]\right)'\ell+\ell'E\left[\varepsilon_{it}\varepsilon_{is}^{\prime}\right]\otimes E\left[u_{it-1}^{*}u_{is-1}^{*\prime}\right]\ell\\&+\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right)\\&=0+\ell'\left(\Omega\otimes E\left[u_{it-1}^{*}u_{is-1}^{*\prime}\right]\right)\ell\cdot 1\left\{t=s\right\}\\&+\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right).\end{aligned}$$

Using these results we deduce that

$$\begin{aligned}&\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}E\left[\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}\right]\\&=3\left(\ell'\left(\Omega\otimes E\left[u_{it-1}^{*}u_{it-1}^{*\prime}\right]\right)\ell\right)^2\\&+6\left(\ell'\left(\Omega\otimes E\left[u_{it-1}^{*}u_{it-1}^{*\prime}\right]\right)\ell\right)\left(\frac{1}{T}\sum_{t,s=1}^{T}\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right)\right)\\&+3\left(\frac{1}{T}\sum_{t,s=1}^{T}\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right)\right)^2\\&+\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}\sum_{j_1,\dots,j_4=1}^{m^2}\left(\prod_{k=1}^{4}\ell_{j_k}\right)\operatorname{cum}_{j_1,\dots,j_4}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_2},\mathbf{z}_{it_3},\mathbf{z}_{it_4}\right)\end{aligned}$$

where the terms involving higher order cumulants are $\frac{1}{T}\sum_{t,s=1}^{T}\text{cum}_{j_1,...,j_4}(u_{it-1}^*,\varepsilon_{it},u_{is-1}^*,\varepsilon_{is}) = O(1)$ and $\sum_{t_1,...t_4=1}^{T}\text{cum}_{j_1,...,j_4}(\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}) = O(T)$ independent of $i, j_1, ..., j_4$. This shows that

$$T^{-2}\sum_{t_1,...t_4=1}^{T} E\left[\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}\right] < \infty$$

uniformly in i and T. Finally consider

$$\begin{aligned}
&E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(I\otimes u_{it-1}^*\right)\varepsilon_{it}\right)^2\right] \\
&= \frac{1}{T}\sum_{t,s=1}^{T} E\left[\left(I\otimes u_{it-1}^*\right)\varepsilon_{it}\varepsilon_{is}'\left(I\otimes u_{is-1}^{*\prime}\right)\right] \\
&= \frac{1}{T}\sum_{t,s=1}^{T}\text{vec}\left(E\left[u_{it-1}^*\varepsilon_{is}'\right]\right)\text{vec}\left(E\left[u_{is-1}^*\varepsilon_{it}'\right]\right) + \frac{1}{T}\sum_{t,s=1}^{T} E\left[\varepsilon_{it}\varepsilon_{is}'\right]\otimes E\left[u_{it-1}^*u_{is-1}^{*\prime}\right] + \frac{1}{T}\sum_{t,s=1}^{T}\mathcal{K}(t,s) \\
&= \Omega\otimes\Upsilon + \mathcal{K} + o(1).
\end{aligned}$$

where $\mathcal{K} = \sum_{t_1=-\infty}^{\infty}\mathcal{K}(t_1,0)$. Note that $\text{vec}\left(E\left[u_{it-1}^*\varepsilon_{is}'\right]\right)\text{vec}\left(E\left[u_{is-1}^*\varepsilon_{it}'\right]\right) = 0$ for all t and s and that $\frac{1}{T}\sum_{t,s=1}^{T} E\left[\varepsilon_{it}\varepsilon_{is}'\right]\otimes E\left[u_{it-1}^*u_{is-1}^{*\prime}\right] = \frac{1}{T}\sum_{t=1}^{T} E\left[\varepsilon_{it}\varepsilon_{it}'\right]\otimes E\left[u_{it-1}^*u_{it-1}^{*\prime}\right] = \Omega\otimes\Upsilon$ by strict stationarity. The last line of the display follows by Cesàro summability and stationarity. The second part of the theorem follows from Lemma (4) which implies that $\mathcal{K}(t_1,t_2) = 0$ for all t_1 and t_2. ■

Lemma 6 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3,*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\left(y_{it-1}-\overline{y}_{i-}\right)\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)\to\mathcal{N}\left(-\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\text{vec}\left(\Omega\right),\Omega\otimes\Upsilon+\mathcal{K}\right).$$

Proof. The result then follows from Lemmas 2, 3, and 5. ■

Lemma 7 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3*

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)' = E\left[\left(y_{it-1}^*-Ey_{it-1}^*\right)\left(y_{it-1}^*-Ey_{it-1}^*\right)'\right] + o_p(1) = \Upsilon + o_p(1).$$

Proof. Available upon request from the authors. ■

B Proof of Theorem 2

We have

$$\begin{aligned}
\sqrt{nT}\,\text{vec}\left(\widehat{\widetilde{\theta}}' - \theta_0'\right) \equiv &\left[I\otimes\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)'\right)^{-1}\right] \\
&\left[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\left(y_{it-1}-\overline{y}_{i-}\right)\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right) + \sqrt{\frac{n}{T}}\left(I\otimes I-\left(I\otimes\widehat{\theta}\right)\right)^{-1}\text{vec}\left(\widehat{\Omega}\right)\right].
\end{aligned}$$

Because $\text{vec}(\Upsilon) = (I - (\theta_0 \otimes \theta_0))^{-1} \text{vec}(\Omega)$, and $\widehat{\theta} = \theta_0 + o_p(1)$, we have

$$\sqrt{\frac{n}{T}} \left(I \otimes I - \left(I \otimes \widehat{\theta} \right) \right)^{-1} \text{vec} \left(\widehat{\Omega} \right) = \sqrt{\rho} \left(I \otimes I - (I \otimes \theta_0) \right)^{-1} \text{vec}(\Omega) + o_p(1). \tag{18}$$

Combining with Lemma 6, we obtain

$$\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes (y_{it-1} - \overline{y}_{i-}) \right) (\varepsilon_{it} - \overline{\varepsilon}_i) + \sqrt{\frac{n}{T}} \left(I \otimes I - \left(I \otimes \widehat{\theta} \right) \right)^{-1} \text{vec} \left(\widehat{\Omega} \right)$$
$$\xrightarrow{d} \mathcal{N}(0, (\Omega \otimes \Upsilon + \mathcal{K})).$$

The conclusion follows by using Lemma 7.

C Proof of Theorem 3

C.1 Framework

For the discussion and derivation of the asymptotic variance bound, we adopt the same framework as in van der Vaart and Wellner (1996, p. 412). For this purpose, we discuss some of their notation. Let H be a linear subspace H of a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let $P_{n,h}$ denote a probability measure on a measurable space $(\mathcal{X}_n, \mathcal{A}_n)$. Consider estimating a parameter $\kappa_n(h)$ based on an observation with law $P_{n,h}$. Now, let $\{\Delta_h : h \in H\}$ be the "iso-Gaussian process" with zero mean and covariance function $E[\Delta_{h_1} \Delta_{h_2}] = \langle h_1, h_2 \rangle$. We say that the sequence $(\mathcal{X}_n, \mathcal{A}_n, P_{n,h})$ is *asymptotically normal* if

$$\log \frac{dP_{n,h}}{dP_{n,0}} = \Delta_{n,h} - \frac{1}{2} \|h\|^2$$

for stochastic processes $\{\Delta_{n,h} : h \in H\}$ such that $\Delta_{n,h}$ converges weakly to Δ_h marginally under $P_{n,0}$. Now, consider the sequence of parameters $\kappa_n(h)$ belonging to a Banach space B, which is *regular* in the sense that $r_n(\kappa_n(h) - \kappa_n(0)) \to \dot{\kappa}(h)$ for every $h \in H$ for a continuous, linear map $\dot{\kappa} : H \to B$ and certain linear maps $r_n : B \to B$. A sequence of estimators τ_n is defined to be *regular* if $r_n(\tau_n - \kappa_n(h))$ converges weakly to the same measure L, say, regardless of h. The bound of the asymptotic variance of a regular estimator can be found from the following theorem due to van der Vaart and Wellner (1996, Theorem 3.11.2):

Theorem 6 *Let the sequence $(\mathcal{X}_n, \mathcal{A}_n, P_{n,h} : h \in H)$ be asymptotically normal and the sequence of parameters $\kappa_n(h)$ and estimators τ_n be regular. Then the limit distribution L of the sequence $r_n(\tau_n - \kappa_n(h))$ equals the distribution of a sum $G + W$ of independent, tight, Borel measurable random elements in B such that $b^* G \sim N(0, \|\dot{\kappa}^* b^*\|)$, for every $b^* \in B^*$. Here, $\dot{\kappa}^* : B^* \to H$ is the adjoint of $\dot{\kappa}$.*

It can be seen that G provides us with the minimal asymptotic distribution for any regular estimator. If G happens to be normal with mean zero, then we can say that its asymptotic variance is the asymptotic variance bound.

We show that our setup is covered by the preceding theorem. Ignoring irrelevant constants, the joint likelihood of the model (1) is given by

$$\mathcal{L} = \frac{nT}{2} \log\det(\Psi) - \frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{T} \operatorname{trace}\left(\Psi Z_{it}(\alpha_i, \theta)\right), \tag{19}$$

where $Z_{it}(\alpha_i, \theta) \equiv \left(y_{it} - \alpha_i - \theta' y_{it-1}\right)\left(y_{it} - \alpha_i - \theta' y_{it-1}\right)'$, and $\Psi = \Omega^{-1}$. We will localize the parameter. Let α denote the sequence $(\alpha_1, \alpha_2, \ldots)$. We will attach subscript 0 to denote the 'truth'. We will localize the parameter around the 'truth', so that $\theta \equiv \theta(n) \equiv \theta_0 + \frac{1}{\sqrt{nT}}\widetilde{\theta}$, $\Psi \equiv \Psi(n) \equiv \Psi_0 + \frac{1}{\sqrt{nT}}\widetilde{\Psi}$, and $\alpha \equiv \alpha(n) \equiv \alpha_0 + \frac{1}{\sqrt{nT}}\widetilde{\alpha}$. Let $h \equiv \left(\widetilde{\alpha}, \widetilde{\theta}, \widetilde{\Psi}\right)$, and let H denote a set of all possible values of $\left(\widetilde{\alpha}, \widetilde{\theta}, \widetilde{\Psi}\right)$. Let $P_{n,h}$ denote the joint probability under parameters characterized by h. Theorems 7, 8, and 9 in the next subsection establishes that, under $P_{n,0}$, we have

$$\log \frac{dP_{n,h}}{dP_{n,0}} = \Delta_{n,h} - \frac{1}{2}\left\|\Delta_{n,h}\right\|^2 + o_p(1),$$

where

$$\begin{aligned}
\Delta_{n,h} &\equiv \Delta_n\left(\widetilde{\alpha}, \widetilde{\theta}, \widetilde{\Psi}\right) \\
&= -\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T} \operatorname{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}' - \Omega\right)\right) + \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta} y_{it-1}^{*}\right)' \Psi \varepsilon_{it}
\end{aligned}$$

converges weakly (under $P_{n,0}$) to $\Delta_h \sim \mathcal{N}\left(0, \|h\|^2\right)$. Here, $\|h\|^2 = \langle h, h\rangle$, and

$$\begin{aligned}
\left\langle \left(\widetilde{\alpha}_1, \widetilde{\theta}_1, \widetilde{\Psi}_1\right), \left(\widetilde{\alpha}_2, \widetilde{\theta}_2, \widetilde{\Psi}_2\right)\right\rangle &\equiv \frac{1}{2}\operatorname{vec}\left(\widetilde{\Psi}_2\right)' \operatorname{vec}(\Omega)\cdot\operatorname{vec}(\Omega)'\operatorname{vec}\left(\widetilde{\Psi}_1\right) + \operatorname{vec}\left(\widetilde{\theta}_2'\right)'\left(\Psi\otimes\Upsilon\right)\operatorname{vec}\left(\widetilde{\theta}_1'\right) \\
&+ \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\widetilde{\alpha}_{1i}'\Psi\widetilde{\alpha}_{2i} + \left(\operatorname{vec}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_i\widetilde{\alpha}_{1i}'\right)\right)'\left(\Psi\otimes\left(I-\theta'\right)^{-1}\right)\operatorname{vec}\left(\widetilde{\theta}_2'\right) \\
&+ \left(\operatorname{vec}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_i\widetilde{\alpha}_{2i}'\right)\right)'\left(\Psi\otimes\left(I-\theta'\right)^{-1}\right)\operatorname{vec}\left(\widetilde{\theta}_1'\right) \\
&+ \left(\operatorname{vec}\left(\widetilde{\theta}_2'\right)\right)'\left(\Psi\otimes\left(I-\theta\right)^{-1}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_i\alpha_i'\right)\left(I-\theta'\right)^{-1}\right)\operatorname{vec}\left(\widetilde{\theta}_1'\right).
\end{aligned} \tag{20}$$

In other words, the sequence $(P_{n,h} : h \in H)$ is *asymptotically normal*.

In order to adapt Theorem 6 to our problem, we consider estimation of the "parameter" $\kappa_n(h) = \kappa_n\left(\widetilde{\alpha}, \widetilde{\theta}, \widetilde{\Psi}\right) = \theta_0 + \frac{1}{\sqrt{nT}}\widetilde{\theta} = \theta$, which is regular because $r_n\left(\kappa_n(h) - \kappa_n(0)\right) = r_n\frac{1}{\sqrt{nT}}\widetilde{\theta} = \widetilde{\theta}$ for $r_n = \sqrt{nT}$: We may write $\sqrt{nT}\left(\kappa_n(h) - \kappa_n(0)\right) \to \dot{\kappa}(h)$ for $\dot{\kappa}(h) \equiv \dot{\kappa}\left(\widetilde{\alpha}, \widetilde{\theta}, \widetilde{\Psi}\right) = \widetilde{\theta}$. We restrict our attention to regular estimators τ_n of θ such that the asymptotic distribution of $\sqrt{nT}\left(\tau_n - \kappa_n(h)\right) = \sqrt{nT}\left(\tau_n - \theta\right)$ under $P_{n,h}$ does not depend on h. Applying Theorem 6, we obtain Theorem 3.

C.2 Technical Lemmas

In this subsection, we omit the 0 subscripts in θ_0, Ψ_0, and α_0 in order to simplify notation. We have

$$\log\frac{dP_{n,h}}{dP_{n,0}} = \frac{nT}{2}\log\det\left(\Psi + \frac{1}{\sqrt{nT}}\widetilde{\Psi}\right) - \log\det\Psi + \frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\Psi Z_{it}\left(\alpha_i, \theta\right)\right)$$
$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\left(\Psi + \frac{1}{\sqrt{nT}}\widetilde{\Psi}\right) Z_{it}\left(\alpha_i + \frac{1}{\sqrt{nT}}\widetilde{\alpha}_i, \theta + \frac{1}{\sqrt{nT}}\widetilde{\theta}\right)\right).$$

Because $Z_{it}\left(\alpha_i, \theta\right) = \varepsilon_{it}\varepsilon_{it}'$ and $\varepsilon_{it} \sim \mathcal{N}\left(0, \Omega\right)$ under $P_{n,0}$, we can write

$$\begin{aligned}
\log\frac{dP_{n,h}}{dP_{n,0}} &= \frac{nT}{2}\log\det\left(\Psi + \frac{1}{\sqrt{nT}}\widetilde{\Psi}\right) - \log\det\Psi + \frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}'\Psi\varepsilon_{it} \\
&\quad -\frac{1}{2}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\left(\Psi + \frac{1}{\sqrt{nT}}\widetilde{\Psi}\right)\left(\varepsilon_{it} - \frac{1}{\sqrt{nT}}\widetilde{\alpha}_i - \frac{1}{\sqrt{nT}}\widetilde{\theta}y_{it-1}\right)\left(\varepsilon_{it} - \frac{1}{\sqrt{nT}}\widetilde{\alpha}_i - \frac{1}{\sqrt{nT}}\widetilde{\theta}y_{it-1}\right)'\right) \\
&\equiv \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5, \qquad (21)
\end{aligned}$$

where

$$\begin{aligned}
\phi_1 &\equiv \frac{nT}{2}\log\det\left(\Psi + \frac{1}{\sqrt{nT}}\widetilde{\Psi}\right) - \frac{nT}{2}\log\det\Psi - \frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}'\widetilde{\Psi}\varepsilon_{it}, \\
\phi_2 &\equiv \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}y_{it-1}\right)'\widetilde{\Psi}\varepsilon_{it}, \\
\phi_3 &\equiv -\frac{1}{2nT\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}y_{it-1}\right)'\widetilde{\Psi}\left(\widetilde{\alpha}_i + \widetilde{\theta}y_{it-1}\right), \\
\phi_4 &\equiv \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}y_{it-1}\right)'\Psi\varepsilon_{it}, \\
\phi_5 &\equiv -\frac{1}{2nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}y_{it-1}\right)'\Psi\left(\widetilde{\alpha}_i + \widetilde{\theta}y_{it-1}\right).
\end{aligned}$$

Lemma 8 *Under $P_{n,0}$, we have*

$$\phi_1 = -\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}' - \Omega\right)\right) - \frac{1}{4}\left(\text{trace}\left(\Psi^{-1}\widetilde{\Psi}\right)\right)^2 + o\left(1\right).$$

Proof. Follows from

$$\frac{nT}{2}\log\det\left(\Psi + \frac{1}{\sqrt{nT}}\widetilde{\Psi}\right) - \frac{nT}{2}\log\det\Psi = \frac{\sqrt{nT}}{2}\text{trace}\left(\Psi^{-1}\widetilde{\Psi}\right) - \frac{1}{4}\left(\text{trace}\left(\Psi^{-1}\widetilde{\Psi}\right)\right)^2 + o\left(1\right),$$
$$\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\widetilde{\Psi}\varepsilon_{it}\varepsilon_{it}'\right) = \frac{\sqrt{nT}}{2}\text{trace}\left(\widetilde{\Psi}\Psi^{-1}\right) + \frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}' - \Omega\right)\right).$$

■

Lemma 9 *Under $P_{n,0}$, $\phi_4 \to \mathcal{N}\left(0, \psi^2\right)$, where*

$$\psi^2 \equiv \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \widetilde{\alpha}_i' \Psi \widetilde{\alpha}_i + \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \widetilde{\alpha}_i' \Psi \widetilde{\theta}' \left(I-\theta\right)^{-1} \alpha_i$$

$$+ \operatorname{trace}\left(\widetilde{\theta}\Psi\widetilde{\theta}' \cdot \Upsilon\right) + \operatorname{trace}\left(\widetilde{\theta}\Psi\widetilde{\theta}' \cdot \left(I-\theta\right)^{-1}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_i\alpha_i'\right)\left(I-\theta'\right)^{-1}\right)$$

Proof. Write

$$\phi_4 = \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\widetilde{\alpha}_i'\Psi\left(\sum_{t=1}^{T}\varepsilon_{it}\right) + \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it-1}'\widetilde{\theta}'\Psi\varepsilon_{it},$$

Apply the same reasoning as in the proof of Lemma 2 to the second term on the right hand side, we obtain

$$\phi_4 = \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}\left(I-\theta\right)^{-1}\alpha_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it} + o_p\left(1\right)$$

Let $\widetilde{\gamma}_i \equiv \widetilde{\alpha}_i + \widetilde{\theta}\left(I-\theta\right)^{-1}\alpha_i$, and $Y_{Tt} \equiv \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it}$. We then may write $\phi_4 = \sum_{t=1}^{T} Y_{Tt} + o_p\left(1\right)$. For each T, Y_{Tt} is a martingale difference sequence. Let σ_{Tt}^2 denote the conditional variance of Y_{Tt} given $\left\{\left(u_{i0}, \varepsilon_{i0}, \ldots, \varepsilon_{it-1}\right), i = 1, \ldots, n\right\}$. We then have

$$\sigma_{Tt}^2 = \frac{1}{nT}\sum_{i=1}^{n}\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right), \qquad \text{and}$$

$$\sum_{t=1}^{T}\sigma_{Tt}^2 = \frac{1}{n}\sum_{i=1}^{n}\widetilde{\gamma}_i'\Psi\widetilde{\gamma}_i + \frac{2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{\gamma}_i'\Psi\widetilde{\theta}u_{it-1}^{*} + \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(u_{it-1}^{*}\right)'\widetilde{\theta}'\Psi\widetilde{\theta}u_{it-1}^{*}.$$

It follows by standard arguments that $\sum_{t=1}^{T}\sigma_{Tt}^2 \to \psi^2$ in probability. By strict stationarity, we have $E\left[Y_{Tt}^2 \cdot 1\left(\left|Y_{Tt}\right| \geq \epsilon\right)\right] = E\left[Y_{Ts}^2 \cdot 1\left(\left|Y_{Ts}\right| \geq \epsilon\right)\right]$ $\forall s, t$. Therefore, if

$$T \cdot E\left[Y_{Tt}^2 \cdot 1\left(\left|Y_{Tt}\right| \geq \epsilon\right)\right] = \sum_{t=1}^{T} E\left[Y_{Tt}^2 \cdot 1\left(\left|Y_{Tt}\right| \geq \epsilon\right)\right] \to 0 \quad \forall \epsilon > 0, \tag{22}$$

then we can use Billingsley (1995, Theorem 35.12) and conclude that

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it} \to \mathcal{N}\left(0, \psi^2\right).$$

Note that

$$T \cdot E\left[Y_{Tt}^2 \cdot 1\left(\left|Y_{Tt}\right| \geq \epsilon\right)\right]$$
$$= E\left[\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it}\right|^2 \cdot 1\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it}\right| \geq \sqrt{T}\epsilon\right)\right].$$

Because

$$E\left[\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it}\right|^2\right] = \frac{1}{n}\sum_{i=1}^{n}E\left[\left|\left(\widetilde{\gamma}_i + \widetilde{\theta}u_{it-1}^{*}\right)'\Psi\varepsilon_{it}\right|^2\right] = O\left(1\right),$$

we obtain (22) by Dominated Convergence. ■

Lemma 10 *Under $P_{n,0}$, we have $\phi_2 = o_p(1)$.*

Proof. By Lemma 9, we have $\phi_4 = O_p(1)$, from which we can easily infer that $\phi_2 = o_p(1)$. ∎

Lemma 11 *Under $P_{n,0}$, we have $\phi_5 = -\frac{1}{2}\psi^2 + o_p(1)$.*

Proof. After applying a reasoning basically the same as in the proof of Lemma 2, we can obtain

$$\phi_5 = -\frac{1}{2n}\sum_{i=1}^{n}\widetilde{\alpha}_i'\Psi\widetilde{\alpha}_i - \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{\alpha}_i'\Psi\widetilde{\theta}\left((I-\theta)^{-1}\alpha_i + u_{it-1}^*\right)$$
$$-\frac{1}{2nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left((I-\theta)^{-1}\alpha_i + u_{it-1}^*\right)'\widetilde{\theta}'\Psi\widetilde{\theta}\left((I-\theta)^{-1}\alpha_i + u_{it-1}^*\right) + o_p(1). \quad (23)$$

It therefore suffices to show

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{\alpha}_i'\Psi\widetilde{\theta}u_{it-1}^* = o_p(1), \qquad \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left((I-\theta)^{-1}\alpha_i\right)'\widetilde{\theta}'\Psi\widetilde{\theta}u_{it-1}^* = o_p(1),$$
$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(u_{it-1}^*)'\widetilde{\theta}'\Psi\widetilde{\theta}u_{it-1}^* = \operatorname{trace}\left(\widetilde{\theta}'\Psi\widetilde{\theta}\cdot\Upsilon\right) + o_p(1).$$

All of them follow quite easily from characterization of means and variances, and details are omitted. ∎

Lemma 12 *Under $P_{n,0}$, we have $\phi_3 = o_p(1)$.*

Proof. Follows easily with Lemma 11. ∎

Theorem 7 *Under $P_{n,0}$, we have*

$$\log\frac{dP_{n,h}}{dP_{n,0}} = -\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\operatorname{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}' - \Omega\right)\right) + \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}(I-\theta)^{-1}\alpha_i + \widetilde{\theta}u_{it-1}^*\right)'\Psi\varepsilon_{it}$$
$$-\frac{1}{4}\operatorname{trace}\left(\left(\Psi^{-1}\widetilde{\Psi}\right)^2\right) - \frac{1}{2n}\sum_{i=1}^{n}\widetilde{\alpha}_i'\Psi\widetilde{\alpha}_i - \frac{1}{n}\sum_{i=1}^{n}\widetilde{\alpha}_i'\Psi\widetilde{\theta}(I-\theta)^{-1}\alpha_i$$
$$-\frac{1}{2}\operatorname{trace}\left(\widetilde{\theta}'\Psi\widetilde{\theta}\cdot\Upsilon\right) - \frac{1}{2}\operatorname{trace}\left(\widetilde{\theta}'\Psi\widetilde{\theta}\cdot(I-\theta)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i\alpha_i'\right)(I-\theta')^{-1}\right) + o_p(1). \quad (24)$$

Proof. Follows from Lemmas 8, 9, 10, 11, and 12. ∎

Theorem 8

$$-\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\operatorname{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}' - \Omega\right)\right) + \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{\alpha}_i + \widetilde{\theta}(I-\theta)^{-1}\alpha_i + \widetilde{\theta}u_{it-1}^*\right)'\Psi\varepsilon_{it}$$
$$\to \mathcal{N}\left(0, \frac{1}{2}\left(\operatorname{trace}\left(\Omega\widetilde{\Psi}\right)\right)^2 + \psi^2\right),$$

Proof. It can be established by the same reasoning as in the proof of Lemma 9, and details are omitted. ∎

Theorem 9 $\Delta_n\left(\widetilde{\alpha}_1, \widetilde{\theta}_1, \widetilde{\Psi}_1\right)$ *and* $\Delta_n\left(\widetilde{\alpha}_2, \widetilde{\theta}_2, \widetilde{\Psi}_2\right)$ *are jointly asymptotically normal with asymptotic covariance as in (20).*

Proof. Joint normality can be established by the same reasoning as in the proof of Lemma 9, and details are omitted. As for the covariance, it can be inferred from the asymptotic variance in Theorem 8 using the formula $\operatorname{Cov}(X, Y) = \frac{1}{2}\left(\operatorname{Var}(X+Y) - \operatorname{Var}(X) - \operatorname{Var}(Y)\right)$. ∎

C.3 Proof of Theorem 3

We may write

$$\begin{aligned}\Delta_n\left(\widetilde{\alpha},\widetilde{\theta},\widetilde{\Psi}\right) &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\left(y_{it-1}^{*}\right)'\widetilde{\theta}'\Psi\varepsilon_{it}\right)-\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}'-\Omega\right)\right)+\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{\alpha}_i'\Psi\varepsilon_{it}\\ &= \text{vec}\left(\widetilde{\theta}'\right)'\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\Psi\otimes y_{it-1}^{*}\right)\varepsilon_{it}-\frac{1}{2\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\text{trace}\left(\widetilde{\Psi}\left(\varepsilon_{it}\varepsilon_{it}'-\Omega\right)\right)+\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{\alpha}_i'\Psi\varepsilon_{it}\\ &\equiv \text{vec}\left(\widetilde{\theta}'\right)'\Delta_{1n}+\Delta_{2n}\left(\widetilde{\alpha},\widetilde{\Psi}\right).\end{aligned}$$

Theorem 10 below implies that the 'minimal' asymptotic distribution is $\mathcal{N}\left(0,\left(E\left[\widetilde{\Delta}_1\widetilde{\Delta}_1'\right]\right)^{-1}\right)$, where $\widetilde{\Delta}_1$ is the residual in the projection of Δ_1 on the linear space spanned by $\left\{\Delta_2\left(\widetilde{\alpha},\widetilde{\Psi}\right)\right\}$. Here, Δ_1 and $\Delta_2\left(\widetilde{\alpha},\widetilde{\Psi}\right)$ denote the 'limits' of Δ_{1n} and $\Delta_{2n}\left(\widetilde{\alpha},\widetilde{\Psi}\right)$. Lemma 13 below establishes that $\left\langle\widetilde{\Delta}_1,\widetilde{\Delta}_1'\right\rangle=\Psi\otimes\Upsilon$. Therefore, the minimum variance of estimation of $\text{vec}(\theta_0)$ is given by the inverse of $\Psi\otimes\Upsilon$, or $\Omega\otimes\Upsilon^{-1}$.

Theorem 10 *Assume that $(P_{n,h}:h\in H)$ is asymptotically shift normal. Also, suppose that (i) $h=(\delta,\Xi)$, $h_0=(0,0)$; (ii) $\kappa_n(h)\equiv\xi_n\equiv\xi_0+\frac{1}{r_n}\delta$ for some $\xi_0\in\mathbb{R}$; and (iii) $\Delta_h\equiv\Delta_1\cdot\delta+\Delta_2(\Xi)$. Further suppose that, with respect to the norm $\|\cdot\|$, (iv) the mapping $\dot{\kappa}:(\delta,\Xi)\to\delta$ is continuous; and (v) H is complete. Then, for every regular sequence of estimators $\{\tau_n\}$, we have*

$$r_n\left(\tau_n-\kappa_n(h)\right)\to\mathcal{N}\left(0,\mathrm{E}\left[\widetilde{\Delta}_1{}^2\right]^{-1}\right)\oplus W$$

for some W, where $\oplus$ denotes convolution, and $\widetilde{\Delta}_1$ is the residual in the projection of Δ_1 on $\{\Delta_2(\Xi):(\delta,\Xi)\in H\}$.[11]

Lemma 13 $\left\langle\widetilde{\Delta}_1,\widetilde{\Delta}_1'\right\rangle=\Psi\otimes\Upsilon.$

Proof. We first establish that

$$\widetilde{\Delta}_1\equiv\begin{bmatrix}e_1'\\ \vdots\\ e_{m^2}'\end{bmatrix}\Delta_1-\begin{bmatrix}\Delta_2\left(D_1\left(I-\theta_0\right)^{-1}\alpha,0\right)\\ \vdots\\ \Delta_2\left(D_{m^2}\left(I-\theta_0\right)^{-1}\alpha,0\right)\end{bmatrix},$$

where, e_j' denotes the jth row of I_{m^2}, and D_j is an $m\times m$ matrix such that $\text{vec}\left(D_j'\right)=e_j$. We minimize the norm of $e_j'\Delta_1-\Delta_2\left(\widetilde{\alpha},\widetilde{\Psi}\right)$ for each j. From (20), we obtain

$$\begin{aligned}\left\|e_j'\Delta_1-\Delta_2\left(\widetilde{\alpha},\widetilde{\Psi}\right)\right\|^2 &= e_j'\left(\Psi\otimes\left(I-\theta_0\right)^{-1}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\alpha_i\alpha_i'\right)\left(I-\theta_0'\right)^{-1}\right)e_j\\ &\quad-2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\widetilde{\alpha}_i'\Psi D_j\left(I-\theta_0\right)^{-1}\alpha_i+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\widetilde{\alpha}_i'\Psi\widetilde{\alpha}_i\\ &\quad+e_j'\left(\Psi\otimes\Upsilon\right)e_j+\frac{1}{2}\left(\text{trace}\left(\Omega\widetilde{\Psi}\right)\right)^2.\end{aligned}$$

[11] This theorem originally appeared in Hahn (1998), but is reproduced here for convenience.

Therefore, the minimum of $\left\| e_j'\Delta_1 - \Delta_2\left(\widetilde{\alpha}, \widetilde{\Psi}\right)\right\|^2$ is attained with $\widetilde{\Psi} = 0$, and $\widetilde{\alpha}_i' = D_j\left(I - \theta_0\right)^{-1}\alpha_i$. Observe that the (j,k)-element of $\left\langle \widetilde{\Delta}_1, \widetilde{\Delta}_1' \right\rangle$ is equal to

$$\left\langle e_j'\Delta_1 - \Delta_2\left(D_j\left(I-\theta_0\right)^{-1}\alpha_i, 0\right), e_k'\Delta_1 - \Delta_2\left(D_k\left(I-\theta_0\right)^{-1}\alpha_i, 0\right)\right\rangle.$$

After some tedious algebra, we can show that it is equal to $e_j'\left(\Psi \otimes \Upsilon\right)e_k$. In other words, $\left\langle \widetilde{\Delta}_1, \widetilde{\Delta}_1' \right\rangle = \Psi \otimes \Upsilon$. ∎

D Proofs of Theorems 4 and 5

Ignore the i subscript whenever obvious. Let $H_T = I_T - \frac{1}{T}\ell_T\ell_T'$, $y = (y_1, \ldots, y_T)'$, $y_- = (y_0, \ldots, y_{T-1})'$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_T)'$. We can write $\sum_{t=1}^T \left(\varepsilon_t - \overline{\varepsilon}\right)\left(y_{t-1} - \overline{y}_-\right) = \varepsilon' H_T y_-$, and $\sum_{t=1}^T \left(y_{t-1} - \overline{y}_-\right)^2 = y_-' H_T y_-$. Here, ℓ_T denotes T-dimensional column vector consisting of ones. Note that

$$y_- = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ T-1 \end{pmatrix}\alpha + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}\varepsilon \equiv \xi_{1T}y_0 + \xi_{2T}\alpha + A_T\varepsilon.$$

Let $D_T \equiv H_T A_T$. We have $H_T\xi_{1T} = 0$, and hence, it follows that

$$\begin{aligned}
H_T y_- &= \left(\xi_{2T} - \frac{T-1}{2}\ell_T\right)\alpha + D_T\varepsilon, \\
\varepsilon' H_T y_- &= \varepsilon'\left(\xi_{2T} - \frac{T-1}{2}\ell_T\right)\alpha + \varepsilon' D_T\varepsilon, \\
y_-' H_T y_- &= \left(\xi_{2T} - \frac{T-1}{2}\ell_T\right)'\left(\xi_{2T} - \frac{T-1}{2}\ell_T\right)\alpha^2 + 2\alpha\varepsilon'\left(\xi_{2T} - \frac{T-1}{2}\ell_T\right) + \varepsilon' D_T' D_T\varepsilon
\end{aligned}$$

In the special case where $\alpha = 0$, we have

$$\varepsilon' H_T y_- = \varepsilon' D_T\varepsilon, \qquad y_-' H_T y_- = \varepsilon' D_T' D_T\varepsilon.$$

Lemma 14

$$\begin{aligned}
&\operatorname{trace}\left(D_T' D_T\right) = \frac{1}{6}T^2 - \frac{1}{6}, \qquad \operatorname{trace}\left[\left(D_T' D_T\right)^2\right] = \frac{1}{90}T^4 + \frac{1}{36}T^2 - \frac{7}{180}, \\
&\operatorname{trace}\left(D_T' D_T^2\right) = -\frac{1}{12}T^2 + \frac{1}{12}, \\
&\operatorname{trace}\left(D_T + D_T'\right) = -T + 1, \qquad \operatorname{trace}\left(\left(D_T + D_T'\right)^2\right) = \frac{1}{6}T^2 + T - \frac{7}{6}, \\
&\operatorname{trace}\left(\left(D_T + D_T'\right)^3\right) = -\frac{1}{4}T^2 - T + \frac{5}{4}, \qquad \operatorname{trace}\left(\left(D_T + D_T'\right)^4\right) = \frac{1}{72}T^4 + \frac{11}{36}T^2 + T - \frac{95}{72}.
\end{aligned}$$

Lemma 15 *As $n, T \to \infty$*

$$\frac{1}{nT^2}\sum_i \varepsilon' D_T' D_T\varepsilon = \frac{\sigma^2}{6} + o_p(1)$$

Proof. Examining the cumulant generating function, we can see that the first two cumulants of $\varepsilon' D_T' D_T \varepsilon$ are equal to $\sigma^2 \operatorname{trace}(D_T' D_T)$, and $2\sigma^4 \operatorname{trace}\left[(D_T' D_T)^2\right]$. Using Lemma 14, we obtain the desired conclusion. ∎

Lemma 16 *As $n, T \to \infty$,*

$$\frac{1}{\sqrt{nT}} \sum_i \left(\varepsilon' D_T \varepsilon - \frac{\sigma^2}{2} (-T+1) \right) \to \mathcal{N}\left(0, \frac{\sigma^4}{12}\right)$$

Proof. We will consider the fourth moment of $\varepsilon' D_T \varepsilon$ and examine the Lyapounov condition. Examining the cumulant generating function, we can show that the first, second and fourth cumulants of $\varepsilon' D_T \varepsilon$ are equal to $\frac{\sigma^2}{2} \operatorname{trace}[D_T + D_T']$, $\frac{\sigma^4}{2} \operatorname{trace}\left[(D_T + D_T')^2\right]$, and $3\sigma^8 \operatorname{trace}\left[(D_T + D_T')^4\right]$. Using Lemma 14 along with the well-known relation between cumulants and central moments, we obtain

$$E\left[(\varepsilon' D_T \varepsilon - E[\varepsilon' D_T \varepsilon])^4\right] = \frac{\sigma^8}{16} \left(T^4 + 22T^2 + 20T - 47 + 4T^3\right).$$

Now, let $\sum_{i=1}^n X_{ni} \equiv \sum_{i=1}^n (\varepsilon' D_T \varepsilon - E[\varepsilon' D_T \varepsilon])$, and $s_n^2 \equiv \sum_{i=1}^n \operatorname{Var}(\varepsilon' D_T \varepsilon) = n\frac{\sigma^4}{2}\left(\frac{1}{6}T^2 + T - \frac{7}{6}\right)$. Because

$$\sum_{i=1}^n \frac{1}{s_n^4} E\left[X_{ni}^4\right] = \frac{1}{n\left(\frac{\sigma^4}{2}\left(\frac{1}{6}T^2 + T - \frac{7}{6}\right)\right)^2} \frac{\sigma^8}{16} \left(T^4 + 22T^2 + 20T - 47 + 4T^3\right) = O\left(\frac{1}{n}\right) \to 0,$$

Lyapounov's condition is satisfied, and hence,

$$\begin{aligned}
\frac{1}{s_n} \sum_{i=1}^n X_{ni} &= \frac{1}{\sqrt{n\frac{\sigma^4}{2}\left(\frac{1}{6}T^2 + T - \frac{7}{6}\right)}} \sum_i (\varepsilon' D_T \varepsilon - E[\varepsilon' D_T \varepsilon]) \\
&= \frac{1}{\sqrt{nT}\sqrt{\frac{\sigma^4}{12} + O\left(\frac{1}{T}\right)}} \sum_i \left(\varepsilon' D_T \varepsilon - \frac{\sigma^2}{2}(-T+1) \right) \to \mathcal{N}(0,1).
\end{aligned}$$

∎

Lemma 17 *As $n, T \to \infty$,*

$$\frac{1}{\sqrt{nT}} \sum_i \varepsilon' \left(D_T + \frac{3}{T+1} D_T' D_T \right) \varepsilon \to \mathcal{N}\left(0, \frac{17\sigma^4}{60}\right).$$

Proof. We first note that the fourth central moment of $\varepsilon' D_T \varepsilon + \frac{3}{T+1} \varepsilon' D_T' D_T \varepsilon$ can be bounded by eight times the sum of the fourth central moments of $\varepsilon' D_T \varepsilon$ and $\frac{3}{T+1} \varepsilon' D_T' D_T \varepsilon$, from which we can conclude that the fourth central moment of $\varepsilon' D_T \varepsilon + \frac{3}{T+1} \varepsilon' D_T' D_T \varepsilon$ is of order T^4. Examining the cumulant generating function, we can see that the first two cumulants of $\varepsilon' \left(D_T + \frac{3}{T+1} D_T' D_T \right) \varepsilon \equiv \varepsilon' G_T \varepsilon$ are given by $\frac{\sigma^2}{2} \operatorname{trace}(G_T + G_T')$, and $\frac{\sigma^4}{2} \operatorname{trace}\left[(G_T + G_T')^2\right]$. Using Lemma 14, we can show that

$$\operatorname{trace}(G_T + G_T') = 0, \qquad \operatorname{trace}\left[(G_T + G_T')^2\right] = \frac{1}{30} \frac{17T^3 - 37T^2 + 37T - 17}{T+1}.$$

Therefore, we have

$$E\left[\varepsilon' \left(D_T + \frac{3}{T+1} D_T' D_T \right) \varepsilon\right] = 0, \qquad \operatorname{Var}\left(\varepsilon' \left(D_T + \frac{3}{T+1} D_T' D_T \right) \varepsilon\right) = \frac{\sigma^4}{60} \frac{17T^3 - 37T^2 + 37T - 17}{T+1}.$$

Because the fourth central moment is of order T^4, and the variance is of order T^2, the Lyapounov condition is satisfied. Therefore, we have

$$\frac{1}{\sqrt{n\frac{\sigma^4}{60}\frac{17T^3-37T^2+37T-17}{T+1}}}\sum_i \varepsilon'\left(D_T+\frac{3}{T+1}D_T'D_T\right)\varepsilon \to \mathcal{N}(0,1),$$

from which the conclusion follows. ■

Lemma 18 *Suppose that* $\lim \frac{1}{n}\sum_{i=1}^n \alpha_i^2 > 0$. *We then have*

$$\frac{1}{n^{1/2}T^{3/2}}\sum_i\left(\varepsilon' H_T y_- - \frac{\sigma^2}{2}(-T+1)\right) \to \mathcal{N}\left(0, \frac{\sigma^2}{12}\lim\frac{1}{n}\sum_{i=1}^n \alpha_i^2\right)$$

Proof. It can be shown that

$$\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)'\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right) = -\frac{1}{12}T+\frac{1}{12}T^3 \tag{25}$$

Therefore,

$$\sum_{i=1}^n \varepsilon_i'\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)\alpha_i \sim \mathcal{N}\left(0,\sigma^2\left(-\frac{1}{12}T+\frac{1}{12}T^3\right)\sum_{i=1}^n\alpha_i^2\right) = O_p\left(n^{1/2}T^{3/2}\right). \tag{26}$$

From Lemma 16, we obtain

$$\sum_i\left(\varepsilon' D_T\varepsilon - \frac{\sigma^2}{2}(-T+1)\right) = O_p\left(n^{1/2}T\right) = o_p\left(n^{1/2}T^{3/2}\right). \tag{27}$$

The conclusion follows from (26) and (27). ■

Lemma 19 *Suppose that* $\lim \frac{1}{n}\sum_{i=1}^n \alpha_i^2 > 0$. *We then have*

$$\text{plim}\frac{1}{nT^3}\sum_i y_-' H_T y_- = \frac{1}{12}\lim\frac{1}{n}\sum_{i=1}^n \alpha_i^2.$$

Proof.

$$y_-'H_Ty_- = \left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)'\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)\alpha^2 + 2\alpha\varepsilon'\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)+\varepsilon' D_T' D_T\varepsilon \tag{28}$$

Using (25), we obtain

$$\sum_i\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)'\left(\xi_{2T}-\frac{T-1}{2}\ell_T\right)\alpha_i^2 = \left(-\frac{1}{12}T+\frac{1}{12}T^3\right)\sum_{i=1}^n\alpha_i^2 = O\left(nT^3\right).$$

From (26) and (27), we can see that the first term on the right hand side of (28) dominates the second and third terms. The conclusion follows. ■

D.1 Proof of Theorem 4

Note that

$$\widehat{\theta}-\theta_0 = \frac{\sum_i \varepsilon' H_T y_-}{\sum_i y_-' H_T y_-} = \frac{\sum_i \varepsilon' D_T\varepsilon}{\sum_i \varepsilon' D_T' D_T\varepsilon} = \frac{\sum_i\left(\varepsilon' D_T\varepsilon - \frac{\sigma^2}{2}(-T+1)\right)+\frac{\sigma^2}{2}n(-T+1)}{\sum_i \varepsilon' D_T' D_T\varepsilon}$$

Therefore, we obtain

$$\sqrt{n}T\left(\widehat{\theta}-\theta_0+\frac{3}{T+1}\right)=\frac{\frac{1}{\sqrt{nT}}\sum_i\left(\varepsilon' D_T\varepsilon-\frac{\sigma^2}{2}\left(-T+1\right)\right)+\frac{1}{\sqrt{nT}}\frac{\sigma^2}{2}n\left(-T+1\right)+\frac{3}{T+1}\frac{1}{\sqrt{nT}}\sum_i\varepsilon' D_T' D_T\varepsilon}{\frac{1}{nT^2}\sum_i\varepsilon' D_T' D_T\varepsilon}$$

By Lemma 17 and 15, we obtain $\sqrt{n}T\left(\widehat{\theta}-\theta_0+\frac{3}{T+1}\right)\to\mathcal{N}\left(0,\frac{51}{5}\right)$.

D.2 Proof of Theorem 5

Note that

$$\widehat{\theta}-\theta_0=\frac{\sum_i\varepsilon' H_T y_-}{\sum_i y_-' H_T y_-}=\frac{\sum_i\left(\varepsilon' H_T y_- -\frac{\sigma^2}{2}\left(-T+1\right)\right)+\frac{\sigma^2}{2}n\left(-T+1\right)}{\sum_i y_-' H_T y_-}$$

Therefore,

$$n^{1/2}T^{3/2}\left(\widehat{\theta}-\theta_0\right)=\frac{\frac{1}{n^{1/2}T^{3/2}}\sum_i\left(\varepsilon' H_T y_- -\frac{\sigma^2}{2}\left(-T+1\right)\right)+\frac{1}{n^{1/2}T^{3/2}}\frac{\sigma^2}{2}n\left(-T+1\right)}{\frac{1}{nT^3}\sum_i y_-' H_T y_-}$$

From Lemmas 18 and 19, we obtain

$$\frac{1}{n^{1/2}T^{3/2}}\sum_i\left(\varepsilon' H_T y_- -\frac{\sigma^2}{2}\left(-T+1\right)\right)\to\mathcal{N}\left(0,\frac{\sigma^2}{12}\lim\frac{1}{n}\sum_{i=1}^n\alpha_i^2\right)$$
$$\operatorname{plim}\frac{1}{nT^3}\sum_i y_-' H_T y_- =\frac{1}{12}\lim\frac{1}{n}\sum_{i=1}^n\alpha_i^2.$$

Furthermore, we have $\lim\frac{1}{n^{1/2}T^{3/2}}\frac{\sigma^2}{2}n\left(-T+1\right)=-\frac{\sigma^2}{2}\lim\sqrt{\frac{n}{T}}$. We therefore obtain

$$n^{1/2}T^{3/2}\left(\widehat{\theta}-\theta_0\right)\to\mathcal{N}\left(-\frac{6\sigma^2\lim\sqrt{\frac{n}{T}}}{\lim\frac{1}{n}\sum_{i=1}^n\alpha_i^2},\frac{12\sigma^2}{\lim\frac{1}{n}\sum_{i=1}^n\alpha_i^2}\right).$$

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Table 1: Performance of Bias Corrected Maximum Likelihood Estimator

			Bias		RMSE	
T	N	θ_0	$\widehat{\theta}_{GMM}$	$\widehat{\widehat{\theta}}$	$\widehat{\theta}_{GMM}$	$\widehat{\widehat{\theta}}$
5	100	0	-0.011	-0.039	0.074	0.065
5	100	0.3	-0.027	-0.069	0.099	0.089
5	100	0.6	-0.074	-0.115	0.160	0.129
5	100	0.9	-0.452	-0.178	0.552	0.187
5	200	0	-0.006	-0.041	0.053	0.055
5	200	0.3	-0.014	-0.071	0.070	0.081
5	200	0.6	-0.038	-0.116	0.111	0.124
5	200	0.9	-0.337	-0.178	0.443	0.183
10	100	0	-0.011	-0.010	0.044	0.036
10	100	0.3	-0.021	-0.019	0.053	0.040
10	100	0.6	-0.045	-0.038	0.075	0.051
10	100	0.9	-0.218	-0.079	0.248	0.085
10	200	0	-0.006	-0.011	0.031	0.027
10	200	0.3	-0.011	-0.019	0.038	0.032
10	200	0.6	-0.025	-0.037	0.051	0.045
10	200	0.9	-0.152	-0.079	0.181	0.082
20	100	0	-0.011	-0.003	0.029	0.024
20	100	0.3	-0.017	-0.005	0.033	0.024
20	100	0.6	-0.029	-0.011	0.042	0.024
20	100	0.9	-0.100	-0.032	0.109	0.037
20	200	0	-0.006	-0.003	0.020	0.017
20	200	0.3	-0.009	-0.005	0.022	0.017
20	200	0.6	-0.016	-0.010	0.027	0.018
20	200	0.9	-0.065	-0.031	0.074	0.034

Simulations are based on 5000 replications. The fixed effects α_i and the innovations ϵ_{it} are assumed to have independent standard normal distributions. Initial observation y_{i0} are assumed to be generated by the stationary distribution $\mathcal{N}\left(\frac{\alpha_i}{1-\theta_0}, \frac{1}{1-\theta_0^2}\right)$.