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***OPTIMAL JOB SEARCH IN A CHANGING WORLD***

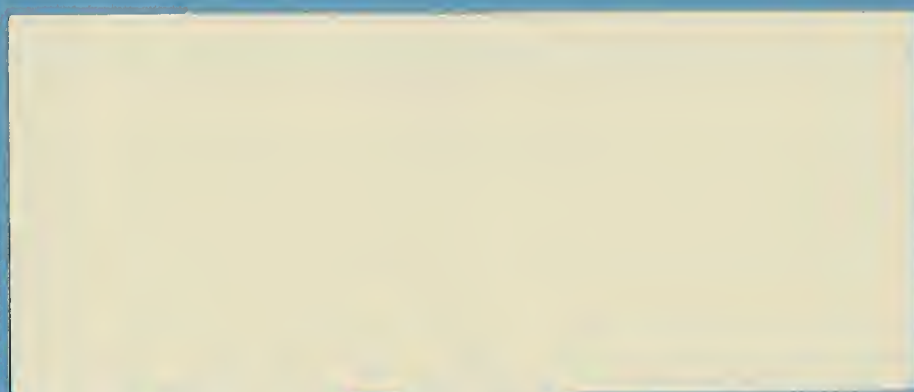
**Lones Smith**

**95-3**

**Jan. 1995**

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# 1. INTRODUCTION

This paper explores some theoretical and economic implications of a nice asymmetry in many classic job search models, whereby individuals are permitted to drop jobs at will, but can retain them at their pleasure (i.e. the reverse cannot occur). To make this more definite, consider the problem faced by an individual who must search for jobs (or ‘prizes’). Suppose that when the individual does not possess a job, she is assumed to be searching for one. While searching, she pays a fixed flow cost (possibly negative, such as unemployment insurance) and enjoys a positive flow arrival rate of wage offers, coming from some nonstationary but deterministically (perhaps discontinuously) evolving distribution. The individual may drop a job at will and start searching again. How then does she behave?

Search theory as a branch of economics has solved a wealth of such problems in both continuous and discrete time, but has largely confined its attention to the *steady-state* theory, where quits do not occur. This paper analyzes the job search problem by supposing, just as in Van Den Berg (1990), that the wage distribution evolves in a nonstationary but deterministic fashion. But unlike that paper, I do not deny searchers the option of ever dropping a job once taken up. This restriction, while briefly justified in the context of his model, is inappropriate for many contexts — for instance, in recent analyses of continuum agent two-sided mutual tenant-at-will employment matching models with heterogeneity, such as Smith (1994a) and Shimer and Smith (1994).

As it turns out, Van Den Berg’s assumption that jobs cannot be dropped also conceals at least two intriguing effects that I wish to explore. For one, the identity between the decision-maker’s reservation wage  $\theta_t$  (or flow utility) and her average unemployed value  $\mathcal{W}_t$  fails, and we must independently derive their respective laws of motion. As it turns out, casual finance theory intuition may then be misleading. Indeed, the reservation wage is *not* the flow return on the Bellman value  $\mathcal{W}_t$  of being unmatched, but has an even more inviting link to  $\mathcal{W}_t$ ; namely, the flow return with an option to renew at the same price! This turns out to be related to the fact that no simple local relationship between  $\theta_t$  and  $\mathcal{W}_t$  exists.

Another major finding is that while  $\mathcal{W}_t$  is necessarily a continuous function of time,  $\theta_t$  need not be, but is always *upper semi-continuous* in time. In fact I show that in more general search models, the optimal acceptance set is a *lower hemicontinuous* correspondence of time. This happens to be a simple example of a very general principle underlying optimal individual behaviour in the presence of sticky state variables. The paper concludes with a simple example of how some well-known comparative statics may fail in a nonstationary environment, and point out some new ones that become apparent.



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## 2. THE MODEL

At any given moment in time, the individual is either searching (waiting) for a prize while experiencing a *flow search cost*  $c$ , or is holding some job with wage  $w$ . The individual can hold at most one at a time.<sup>1</sup> Wage offers arrive at Poisson time intervals with parameter  $\rho > 0$ . The wage distribution facing the individual is deterministically evolving, with time-dependent c.d.f.  $\{F_t(w) | t, w \in \mathbb{R}_+\}$ . Assume that  $F_t$  has support on  $[0, \infty)$  with uniformly bounded finite mean for fixed  $t$ ; further let  $F_t(w)$  be a piecewise continuous function of time  $t$  for fixed  $w$ . That is, the wage distribution can shift discontinuously at possibly a countable number of distinct times.

The individual cannot search while holding a job,<sup>2</sup> nor is she permitted to hold more than one job at a time. But any job may be retained forever if desired, or may be *freely* dropped at will.<sup>3</sup> While I allow searchers the additional latitude to drop existing jobs, for definiteness, I do not permit them to ‘drop out’ of the search market altogether, and thus forego any search costs.

At any time  $t$ , the individual seeks to maximize the expected present value of all future jobs held less search costs borne, namely

$$\int_t^\infty \beta e^{-\beta(s-t)} m(s) ds$$

where  $m(s)$  is the income flow at time  $s$ , and the expectation is taken with respect to arrival times and job realizations.

A *strategy* is a time-dependent sequence of *acceptance sets*  $\mathcal{A} = (\mathcal{A}_t \subseteq \mathbb{R}_+ | t \in [0, \infty))$  for each time. That is, the individual is willing to accept or retain a job  $x$  iff its wage  $w$  belongs to  $\mathcal{A}_t$ .<sup>4</sup> Note that  $\mathcal{A}_t$  can only possibly be well-defined up to measure zero equivalence, in the sense that any  $\tilde{\mathcal{A}}_t$  that *contains*  $\mathcal{A}_t$  on a set of times of measure zero yields the same payoff. Although a strategy might possibly also depend on whether an individual currently holds a job, an optimal strategy  $\mathcal{A}^*$  cannot. This must be true, since both the costs and outside options of either retaining a current job, or accepting a new one, are identical.

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<sup>1</sup>Or, more generally, one could modify the model to suppose that the individual is constrained in the number of prizes she can simultaneously enjoy.

<sup>2</sup>Less stringently, one could suppose that one experiences a lower flow arrival rate of job offers while matched than while unmatched.

<sup>3</sup>If there are lump-sum quitting costs, then reservation wages will no longer exist; that is, an employed individual will have a lower ‘reservation wage’ than will her unemployed counterpart. This is somewhat analogous to the effect of switching costs in the multi-armed bandit, where Gittin’s indices no longer exist. See Banks and Sundaram (1994).

<sup>4</sup>This implicit restriction to pure strategies is without loss of generality, given a mild generalization of a result due to Dvoretzky, Wald and Wolfowitz (1951).



### 3. THE MAIN RESULTS

#### 3.1 Optimal Strategies

Let  $\mathcal{W}_t$  be the time- $t$  (Bellman) optimal *average value* of being unemployed, and  $\mathcal{W}_t(w)$  the analogous value with wage  $w \geq \theta_t$  in hand. While it would be nice to say that  $w \in \mathcal{A}_t$  exactly when  $\mathcal{W}_t(w) \geq \mathcal{W}_t$ , since quitting ‘immediately’ is an option, the inequality  $\mathcal{W}_t(w) \geq \mathcal{W}_t$  is necessarily always true that. So say that a wage  $w$  at time  $t$  is *strictly agreeable* if  $\mathcal{W}_t(w) > \mathcal{W}_t$ . Then wage  $w$  is called *agreeable* and  $w \in \mathcal{A}_t$  exactly when  $\mathcal{W}_t(w) > \mathcal{W}_t$ , or if  $\mathcal{W}_t(w) = \mathcal{W}_t$  but an optimal strategy entails retaining that wage for a positive period of time. (It will follow that any agreeable wage  $w = \lim_{n \rightarrow \infty} w_n$  for some sequence of strictly agreeable wages  $w_n$ .)

Notice further that an optimal strategy entails monotonic preferences at every instant in time: If  $w \in \mathcal{A}_s^*$  for all  $s \in [t, t + \Delta]$ , then at least in the same time interval so must  $w' \in \mathcal{A}_s^*$  for  $w' > w$ . As I assume that all agreeable wages are accepted,  $\mathcal{A}_t^*$  is closed in  $\mathbb{R}$  for all  $t$ . Hence,

**Lemma 1 (Monotonicity)** *The optimal strategy is  $\mathcal{A}_t^* = [\theta_t, \infty)$  at time  $t$ .*

I shall refer to  $\theta_t$  as the individual’s *reservation wage*; in the more general model considered in subsection 3.4, a more neutral term might be the (lower) *threshold*. I shall soon argue that this is conceptually distinct from her *flow value* of being unmatched at time  $t$ .

Given  $\langle \mathcal{W}_t \rangle$ , the determination of the threshold time path  $\langle \theta_t \rangle$  is an exercise in *optimality*, while the calculation in the reverse direction is a simple exercise in accounting. Suppose that  $\langle \theta_t \rangle$  is given. In order to compute  $\langle \mathcal{W}_t \rangle$ , I must know the next time after  $t$  that  $\theta_t$  will assume (or exceed, if it jumps) the level  $w$ . That is, define  $\theta_t^{-1}(w) \equiv \inf\{s > t \mid \theta_s \geq w\}$ . Then

$$\mathcal{W}_t(w) = \begin{cases} w + e^{-\beta(\theta_t^{-1}(w)-t)}(\mathcal{W}_{\theta_t^{-1}(w)} - w) & \text{if } w \geq \theta_t \\ \mathcal{W}_t & \text{else} \end{cases} \quad (1)$$

Finally,  $\langle \mathcal{W}_t \rangle$  must solve the following recursive *accounting* equation:

$$\mathcal{W}_t = \int_t^\infty \rho e^{-\rho(s-t)} \left( -c(1 - e^{-\beta(s-t)}) + e^{-\beta(s-t)} \int_0^\infty \mathcal{W}_s(w) dF_s(w) \right) ds \quad (2)$$

The functional form of (2) fundamentally differs from that in Van Den Berg (1990), on account of my allowing jobs to be dropped. On the other hand, Van Den Berg’s proof that a continuous value function uniquely exists, by means of a standard contraction-mapping argument, applies here with few changes,<sup>5</sup> and is thus omitted.

<sup>5</sup>His proof, for instance, assumed stationarity after some finite time  $T$ ; however, a continuous

**Lemma 2 (The Value Function)** *There exists a unique continuous solution  $\langle \mathcal{W}_t \rangle$  to (2), that is continuous and a.e. differentiable.*

Observe that by the Fundamental Theorem of Calculus applied to (2), regardless of the functional form of  $\theta_t^{-1}$  or  $F$ ,  $\mathcal{W}$  is a continuous and differentiable function of time. In fact, together with (1), we have

$$\dot{\mathcal{W}}_t = \beta \mathcal{W}_t + \beta c - \rho \int_{\theta_t}^{\infty} (w + e^{-\beta(\theta_t^{-1}(w)-t)} (\mathcal{W}_{\theta_t^{-1}(w)} - w) - \mathcal{W}_t) dF_t(w) \quad (3)$$

### 3.2 Average Values and Thresholds

I now must characterize optimal behaviour, and compute  $\langle \theta_t \rangle$  given  $\langle \mathcal{W}_t \rangle$ . There are many approaches to this problem. For instance, one could treat  $\langle \theta_t \rangle$  as an infinite-horizon control, and use the calculus of variations.<sup>6</sup> But a less ambitious route is much more transparent. For ease of notation, define  $\theta^{-1}(t) \equiv \theta_t^{-1}(\theta_t)$ , namely the next time that the reservation wage will be at least as high as its current level. This is the moment that a maximizing individual will drop a job paying  $\theta_t$ . I now claim that optimal behaviour yields

**Theorem 1 (Reservation Wages)** *For any  $c \geq 0$ , and given time path of average value functions  $\langle \mathcal{W}_t \rangle$ , the reservation wage satisfies  $\theta_t \leq \mathcal{W}_t$  for all  $t$ . Moreover,  $\theta_t = \lim_{s \rightarrow \theta^{-1}(t)} (\mathcal{W}_t - e^{-\beta(s-t)} \mathcal{W}_s) / (1 - e^{-\beta(s-t)})$ , or equivalently*

$$\theta_t = \begin{cases} \mathcal{W}_t - \dot{\mathcal{W}}_t / \beta & \theta^{-1}(t) = t \\ \mathcal{W}_t - e^{-\beta(\theta^{-1}(t)-t)} (\mathcal{W}_{\theta^{-1}(t)} - \theta_t) & \theta^{-1}(t) > t \\ \mathcal{W}_t & \theta^{-1}(t) = \infty \end{cases}$$

*Proof:* The inequality  $\theta_t \leq \mathcal{W}_t$  follows from (1) and the identity  $\mathcal{W}_t \equiv \mathcal{W}_t(\theta_t)$ .

The result for  $\theta^{-1}(t) = \infty$  is essentially Van Den Berg (1990). Next, suppose  $\theta^{-1}(t) = T \in (t, \infty)$ . Then optimal behaviour demands that the job paying  $\theta_t$  be dropped at time  $T$ . Hence,  $\mathcal{W}_t = \theta_t + e^{-\beta(T-t)} [\mathcal{W}_T - \theta_t]$ , yielding the result for  $\theta^{-1}(t) > t$ . Finally, suppose that  $\theta^{-1}(t) = t$ . By monotonicity, if  $w > \theta_t$  then it must be optimal to hold the job paying  $w$  for some positive period of time, i.e.  $\mathcal{W}_t < w + e^{-\beta(s-t)} (\mathcal{W}_s - w)$  for some  $s > 0$ ; conversely, if  $w < \theta_t$  then it is strictly better to stay unemployed, and the reverse inequality holds for all  $s > 0$ . Thus,  $\theta_t = \lim_{s \rightarrow t} (\mathcal{W}_t - e^{-\beta(s-t)} \mathcal{W}_s) / (1 - e^{-\beta(s-t)})$ . In that case,  $\theta_t = \mathcal{W}_t - \dot{\mathcal{W}}_t / \beta$ , by l'Hôpital's rule.  $\diamond$

function on  $[0, T]$  is uniformly continuous, and so a simple limiting exercise suffices to complete the argument.

<sup>6</sup>Van Den Berg's 'no-quitting' assumption meant that he did not have to independently derive the law of motion for  $\langle \theta_t \rangle$  and  $\langle \mathcal{W}_t \rangle$ . The analysis that follows thus did not arise in his paper.

In words, when the reservation wage is rising, so that  $\theta^{-1}(t) = t$ , its relationship to the unmatched value is purely local (and has a natural financial interpretation, as we shall see); when it is weakly monotonically decreasing on  $[t, \infty)$ , then  $\theta_s = \mathcal{W}_s$  for all  $s \geq t$ , again a purely local relationship arises. But when it is falling, and later to rise higher, global optimality comes into play.

### 3.3 Some Financial Intuition

As done in Shimer and Smith (1994), I can define the *spot market wage* system  $\langle \psi_t \rangle$  implicitly through  $\mathcal{W}_t \equiv \int_t^\infty \beta e^{-\beta(s-t)} \psi_s ds$ . Interpret  $\psi_t$  as the reservation wage at time  $t$  in a hypothetical spot market. An individual is always indifferent between remaining unemployed and accepting her spot market reservation wage for an arbitrarily short period of time, during which she cannot search. The more explicit formula  $\psi_t = \mathcal{W}_t - \dot{\mathcal{W}}_t/\beta$  follows by the Fundamental Theorem of Calculus. Rewriting this as  $\beta(\mathcal{W}_t/\beta) = \psi_t + \dot{\mathcal{W}}_t/\beta$  makes it clear that  $\psi_t$  is *flow value* or the *flow return* and  $\dot{\mathcal{W}}_t$  the associated instantaneous capital gain of the asset ‘being unemployed’, whose present value is  $\mathcal{W}_t/\beta$ .

But Theorem 1 establishes that  $\psi_t \neq \theta_t$  in general. It turns out that  $\theta_t$  is the flow return *with an option to renew at the same rate*. Intuitively, this follows from the fact that an individual taking a job does so knowing that she may remain in it for as long as she wishes at the same wage. That is, at time  $t_0$ , she is indifferent about exchanging the variable return on the asset  $\langle \mathcal{W}_t, t \geq t_0 \rangle$  for the flow return  $w = \theta_{t_0}$  that is fixed (for as long as she wishes). If the option is never exercised, i.e.  $\theta^{-1}(t) = t$ , then  $\theta_t = \psi_t$ . Otherwise, the renewal option has value, and so  $\theta_t > \psi_t$ . When it is forever exercised, in the case  $\theta^{-1}(t) = \infty$ , then the unemployed asset  $\mathcal{W}_t$  is effectively sold, and so  $\theta_t = \mathcal{W}_t$ . If the renewal option is not exercised at some future date, then no purely local relationship between  $\theta_t$  and  $\mathcal{W}_t$  can exist.

### 3.4 A General Continuity Result

I temporarily abstract away from the model discussed above, in order to derive a somewhat more penetrating result. Treat the jobs as prizes  $x$  providing flow utility  $u(x)$  to the owner, where  $x$  is a vector of prize attributes. For instance, in the job search problem, there may be a multidimensional job characteristics space, with wage not the only consideration. Suppose that there is given a Borel space of prizes  $x \in \mathbb{R}^n$  whose distribution follows a known deterministic evolution. As above, one either possesses a prize or must search for one.

Observe the following simple property of the graph of the optimal acceptance sets  $\langle \mathcal{A}_t^* \rangle$  over time.



**Theorem 2 (Lower Hemicontinuity)** *If the utility function  $u$  is nonnegative, continuous, and satisfies local non-satiation, then  $\mathcal{A}_t^*$  is a lower hemi-continuous correspondence of time.*

*Proof:* Assume  $c = 0$ , for positive search costs will not invalidate the arguments below. Note that it is obvious that  $\mathcal{A}_t^* \neq \emptyset$ , for that strategy is dominated by  $\mathcal{A}_t^* = \mathbb{R}^n$ , since  $u \geq 0$  for all utilities.

Suppose that  $x \in \mathcal{A}_t^*$ . We must show that for all sequences of times  $t_n \rightarrow t$ , there is a sequence of prizes  $x_n \rightarrow x$  such that  $x_n \in \mathcal{A}_{t_n}^*$  for all large enough  $n$ . For definiteness, it suffices to argue that this limit separately holds for both increasing and decreasing sequences of times. Consider the first case, with  $t_n \downarrow t$ . If  $\mathcal{W}_t(x) = \mathcal{W}_t$  even though  $x \in \mathcal{A}_t^*$ , then  $x \in \mathcal{A}_{t_n}^*$  for all large enough  $n$  by definition. So suppose that  $\mathcal{W}_t(x) > \mathcal{W}_t$ . I shall employ the sequence  $x_n \equiv x$ . Due to the accounting identity (1), prize  $x$  must be held until some time  $\theta_t^{-1}(u(x)) > t$  if  $\mathcal{W}_t(x) > \mathcal{W}_t$ . So  $\mathcal{W}_s$  and  $\mathcal{W}_s(x)$  are both right continuous in time, yielding  $\mathcal{W}_s(x) > \mathcal{W}_s$  for  $s$  close enough to  $t$ ,  $s > t$ . Hence,  $x \in \mathcal{A}_{t_n}^*$  for all large enough  $n$ .

Next, consider the second case, with  $t_n \downarrow t$ . Here we must appeal to simple optimality considerations. Suppose that  $\mathcal{W}_t(x) > \mathcal{W}_t$ , and consider the sequence  $x_n \equiv x$ . For  $s < t$ , let  $\hat{\mathcal{W}}_s(x)$  be the not necessarily optimal average value of holding onto prize  $x$  until time  $t$  and then behaving optimally. Then  $\mathcal{W}_s(x) \geq \hat{\mathcal{W}}_s(x) > \mathcal{W}_s$  for  $s$  close to  $t$ , and so  $x \in \mathcal{A}_{t_n}^*$  for all large enough  $n$ .

Finally, suppose that  $\mathcal{W}_t(x) = \mathcal{W}_t$  even though  $x \in \mathcal{A}_t^*$  (so that  $x \in \mathcal{A}_s^*$  for  $s > t$  close enough to  $t$ , by definition). Here must appeal to the properties of  $u$ . By local nonsatiation, there exists a prize  $y$  close to  $x$  yielding utilities  $u(y) > u(x)$ . Write

$$\mathcal{W}_{t_n}(y) - \mathcal{W}_{t_n}(x) = [\mathcal{W}_{t_n}(y) - \hat{\mathcal{W}}_{t_n}(y)] + [\hat{\mathcal{W}}_{t_n}(y) - \hat{\mathcal{W}}_{t_n}(x)] + [\hat{\mathcal{W}}_{t_n}(x) - \mathcal{W}_{t_n}(x)]$$

Then the outer terms on the right side must vanish as  $n \rightarrow \infty$ , while the inner one has positive liminf because both  $x, y \in \mathcal{A}_s^*$  for  $s > t$  close enough to  $t$  (by assumption for  $x$  and by monotonicity for  $y$ ). Thus, for large enough  $n$ , we have  $\mathcal{W}_{t_n}(y) - \mathcal{W}_{t_n}(x) > 0$ . By continuity and local nonsatiation, there must exist prizes  $x_n \rightarrow x$  yielding utilities  $u(x_n) > u(x)$  and such that  $\mathcal{W}_{t_n}(x_n) - \mathcal{W}_{t_n}(x) > 0$ , i.e.  $x_n \in \mathcal{A}_{t_n}^*$ , as required.  $\diamond$

**Corollary (Upper Semicontinuity)** *In the job search problem, the reservation wage  $\theta_t$  is an upper semicontinuous function of time.*

This follows from Theorem 2 and the definition of what an u.s.c. function is; namely, that for all  $\varepsilon > 0$ ,  $\theta_{t_0} > \theta_t - \varepsilon$  on some small enough neighbourhood of  $t_0$ . So while the average unmatched value  $\mathcal{W}_t$  is a continuous function of time, Theorem 1 illustrates how the reservation wage can discontinuously jump up when  $\mathcal{W}_t$  jumps.

Intuitively, the acceptance set may suddenly ‘implode’ but can never ‘explode’, and the reservation wage can ‘jump up’ but can never ‘jump down’. I should underscore how general is the underlying idea here. Indeed, Smith (1994b) considers a class of models where a *heterogeneous* stock variable satisfies an ‘upward inertia’ property (hard to build up, but easy to destroy). If preferences over the *stock* composition have a deterministic evolution, then the optimal policy entails *flows* that are lower hemi-continuous in time: One might well suddenly wish to discretely destroy a discrete portion of the stock, but one only continuously builds it up. In this search model, think of the stock of capital in set  $A$  as the probability one has a prize  $x \in A$ , or a job with wage  $w \in A$ ; the flow is the acceptance set at any time.

Other typical illustrations of this idea include a firm’s behaviour with respect to its diverse capital stock, or individual behaviour with respect to one’s reputation. Smith (1994b) explores the general proof of this property and its applications.

#### 4. AN ILLUSTRATIVE EXAMPLE

I conclude with an example showing how this nontrivially generalizes previous work on nonstationary job search by Van Den Berg (1990). I shall also show how some basic comparative statics insights change. Suppose, as in figure 1, that wages are uniform on  $(0, 1)$  until time  $T > 0$  and thereafter uniform on  $(0, ae^{-\gamma(t-T)})$ , for some  $a > 1$  and  $\gamma \geq 0$ . This corresponds to a foreseen wage distribution ‘spike’ when  $\gamma > 0$ , and a foreseen one-shot upward shift in the wage distribution when  $\gamma = 0$ . Job offers have an arrival rate  $\rho > 0$ , the interest rate is  $\beta > 0$ , and the (inescapable) search costs are  $c \geq 0$ .

**Case 1:**  $t \geq T$

Since the prospects are deteriorating over time after  $T$ , we have  $\mathcal{W}_t = \theta_t$ , and a job once accepted will never be dropped. Thus, equations (1) and (2) reduce to

$$\mathcal{W}_t = -c + \int_t^\infty \rho e^{-(\beta+\rho)(s-t)} \left[ c + \left( \frac{ae^{-\gamma(s-T)} - \theta_s}{ae^{-\gamma(s-T)}} \right) \frac{ae^{-\gamma(s-T)} + \theta_s}{2} + \frac{\theta_s}{ae^{-\gamma(s-T)}} \mathcal{W}_s \right] ds$$

Since Theorem 1 asserts that  $\theta_t = \mathcal{W}_t$  for  $t \in [T, \infty)$ , we have

$$\theta_t = -c + \int_t^\infty \rho e^{-(\beta+\rho)(s-t)} \left[ c + (a^2 e^{-\gamma(s-T)} + \theta_s^2 e^{\gamma(s-T)}) / 2a \right] ds$$

Thus, so long as  $\theta_t > 0$ , we have

$$\dot{\theta}_t = -(\rho/2a)[a^2 e^{-\gamma(t-T)} + \theta_t^2 e^{\gamma(t-T)}] + (\rho + \beta)\theta_t + \beta c \quad (4)$$



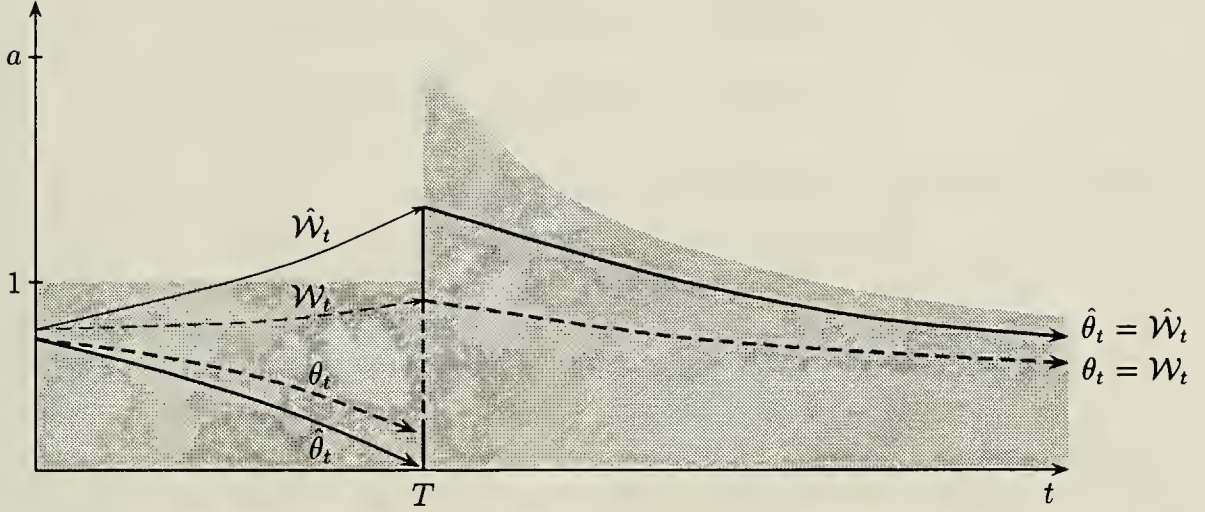


Figure 1: **Optimal Behaviour given a Foreseen Wage Distribution Spike.** The reservation wage (thicker lines) and unmatched value are depicted for a wage distribution that is known to be uniformly distributed at any moment in time (on the vertical slices) on the shaded region. Depending on the parameters  $\rho$  and  $\beta$ , either the solid or the dashed paths obtain.

For reasons of tractability, I shall confine attention to two cases. First, if  $\gamma = 0$  and  $c \geq 0$ , then the problem is stationary, and so optimally  $\dot{\theta}_t = 0$ . Hence,

$$\theta_t = \max \left\langle 0, a(\rho + \beta)/\rho - \sqrt{a^2(\rho + \beta)^2/\rho^2 - (a^2 - 2ac\beta/\rho)} \right\rangle$$

for all  $t \geq T$ . Observe that  $\theta_t > 0$  (and thus the equations are well-defined) precisely when  $c < a\rho/2\beta$ , i.e. when flow search costs are not too high.

Second, if  $\gamma > 0$  and  $c = 0$ , then (4) admits the solution  $\theta_t = e^{-\gamma(t-T)}\theta_T$ , where

$$\theta_t = \max \left\langle 0, a(\rho + \beta + \gamma)/\rho - \sqrt{a^2(\rho + \beta + \gamma)^2/\rho^2 - a^2} \right\rangle$$

### Case 2: $t < T$

If  $\theta_T \geq 1$ , then the individual will drop any job agreed to in  $[0, T)$ ; thus, she solves a de facto finite horizon problem on  $[0, T)$ , subject to  $\theta_{T-} = 0$ . Otherwise, if  $\theta_T < 1$ , we can act as if the problem is finite horizon with a terminal job  $\max(0, w -$

$\theta_T$ ) if wage  $w$  is held at time  $T$ . In general, the Bellman equation is<sup>7</sup>

$$\begin{aligned}
[1 - e^{-\beta(T-t)}]\theta_t &= \int_t^T \rho e^{-\theta(s-t)} \left( -c(1 - e^{-\beta(s-t)}) \right. \\
&\quad \left. + e^{-\beta(s-t)}(1 - e^{-\beta(T-s)})[(1 - \theta_s)(1 + \theta_s)/2 + \theta_s^2] \right) ds \\
&\quad + \max\langle 0, e^{-\beta(T-t)}(1 - \theta_T)/2 \int_t^T \rho(1 - \theta_r)e^{-\int_t^r \rho(1 - \theta_r)dr} ds \rangle
\end{aligned} \tag{5}$$

For  $\theta_T \geq 1$ , the last term is zero, and so

$$\dot{\theta}_t = \left( \frac{e^{-\beta(T-t)}}{1 - e^{-\beta(T-t)}} + \rho \right) \theta_t - \rho(\theta_t^2 + 1)/2 + \frac{\beta c}{1 - e^{-\beta(T-t)}}$$

As might be guessed, this differential equation admits no neat closed form solution, but can be numerically evaluated. By the same token, only a numerical solution is available for  $\theta_T < 1$ ; however, we can apply l'Hôpital's rule to (5), and deduce that  $\theta_{T-} \equiv \lim_{t \uparrow T} \theta_t = \rho(1 - \theta_T)^2/2\beta > 0$ . Thus, the greater is  $\theta_T < 1$ , the lower is  $\theta_{T-}$ .

### Comparative Statics

This simple example offers some useful explanatory power.

1. **[Improved Wages: FSD Shifts]** First, the standard reservation wage monotonicity with respect to improvements with respect to first order stochastic dominance (FSD) fails: Consider  $\gamma = 0$ . In that case, if the wage spike occurs earlier at time  $T' < T$ , then  $\theta$  shifts down on  $[T' - \varepsilon, T')$ , for some  $\varepsilon > 0$ . But it is still true that the unmatched value  $\langle \mathcal{W}_t \rangle$  uniformly increases on  $[0, T)$ , and in fact many of our stylized steady-state intuitions are best interpreted as extending to average rather than flow values.
2. **[Increased Risk: SSD Shifts]** While the example cannot illustrate this, it is also true that increases in risk (i.e. a mean-preserving spread) need not uniformly increase  $\langle \theta_t \rangle$ . This is intuitive if one imagines just as above that the increase is concentrated starting at some time.
3. **[Policy Duration]** A 'more permanent' change (i.e. smaller  $\gamma$ ) cannot decrease and can increase reservation wage volatility around the regime shift. Indeed, if  $\theta_T < 1$ , then a fall in  $\gamma$  pushes up  $\theta_T$ , and hence down  $\theta_{T-}$ . I think this is an interesting addendum to macro arguments on the consequences of a permanent versus a transitory government policy shift.

By means of a different example, I could also show that if the wage distribution were cyclical, say uniform on  $[0, 1]$  at times  $[0, 1) \cup [2, 3) \cup \dots$  and uniform on  $[0, a]$

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<sup>7</sup>The non-zero maximand consists of the average terminal value if the wage that is accepted exceeds  $\theta_T$  times (the integral) the chance that the first accepted wage exceeds  $\theta_T$ .

at times  $[1, 2) \cup [3, 4) \cup \dots$ , then an optimizing individual will also employ a cyclical reservation wage rule. This will lead to a stochastic number of voluntary unemployment spells and eventual ‘permanent’ position with a sufficiently high wage. I shall abstain from providing the mathematics, but the intuition should be clear.

## 5. CONCLUSION

There are two modelling assumptions in this paper worth commenting on. First, there is no distributional uncertainty here: I assume that the wage distribution follows a deterministic evolution that is known *ex ante*. It is not clear how to relax this assumption, but it would certainly entail much messier analysis, in which one might obtain declining reservation wages as in Burdett and Vishwanath (1988). Second, I have assumed that it is the outside environment, and not the current job characteristics, that are variable. For if one’s own wage were subject to change, then it might well be true that no current reservation wage rule alone could optimally serve as a quitting yardstick (see Shimer and Smith (1994)). In that case, or if one were slowly learning one’s preferences about the job, then there might exist some (eg. Gittins) index that would serve the same role as the wage in this paper.

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