working paper
department
of economics

QUANTITY CONSTRAINTS, SPILLOVERS,
AND THE HAHN PROCESS

By

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Number 169 November 1975

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1. Introduction

In recent years there has been considerable interest in general equilibrium (or disequilibrium) models in which agents perceive that they cannot complete their desired transactions and hence choose their demands by optimizing subject to quantity constraints on their trading activity.[2] Such constraints affect not only the markets in which they occur but "spillover" in their effects on other markets because they can alter demands for all goods. The leading case, of course, is that in which the realization of unemployment affects demands by consumers, and, indeed, much of the literature has concentrated on such phenomena as an explanation for the existence of Keynesian underemployment equilibria. However, the phenomenon involved can be much more general and one should expect that it leads to equilibria which are not Walrasian (competitive) equilibria in a general model as quantity-constrained demands all get satisfied.[3]

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[1] Research for this paper was supported by National Science Foundation Grant GS 43185. I am indebted to Hal Varian for helpful conversations but retain responsibility for error.

[2] Examples are Clower [8], Patinkin [24], Leijonhufvud [21], [22], Barro and Grossman [2], Benassy [3], [4], [5], [6], Frevert [16], Hayashi [18], Varian [25], [26] and Veendorp [27].

[3] Probably the most satisfactory proofs of existence are those of Benassy [3], [4], [5], [6].

Despite the fact that the subject is one of disequilibrium behavior, however, most of the work has concentrated on showing the existence of such non-Walrasian equilibria and relatively little is known about the dynamic stability of adjustment processes in such models. Such analyses as are available[1] typically involve relatively simple models or local stability or very strong restrictions of the gross substitute type. One principal purpose of the present paper, therefore, is to consider a relatively general model of stability under quantity constraints.

If satisfactory discussions of stability are largely lacking from the quantity constraint literature, however, that literature does provide an important feature the lack of which makes more traditional discussions of stability most unsatisfactory indeed -- the consciousness on the part of the agents that they are in disequilibrium. The stability literature typically proceeds by assuming that agents expect their transactions to be completed and that they formulate their demands accordingly, taking prices as given and (usually) unchanging.[2] I shall not here be concerned with the assumptions as to prices[3] but shall concentrate on the completion-of-transactions aspect which is clearly unsatisfactory in itself in models that have long since passed beyond the tâtonnement stage

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[1]For example, Frevert [16], Hayashi [18], Varian [25], [26], and Veendorp [27].

[2]See Fisher [14] for a recent survey and discussion. Earlier surveys are given in Negishi [23] and Arrow and Hahn [1].

[3]For some work on price adjustment by individual agents, see Fisher [9], [10], [11], and [15].

to allow disequilibrium trading to occur. Thus the second principal purpose of this paper is to consider how existing stability models can be modified to accommodate the kind of consciousness of disequilibrium which is expressed in the use of quantity constraints.

As it happens, this turns out to be quite easy to do as regards the class of stability models using what I consider to be the most satisfactory non-tâtonnement adjustment process so far developed, the Hahn Process.[1] Indeed, to a certain extent this has already been done by Arrow and Hahn [1, Ch. XIII]. They at least localize the problem that one must offer something of value before one can buy by developing a model in which all purchases must be made for a particular commodity, "money," and demands not backed up with money are not effective ("active") demands but only planned ("target") demands. Even in that model, however, agents act as though they expect their transactions to be completed and formulate active demands taking into account only their cash constraint and without regard for the fact that their transactions on other markets may be limited. The present paper shows one way in which such quantity restrictions can be accommodated along the lines of Arrow and Hahn [1, Ch. XIII]. It remains necessary, however, as in that analysis, to assume that agents never run out of money out of equilibrium -- an assumption which is pretty strong in contexts such as the present one and to which I shall return below.[2]

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[1] So named by Negishi [23]. The original paper is Hahn and Negishi [17]. Arrow and Hahn [1, Ch. XIII] develop the model for a monetary economy with cash constraints on effective demand; Fisher [10] puts in individual price adjustment; firms are introduced in [12], and disequilibrium production and consumption (as opposed to merely disequilibrium trading) in [13] and [15].

[2] That assumption has somewhat more plausibility when prices are set by individuals conscious of disequilibrium than it does in other contexts. See [10].

In order to concentrate on the problem at hand, the incorporation of quantity constraints and spillovers into the Hahn Process, I shall deal with a version of that process in which the monetary matters introduced by Arrow and Hahn are only implicit rather than giving them explicit notational treatment. Further, although consideration of quantity constraints obviously makes most sense when consumption and production are going on out of equilibrium, the complications involved in allowing such activities are large and not particularly germane to the issues here discussed.[1] Hence I shall deal with the case in which the only disequilibrium activity is the trading of commodities or, in the case of firms, of commitments to buy inputs and sell outputs, production and consumption being deferred until equilibrium is reached. Moreover, since most of the issues involved can be discussed in the context of pure exchange, I give explicit treatment only to that case, briefly discussing the differences involved in the incorporation of firms after the technical results.[2]

The results obtained are fairly strong. We find that a seemingly plausible specification of how quantity constraints are perceived and how they affect behavior leads immediately to the quasi-stability of the equilibrium set (which includes non-Walrasian equilibria). Moreover, extension of that specification in what seems a reasonable way to allow experimentation with constraints when markets continue to clear generates

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[1] See Fisher [13] and [15].

[2] The exchange model to be discussed is closest to that of Arrow and Hahn [1, Ch. XIII]. The model with firms is closest to that of Fisher [12]. In both cases the present paper suppresses the explicit treatment of money.

a far stronger result, namely, that the adjustment process involved is globally stable in that from any set of initial conditions it converges to some Walrasian equilibrium.

Clearly, the question of the plausibility of the assumptions which lead to such results becomes one of importance. At the end of the paper I argue that the assumptions are more likely to be plausible close to Walrasian equilibrium than away from it, so that perhaps the results should be taken as providing reasons for believing that the adjustment process is locally stable and converges locally to Walrasian equilibrium even when there are quantity constraints.

2. Pure Exchange and Quasi-Stability

There are n+1 commodities, numbered 0, 1, ..., n, with commodity 0 being money whose price is always unity. The price of commodity i is denoted by p_i and the n+1 component vector of the p_i (first element unity) by p.

Households are subscripted h. The hth household has a strictly quasi-concave, differentiable utility function, $U^h(x_h)$ where x_h denotes a vector of demands. The household's actual possessions are denoted by $\bar{x}_h$. Both x_h and $\bar{x}_h$ are functions of time, t, as are prices, but we generally suppress the time argument.

The household's Walrasian demands are obtained by maximizing its utility function subject only to its budget constraint:

$$p'(x_h - \bar{x}_h) = 0 \qquad (2.1)$$

where the prime denotes transposition. We denote its Walrasian excess demands by z_h ($= x_h - \bar{x}_h$). I shall refer to the components of z_h as target excess demands.

Now, denote the vector of the active excess demands (the demands it attempts to exercise in the market place) by a_h. The household goes without saying that if $z_{hi} \leq 0$ for all i so that (in view of (2.1)) the household is able to fulfill its unconstrained plans, with the possible exception of disposing of free goods, then $a_{hi} = z_{hi} = 0$ for all goods with positive prices and $a_{hi} \leq 0$ for all free goods. However, in general, it will not be the case that the household's target excess demands and active excess demands coincide. There are at least two reasons for this.

In the first place, it is natural to assume (following Arrow and Hahn [1, Ch. XIII]) that purchases must be made for money. Hence, since the household may not always have enough money to effect its desired purchases, it may have to limit its positive excess demands to amounts which can be backed up with money. I shall assume that the household does this in some appropriately continuous way and that it never runs out of money.[1]

The second reason that active and target excess demands can differ is the one with which we are concerned in this paper. If the household realizes that it may not be able to complete all its transactions it may adjust its demands to take account of the fact that its trading on

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[1]For more extended discussion of these issues, see Arrow and Hahn [1] and Fisher [10] and [14]. I return to No Bankruptcy later. The two assumptions are implicitly involved in the explicit ones below.

particular markets is limited. One way of modeling such behavior is to assume that there are certain quantity constraints perceived by the household and that it chooses its active excess demands by maximizing its utility function subject to those constraints.

Such a maximization process can be consistent with the somewhat different approach taken here. It is somewhat more convenient, however, to proceed in a somewhat different way by supposing that the household first chooses its target excess demands and then looks at the quantity constraints and reallocates its demands to be active excess demands using some appropriately continuous rule consistent with the assumptions below. Whether the result is or is not consistent with utility maximizing behavior depends in large part on how the household perceives the quantity constraints.

One such way in which the constraints can be perceived which leads to the desired result is as follows. After households initially attempt to exercise their target excess demands, each market (for a good other than money) develops a long and a short side. Households who are on the long side of a given market and who are considering moving to the short side see that there is already a queue of unsatisfied agents ahead of them and do not expect to be able to trade if they make the move.

As this suggests, the crucial assumption turns out to be that households never move from the long to the short side of a market as a result of the imposition of quantity constraints. More formally, we shall assume that trade takes place instantaneously, so that excess demands are evaluated post-trade. Let A_i denote the total active excess demand for commodity i, that is, the sum of the a_{hi} over h. We assume:

Assumption 2.1 (Modified Hahn Process): For every h and every $i=1,\ldots,n$, it is always the case post-trade that $z_{hi}A_i \geq 0$.

Obviously this requires some discussion beyond that already given. Thus, consider, for example, a household which would like (in the target sense) to buy coffee and sell tea. Suppose that such a household finds that coffee is in short supply. It seems evident that such a constraint, if it is the only one, will not cause the household to turn around and wish to sell coffee, but, in any case, this is not the issue, for shifting from the short side to the long side of a market is not ruled out by the Modified Hahn Process Assumption. The crucial question concerns the spillover effect of the coffee constraint on the household's behavior on the tea market. It is perfectly possible that the household, deprived of the opportunity to obtain as much coffee as it wishes will turn around and decide to acquire tea. What is ruled out by Assumption 2.1 is the possibility that the household will try to do this even though it knows that tea is also in short supply. This seems pretty reasonable.

To put it another way, Assumption 2.1 states that if, post-trade, the household has an unsatisfied target excess demand for some commodity, it must be because that target excess demand is on the short side of the market and the household faces a binding constraint for that commodity. Note again that this does not prevent the imposition of quantity constraints from causing a household with a target excess demand on the short side of a particular market from exercising and fulfilling an active demand on the long side.

There is, of course, somewhat more to it than that, although the discussion just given is really the heart of the matter. Implicit in that discussion is the assumption (required anyway for the satisfaction of Lipschitz Conditions) that things change smoothly enough that there is continuity as to which side of the market is the short side. More important, embedded in the discussion is the basic Hahn Process idea that unsatisfied active demands occur only on one side of the market, that is, that willing buyers and sellers can readily find each other so that, after trade, there are not unsatisfied agents on both sides of the market.[1] Indeed, one way of generating Assumption 2.1 is by assuming that the household's target and active excess demands for commodity i are (weakly) of the same sign and then adding the traditional Hahn Process Assumption that all active excess demands for commodity i have the same sign. Such an assumption would be plausible as applied to <u>post-trade</u> target and excess demands (although not to pre-trade ones where it would prevent switching from the short to the long side of a market) and is stronger than Assumption 2.1.

I now turn to a discussion of the dynamics of the model. I assume the traditional price-adjustment process, applied to active demands, namely:

(2.2) $$\dot{p}_i = G^i(A_i) \text{ unless } p_i = 0 \text{ and } A_i < 0, \text{ in which case } \dot{p}_i = 0 \quad (i=1,\dots,n)\ ,$$

where $G^i(.)$ is a continuous and sign-preserving function which does not approach zero except as A_i does.

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[1]I argued in [10] that such an assumption is almost compelled in models in which individuals adjust their own prices.

The other fundamental variables of the model are the actual stocks held by households, the $\bar{x}_h$. It is not necessary to be precise about the rules whereby these change except to assume that trade takes place only if active demands are non-zero[1] and that trading rules satisfy Lipschitz Conditions and two further conditions, namely:[2]

$$\sum_h \dot{\bar{x}}_h = 0 \tag{2.3}$$

and

$$p'\dot{\bar{x}}_h = 0 \,, \quad \text{all } h. \tag{2.4}$$

The first of these rules (2.3) is an immediate consequence of the fact that we are in pure exchange, so that the total amount of each commodity is fixed. The second (2.4) is a No-Swindling Assumption which states that a household cannot alter its wealth through trading at constant prices.

We assume that the adjustment process just described determines prices and stocks (and implicitly all other variables) as single-valued and continuous functions of time t and initial conditions in the usual way.[3]

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[1] The only interesting (if unlikely) possibility that is eliminated by this is that at a non-Walrasian equilibrium quantity constraints get adjusted in such a way that trade takes the economy to a Walrasian equilibrium with prices unchanged and post-trade active demands zero. Handling of this sort of thing is not difficult but explicitly allowing for it unduly complicates the exposition. This note should be borne in mind when considering Assumption 3.1 below.

[2] Arrow and Hahn [1, Ch. XIII] give one example of such rules.

[3] The violation of Lipschitz Conditions in (2.2) is not a problem here. See Henry [19] and [20] and Champsaur, Drèze, and Henry [7].

The dynamic system just described is in equilibrium wherever all active demands are non-positive with active demands zero for all positively priced goods. It is in Walrasian equilibrium wherever the same thing applies to target demands.

It is now easy to prove:

Theorem 2.1: Under Assumption 2.1, the process is quasi-stable. That is, every limit point of the path of prices and stocks is a (possibly non-Walrasian) equilibrium.

Proof: The proof is essentially the same as for the standard Hahn Process models but is so short that it may as well be given here.

The Lagrangian for the household's target maximization problem can be written:

$$(2.5) \qquad L_h = U^h(x_h) - \lambda_h p'(x_h - \bar{x}_h) ,$$

where λ_h is a Lagrange multiplier equal to the marginal utility of wealth. By the Envelope Theorem, we can evaluate $\dot{U}^h$ by differentiating (2.5) with respect to the things not wholly under the household's control, namely, prices and stocks. Doing so, we obtain:

$$(2.6) \qquad \dot{U}^h = - \lambda_h \dot{p}' z_h + \lambda_h p' \dot{\bar{x}}_h \quad .$$

However, the second term on the righthand side of (2.6) is zero by (2.4). Moreover, λ_h is positive, and every term in the inner product in the first term on the right-hand side is non-negative, since individual target excess demands have (weakly) the same sign as total active excess demands, by Assumption 2.1, and so do price changes, by (2.2). Since utility can be taken as bounded below by the utility which would be obtained by consuming initial stocks, it is evident that the sum of target utilities over households is a Lyapounov Function.

3. Convergence to Walrasian Equilibrium

To get further than quasi-stability and obtain global stability of the adjustment process requires two more steps. The first is a proof of boundedness and the second a proof either that equilibria are locally isolated or that all limit points starting from given initial conditions are the same.

Boundedness presents no problem as regards stocks, in view of (2.3). The issue as regards prices is somewhat trickier. There are two known ways to obtain boundedness of prices in Hahn Process models (other than by assuming it directly).[1] The first of these is to assume that if relative prices become too high, the highest priced good is in non-positive excess demand. Such a method seems as applicable to active as to target excess demands. The other method for obtaining boundedness is to assume that the adjustment functions, $G^i(.)$, are bounded above by rays through the origin and that money is in positive excess demand if its relative price is sufficiently low. This method, however, requires homogeneity of degree zero and Walras' Law, so its applicability in the present case depends in part on whether we require that active demands are derived

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1 Details may be found in Fisher [12].

from maximization which includes the budget constraint.[1] In any case, I shall henceforward assume boundedness of relative prices.

The remainder of a global stability proof -- a demonstration that equilibria are locally isolated or that all limit points are the same -- is clearly not available in the present analysis without more explicit treatment of the way in which active demands are generated. Indeed, in view of Varian [26], it is clear that local isolation is not an inevitable result. Hence one cannot proceed further down this line in a model as general as the present one, although specific analyses may be able to do so.

In any case, it seems to me to be more interesting to examine the implications of some further assumptions to which one might be led by consideration of the sort of argument leading to Assumption 2.1. As it turns out, such further assumptions do lead to global stability of the adjustment process (given boundedness); moreover, they lead to convergence to a Walrasian equilibrium. Thus consideration of them is interesting if only to examine the differences which keep the system from getting stuck in a non-Walrasian place -- or allow it to do so.

The first such assumption (which I shall argue below is not the plausible one) is a strengthening of Assumption 2.1 to:

Assumption 2.1' (Strong Modified Hahn Process): For every h and every $i=1,\ldots,n$, it is always the case post-trade that $z_{hi}A_i \geq 0$ and $z_{hi}A_i > 0$ unless $z_{hi} = 0$.

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[1] Or on more precise specification of the relations between an individual's active and target excess demands as in Arrow and Hahn [1, Ch. XIII].

This is much too strong. What it says that is not contained in the weaker Assumption 2.1 is that any non-zero target excess demand post-trade must result in a non-zero active demand of the same sign.[1] This means that quantity constraints can alter demands but not by so much that constrained demands balance while unconstrained ones are satisfied. This is not particularly plausible as an assumption holding at every moment of time and it is not surprising that it rules out non-Walrasian equilibria as limit points to the adjustment process as it clearly does (It also implies global stability as will be seen below.). It may be instructive, however, to compare Assumption 2.1' to Assumption 2.1 above and to Assumption 3.1 below as an aid to assessing the plausibility of the latter two assumptions.

Now, not only is Assumption 2.1 implausibly strong, but it is also stronger than necessary to produce the results of interest. If we seek for an assumption which will keep the adjustment process from getting stuck at a non-Walrasian equilibrium, it is not necessary to have active and target demands strongly of the same sign for all households, all commodities, and all moments of time. Rather one can keep the system moving by giving any appropriate market a slight nudge.

Consider, then, the way in which households perceive quantity constraints and the sort of argument that makes Assumption 2.1 seem fairly plausible. We said, in considering that assumption, that it made

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[1] It is probably worth remarking that the requirement that target excess demands result in non-zero action goes very deep in stability models. It is assumed without comment in Arrow and Hahn [1, Ch. XIII] but plays a major role in what I have termed the "Present Action Postulate" which comes to the fore when disequilibrium production and consumption are introduced. See Fisher [13], [14], and [15].

sense for households not to switch from the long to the short side of markets as a result of quantity constraints, possibly because they would regard it as useless to do so. When total active demand in a particular market is zero, however, a household considering what to do will no longer see a short and a long side. If he now wishes to switch sides he will not perceive anyone in the queue ahead of him. Accordingly, there would be some plausibility to requiring that when total active demand in some market is zero, but there is non-zero target demand, then at least one household will attempt to exercise that target demand by making it active.

In fact, we can get by with a substantially weaker version of the same thing. We need only require that if the system is in a non-Walrasian equilibrium for a sufficiently long time, so that total active demands are non-positive and are zero for positively priced goods, then eventually some household wakes up to the fact that there is no longer a shortage and attempts to exercise some of its unsatisfied target excess demand on at least one market. Formally, we retain Assumption 2.1 (instead of Assumption 2.1') and add to it the following:

<u>Assumption 3.1 (Eventual Experimentation)</u>: There exist two finite time intervals, Δt_1 and Δt_2, with $\Delta t_2 > \Delta t_1 \geq 0$, such that, for any t^*, if for all t in the interval $[t^*, t^* + \Delta t_1]$, $A_i \leq 0$ for all i and $\sum_{i+1}^{n+1} p_i A_i = 0$, but for some h and i, $z_{hi} > 0$, then there exists a household, k, and a commodity j, such that at time $t^* + \Delta t_2$, $z_{kj} A_j > 0$, and not both $p_j = 0$ and $A_j < 0$.

This apparently elaborate assumption simply makes the statement about eventual experimentation already given, the last clause merely ensuring that the experiment is not only that of trying to dispose of free goods. Four further comments are in order.

First, as already remarked, this is weaker then requiring experimentation on a particular market whenever that market has had a zero active and a non-zero target excess demand for a long enough time.

Second, we could have stated the assumption directly in terms of having at time $t^* + \Delta t_2$ a non-zero A_j which is not just an active attempt to dispose of a free good. While this is the direct implication of Assumption 3.1, to state it that way would not capture the flavor of experimentation by at least one household on some market.

Third, a way to obtain Assumption 3.1 which seems eminently reasonable but which is stronger than necessary would be to assume that for k and j at $t^* + \Delta t_2$, $z_{kj}a_{kj} > 0$, so that post-trade target and active excess demand have <u>strictly</u> the same sign and then add the usual Hahn Process Assumption that all unsatisfied active excess demands have the same sign.

Finally, as observed in a previous footnote, we have already ruled out the possibility that the system attains a non-Walrasian equilibrium which just happens to have the same prices as a Walrasian equilibrium and then reaches the Walrasian equilibrium without price change. It would do so because quantity constraints happen to adjust in just such a way as to allow trading to the Walrasian equilibrium to take place with post-trade active demands remaining zero for all positively priced,and non-positive for all free goods. Such a possibility could also be handled but does not seem very interesting.

As will be evident in the proof of global stability below, Assumption 2.1 and 3.1 together ensure that the limit points of the system are all Walrasian equilibria, thus strengthening the result of Theorem 2.1. However, in order to proceed to global stability, rather than quasi-stability, one more assumption is needed which has nothing to do with the issues we have been discussing. It is:

Assumption 3.2 (Indecomposability of Walrasian Equilibrium): At any Walrasian equilibrium, let S be the set of commodities (including money) which have strictly positive prices. For any proper subset of S, say S', there exists a pair of commodities, $i \in S'$ and $j \in (S - S')$, such that for some household the marginal rate of substitution between i and j is not at a corner and is equal to the ratio of their prices, p_i/p_j.

This assumption prevents the possibility that the economy breaks into two parts at a Walrasian equilibrium with one group of agents holding one set of commodities and having corner solutions with respect to a second set and the remaining agents holding the second set of commodities and having corner solutions with respect to the first set. It is required in the proof below to establish the global stability of prices but not of stocks. Note that it would make no sense to require such indecomposability out of Walrasian equilibrium.

We can now prove:

Theorem 3.1: Under Assumptions 2.1, 3.1, and 3.2, the economy is globally stable. That is, from any initial conditions, prices and stocks converge to some Walrasian equilibrium (with the possible exception that holdings of free goods may not converge).[1]

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[1]Note that since Assumption 2.1' implies Assumption 2.2, it would yield the same result.

<u>Proof</u>: The proof is very similar to that for standard Hahn Process models, but is not identical.

The principal difference which does occur is that we must first show that Assumption 3.1 rules out non-Walrasian equilibria as limit points, so that the quasi-stability result of Theorem 2.1 becomes applicable to the proof here. This is easily done by a slight modification of the usual proofs of Lyapounov's Second Method.

Let V denote the sum of target utilities over households. Then V is a function of prices and stocks. Denote the vector of all prices and all stocks at time t by y(t) and write $V = V(y(t))$. Now, for any t, consider $y(t + \Delta t)$, where $\Delta t > 0$. Evidently, $y(t + \Delta t)$ is a continuous function of y(t) and Δt. Define $D \equiv V(y(t + \Delta t)) - V(y(t))$. Then D is a continuous function of y(t) and Δt, and we can write $D = D(y(t), \Delta t)$. Now fix Δt at t_2, where Δt_2 is defined in Assumption 3.1 and write $F(y(t)) \equiv D(y(t), \Delta t_2)$. It is clear from the proof of Theorem 2.1 and Assumption 2.2 that $F(y(t)) \leq 0$, and $F(y(t)) < 0$ unless y(t) is a Walrasian equilibrium. Moreover, since V converges, F(y(t)) must converge to zero.

Now, let $\{y(t_\lambda)\}$ be a convergent subsequence of points approaching a limit y*. Evidently:

$$F(y^*) = \lim_{\lambda \to \infty} F(y(t_\lambda)) = 0 \tag{3.1}$$

so that, by what has been said, y* must be a Walrasian equilibrium.

The common sense of this is that Lyapounov's Second Method does not require that the Lyapounov Function always decline out of equilibrium, but only that it be everywhere non-increasing and <u>ultimately</u> declining out of equilibrium. Finite flat stretches can be readily accommodated.

Since we have already assumed boundedness, it remains to show that all limit points are the same. Suppose that there are two limit points, denoted by * and **. Since U^h is non-increasing and bounded below it approaches a limit which must be the same at both limit points. Hence $U^h(x_h{*}) = U^h(x_h{**})$. Because * is a Walrasian equilibrium, however, the strict quasi-concavity of U^h implies that:

(3.2) $$p^{*\prime}x_h{*} \leq p^{*\prime}x_h{**}$$

with the strict inequality holding unless $x_h{*}$ and $x_h{**}$ coincide for all commodities with a positive price at *. (Note that this statement would not hold for non-Walrasian equilibria; this is the principal place at which an attempt to prove global stability simply from Theorem 2.1 would break down, another being Assumption 3.2.)

Now, since * and ** are equilibria, the left-hand side of (3.2) is the value of actual holdings at *, while the right-hand side cannot exceed the value of actual holdings at **, valued in the prices of *. Hence,

(3.3) $$p^{*\prime}\bar{x}_h{*} \leq p^{*\prime}\bar{x}_h{**} \quad ,$$

with the strict inequality holding under the same conditions as before. Since we are in pure exchange, however, the sum of actual stocks over households is the same at both equilibrium points, hence summing (3.3) over households will yield a contradiction unless both x_h and $\bar{x}_h$ are the same at * and ** for every household as regards all goods with positive prices at *. It is but a matter of interchanging the notation to show that the same must be true for all goods which have positive prices at **.

It remains to show that p* and p** are the same. To do this, observe that if money is the only non-free good at both * and **, then there is nothing to prove. We may thus suppose that there is some other good with a positive price at one of the equilibria, say *. By Assumption 3.2, there exists a commodity j, other than money, which has a positive price at *, such that the marginal rate of substitution between j and money for some household is not at a corner and is equal to the price ratio. Since x_h* and x_h** coincide in <u>all</u> non-free goods, that marginal rate of substitution must be the same at ** as at *, whence the price of j must be the same also. If there is another commodity with a positive price at * or **, then there is a commodity, i, with a positive price such that, for some household, the marginal rate of substitution between i and j or between i and money is not at a corner and is equal to the price ratio. Since marginal rates of substitution are the same at the two equilibria, so is the price of i. Proceeding in this way we can reach every positive price and the theorem is proved.

4. Introducing Firms

I now briefly indicate how the above results can be adapted to models involving firms. The formalities involved are really no different than what has gone before as regards the problems at issue here and so I shall not give a formal model (The interested reader can consult Fisher [12] for the simplest model adaptable for the present discussion.). There is some point in discussing firms, however, not merely to indicate how to adapt the analysis, but because such discussion points up some issues regarding the plausibility of the assumptions leading to our rather strong results.

There seems no reason why Assumptions 2.1 and 3.1 should not apply to profit-maximizing firms as well as to utility-maximizing households. If Assumption 2.1 applies to firms, then, essentially as in the proof of Theorem 2.1, firms will find that their target profits (the profits they would make if they could complete all their transactions) are declining out of equilibrium. This is because the things they would like to sell but cannot sell are getting cheaper, while the things which they would like to buy but cannot buy are getting more expensive.

Given that target profits of firms are declining out of equilibrium, target utilities of households will decline also. For households there are now two reinforcing effects. First, there is the direct Hahn Process effect as for firms, essentially that involved in Theorem 2.1, which means that prices move so as to reduce the household's target utility. Second, the household's resources will decline because of the decline in its share of target profits. Thus the sum of household utilities can again be taken as a Lyapounov function and quasi-stability established.

Finally, if Assumption 3.1 is adapted to include firms, convergence to Walrasian equilibrium can be obtained by adapting the proof in Fisher [12]. The principal addition to the proof of Theorem 3.1 is the use of profit-maximization for firms together with expenditure minimization for households to force the kind of contradiction involved in summing (3.3) above.

Thus our results can be extended to firms, although it should be observed that, for reasons unrelated to the present paper, constant returns cannot be accommodated. (See [12].)

5. Target Profits, No Bankruptcy, and the Modified Hahn Process

Consideration of firms, however, points up a central issue involved in our assumptions. In deciding on their target excess demands, consumers in the model just discussed take the target profits of firms as part of their resources. Assumptions 2.1 and 3.1 in different ways assume that the target excess demands so generated are related to active excess demands in particular ways. In an economy in which agents recognize that they are in disequilibrium, is this plausible?

In fact, the issue involved here is not greatly different from that involved under pure exchange where the wealth involved in computing a household's target excess demands is that which he would receive if he could sell all his resources including his labor. In both cases, the central question is how restrictions on spendable wealth are to be taken into account.

This brings us back to an assumption implicitly made throughout, namely, that no agent runs out of money in the course of the adjustment process. Without that assumption, positive target excess demands cannot be exercised at all and there is no reason that prices cannot move so as to increase target utilities. It is generally very hard, moreover, to ensure that the No Bankruptcy condition holds, although it is easier when individuals adjust prices.[1] When agents are conscious of disequilibrium,

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[1] See Arrow and Hahn [1] and Fisher [10].

however, such an assumption becomes somewhat more plausible. If I know that I will have difficulty finding employment, for example, I am likely to be rather careful about spending all my money while looking for work. Certainly, I will be less likely to spend it all deliberately than I would in a world in which I erroneously thought I could find work whenever I wanted and could thus plan to live off current earnings rather than dipping into capital.

If we accept the No Bankruptcy assumption, however, there remains the question of whether Assumption 2.1 (and 3.1) are very plausible. The issue here turns, in a way, on how far from Walrasian equilibrium the system is perceived to be. If it is very far, then agents may feel that the target demands which they would like to exercise if they had the resources which generate them have little to do with the real world in which they live. Assumption 2.1 makes sense only if agents take their target excess demands as a reasonable starting point in constructing their active ones. If they regard quantity constraints as so severe that target excess demands are wholly unrealistic, then our analysis will not go through. This may be particularly likely where target excess demands involve spending the undistributed and unachieved profits of firms which are themselves far from being able to complete their transactions, but it can occur in pure exchange as well.

On the other hand, if target excess demands are a good starting point, then Assumptions 2.1 and 3.1 seem pretty plausible. In that case, the system is at least quasi-stable and, if the sort of experimentation involved in Assumption 3.1 is engaged in, will even converge to Walrasian equilibrium. Since target excess demands may be good starting points if the system is close to Walrasian equilibrium, this may be a reason for supposing that the process is locally, if not globally, stable and converges locally to Walrasian equilibrium.[1]

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[1] Such an argument is only suggestive, however. A full proof of local stability would require showing that if the process starts in a region where Assumptions 2.1 and 3.1 are plausible, then it always remains in such a region.

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