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A Simple Model of Optimum Life-Cycle Consumption
With Earnings Uncertainty

by

Eytan Sheshinski

Number 409

January 1986

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*Support from NSF Grant No. SES-8105184 is gratefully acknowledged. I am indebted to Peter Diamond for a helpful discussion.

Abstract

We analyze the optimum consumption path of an individual who maximizes expected discounted utility over an infinite horizon, when savings earn a fixed return and wage earnings follow a two-state random process. It is assumed that no insurance is available against earnings uncertainty. We show that optimum consumption always changes discontinuously with the level of earnings. The optimum consumption path is characterized for the case where the subjective discount rate is equal to the interest rate. While under certainty consumption is constant in this case, it is shown that here consumption strictly increases or decreases over time. Specifically, when the 'Hazard-Rate' non-decreases with time, consumption approaches infinity or zero, respectively. The discontinuity in consumption, however, is always smaller than the difference in earnings across states. Comparison to the full insurance case when earnings can be annuitized is also provided.

These qualitative results may explain some recent findings (Hall (1982)) concerning the volatility of consumption and point to the appropriate life-cycle modelling in the absence of earnings insurance (Diamond-Mirrlees (1982)).

1. Introduction

We consider a consumer who earns a wage income that may change stochastically over time while savings earn a fixed return. The consumer chooses a consumption path that maximizes expected discounted utility over an infinite horizon. Our objective is to characterize the optimum consumption path, focusing on the dependence of consumption on future earnings and on the corresponding relation between changes in consumption and changes in actual earnings. Clearly, the relation between consumption and current earnings depends on the nature of the underlying stochastic process.

When earnings over-time are identically distributed random variables (as assumed, for example, in Yaari (1976) and Schechtman (1976)) or, more generally, if earnings follow a stationary stochastic process (as in Bewley (1977), (1980)), changes in earnings are regarded as transitory and hence do not convey information with regard to future earnings. This explains the asymptotic convergence results - of the expected optimum consumption path in the absence of discounting and, more generally, of expected marginal utility^{1/} which were taken as the appropriate analogue of the 'permanent-income' (Friedman (1957)) concept under stochastic conditions.

We adopt a different approach. Changes in earnings are assumed to contain a large permanent component. Obvious applications are to cases of disability (Diamond-Mirrlees (1982))^{2/} or occupational positions, where events have significant long-term effects. In fact, we make the extreme assumption that once earnings change this change is permanent. Consequently, consumption changes discontinuously (and non-negligibly) with earning.

In general, consumption at any point in time depends on the level of assets and on expectations concerning future earnings. Thus, even when actual earnings do not change and the rate of interest is equal to the subjective rate of time preference (which, under certainty, yields a constant optimum consumption path), optimum consumption changes over time. Specifically, when earnings may fall at some future date, savings are positive and consumption rises over time until the occurrence of the fall in earnings. Afterwards savings cease and the consumption remains constant. The opposite pattern is found when earnings are expected to rise. Optimum consumption in the initial state increases indefinitely in the former case and approaches zero in the latter case. There is therefore no obvious interpretation of the 'permanent-income' concept in this context.

At any time, optimum consumption in the initial state is bounded between total income - i.e., the return on assets plus wage earnings - when earnings are high and total income when earnings are low. Thus, the magnitude of the discontinuity in consumption as the level of earnings changes is bounded by the difference in earnings. This, (and other aspects of the model) could provide a testable hypothesis to the volatility of consumption question analyzed by Hall (1978) and Hall and Mishkin (1982).

2. The Model

An individual may earn a wage of w_0 or w_1 , labeled state 0 and state 1, respectively. Time, t , is taken to be continuous. At time $t = 0$, the individual is in state 0, earning w_0 . For a small dt , there is a probability of $f(t)dt$ that in the interval $(t, t+dt)$ a switch to state 1 will occur. Once such a switch occurs the individual stays forever in state 1. It is assumed that there is no possibility for insurance against such event.

There is, however, a perfect capital market with a fixed return, r , on savings. The individual's objective is to maximize his expected discounted utility over an infinite horizon.

Once state 1 occurs there is no further uncertainty. Hence, as in standard dynamic programming, we solve the individual's problem backwards. Suppose that state 1 occurs at time t . The individual's problem is then to choose the consumption path in state 1, $c_1(s,t) \geq 0$, for all $s \geq t$, that maximizes

$$\int_t^{\infty} u(c_1(s,t)) e^{-\delta(s-t)} ds \quad (1)$$

subject to the intertemporal budget constraint on assets, $a(s)$,

$$\dot{a}(s) = ra(s) + w_1 - c_1(s,t), \quad s \geq t, \quad (2)$$

for a given initial $a(t)$, where the utility function, $u(c)$, is assumed to be strictly increasing and concave, bounded above, twice continuously differentiable for all $c > 0$, with $u'(0) = \infty$. The interest rate, r , and the subjective time-preference rate, δ , are assumed to be positive constants. The boundedness of u ensures that (1) is finite for any feasible consumption path satisfying (2).

The Euler first-order conditions for a maximum of (1) subject to (2) are

$$\dot{c}_1(s,t) = - \frac{(r-\delta) u'(c_1(s,t))}{u''(c_1(s,t))}, \quad s \geq t \quad (3)$$

and the transversality condition $\lim_{s \rightarrow \infty} a(s)e^{-rs} = 0$, where $\dot{c}_1 = \frac{dc_1}{ds}$.

Integrating (2) and taking $s \rightarrow \infty$, the latter is equivalent to

$$\int_t^{\infty} c_1(s, t) e^{-r(s-t)} ds = a(t) + w_1 \int_t^{\infty} e^{-r(s-t)} ds, \quad (4)$$

i.e., the present discounted value of consumption equals the present discounted value of future wage earnings plus initial assets. Conditions (3) and (4) determine the optimum c_1 , denoted $\hat{c}_1$, which can be written $\hat{c}_1(s, t) = \hat{c}_1(t) h(s-t)$, where $h(0) = 1$ and, by (3), for all $v \geq 0$, $h'(v) \geq 0$ as $r - \delta \geq 0$. Optimum consumption at the onset of state 1, $\hat{c}_1(t)$, is determined by (4),

$$\hat{c}_1(t) = \frac{1}{\phi} (ra(t) + w_1) \quad (5)$$

where $\phi = r \int_t^{\infty} h(v-t) e^{-r(v-t)} dv$ is a constant independent of t .

Denote by $U(a(t))$ the maximized value of (1),

$$U(a(t)) = \int_t^{\infty} u(\hat{c}_1(s, t)) e^{-\delta(s-t)} ds. \quad (6)$$

It is easy to verify that (3) implies that

$$U'(a(t)) = u'(\hat{c}_1(t)). \quad (7)$$

Expected discounted utility, V , is now given by

$$V = \int_0^{\infty} [u(c_0(t))(1-F(t)) + U(a(t))f(t)]e^{-\delta t} dt \quad (8)$$

where $c_0(t) \geq 0$ is consumption in state 0 at time t and $F(t)$ is the cumulative distribution of $f(t)$. The intertemporal budget constraint in state 0 is

$$\dot{a}(t) = ra(t) + w_0 - c_0(t), \quad t \geq 0. \quad (9)$$

The individual's objective is to choose $c_0(t)$ so as to maximize (8) subject to (9), for a given $a(0) \geq 0$.^{3/} The Euler first-order condition, using (7), is

$$\dot{c}_0(t) = - \frac{(r-\delta) u'(c_0(t))}{u''(c_0(t))} - \left(\frac{u'(\hat{c}_1(t)) - u'(c_0(t))}{u''(c_0(t))} \right) \alpha(t) \quad (10)$$

where $\alpha(t) = f(t)/(1-F(t))$, termed the 'Hazard Rate', is the conditional probability of state 1 occurring at time t . In addition, we have the transversality condition $\lim_{t \rightarrow \infty} a(t)e^{-rt} = 0$. Integrating (9), letting $\tau \rightarrow \infty$, the latter is equivalent to

$$\int_0^{\infty} c_0(t)e^{-rt} dt = a(0) + w_0 \int_0^{\infty} e^{-rt} dt, \quad (11)$$

which equates the present discounted value of consumption to that of wage earnings plus initial assets at time 0. Denote the optimum consumption path by $\hat{c}_0(t)$. Conditions (5), (10) and (11) fully determine $\hat{c}_0(t)$ for all $t \geq 0$.

Uncertainty is spurious in two obvious cases, in both of which consumption is equal to its level under certainty:

(1) $w_0 = w_1$. Observe that in this case conditions (4) and (11) are the same (at $t=0$), and hence setting $\hat{c}_0(t) = \hat{c}_1(t)$ for all t also satisfies conditions (3) and (10).

(2) $F(t) = 0, 0 \leq t < t_0$, and $F(t) = 1, t_0 \leq t < \infty$. This is the case in which the onset of state 1 is certain to occur at time t_0 . The 'Hazard-Rate', $\alpha(t)$, is then identically zero for all $t \geq 0$. Hence, equations (10) and (3) have the same functional form. Setting $\hat{c}_0(t) = \hat{c}_1(t) = \hat{c}(t)$, is seen to solve (3) and (10) subject to the budget constraint

$$\int_0^{\infty} \hat{c}(t) e^{-rt} dt = w_0 \int_0^{t_0} e^{-rt} dt + w_1 \int_{t_0}^{\infty} e^{-rt} dt. \quad (12)$$

We shall henceforth assume that $w_0 \neq w_1$ and that $\alpha(t)$ is not identically zero for all $t \geq 0$.

We can now address the question of discontinuous changes in the optimum consumption path when the state changes. We shall prove that, generically, consumption is never continuous:

Theorem 1: $\hat{c}_0(t) = \hat{c}_1(t)$ for all $t \geq 0$ is impossible.

Proof: By contradiction. Suppose that $\hat{c}_0(t) = \hat{c}_1(t)$ for all $t \geq 0$. Then $\hat{c}_0(t) = \hat{c}_1(0)h(t)$, where, by (5), $\hat{c}_1(0) = \frac{1}{\psi}(ra(0) + w_1)$. However, substituting $\hat{c}_1(0)h(t)$ for $\hat{c}_0(t)$ in (11) yields $\hat{c}_1(0) = \frac{1}{\psi}(ra(0) + w_0)$, which is impossible since $w_0 \neq w_1$.

To further characterize the difference between the uncertainty and the certainty case, we shall focus on the case where $r - \delta = 0$. It is well-known, (10), that under certainty optimum consumption is constant in this case.

3. Optimum Consumption When $r - \delta = 0$

In this case, $h(t) = 1$ for all t and hence $\psi = 1$. By (5), $\hat{c}_1(t) = ra(t) + w_1$, for all $t \geq 0$. Differentiating, using (9),

$$\dot{\hat{c}}_1(t) = r(w_0 - w_1 - \hat{c}_0(t) + \hat{c}_1(t)) . \quad (13)$$

Condition (10) now becomes

$$\dot{\hat{c}}_0(t) = - \left(\frac{u'(\hat{c}_1(t)) - u'(\hat{c}_0(t))}{u''(\hat{c}_0(\tau))} \right) \alpha(t) \quad (14)$$

Equations (13)-(14) are two differential equations that, together with the transversality condition, (11), determine the optimum path $(\hat{c}_0(t), \hat{c}_1(t))$.

We have not yet made any specific assumption concerning the 'Hazard Rate', $\alpha(t)$, as a function of time. We shall assume that the probability for the onset of state 1 does not decrease the longer state 0 prevails. Formally,

$$\alpha(t) > 0 \text{ non-decreases for all } t \geq 0. \quad (15)$$

If for some $t_0 > 0$, $\alpha(t) = 0$ for all $t \geq t_0$, this means that state 1 occurs with certainty before or at t_0 . The optimum path in this case can easily be inferred from the following analysis. Condition (15) is satisfied by the most common distribution functions, such as the exponential (Poisson).

We can now provide a characterization of the optimum consumption path.

Theorem 2: Under (15),

(a) if $w_0 - w_1 > 0$ then $w_0 - w_1 > \hat{c}_0(t) - \hat{c}_1(t) > 0$ and $(\hat{c}_0(t), \hat{c}_1(t))$ strictly increase for all t , with $\lim_{t \rightarrow \infty} \hat{c}_0(t) = \lim_{t \rightarrow \infty} \hat{c}_1(t) = \infty$;

(b) if $w_0 - w_1 < 0$ then $w_0 - w_1 < \hat{c}_0(t) - \hat{c}_1(t) < 0$ and $(\hat{c}_0(t), \hat{c}_1(t))$ strictly decrease for all t , with $\lim_{t \rightarrow \infty} (\hat{c}_0(t) - \hat{c}_1(t)) = w_0 - w_1$.

Proof: Consider the case $w_0 - w_1 > 0$. We first show that $\hat{c}_0(t) - \hat{c}_1(t) > 0$ for all $t \geq 0$. Suppose, to the contrary, that there exists a $t_0 \geq 0$ for which $\hat{c}_0(t_0) - \hat{c}_1(t_0) \leq 0$. Then, by (13) and (14), $\dot{\hat{c}}_0(t) < 0$ and $\dot{\hat{c}}_1(t) > 0$ for all $t > t_0$. Also, $\dot{a}(t) = ra(t) + w_0 - \hat{c}_0(t) = w_0 - w_1 - \hat{c}_0(t) + \hat{c}_1(t) > 0$ for all $t > t_0$. In fact, $\dot{a}(t) > 0$ for all $t \geq 0$. For if not, $w_0 - w_1 - \hat{c}_0(t) + \hat{c}_1(t) < 0$ for some $0 \leq t < t_0$, which, by (13) and (14), implies that $\hat{c}_0(t) - \hat{c}_1(t) > 0$ and $\dot{\hat{c}}_0(t) > 0$, $\dot{\hat{c}}_1(t) < 0$ thereafter. But then there could not exist a time t_0 for which $\hat{c}_0(t_0) - \hat{c}_1(t_0) \leq 0$. Thus, $a(t_0) > 0$. Since $\hat{c}_0(t)$ is strictly decreasing with t , it has a limit. In fact, $\lim_{t \rightarrow \infty} \hat{c}_0(t) = 0$, because, by (14) and (15), at any positive $\hat{c}_0(t)$,

$\dot{\hat{c}}_0(t) < 0$ is bounded away from 0.

Now, given that $\lim_{t \rightarrow \infty} \hat{c}_0(t) = 0$, there exists a $t_1 \geq t_0$, such that $w_0 - \hat{c}_0(t) \geq 0$ for all $t \geq t_1$. Since

$$a(t)e^{-rt} = \int_{t_1}^t (w_0 - \hat{c}_0(s))e^{-rs}ds + a(t_1)e^{-rt_1} \quad t \geq t_1, \quad (16)$$

and $a(t_1) > 0$, $\lim_{t \rightarrow \infty} a(t)e^{-rt} > 0$, violating the transversality condition.

We now prove that $w_0 - w_1 > \hat{c}_0(t) - \hat{c}_1(t)$ for all $t \geq 0$. Suppose, to the contrary, that there exists some $t_0 \geq 0$ for which $\hat{c}_0(t) - \hat{c}_1(t) \geq w_0 - w_1$. By (13) and (14), $\dot{\hat{c}}_0(t) > 0$ and $\dot{\hat{c}}_1(t) < 0$ for all $t > t_0$. Furthermore, from (13), $\hat{c}_1(t)$ decreases with t . Hence, there exists a $T(> t_0)$ such that $\hat{c}_1(T) = 0$. This path, however, cannot be optimal because, by assumption, $u'(0) = \infty$.

Consider next the case $w_0 - w_1 < 0$. We first show that $\hat{c}_0(t) - \hat{c}_1(t) < 0$ for all t . Suppose, to the contrary, that there exists a $t_0 \geq 0$ such that $\hat{c}_0(t_0) - \hat{c}_1(t_0) \geq 0$. It follows from (13) and (14) that $\dot{\hat{c}}_0(t) > 0$ and $\dot{\hat{c}}_1(t) < 0$ for all $t > t_0$. Furthermore, from (13), $\hat{c}_1(t)$ is decreasing with t . Hence, there exists a $T(> t_0)$ such that $\hat{c}_1(T) = 0$, which cannot be optimal.

Now we show that $w_0 - w_1 < \hat{c}_0(t) - \hat{c}_1(t)$ for all $t \geq 0$. Suppose that there exists a $t_0 \geq 0$ such that $w_0 - w_1 \geq \hat{c}_0(t_0) - \hat{c}_1(t_0)$. Then, by (13) and (14), $\dot{\hat{c}}_0(t) < 0$ and $\dot{\hat{c}}_1(t) > 0$ for all $t \geq t_0$. Furthermore,

$\lim_{t \rightarrow \infty} \hat{c}_1(t) = \infty$ and $\lim_{t \rightarrow \infty} \hat{c}_0(t) = 0$. Since $\hat{c}_1(t) = ra(t) + w_1$, there exists a t_1 ($\geq t_0$) such that $a(t_1) > 0$ and $\hat{c}_0(t) < w_0$ for all $t \geq t_1$. As was shown before, this violates the transversality condition $\lim_{t \rightarrow \infty} a(t)e^{-rt} = 0$.

Finally, to prove that $\lim_{t \rightarrow \infty} \hat{c}_0(t) - \hat{c}_1(t) = w_0 - w_1$, observe that if $\lim_{t \rightarrow \infty} (\hat{c}_0(t) - \hat{c}_1(t)) > w_0 - w_1$, then $\dot{\hat{c}}_1(t) < 0$ is bounded away from 0, which implies that there exists a finite T for which $c_1(T) = 0$. Since $u'(0) = \infty$, this cannot be optimal #.

Theorem 2 proves that consumption in state 0, $\hat{c}_0(t)$, and initial consumption in state 1, $\hat{c}_1(t)$, are co-monotone, that there is always a jump in consumption in the direction of the change in earnings upon the onset of state 1 and that the absolute size of the jump is always less than the difference in earnings in the two states. Clearly, these are all empirically testable hypotheses.

The phase diagram for $(\hat{c}_0(t), \hat{c}_1(t))$ is drawn in Figure 1a for the case $w_0 - w_1 > 0$ and in Figure 1b for the case $w_0 - w_1 < 0$. Note that in both cases $\hat{c}_1(0) = ra(0) + w_1$.

Example: Let $u(c) = \log c$. Then equation (14) becomes

$$\frac{\dot{\hat{c}}_0(t)}{\hat{c}_0(t)} = \left(\frac{\hat{c}_0(t)}{\hat{c}_1(t)} - 1 \right) \alpha(t) \quad (17)$$

There is no explicit solution for (13) and (17). However, for the case $w_0 - w_1 > 0$, for large t we use Theorem 2 which implies that $\hat{c}_0(t)/\hat{c}_1(t) \sim 1$.

denoting $x(t) = \hat{c}_0(t) - \hat{c}_1(t)$, (13) and (17) yield

$$\dot{x}(t) = (r+\alpha(t))x(t) - r(w_0 - w_1) \quad (18)$$

Further assuming that $\alpha(t) = \alpha$ constant, the only feasible solution is $\dot{x}(t) = 0$, i.e., $x(t) = \bar{x} = \frac{r}{r+\alpha}(w_0 - w_1)$. Hence the (approximate) steady-state solution is (Figure 2)

$$\begin{aligned} \hat{c}_0(t) &= ra(t) + \frac{r}{r+\alpha} w_0 + \frac{\alpha}{r+\alpha} w_1 \\ \hat{c}_1(t) &= ra(t) + w_1 \end{aligned} \quad (19)$$

In state 0, the relevant variable for consumption is a weighted average of the wages in state 0 and in state 1. As expected, the higher the 'Hazard-Rate', α , the higher is the weight of (the lower wage), w_1 .

3. Full Insurance

Suppose that earnings can be insured on an actuarially fair basis. If the switch from state 0 to 1 occurs at time t , the present discounted value of earnings is given by

$$\frac{1}{r} [(1-e^{-rt})w_0 + e^{-rt}w_1]$$

Hence, the 'human-capital' annuity value is

$$\frac{1}{r} \int_0^{\infty} [(1-e^{-rt})w_0 + e^{-rt}w_1]f(t)dt = \frac{1}{r} [(1-\alpha(r))w_0 + \alpha(r)w_1] \quad (20)$$

where $\alpha(r) = \int_0^{\infty} e^{-rt}f(t)dt$ is the 'Laplace transform' of $f(t)$. Optimum consumption (for the case $\delta = r$) is constant, c^* , and equal to:

$$c^* = ra(0) + (1-\alpha(r))w_0 + \alpha(r)w_1. \quad (21)$$

When $f(t) = \alpha e^{-\alpha t}$. Equation (21) is the same as c_0 in (19),

$$c^* = ra(0) + \frac{r}{r+\alpha} w_0 + \frac{\alpha}{r+\alpha} w_1, \quad (22)$$

point E in Figure 2. Compared to (19), therefore, $\hat{c}_0$ is always larger than c^* , while $\hat{c}_1$ is initially lower and then higher for large t .

Footnotes

^{1/}— The most general results on asymptotic properties of optimum stochastic consumption are in Chamberlin and Wilson (1984).

^{2/}— In fact, our two-state earnings model is based on the continuous time version of the Diamond-Mirrlees (1982) paper on social insurance.

^{3/}— Note that no non-negativity restriction is imposed on $a(t)$.

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Figure 1a : $w_0 - w_1 > 0$

Figure 1b: $w_0 - w_1 < 0$

Figure 2

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