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**The Tractable “Quadratic” Class of  
Growth and Interest Rate Processes**

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# The Tractable “Quadratic” Class of Growth and Interest Rate Processes

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Very Preliminary and Incomplete – Comments Welcome.

## Abstract

We propose a new class of growth and interest rate processes with appealing properties for the paper-and-pencil theorist. It yields simple closed-form solutions for stocks, bonds, and perpetuities, and can accommodate an arbitrary number of factors. Expressions are linear in the state variables, such as the interest rate, the equity premium and the stock’s growth rate. Surprisingly, one can change the volatility of the process, or force the interest rate to be positive, without changing bond prices. The process generalizes to discrete-time settings, and has a number of economic modeling applications. These include (i) macroeconomic situations with changing trend growth rates, (ii) asset pricing with time-varying risk premia or time-varying dividend growth rates, and (iii) yield curve analysis that allows flexibility and transparency. (JEL: G12, G13)

Keywords: Long term risk, Growth rate risk, Perpetuity, Modified Gordon growth model, Yield curve, Interest rate processes, Stochastic Discount Factor, Quadratic processes.

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# 1 Introduction

This paper proposes a new growth and interest rate process that has a number of attractive properties. Most notably, it is simple and tractable, allowing closed-form solutions for the prices of stocks, bonds and perpetuities. It is also flexible, can be applied to an arbitrary number of factors with a rich correlation structure, and generalizes well to discrete time. All of these features make it particularly accessible to the paper-and-pencil theorist.

Quadratic processes have the drift of an Ornstein-Uhlenbeck processes, with a squared term added. This squared term does not change the drift much, so that quadratic processes are in effect fairly close to an Ornstein-Uhlenbeck process. However, adding the quadratic terms greatly simplifies the expressions of expected values of stocks and bonds. The expressions are linear in the factors. We obtain closed forms for the price of stocks with a time-varying growth rate, and the price of perpetuities. Such closed forms for perpetuities is not available with the available processes, such as those of Ornstein-Uhlenbeck / Vasicek (1977), Cox, Ingersoll, Ross (1985), the Heath, Jarrow, Morton (1992), or models in the affine class (Duffie and Kan 1996).

Quadratic processes are meant to be a practical tool for several topics in economics. The quadratic process is likely to be useful in: (i) macroeconomics, for models in which GDP growth rate is time-varying; (ii) asset pricing, for models where the equity premium is time-varying, and models where the trend growth rate of dividends is varying; and (iii) fixed-income analysis. Also, the process may be useful for thinking about other situations where the uncertainty in the discount factor matters, e.g., in environmental economics.

Several literatures motivate the need for a tool such as the quadratic process. Many recent studies investigate the importance of long-term risk for asset pricing and macroeconomics, e.g., Bansal and Yaron (2004), Barro (2006), Croce, Lettau and Ludvigson (2006), Gabaix and Laibson (2002), Hansen Heaton and Li (2005), Hansen and Scheinkman (2005), Julliard and Parker (2004), Parker (2001). The quadratic process offers a way to model long-term risk, while keeping a closed form for stock prices.

In addition, there is debate about the existence and mechanism of the time-varying expected stock market returns, e.g., Campbell and Shiller (1988), Cochrane (2006), and many others. Because of the lack of closed forms, the literature relies on simulations and approximations. The quadratic process offers closed forms for stocks with time-varying equity premium, which is useful for thinking about those issues.

Finally, we contribute to the vast literature on interest rate processes, by presenting a new, flexible process. It is as flexible as the affine class of Duffie and Kan (1996), which includes the Vasicek (1978) and the Cox, Ingersoll, Ross (1985) process as special cases. Section 4.3 develops the link between the quadratic class and the affine class.

This paper stipulates a process for finding desirable properties for the pricing kernel, and in this follows a productive literature represented by, e.g., Campbell and Cochrane (1999), Cox, Ingersoll, Ross (1985), and particularly Menzly, Santos and Veronesi (2004) and Pastor and Veronesi (2005).<sup>1</sup> The last two papers are the closest to this paper, as they contain several useful closed forms.

The core insight, that the quadratic process (72) yields particularly tractable formulas, appears to be new to the literature.<sup>2</sup>

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<sup>1</sup> Those papers do provide an economic foundation (via external habit formation) for the postulated processes.

<sup>2</sup> The Fisher-Wright process (e.g., Karlin and Taylor 1982) does contain a quadratic term, but it has not been applied

Section 2 presents the basic one-factor process. Section 3 applies the one-factor process to stocks. Section 4 presents the multifactor version of the quadratic process, which is useful for talking about processes with fast and slowly mean-reverting components, or interest rate processes with short and long rates. Section 5, which is more technical, studies the range of admissible initial conditions. Section 6 presents the discrete-time version of the process. Section 7 concludes.

## 2 The one-factor quadratic process

### 2.1 Definition and basic properties

We fix a probability space  $(\Omega, \mathcal{F}, P)$  and an information filtration  $\mathcal{F}_t$  satisfying the usual technical conditions (see, for example, Karatzas and Shreve (1991)). We start from a simple Ornstein-Uhlenbeck process, but add a quadratic term:

$$dr_t = -(\beta - \alpha)(r_t - \alpha)dt + (r_t - \alpha)^2 dt + dM_t,$$

with  $\alpha < \beta$ , for  $r_t \leq \bar{r} \in (\alpha, \beta)$  where  $r_t$  is the short-term interest rate.

$-(\beta - \alpha)(r_t - \alpha)dt$  is the standard term which leads to mean reversion to the long run value  $\alpha$ , with a speed  $\beta - \alpha$ . The second term,  $(r_t - \alpha)^2$ , represents a small departure from a classical Ornstein-Uhlenbeck process. When  $r_t$  is not too far from  $\alpha$ , the drift is little changed from the standard  $-(\beta - \alpha)(r_t - \alpha)dt$ . However, it turns out that the correction  $(r_t - \alpha)^2 dt$  substantially improves the tractability of the process.

We assume that the martingale  $\{M_t, t \geq 0\}$  has properties that ensure the interest rate remains below some upper bound  $\bar{r} \in (\alpha, \beta)$ , and thus does not explode. One example is  $dM_t = \sigma(r_t)dz_t$  where  $z_t$  is a Brownian process and  $\sigma(r) \sim k(\bar{r} - r)^\kappa$ , for  $r$  in a left neighborhood of  $\bar{r}$ ,  $\kappa > 1/2$  and  $k > 0$ . Given the drift is negative around  $\bar{r}$ , that will ensure that  $\bar{r}$  is a natural boundary, and  $\{\forall t, r_t \leq \bar{r}\}$  almost surely.<sup>4</sup> We also have the option to set a lower bound  $\underline{r}$  to the interest rate, by adding an additional restriction on  $dM_t$ . For instance, if one wishes to ensure  $r_t \geq \underline{r}$  for some  $\underline{r} < \alpha$ , we could have  $dM_t = \sigma(r_t, t)dz_t$ , with  $\sigma_t \sim k'(r - \underline{r})^{\kappa'}$ ,  $\kappa' > 1/2$  for  $r$  in a right neighborhood of  $\underline{r}$ , and  $k' > 0$ .

The above process can be rewritten:

$$dr_t = (r_t - \alpha)(r_t - \beta)dt + dM_t. \tag{1}$$

Appendix A contains a formal statement and justification of these assumptions, and further purely technical assumptions. Given these assumptions, and an initial value  $r_0$ , the stochastic differential equation (1) has a unique strong solution for  $r_t$ , defined for all times  $t \geq 0$ .

The following Theorem gives a basic result. Its proof is in Appendix B.

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to the pricing bonds or stocks. Also, it is more special than the quadratic class, because it imposes a specific functional form on the variance. Driessen, Maenhout and Vilkov (2005) apply the Fisher-Wright process to options.

<sup>3</sup>In many applications, it is plausible that  $|r_t - \alpha| < 5\%$ , and thus the quadratic term  $(r_t - \alpha)^2$  is less than 0.25% per year.

<sup>4</sup>See Karlin and Taylor (1981, Lemma 15.6.3, Table 15.6.2), and the appendix of Ait-Sahalia (1996) for a concise treatment.

**Theorem 1** *The price at time  $t$  of a zero-coupon bond of maturity  $T$ ,  $Z_t(T) = E_t \left[ \exp \left( - \int_t^{t+T} r_s ds \right) \right]$ , is:*

$$Z_t(T) = \frac{\beta - r_t}{\beta - \alpha} e^{-\alpha T} + \frac{r_t - \alpha}{\beta - \alpha} e^{-\beta T}. \quad (2)$$

Expression (2) is particularly simple, as it is linear in the current rate  $r_t$ . Remarkably, the bond price does not depend on the noise  $dM_t$  and thus the volatility of the process.<sup>5</sup> The intuition is as follows. Consider the case where the process is deterministic, i.e.,  $dM_t \equiv 0$  for all  $t$ . Theorem 1 still applies, and we rewrite Eq. 2 as:  $Z(T, r) = A + Br$ , where  $A$  and  $B$  are constants that depend on  $T$  but not on  $r$ . Suppose that at time  $t$  the interest rate is  $r_t$ , and apply a shock to the interest rate at time  $t + dt$  so that  $r_{t+dt} = r_t + u_t$ , where  $u_t$  is a mean-zero random variable. After  $t + dt$ , the process remains deterministic, and evolves according to (1) with  $dM_t = 0$ . Given an initial value  $r_t + u_t$  of the interest rate, the bond price at  $t + dt$  is  $Z(T, r_t + u_t) = Z(T, r_t) + Bu_t$ , because  $Z$  is affine in  $r$ . Hence the value at  $t$  is  $E[Z(T, r_t + u_t)] = Z(T, r_t)$ . The expected value of the impact of  $u_t$  is 0, because the bond price is linear in  $u_t$ . A zero mean addition of volatility does not affect the bond price. The crux is that, in the deterministic case ( $\forall t, M_t = 0$ ), the bond price is an affine function of the interest rate:  $Z(T, r_t) = A(T) + B(T)r_t$ . By contrast, intuition based on the expression  $E_t \left[ \exp \left( - \int_t^{t+T} r_s ds \right) \right]$  would suggest a convex function.

The independence of bond prices from volatility greatly simplifies the analysis. In particular,  $dM_t$  could have jumps, which model a decision by the central bank. One does not need to specify the volatility process to get the prices of bonds: only the drift part is necessary. This leaves a high margin of flexibility to calibrate volatility, for instance on interest rate derivatives, a topic we do not pursue here.

Equally surprisingly, we can impose a lower bound without affecting bond prices, provided the lower bound is less than  $\underline{r} < \alpha$ . The intuition is the following. We can make  $\underline{r}$  a lower bound for the process simply by setting the variance to be 0 below  $\underline{r}$ . Since variance does not affect bond prices, the new lower bound does not affect bond prices.

Existing models, such as Vasicek and Cox-Ingersoll-Ross, are able to generate closed-form solutions for zero-coupon bonds. However, the quadratic process introduced by this paper can also generate closed-form solutions for perpetuities.

**Proposition 1** *The price of a perpetuity is*

$$V = E_t \left[ \int_0^\infty e^{-\int_t^{t+T} r_s ds} dT \right] = \frac{\alpha + \beta - r_t}{\alpha\beta} = \frac{1}{\alpha} + \frac{\alpha - r_t}{\alpha\beta}. \quad (3)$$

**Proof.** This follows directly from Eq. 2 and

$$V = \int_0^\infty Z_t(T) dT = \frac{\beta - r_t}{\beta - \alpha} \frac{1}{\alpha} + \frac{r_t - \alpha}{\beta - \alpha} \frac{1}{\beta} = \frac{\alpha + \beta - r_t}{\alpha\beta}.$$

■

We now analyze the dynamics of the process in some further detail. If we start from  $r_t = \alpha$ , then  $Z_t(T) = e^{-\alpha T}$ , which is the value the bond would have if  $r_t$  remained pinned at  $\alpha$ . However,  $r_t$  is

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<sup>5</sup>Hence the process exhibits “unspanned volatility” in the sense of Collin-Dufresne and Goldstein (2002).



stochastic. With  $\alpha < \beta$ , Eq.2 shows that  $Z_t(T) \sim \frac{\beta-r_t}{\beta-\alpha} e^{-\alpha T}$  for  $T \rightarrow \infty$ . Hence  $\alpha$  is the long run interest rate,  $\alpha = -\lim_{T \rightarrow \infty} Z'(T)/Z(T)$ . Given the drift process (1),  $E[dr_t] > 0$  if  $r$  is below  $\alpha$ , and  $E[dr_t] < 0$  if  $r$  is above  $\alpha$ . Hence,  $r$  is mean-reverting towards  $\alpha$ .

$\beta$  has two roles. First, it represents an upper bound to the interest rate:  $r_t \leq \bar{r} < \beta$ . Second,  $\beta - \alpha$  is the speed of mean-reversion of the process. When  $\beta$  is high, the process mean-reverts faster.

The process is defined only for  $r_t \leq \bar{r} < \beta$ . If  $r_t > \bar{r}$ , the process may explode in finite time. Indeed, when the process is deterministic ( $M = 0$ ) and  $r_t > \bar{r}$ , then the process explodes in finite time, i.e., there is  $t_1 > t$  such that  $\lim_{s \rightarrow t_1} r_s = \infty$ . The assumption  $r_t < \beta$  is needed in the proof to ensure that the process is defined at times in  $[t, t+T]$ . This is guaranteed if  $r_t < \beta$ , under the conditions of Appendix A.

Finally, we require the leading terms to be  $r_t^2$  in the definition (1) of the quadratic process, not  $kr_t^2$  for a  $k \neq 1$ .

The empirical relevance of the quadratic term is unclear. It could be, for instance, that the quadratic term exists under the risk-neutral probability, though not in the physical probability. Also, of course all processes are an approximation of reality, and in many cases, for instance for options, the major counterfactual assumption is to assume Gaussian innovations rather than fat-tailed ones (see Gabaix et al. 2003, 2006). Quadratic processes allow for jumps and fat-tailed innovations, as the shape of the innovations (the  $dM_t$  term) does not affect the prices.

The rest of this section considers more specialized topics in the one-factor quadratic process.

## 2.2 Sum of two quadratic processes

This section studies the sum of two quadratic processes. This application is useful, for instance, when both the equity premium and the dividend growth rate are time-varying.

**Proposition 2** (*Sum of two independent quadratic processes*) *Consider two one-factor quadratic processes:*

$$\begin{aligned} dr_t &= -\phi(r_t - r_*)dt + (r_t - r_*)^2 dt + dn_t \\ dR_t &= -\Phi(R_t - R_*)dt + (R_t - R_*)^2 dt + dN_t \end{aligned}$$

where  $n$  and  $N$  are martingales with uncorrelated innovations,  $d\langle n, N \rangle_t = 0$ , and such that  $(n, N)$  is a martingale. Then,

$$E_t \left[ \exp \left( - \int_t^{t+T} (r_s + R_s) ds \right) \right] = E_t \left[ \exp \left( - \int_t^{t+T} r_s ds \right) \right] E_t \left[ \exp \left( - \int_t^{t+T} R_s ds \right) \right] \quad (4)$$

and the price of a perpetuity with interest rate  $r_t + R_t$ ,  $P_t = E_t \left[ \int_0^\infty \exp \left( - \int_t^{t+T} (r_s + R_s) ds \right) dT \right]$ , is, calling  $S_* = r_* + R_*$ ,

$$P_t = \frac{1}{S_*} \left[ 1 - \frac{r_t - r_*}{S_* + \phi} - \frac{R_t - R_*}{S_* + \Phi} + \frac{(r_t - r_*)(R_t - R_*)(2S_* + \phi + \Phi)}{(S_* + \phi)(S_* + \Phi)(S_* + \phi + \Phi)} \right]. \quad (5)$$

**Proof.** The first part is immediate using Ito's lemma. The second part comes from the calculation of:

$$P_t = \int_0^\infty e^{-(r_* + R_*)T} \left( 1 + (r_t - r_*) \frac{e^{-\phi T} - 1}{\phi} \right) \left( 1 + (R_t - R_*) \frac{e^{-\Phi T} - 1}{\Phi} \right) dT.$$

■

### 3 Application to the pricing of stocks with a stochastic growth rate

This section applies the quadratic process to model the dividend growth rate of stocks. We start with a stochastic trend growth rate of dividends, continue with a stochastic equity premium, and finally analyze more complex examples.

#### 3.1 Time-varying dividends

The stock has a dividend  $D_t$ , which grows at a stochastic rate  $g_t$ :

$$\frac{dD_t}{D_t} = g_t dt + dN_t \quad (6)$$

$$dg_t = \phi (g_* - g_t) dt - (g_t - g_*)^2 dt + dM_t \quad (7)$$

$$\langle dM_t, dN_t \rangle / dt = cg_t + b \quad (8)$$

$$g_t > \underline{g} \geq \beta \quad (9)$$

where  $\beta < \alpha$ ,  $N_t$  is a martingale, and  $M_t$  is a martingale that ensures  $g_t \geq \underline{g}$  for some  $\underline{g} \geq \beta$ . When  $b = c = 0$ ,  $g_*$  is the long-run growth rate of the stock.

Innovations to dividends,  $dN_t$ , can be correlated with innovations to the growth rate,  $dM_t$ . The strength of that correlation can vary with the level of the growth rate  $g_t$ .

The discount factor is constant at  $R$ . The stock price is:

$$V_t = E \left[ \int_t^\infty e^{-R(t-s)} D_s ds \right].$$

We start with the case  $b = c = 0$ , i.e., of no correlation between dividend level and growth rate.

**Proposition 3** (*Modified Gordon growth model, with time-varying dividend growth rate*) *When there is no correlation between innovation the level of dividend and innovations to their growth rate ( $\langle dM_t, dN_t \rangle = 0$ ), the price/dividend ratio of the stock is:*

$$V_t/D_t = \frac{1}{R - g_*} \left( 1 + \frac{g_t - g_*}{R - g_* + \phi} \right) \quad (10)$$

where  $g_*$  is the long term growth of dividends, and  $g_t - g_*$  is the deviation of the current growth rate ( $g_t$ ) from the trend ( $g_*$ ).

The first term,  $R - g_*$ , is the P/D ratio from the Gordon growth model, since  $g_*$  is the long term growth rate of the dividend. The second term,  $g_t - g_*$ , is the deviation of the current growth rate ( $g_t$ ) from the trend ( $g_*$ ). It increases the P/D ratio, by an duration factor  $1/(R - g_* + \phi)$  that is decreasing in the discount rate  $R$ , and the speed of mean reversion  $\phi$ .

We now consider the case where  $\langle dM_t, dN_t \rangle / dt = cg_t + b$ .



**Proposition 4** (*Modified Gordon growth model, with correlations between innovations to the level and growth rate of dividends*) The price/dividend ratio of the stock is:

$$V_t/D_t = \frac{R - g_* + \phi + c + (g_t - g_*)}{(R - g_*)(R - g_* + \phi) - cR - b}. \quad (11)$$

The stock price is increasing in the correlation between the level of dividend and its trend growth rate.

### 3.2 Time-varying equity premium

Consider the stochastic discount factor:

$$Q_t = \exp \left( -rt - \int_0^t \frac{\lambda_s^2}{2} ds - \int_0^t \lambda_s dB_s \right). \quad (12)$$

The price at time  $t$  of any asset yielding a dividend  $D_s ds$  at time  $s$  is:  $V_t = E_t [\int_t^\infty Q_s D_s ds] / Q_t$ . We seek to price a stock with dividend:

$$D_t = \exp \left( gt - \int_0^t \frac{\sigma_s^2}{2} ds + \int_0^t \sigma_s dB_s \right) D_0. \quad (13)$$

Assume that the risk premium,  $\pi_t = \lambda_t \sigma_t$ , follows a one-factor quadratic process:

$$d\pi_t = -\phi(\pi_t - \pi_*) + (\pi_t - \pi_*)^2 dt + dN_t$$

where  $N_t$  is a martingale that ensures that the process is well-defined ( $\pi_t \leq \bar{\pi} < \pi_* + \phi$ ). Assume that processes  $N_t$  and  $B_t$  are independent. Then, the price of a stock is, calling  $\mathcal{F}_t^N$  the filtration associated with  $N_t$ ,

$$\begin{aligned} V_0 &= E_0 \left[ \int_0^\infty Q_t D_t dt \right] / Q_0 \\ &= E_0 \left[ \int_0^\infty \exp \left( (g - r)t - \int_0^t \frac{\lambda_s^2 + \sigma_s^2}{2} ds + \int_0^t (\sigma_s - \lambda_s) dB_s \right) dt \right] \\ &= E_0 \left[ E \left[ \int_0^\infty \exp \left( (g - r)t - \int_0^t \frac{\lambda_s^2 + \sigma_s^2}{2} ds + \int_0^t \frac{(\sigma_s - \lambda_s)^2}{2} ds \right) dt \mid \mathcal{F}_t^N \right] \right] \\ &= E_0 \left[ \int_0^\infty \exp \left( (g - r)t - \int_0^t \lambda_s \sigma_s ds \right) dt \right] = E_0 \left[ \int_0^\infty \exp((g - r - \pi_t)t) dt \right] \end{aligned}$$

so that, applying Proposition 1 to  $r_t = -g + r + \pi_t$ , we get:

$$V_t/D_t = \frac{1}{r + \pi_* - g} \left( 1 - \frac{\pi_t - \pi_*}{r + \pi_* - g + \phi} \right). \quad (14)$$

This is a Gordon growth formula, with time-varying risk-premium  $\pi_t$ . When the risk premium  $\pi_t$  is equal to its central value,  $\pi_*$ , we get the traditional formula,  $V_t/D_t = 1/(r + \pi_* - g)$ . When  $\pi_t$

is different from  $\pi_*$ , we get an adjustment, equal to  $\pi_t - \pi_*$  times the duration of that term, which is the effective rate of discounting,  $r + \pi_* - g$ , plus the speed  $\phi$  of mean-reversion of the risk-premium.

### 3.3 Time-varying equity premium and dividend growth rate

We now merge the two previous examples, to incorporate both a time-varying equity premium and a time-varying dividend growth rate. The stochastic discount factor and dividend are given as follows:

$$Q_t = \exp \left( -rt - \int_0^t \frac{\lambda_s^2}{2} ds - \int_0^t \lambda_s dB_s \right), \quad (15)$$

$$D_t = \exp \left( \int_0^t g_s ds - \frac{\sigma_s^2}{2} dt + \sigma_s dB_s \right) D_0,$$

where  $g_t$  follows the quadratic process

$$dg_t = -\Phi (g_t - g_*) dt - (g_t - g_*)^2 dt + dN_t.$$

The risk premium,  $\pi_t = \lambda_t \sigma_t$ , follows a one-factor quadratic process:

$$d\pi_t = -\phi (\pi_t - \pi_*) dt + (\pi_t - \pi_*)^2 dt + dn_t$$

where  $n_t, N_t$  are martingales satisfying the familiar quadratic conditions. Assume that the processes  $n_t, N_t$  and  $B_t$  are independent. Then, the price of a stock,  $V_t = E_0 [\int_0^\infty Q_t D_t dt] / Q_0$ , is, by the reasoning of the previous section,

$$V_t = E_t \left[ \int_{s=t}^\infty \exp \left( - \int_{u=t}^s (r + \pi_u - g_u) du \right) ds \right].$$

Define  $S_* = r + \pi_* - g_*$ . Formula (5) gives:<sup>6</sup>

$$V_t/D_t = \frac{1}{S_*} \left[ 1 - \frac{\pi_t - \pi_*}{S_* + \phi} + \frac{g_t - g_*}{S_* + \Phi} - \frac{(\pi_t - \pi_*)(g_t - g_*)(2S_* + \phi + \Phi)}{(S_* + \phi)(S_* + \Phi)(S_* + \phi + \Phi)} \right]. \quad (16)$$

The central value is again the Gordon formula,  $V_t/D_t = 1/(1 + \pi_* - g_*)$ . It is modified by the current level of the equity premium, and the growth rate of the stock. A stock with a currently high growth rate  $g_t$  exhibits a higher price-dividend ratio, and this is amplified when the equity premium is low, as shown by the term  $(\pi_t - \pi_*)(g_t - g_*)$ .

In the above process,  $g_t$  mean-reverts, so the price/dividend ratio mean-reverts in (11). There is much recent literature on the predictability of returns, in particular the predictability of the P/D ratio (Campbell and Shiller 1988; Cochrane 2006). The above setup may serve as a useful benchmark to evaluate the debate.

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<sup>6</sup> Menzly, Santos and Veronesi (2004, Eq. 20) have a similar expression, even though the processes are quite different in their details.

### 3.4 Correlation between the dividend and the discount factor

We next study an example where the innovations to the dividend are correlated with the discount factor. We suppose:

$$\frac{dD_t}{D_t} = gdt + dN_t$$

with  $\gamma$  is a fixed number, and  $N_t$  a martingale. We assume a quadratic process:  $dr_t = (r_t - \lambda)(r_t - \mu)dt + dM_t$ , and

$$\left\langle \frac{dD_t}{D_t}, dr_t \right\rangle / dt = ar_t + b$$

for two constants  $a$  and  $b$ . Dividend growth may be correlated with the interest rate.

**Proposition 5** *The price at time  $t$  of a claim at date  $s \geq t$  is:*

$$E \left[ e^{-\int_t^s r_u du} D_s \right] = \left( e^{-\lambda'(s-t)} \frac{\mu' - r_t}{\mu' - \lambda'} + e^{-\mu'(s-t)} \frac{r_t - \lambda'}{\mu' - \lambda'} \right) D_t \quad (17)$$

where  $\lambda'$  and  $\mu'$  satisfy

$$\lambda' + \mu' = \lambda + \mu - g - a \quad (18)$$

$$\lambda' \mu' = (\lambda - g)(\mu - g) + b \quad (19)$$

i.e., are the solution of

$$\xi^2 - [\lambda + \mu - g - a]\xi + (\lambda - g)(\mu - g) + b = 0.$$

**Proposition 6** *The price at time  $t$  of a dividend claim yield  $D_s$  at every  $s \geq t$  is:*

$$E \left[ \int_{s=t}^{\infty} e^{-\int_t^s r_u du} D_s ds \right] = \frac{\lambda + \mu - g - a - r_t}{(\lambda - g)(\mu - g) + b} D_t.$$

**Proof.** By direct integration of (17). ■

### 3.5 A quantitative illustration

When  $g_t$  is at its long run value,  $g_*$ , Eq. 10 gives the simple price dividend ratio  $1/(R - g_*)$ . However the P/D ratio also varies with the growth rate. To assess the quantitative importance of this effect, consider the following calibration using annualized values:  $R = 8\%$ , which reflects a risk-free rate of 2% and an equity premium of 6%;  $g_* = 3\%$ , the long-run growth rate, which is proportional to typical values of long-run GDP growth. This yields a central P/D ratio of:  $1/(R - g_*) = 20$ . We also take  $\phi = 5\%$ , which gives a half-life of mean reversion of  $\ln 2/\phi = 14$  years. This gives

$$V_t/D_t = \frac{1}{R - g_*} + \frac{g_t - g_*}{(R - g_*)(R - g_* + \phi)} = 20 + 200(g_t - \alpha).$$

With a volatility  $\sigma_{g_t} = 2\%$ , we get  $\sigma(V/D) = 200\sigma_g = 4$ . The P/D ratio has an annual standard deviation of 4, and the volatility of returns is, at  $g_t = g_*$

$$\sigma_{\ln(V_t/D_t)} = \frac{\sigma_g}{R - g_* + \phi} = 10\sigma_g = 20\%.$$

### 3.6 Application: Long term growth risk

Consider  $dC_t/C_t = g_t$ , with  $g_t$  stochastic. Thus  $C_{t+T}/C_t = \exp\left(\int_t^{t+T} g_s ds\right)$ . The price of a stock in such a model is:

$$V_t/C_t = \int_0^\infty e^{-\rho T} \left(\frac{C_{t+T}}{C_t}\right)^{1-\gamma} dT = \int_0^\infty \exp\left(-\rho s - (\gamma - 1) \int_t^{t+T} g_s ds\right) dT.$$

If we define:

$$r_t = \rho + (\gamma - 1) g_t,$$

we obtain  $V_t/C_t = \int_0^\infty e^{-\int_t^{t+T} r_s ds} ds$ , the perpetuity price.

We study the process in two representative cases. For a quadratic process,  $dr_t = (r_t - \lambda)(r_t - \mu)dt + \sigma_t dz_t$ , with  $\lambda = \rho + \gamma g^* < \mu$  we have:

$$W_t = \frac{1}{\lambda} \left(1 + \frac{\lambda - r_t}{\mu}\right).$$

Therefore, the volatility of the P/E ratio is, at the central value  $r_t = \lambda$ :

$$\sigma_{dW_t/W_t} = \frac{\sigma_r|_{r=\lambda}}{\mu}.$$

We calibrate using  $\gamma = 3$ ,  $g = 2\%$ ,  $\rho = 1\%$ , which yields  $\lambda = 7\%$ . We choose a central P/E ratio of  $1/\lambda$ , which is 14.3. Around  $\lambda$ , the process behaves like a Ornstein-Uhlenbeck process,  $dr = -(\mu - \lambda)(r - \lambda)dt + \sigma_\lambda dz_t$ . We select  $\mu = 15\%$  and a speed of mean-reversion  $\mu - \lambda = 8\%$ , i.e., a time scale of  $1/(\mu - \lambda) = 12.5$  years.

We pick  $\sigma_r = 0.5\%$ . This yields a standard deviation of growth rate of approximately:  $\sigma_r (2(\mu - \lambda))^{-1/2} = \sigma_r (2 \cdot 8\%)^{-1/2} = \sigma_r / 0.4$ . Taking 1.5% as the standard deviation of the growth rate, we have:  $\sigma_r = 0.4 \cdot 1.5\% = 0.6\%$ . With a growth rate volatility of  $\sigma_g = 0.5\%$ , we obtain interest rate volatility of  $\sigma_r = \gamma\sigma_g = 1.5\%$ . Choosing  $\mu = 15\%$ , this yields a stock price volatility  $\sigma_W = \frac{1.5\%}{15\%} = 10\%$ .

## 4 The multifactor quadratic process

Several factors are needed to capture the dynamics of bonds (Litterman and Scheinkman 1991) and stocks (Fama and French 1996). Accordingly, we study the multifactor version of the quadratic process.

## 4.1 Definition and basic properties

A quadratic process  $X_t \in \mathbb{R}^n$  has the form

$$dX_t = (b - AX_t)dt + (r_t - r_*)(X_t - X_*)dt + dM_t \quad (20)$$

$$r_t = \beta'(X_t - X_*) + r_* \quad (21)$$

with:  $A$  is a  $n \times n$  matrix with positive eigenvalues  $(\delta_i)_{i=1 \dots n}$ .  $b, \beta, \in \mathbb{R}^n$ ,  $X_* = A^{-1}b \in \mathbb{R}^n$ ,  $r_* \in \mathbb{R}$ .  $M_t \in \mathbb{R}^n$  is a martingale, so  $E[dM_t] = 0$ , but its component  $dM_{it}$ ,  $dM_{jt}$  can be correlated. As in the one-factor process, the volatility of  $dM_t$  must go to zero in some limit regions for the process to be well-defined. We defer this more technical issue until later.

We calculate the price of a zero coupon,  $Z_t(T) = E_t \left[ \exp \left( - \int_t^{t+T} r_s ds \right) \right]$ .

**Theorem 2** (*Price of a bond for a  $n$ -factor process*) The price at time  $t$  of a zero coupon bond of maturity  $T$ , given the state  $X_t$ , is:

$$Z_t(T) = e^{-r_* T} + e^{-r_* T} \beta' \frac{e^{-AT} - 1}{A} (X_t - X_*). \quad (22)$$

We use a function  $f(A)$  for a matrix. The notation is standard. If  $f$  is analytic and admits the expansion  $f(x) = \sum f_n x^n$ , then  $f(A) = \sum f_n A^n$ . If  $A$  is in diagonal form,  $A = \Omega \text{Diag}(\delta_1, \dots, \delta_n) \Omega^{-1}$ , then  $f(A) = \Omega \text{Diag}(f(\delta_1), \dots, f(\delta_n)) \Omega^{-1}$ .

**Proposition 7** (*Price of a perpetuity for a  $n$ -factor process*) The price of a perpetuity,  $P_t = \int_0^\infty Z_t(T) dT$ , is, with  $I_n$ , the identity matrix of dimension  $n$ :

$$P_t = \frac{1}{r_*} - \frac{1}{r_*} \beta' (A + r_* I_n)^{-1} (X_t - X_*). \quad (23)$$

## 4.2 Some examples

**Example 1.** For  $n = 1$ , we obtain the expressions of the one-factor quadratic process.

**Example 2.** Dividend growth rate as a sum of mean-reverting processes (e.g., a slow and a fast process)

$$\begin{aligned} g_t &= g_* + \sum_{i=1}^n X_{it} \\ dX_{it} &= -\phi_i X_{it} dt - (g_t - g_*) X_{it} dt + dM_{it} \end{aligned}$$

The growth rate  $g_t$  is a steady state value  $g_*$ , plus the sum of mean-reverting processes  $X_{it}$ . Each  $X_{it}$  mean-reverts with speed  $\lambda_i$ , and also has the quadratic perturbation  $(g_t - g_*) X_{it} dt$ .

We apply the Theorem 2, with  $\beta' = (1, \dots, 1)$ ,  $A = \text{Diag}(\phi_1, \dots, \phi_n)$ . The expected value of a dividend  $T$  periods ahead is:

$$E_t[D_{t+T}] / D_t = e^{g_* T} + e^{g_* T} \sum_{i=1}^n \frac{1 - e^{-\phi_i T}}{\phi_i} X_{it}$$



and, for the price-dividend ratio of a stock:

$$P_t/D_t = \frac{1}{R - g_*} \left( 1 + \sum_{i=1}^n \frac{X_{it}}{R - g_* + \phi_i} \right). \quad (24)$$

Each component  $X_{it}$  perturbs the baseline Gordon expression  $1/(R - g_*)$ . The perturbation is  $X_{it}$ , times the duration of  $X_i$ , discounted at rate  $R - g_*$ , which is the term  $1/(R - g_* + \phi_i)$ .

**Example 3. A short rate and a long rate**

$$\begin{aligned} dr_t &= \left[ \phi (L_t - r_t) + (r_t - r_*)^2 \right] dt + dM_t^1 \\ dL_t &= [\psi (r_* - L_t) + (r_t - r_*) (L_t - r_*)] dt + dM_t^2 \end{aligned}$$

$r_t$  is the short term rate.  $L_t$  is the long term rate. Because of the first equation,  $r_t$  is drawn to  $\hat{L}_t$ . The second equation expresses that  $L_t$  goes to a steady-state value  $r_*$ . The process is enriched by the terms  $(r_t - r_*)^2$  and  $(r_t - r_*) (L_t - r_*)$ , which make the process quadratic, hence making bond prices tractable. In practice, it is reasonable to expect  $|r_t - r_*|$  and  $|L_t - r_*|$  both less than 5%, so that the magnitude of the quadratic terms is less than  $5\%^2 = 0.25\%$  per year.

We apply the Theorem 2, with  $X'_t = \begin{pmatrix} r_t \\ L_t \end{pmatrix}$ ,  $\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $A = \begin{pmatrix} \phi & -\phi \\ 0 & \psi \end{pmatrix}$ ,  $b = \begin{pmatrix} 0 \\ L_* \end{pmatrix}$ . We get, with  $r'_t = r_t - r_*$ ,  $L'_t = L_t - r_*$ ,

$$\begin{aligned} Z_t(T) &= e^{-r_* T} \left( 1 - \frac{r'_t}{\phi} - \frac{L'_t}{\psi} \right) + e^{-(r_* + \psi)T} \cdot \frac{\phi}{\psi} \frac{L'_t}{\phi - \psi} + e^{-(r_* + \phi)T} \left( \frac{r'_t}{\phi} - \frac{L'_t}{\phi - \psi} \right) \\ P_t &= \frac{1}{r_*} - \frac{r'_t}{r_* (r_* + \phi)} - \frac{\phi L'_t}{r_* (r_* + \phi) (r_* + \psi)}. \end{aligned}$$

**Example 4.  $r_t$  having a time-varying trend** In the post-Volcker era, interest rates tended to have predictable trends of increase or decrease, which may be captured by the following trend growth rate  $s_t$  of the interest rate:

$$\begin{aligned} dr_t &= s_t dt + (r_t - r_*)^2 dt + dM_{1t} \\ ds_t &= [-\lambda \mu (r_t - r_*) - (\lambda + \mu) s_t] dt + (r_t - r_*) s_t dt + dM_{2t} \end{aligned}$$

with  $\lambda, \mu \geq 0$ , and  $\lambda + \mu > 0$ . Economically,  $s_t$  is the predicted trend in interest rates, as per the first expression.  $s_t$  mean-reverts for two reasons: first, because of the  $-\lambda \mu (r_t - r_*)$  term ( $s_t$  becomes negative if interest rates are too high); second, because of the  $-(\lambda + \mu) s_t$  term.

We apply Theorem 2, with  $X'_t = (r_t, s_t)$ ,  $\beta' = (1, 0)$ ,  $A = \begin{pmatrix} 0 & -1 \\ \lambda\mu & \lambda + \mu \end{pmatrix}$ . We get:

$$\begin{aligned} Z(T) &= e^{-r_*T} \left[ 1 + \left( e^{-\lambda T} - 1 \right) \frac{s_t + \mu(r_t - r_*)}{\lambda(\mu - \lambda)} - \left( e^{-\mu T} - 1 \right) \frac{s_t + \lambda(r_t - r_*)}{\mu(\mu - \lambda)} \right]. \\ P_t &= \frac{1}{r_*} - \frac{s_t + (r_* + \lambda + \mu)(r_t - r_*)}{(r_* + \lambda)(r_* + \mu)}. \end{aligned}$$

Those examples show it is quite easy to get closed forms with easy to interpret processes.

### 4.3 Comparison with existing processes

**Comparison with the affine class** The affine class (Duffie and Kan 1996; Duffie, Pan and Singleton 2000; Duffie 2002) comprises processes of the type:

$$\begin{aligned} dX_t &= (b - AX_t)dt + w_t dz_t \\ w_t w'_t &= \sigma^2 (H'_1 X_t + H_0) \end{aligned}$$

with  $r_t = r_* + \beta'(X_t - X_*)$ .  $b, X_t \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $(H_0, H_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$ ,  $\sigma \in \mathbb{R}$ . We note, as before,  $X_* = A^{-1}b$ , assumed to exist.

Under mild technical conditions, bond prices have the expression:

$$Z_t^{\text{Aff}}(T) = \exp(-r_*T + \Gamma(T)'(X_t - X_*) + \sigma^2 a(T))$$

with  $r_*$  where  $a(T)$  and  $\Gamma(T)$  satisfy a typically complicated ordinary differential equation. The situation is simpler if  $H_1 = 0$ . In that case,  $\Gamma(T) = \gamma(T)$ , with  $\gamma(T)' = \beta'(e^{-AT} - 1)/A$ . Then:

$$Z_t^{\text{Aff}}(T) = \exp(-r_*T + \gamma(T)'(X_t - X_*) + \sigma^2 a(T)). \quad (25)$$

The express can be contrasted with the expression for the quadratic process (20):

$$Z_t^{\text{QP}}(T) = e^{-r_*T} (1 + \gamma(T)'(X_t - X_*)). \quad (26)$$

If  $\gamma(T)'X_t$  is small, the two expressions are the same, up to terms of second order in  $\gamma(T)'X_t$ , and second order in  $\sigma$ . Hence, a quadratic process is a good approximation if the underlying process is in fact affine, and vice-versa. In most cases, the two values are likely to be close, so that existing estimates of parameters in the affine class can be used to calibrate quadratic processes.<sup>7</sup>

**Comparison with Heath, Jarrow and Morton** In the Heath, Jarrow and Morton model, forward rates are factors. In the quadratic process, bond prices are factors. The HJM does not

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<sup>7</sup>That equivalence gives a useful way to calculate easily functionals of quadratic processes, that can be expressed as a linear combination of bonds. One first works with the affine process, setting volatility to 0, doing a first order Taylor expansion of terms in  $(X_t - X_*)$ . One gets an expression:  $P^{\text{Aff}} = a + b(X_t - X_*) + o(X_t - X_*) + o(\sigma^2)$ , for some constant  $a, b$ . Then, one knows that for the corresponding quadratic process, the value of the asset is:  $P^{\text{QP}} = a + b(X_t - X_*)$ , exactly.

yield essentially any closed forms, but has proven very useful to price options. Applications of the quadratic process to options are not investigated in this paper.

#### 4.4 Additional remarks

The following relation highlights why the quadratic process yields linear expressions.

**Lemma 1** *Suppose that process  $y_t \in \mathbb{R}^q$  satisfies  $E_t[dy_t]/dt = Cy_t dt + r_t y_t dt$ , for some  $q \times q$  matrix  $C$ . Then,  $\forall T \geq 0$ ,*

$$E_t \left[ e^{-\int_t^{t+T} r_s ds} y_{t+T} \right] = e^{CT} y_t. \quad (27)$$

**Proof.** Call  $L_S = E_t \left[ e^{-\int_t^S r_s ds} y_S \right]$ .

$$dL_S = E_t \left[ e^{-\int_t^S r_s ds} (-r_S y_S dS + C y_S dS + r_S y_S dS) \right] = CL_S$$

As  $L_t = y_t$ , we get  $L_S = e^{(S-t)C} y_t$ . ■

As a corollary, consider which would have a constant price. We can easily see, using the above Lemma, that an asset yielding a dividend  $D_s = r^* (1 - \beta' A^{-1} X_s)$  at each date  $s$ , has a constant price of 1.

We close with the following Conjecture. If true, the quadratic term is the only way to obtain linear bond prices.

**Conjecture 1** *Suppose that  $X_t$  is a Markov process with values in  $\mathbb{R}^n$  such that the interest rate is a function  $r = \rho(X_t)$ , and bond prices are affine in  $X_t$ , i.e., there are deterministic functions  $a(T)$  and  $\Gamma(T)$  with values in  $\mathbb{R}$  and  $\mathbb{R}^n$  respectively, such that, for all  $X$  in the domain of definition,  $\forall T \geq 0$ ,  $E \left[ \exp \left( - \int_t^{t+T} \rho(X_s) ds \right) \mid X_t = X \right] = a(T) + \Gamma(T)' X$ . Then,  $X_t$  follows a quadratic process.*

## 5 Conditions to keep the process well-defined

### 5.1 The practical bottom line

With one factor, the process is well-defined if it stays within  $r \leq \bar{r}$ , with  $\bar{r} < \mu$ . For an  $n$ -factor process, the following algorithm gives the conditions for the initial state vector to yield a well-defined process.

First, one diagonalizes the matrix  $A$ , i.e., sets  $A = \Omega \Delta \Omega^{-1}$ , with  $\Delta = \text{Diag}(\delta_1, \dots, \delta_n)$ , with the root not necessarily ordered. The eigenvector corresponding to eigenvalue  $\delta_j$  is  $(\Omega_{ij})_{i=1 \dots n}$ .

Then, one sets:<sup>8</sup>  $Q = \text{Diag} \left( (\Omega' \beta)_j / \delta_j \right) \cdot \Omega^{-1}$ .

The process is well-defined, if the initial value of  $X_t$  satisfies the  $n$  inequalities:

$$\forall i = 1 \dots n, \sum_{j=1}^n \frac{\min(\delta_i, \delta_j)}{\delta_i} Q_{jk} (X_k - X_{*k}) < 1. \quad (28)$$

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<sup>8</sup>This rescales the eigenvectors, so that in the diagonal representation,  $r = \delta' X$ .

which we call define being in the “Root” region.

In matrix form, one defines  $\Gamma_{ij} = \left( \frac{\min(\delta_i, \delta_j)}{\delta_i} \right)$ , and the criterion is:  $\Gamma Q (X_t - X_*) < (1, \dots, 1)$ , in the sense that the inequality holds for each of the  $n$  coordinates.

If the initial value of  $X_t$  satisfies (28), and increments are continuous, then all future  $X_{s>t}$  also satisfy (28).<sup>9</sup>

The above conditions are simply sufficient, but one conjectures that they are actually optimal, in the domain of conditions requiring only  $n$  linear inequalities.

**Some examples** For  $n = 1$ , the condition is just  $r_t < \mu$ , i.e.  $r_t - r_* < \phi$ . The equivalent condition for the stock with time-varying growth rate is:  $g_t - g_* > -\phi$ .

With  $n$  factors, if the process is diagonal, i.e.,  $A = \text{Diag}(\delta_1, \dots, \delta_n)$ , with  $r = \sum \beta_i X_i$ , then  $\Omega$  is the identity matrix,  $Q = \text{Diag}(\beta_j / \delta_j)$ , and the condition is:  $\forall i, \sum_{j=1}^n \frac{\min(\delta_i, \delta_j)}{\delta_i} \frac{\beta_j}{\delta_j} X_j < 1$ , i.e.,  $\forall i, \sum_{j=1}^n \frac{\beta_j X_j}{\max(\delta_i, \delta_j)} < 1$ .

When  $n = 2$ , with  $A = \text{Diag}(\delta_1, \delta_2)$  and  $r = \sum_{i=1}^2 \delta_i X_i$ , the root region is:

$$\omega_\rho = \left\{ X \mid X_1 + X_2 < 1 \text{ and } \frac{\delta_1}{\delta_2} X_1 + X_2 < 1 \right\}.$$

In Example 2 of section 4.2 with a dividend growth rate  $g_t = g_* + \sum_{i=1}^n X_{it}$ , one applies the criterion to  $r_t = R - g_t$ , with  $\beta_i = -1$ . Therefore, the condition is:

$$\forall i = 1 \dots n, \sum_j \frac{X_j}{\max(\delta_i, \delta_j)} > -1.$$

In Example 3 of section 4.2, with the short rate and the long rate, we get  $\delta' = (\phi, \psi)$ ,  $\Omega = \begin{pmatrix} 1 & \phi \\ 0 & \phi - \psi \end{pmatrix}$ ,  $Q = \text{Diag}(1/\phi, \phi/\psi) \Omega^{-1}$ . When  $\phi > \psi$  (the short rate moves faster than the long rate, so to speak),  $\Gamma = \begin{pmatrix} 1 & \psi/\phi \\ 1 & 1 \end{pmatrix}$ , and the conditions become:

$$\frac{r - r_*}{\phi} < 1 \text{ and } \frac{r - r_*}{\phi} + \frac{L - r_*}{\psi} < 1.$$

Intuitively,  $r$  and  $L$  cannot deviate from their central value,  $r_*$ , by a value order of magnitude the speed of their mean-reversion.

The rest of this section justifies these claims. It may be skipped in a first reading.

## 5.2 Notation and motivations

We consider a diagonal process, i.e.,  $A = \text{Diag}(\delta_1, \dots, \delta_n)$ , with  $\delta_1 \leq \dots \leq \delta_n$ , and  $n > 1$ . In a dense open subset of cases, we can rely on this case after diagonalizing  $A$ . We define  $\delta_0 = 0$  and  $\delta_{n+1} = \delta_n$ . We rescale  $X$  if needed, so that  $\beta = \delta$ , and so that  $r = \sum \delta_i X_i$ . We reason first in the deterministic version of the process.

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<sup>9</sup>One can show that another, strictly more stringent, sufficient condition, is  $\forall i = 1 \dots n, \sum_{j=1}^n 1_{\delta_i \leq \delta_j} Q_{jk} (X_k - X_{*k}) < 1$ .



We use  $n$  tests vectors,  $\Gamma^{(1)}, \dots, \Gamma^{(n)}$ , with  $\forall i = 1 \dots, n, \Gamma^{(i)} \in \mathbb{R}^n$ . We study regions of the type:

$$\omega_\Gamma = \left\{ X \in \mathbb{R}^n \mid \forall i = 1 \dots n, \Gamma^{(i)} \cdot X < 1 \right\}. \quad (29)$$

We want to make sure that, if the process hits the boundary of  $\omega$ , noted  $\partial\omega$ , then it goes back inside  $\omega$ . To express this, call  $m(X) = E[dX_t/dt \mid X_t = X] = (r - A)X$  the drift of  $X_t$  at  $X$ . If  $X \in \partial\omega$ , then call  $j$  the coordinate such that  $\Gamma^{(j)} \cdot X = 1$ ; then, we want to ensure  $\Gamma^{(i)} \cdot m(X) \leq 0$ . We calculate:

$$\begin{aligned} \Gamma^{(i)} \cdot m(X) &= \Gamma^{(i)'} (r - A)X = r\Gamma^{(i)'}X - \Gamma^{(i)'}AX \\ &= r \cdot 1 - \left(A'\Gamma^{(i)}\right)'X = \left(\delta - A\Gamma^{(i)}\right) \cdot X. \end{aligned}$$

So, we require, for all  $i > 1$ :  $\{\forall j \neq i, \Gamma^{(j)} \cdot X < 1 \text{ and } \Gamma^{(i)} \cdot X = 1\} \implies (\delta - A\Gamma^{(i)}) \cdot X < 0$ .

We impose this condition only for  $i > 1$ , because in practice, a boundary of choice is  $\Gamma^{(1)} = (1, \dots, 1)'$ , which requires a special treatment. Call  $S_t = \Gamma^{(1)} \cdot X_t = \sum_j X_{jt}$ , then  $dS_t/dt = r_t(S_t - 1)$ . So,  $S_t$  cannot cross 1, and if  $S_{t_0} < 1$ , at an initial date  $S_{t_0} < 1$ , then for all  $t \geq t_0$ ,  $S_t < 1$ . So, the region  $\{X \in \mathbb{R}^n \mid \Gamma^{(1)} \cdot X < 1\}$  is automatically non-repelling.

The following Lemma completes the reasoning, by providing a sufficient condition for the region  $\omega_\Gamma$  to be non-repelling.

**Lemma 2** (*Sufficient condition to get a non-repelling region*) Suppose  $\Gamma^{(1)} = (1, \dots, 1)'$ , and that for each  $i = 2 \dots n$ , there is a  $\alpha_i > 0$  and a non-negative vector  $\psi^{(i)} \in \mathbb{R}_+^n$ , with  $\sum_j \psi_j^{(i)} = 1$ ,  $\psi_i^{(i)} < 1$ , such that:

$$\alpha_i \delta + (1 - \alpha_i A) \Gamma^{(i)} = \sum_j \psi_j^{(i)} \Gamma^{(j)}. \quad (30)$$

Then, the process does not escape from region  $\omega$ .

**Proof.** Indeed, then

$$\begin{aligned} \alpha_i (\delta - A\Gamma^{(i)}) \cdot X &= \left( -\Gamma^{(i)} + \sum_j \psi_j^{(i)} \Gamma^{(j)} \right) \cdot X \\ &= -1 + \sum_j \psi_j^{(i)} \Gamma^{(j)} \cdot X < -1 + \sum_j \psi_j^{(i)} = 0. \end{aligned}$$

■

### 5.3 The Root region

We consider the region  $\omega_\Gamma$  defined by  $\Gamma_j^{(i)} = \delta_{i \wedge j} / \delta_i$ , where  $i \wedge j = \min(i, j)$ . Because it involves the eigenvalues of  $A$ , we call it the “Root region”. Heuristic arguments lead to the conjecture that the Root region is optimal, in the sense that it may be the largest non-repelling region that can be defined by the intersection of no more than  $n$  inequalities expressed as linear combinations of the factors.



**Theorem 3** *The Root region  $\omega_\Gamma$ , defined by the boundary vectors  $\Gamma_j^{(i)} = \delta_{i \wedge j} / \delta_i$  and*

$$\omega_\Gamma = \left\{ X \in \mathbb{R}^n \mid \forall i = 1 \dots n, \Gamma^{(i)} \cdot X < 1 \right\},$$

*is non-repelling.*

#### 5.4 Conditions on volatility

In dimension 1, we require that  $\text{var}(dM_t) = 0$  near the boundary. It can be 0 in a region  $[\mu - \varepsilon, \mu]$ , for some  $\varepsilon > 0$ , or  $\sigma(r) \sim k(\bar{r} - r)^\alpha$  with  $\alpha > 1/2$  or  $\alpha = 1/2$  and  $k$  large enough.

In dimension  $n$ , the analog is that when  $\Gamma^{(i)'} X_t = 1$ , (while  $X \in \bar{\omega}_\Gamma$ ), then  $\text{var}(\Gamma^{(i)'} \cdot dM_t)$  should be 0 near the boundary. The simplest condition is that, for each  $i = 1 \dots n$ , there is an open neighborhood  $\nu^{(i)}$  of  $\bar{\omega}_\Gamma \cap \{X \mid \forall i, \Gamma^{(i)} \cdot X = 1\}$ , such that

$X_t \in \nu^{(i)} \Rightarrow \text{var}(\Gamma^{(i)'} \cdot dM_t) = 0$ . In other terms, the noise along the projection on  $\Gamma^{(i)}$  becomes 0.

A simple way to ensure that the process is well-defined is the following. For an  $\varepsilon > 0$ , in practice very small, we define the killing function

$$K_\varepsilon(X) = 1 \left\{ \max_{i=1 \dots n} \Gamma^{(i)} \cdot X < 1 - \varepsilon \right\} \quad (31)$$

where, as always,  $1\{A\}$  is equal to 1 if  $A$  is true, or 0 otherwise.

We start from an arbitrary continuous martingale process,  $d\widetilde{M}_t$ , and then define a new continuous martingale:

$$dM_t = K_\varepsilon(X_t) d\widetilde{M}_t. \quad (32)$$

In other terms, all variances become 0 if  $X$  is within a small neighborhood of the limit of the root region  $\omega_\Gamma$ . Otherwise, the process is not modified. This “killing” procedure might be the most convenient to use in practice.

#### 5.5 Condition with diagonalizable matrices

In the general case, given a diagonalizable matrix  $A$ , we diagonalize it,  $A = \Omega \Delta \Omega^{-1}$ , with  $\Delta = \text{Diag}(\delta_1, \dots, \delta_n)$ . We set  $Q = \text{Diag}((\Omega' \beta)_i / \delta_i) \cdot \Omega^{-1}$ , and define the new state vector to be:  $Y_t = Q X_t$ . Then:

$$\begin{aligned} \delta' Y_t &= \delta' \cdot \text{Diag}((\Omega' \beta)_i / \delta_i) \Omega^{-1} X_t = \text{Diag}((\Omega' \beta)_i) \Omega^{-1} X_t \\ &= \beta' \Omega \cdot \Omega^{-1} X_t = \beta' X_t = r_t. \end{aligned}$$

As  $r_t = \delta' Y_t$ ,  $Y_t$  satisfied the conditions of this section for the tests, and the Root region is:  $\forall i = 1 \dots, n$ ,  $\Gamma^{(i)} \cdot Y < 1$ . As  $\Gamma^{(i)} \cdot Y_t = \Gamma^{(i)'} Q X_t = \sum_j \frac{\delta_i \wedge j}{\delta_i} Q_{jk} X_k$ , the condition can be re-expressed in a way that does not depend on the prior ordering of the roots:  $\forall i, \sum_j \frac{\delta_i \wedge j}{\delta_i} Q_{jk} X_k < 1$ , which gives the condition announced in (28).

## 6 The process in discrete time

This section studies the discrete-time version of the quadratic process, which is useful for several reasons. In many applications, discrete-time is simpler than continuous time. The technical issues are clarified by discrete-time examples.

The key object is the stochastic discount factor  $m_t \in \mathbb{R}_+^*$ . The price at time  $t$  of a claim yielding a stochastic dividend  $D_S$  at date  $S \geq t$  is:  $P_t = E[m_{t+1} \dots m_S D_S]$ . The price of a zero coupon bond of maturity  $T$  is:

$$Z_t(T) = E_t[m_{t+1} \dots m_{t+T}]. \quad (33)$$

For instance, the gross short term interest from  $t$  to  $t+1$  is  $1/E_t[m_{t+1}]$ . In terms of the continuous time process,  $m_t = \exp\left(-\int_{t-1}^t r_u du\right)$ .

### 6.1 Discrete time, one factor

The most obvious generalization of the process,  $E m_{t+1}$  quadratic in  $m_t$ , does not work. Instead, the generalization we propose is:

$$E_t m_{t+1} = \frac{1}{a} + \frac{1}{b} - \frac{1}{abm_t} \quad (34)$$

with  $0 < a < b$ . We assume that the process is such that for all times,  $m_t > 1/b$ .<sup>10</sup>

The expected value of  $m_{t+1}$  is increasing in  $m_t$ , so the process has a positively autocorrelation. On the other hand, if there is no noise, then  $m_t$  goes to the attractive root,  $1/a$ . Indeed,  $E_t[m_{t+1}] - 1/a = \frac{1}{m_t b} (m_t - \frac{1}{a})$ , so that, for  $m_t$  in a neighborhood of  $1/a$ ,

$$\begin{aligned} E_t \left[ m_{t+1} - \frac{1}{a} \right] &= \rho \left( m_t - \frac{1}{a} \right) + o \left( m_t - \frac{1}{a} \right) \\ \rho &= a/b \in (0, 1). \end{aligned}$$

Hence (34) behaves like a autoregressive process AR(1), with central value  $m_* = 1/a$ , and autocorrelation coefficient  $\rho = a/b$ . However, its functional form yields tractable bond and perpetuity prices, as shown by the following Proposition.<sup>11</sup>

**Proposition 8** (*Price of a zero-coupon bond in the 1-factor, discrete time case*) *The price of a zero*

<sup>10</sup>There are many ways to ensure that  $m_t$  remains above a lower bound  $\underline{m} \in [1/b, 1/a)$ . The simplest one is to get series of independent, positive random variables  $\eta_t$ , with  $E[\eta_t] = 1$ , and set:  $m_{t+1} = \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{abm_t} - \underline{m}\right) \eta_t + \underline{m}$ . As  $m_t > \underline{m} \in [1/b, 1/a)$ ,  $\frac{1}{a} + \frac{1}{b} - \frac{1}{abm_t} - \underline{m} > 0$  and  $m_{t+1} > \underline{m}$ . The reason to impose  $m_t > 1/b$  is that, if we had  $m_t < 1/b$  for some  $t$ , in the deterministic version of the process, there would be a time  $s > t$  such that  $m_s \leq 0$ , which is not allowed for a discount factor.

<sup>11</sup>The correspondence with the one-factor process is the following: for a small  $\tau$ ,  $a = 1/(1 - \tau\alpha)$ ,  $b = 1/(1 - \tau\beta)$ ,  $m_t = 1 - \tau r_t$ . Then, (34) reads:

$$\tau E_t[r_{t+1} - r_t] = -(E[m_{t+1}] - m_t) = \left(\frac{1}{a} - m_t\right) \left(\frac{1}{b} - m_t\right) \frac{1}{m_t} = (r_t - \alpha)(r_t - \beta) \frac{\tau^2}{1 - \tau r_t}$$

i.e.,  $E[r_{t+1} - r_t] = (r_t - \alpha)(r_t - \beta)\tau + O(\tau)$ , i.e.,  $E[dr_t] = (r_t - \alpha)(r_t - \beta)dt$ .

coupon bond of maturity  $T$  is:

$$Z_t(T) = a^{-T} \frac{b - m_t^{-1}}{b - a} + b^{-T} \frac{m_t^{-1} - a}{b - a}. \quad (35)$$

Summing over the maturities, we get the price of a perpetuity.<sup>12</sup>

**Proposition 9** (*Price of a perpetuity in the 1-factor, discrete time case*) The price of a perpetuity is:

$$P_t = \frac{a + b - 1 - m_t^{-1}}{(a - 1)(b - 1)}. \quad (36)$$

The above discount factor can be enriched by a stochastic component, as in Eq. 15.

## 6.2 Discrete time, $n$ factors

This section studies the case with a number of factors  $n \geq 1$ . A quadratic process is given by: (i) a stochastic discount factor  $m_t \in \mathbb{R}_+^*$ ; (ii) a state vector  $X_t \in \mathbb{R}^n$ ; (iii) a matrix  $\Gamma \in \mathbb{R}^{n \times n}$ , with eigenvalues in  $(-1, 1)$ ; (iv) a vector  $\beta \in \mathbb{R}^n$ ; and, (v) a central value of the gross interest rate  $R_* = 1 + r_* > 0$ , with:

$$\begin{aligned} E_t[X_{t+1}] &= \frac{\Gamma X_t}{R_* m_t} \\ m_t &= \frac{1}{R_*} (1 - \beta' X_t). \end{aligned}$$

We assume that the process is well-defined for all  $t$ 's. The conditions that ensure this are analogous to the continuous case, and will be specified in the next iteration of the paper.

To get some intuition for the process, we first consider the steady-state. As  $\Gamma$  is contractive, the steady state value of  $X$  is 0, so the steady-state value of  $m_t$  is  $1/R_*$ . In general, if  $\Gamma$  has positive roots, then  $X_t$  is positively autocorrelated, with a small modulation introduced by the  $m_t$  term.

**Lemma 3** Suppose that process  $y_t \in \mathbb{R}^q$  satisfies  $E_t[y_{t+1}] = Cy_t/m_t$ , for some  $q \times q$  matrix  $C$ . Then,  $\forall T \geq 0$ ,

$$E_t[m_{t+1} \dots m_{t+T-1} y_{t+T}] = C^T \frac{y_t}{m_t}. \quad (37)$$

**Proof.** Call  $L_T$  the left-hand side.  $L_1 = E[y_{t+1}] = Cy_t/m_t$ , and

$$\begin{aligned} L_{T+1} &= E_t[m_{t+1} \dots m_{t+T} y_{t+T+1}] = E_t[m_{t+1} \dots m_{t+T-1} E_{t+T}[m_{t+T} y_{t+T+1}]] \\ &= E_t[m_{t+1} \dots m_{t+T-1} C y_{t+T}] = C E_t[m_{t+1} \dots m_{t+T-1} y_{t+T}] = C L_T. \end{aligned}$$

■

<sup>12</sup>Alternatively, one can try a price process of the type  $P_t = A + B/m_t$ , and solve for  $A, B$  that satisfy the functional equation:  $0 = P_t - E_t[m_{t+1}(1 + P_{t+1})]$ .

**Theorem 4** (*Price of a zero-coupon bond in the multifactor, discrete time case*) The price of a zero-coupon bond of maturity  $T$  is, with  $I_n$  the identity matrix of dimension  $n$ :

$$Z_t(T) = R_*^{-T} - R_*^{-T-1} \beta' \Gamma (I_n - \Gamma)^{-1} (I_n - \Gamma^T) \frac{X_t}{m_t}. \quad (38)$$

**Proposition 10** (*Price of perpetuity in the multifactor, discrete time case*) Assume  $R_* > 1$ . The price of a perpetuity,  $P_t = \sum_{T=1}^{\infty} Z_t(T)$ , is, with  $I_n$  the identity matrix of dimension  $n$ :

$$P_t = \frac{1}{R_* - 1} - \frac{1}{R_* - 1} \beta' \Gamma (R_* I_n - \Gamma)^{-1} \frac{X_t}{m_t} \quad (39)$$

**Proof.** One simply sums (38). Alternatively, one can try a price process of the type:  $P_t = a + b' X_t / m_t$ , and solve for  $a \in \mathbb{R}$  and  $b \in \mathbb{R}^n$  that satisfy the functional equation:  $0 = P_t - E_t[m_{t+1} (1 + P_{t+1})]$ . ■

## 7 Conclusion

Quadratic processes are very tractable, as they yield closed forms for perpetuities, and prices that are linear in factors. They are likely to be useful in several parts of economics, when trend growth rates, or risk premia, are time-varying. Hence quadratic processes might be a useful addition to the economists' toolbox.



## Appendix A. Regularity conditions for the one-factor process

This appendix details conditions for the existence and uniqueness of the solutions. We recommend Katatzas and Shreve (1991 Chapter 5.5) and Revuz and Yor (1999, Chapter IX) for systematic treatments, and Ait-Sahalia (1996, Appendix) and Duffie (2001, Appendix E) for pedagogical overviews. We call  $\mathcal{D} = (\underline{r}, \bar{r})$  the domain of existence of  $r$ , and  $c$  an arbitrary point in  $\mathcal{D}$ . We call  $\mu(r)$  the drift of  $r$ , and assume  $dM_t = \sigma(r_t) dz_t$ . We make the following assumptions.

- (i) The drift and diffusion functions are continuously differentiable in  $r$  in  $\mathcal{D}$ , and  $\sigma^2(r) > 0$  in  $\mathcal{D}$ .
- (ii) The integral of  $m(r) = \exp(\int_c^r 2\mu(u)/\sigma^2(u) du) / \sigma^2(r)$  converges at both boundaries of  $\mathcal{D}$ .
- (iii) The integral of  $s(r) = \exp(-\int_c^r 2\mu(u)/\sigma^2(u) du)$  diverges at both boundaries of  $\mathcal{D}$ .
- (iv)  $\mu$  is Lipschitz continuous, and there is a function  $\rho(x) : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ , with  $\rho(0) = 0$ , such that for any  $\varepsilon > 0$ ,  $\int_{(0,\varepsilon)} \rho(x)^{-2} dx = +\infty$ , and  $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$ .

If conditions (i)-(iv) are satisfied, then there is a unique Itô process  $\{r_t, t \geq 0\}$  which is a strong solution of the stochastic differential equation (1) with initial condition  $r_0 = r$ . Moreover,  $\{r_t, t \geq 0\}$  is Markov.

The key substantive point is that the process is defined for all  $t \geq 0$ , and does not explode. This condition is crucial, as if we started with  $r_0 > \beta$ , the process would explode in finite time with positive probability, so that the process would not be defined for all times.

Conditions (i), (ii) and (iv) guarantee the existence and uniqueness of the solution up to the variable may hit the boundaries. Condition (iii) implies that the boundaries are actually not reached. The intuition is as follows. Consider the correct boundary. Condition (iii) implies  $\mu(\bar{r}) < 0$ , so that the process tends to return inside  $\mathcal{D}$ , and also requires that  $\sigma^2(r)$  tends to 0 fast enough as  $r \uparrow \bar{r}$ .

**Sufficient conditions to ensure (i)-(iv)** Conditions (i) and (ii) guarantee that the stochastic differential equation (1) admits a unique strong solution. Those conditions are verified in the following cases. Condition (iii) guarantees that the end points  $\mathcal{D}$  of are natural boundaries.

We assume  $\mu(\bar{r}) < 0$  and  $\lim_{r \rightarrow \underline{r}} \mu(r) > 0$ , so that close to the end points of  $\mathcal{D}$ , the process tends to go back inside  $\mathcal{D}$ . In the case  $\mu(r) = (r - \alpha)(r - \beta)$ , with  $\alpha < \beta$ , this corresponds to  $\bar{r} \in (\alpha, \beta)$  and  $\bar{r} \in [-\infty, \alpha)$ .

Conditions (ii) and (iii) are verified if the following conditions (C-D) hold. For  $r$  in a left-neighborhood of  $\bar{r}$ ,  $\sigma^2(r) \sim k(\bar{r} - r)^\kappa$ , with  $(\kappa > 1$  and  $k > 0)$  or  $(\kappa = 1$  and  $0 < k < -2m(\bar{r}))$ . If  $\underline{r} > -\infty$ , for  $r$  in a right neighborhood of  $\underline{r}$ ,  $\sigma^2(r) \sim k'(r - \underline{r})^{\kappa'}$ , with  $(\kappa' > 1$  and  $k' > 0)$  or  $(\kappa' = 1$  and  $0 < k' < 2m(\underline{r}))$ . If  $\underline{r} = -\infty$ , then  $\underline{r}$  is a natural boundary if, for  $r$  in a neighborhood of  $-\infty$   $\sigma^2(r) \sim k|r|^\beta$ , with  $k > 0$  and  $\beta < 3$ . Those last conditions imply assumptions (ii), (iii).

For  $\underline{r} = -\infty$ , the situation is complex for condition (iv), as the standard conditions found in textbooks do not apply.  $\mu(r)$  is not Lipschitz continuous, as  $\mu'(r)$  is unbounded. We conjecture that a simple weakening of condition (iv) will allow the case  $\underline{r} = -\infty$ .

If  $\underline{r} > -\infty$ , the above conditions (C-D) also imply (iv), as one can take  $\rho(x) = K \max(x^{\kappa/2}, x^{\kappa'/2}, x)$ , for a large enough constant  $K$ .



## Appendix B. Proofs

### 7.1 Proof of Theorem 1

It is sufficient to prove the statement for an initial time equal to 0. We fix  $T > 0$  and define  $V_t = E_t \left[ \exp \left( - \int_t^T r_s ds \right) \right]$ , for  $t \in [0, T]$ . The arbitrage equation of  $V_t$  is:

$$-r_t V_t + E[dV_t]/dt = 0 \quad (40)$$

We seek a solution  $V_t = W_t$  with the functional form:

$$W_t = w(r_t, t) = A(t) r_t + B(t) \quad (41)$$

The boundary condition  $W_T = 1$  implies  $A(T) = 0$  and  $B(T) = 1$ . We call  $\mathcal{A}$  the infinitesimal diffusion operator of the process, which in the case  $dM_t = \sigma(r_t) dz_t$  is:

$$\mathcal{A}f(r) = (r - \alpha)(r - \beta) f'(r) + \frac{\sigma^2(r)}{2} f''(r)$$

The PDE (40) becomes:

$$-rw(r, t) + \mathcal{A}w(r, t) + \frac{\partial}{\partial t} w(r, t) = 0$$

Given  $Ar = (r - \alpha)(r - \beta)$ , we have  $\mathcal{A}w(r, t) = A(t)(r - \alpha)(r - \beta)$ . Importantly, because  $\partial^2 w / \partial r^2 = 0$  in the functional form (41), the Ito second-order term (equal to  $\frac{\sigma^2(r)}{2} \partial^2 w / \partial r^2$  in the case of a diffusion) is 0. The PDE is  $Q = 0$  with:

$$\begin{aligned} Q &= -rw(r, t) + \mathcal{A}w(r, t) + \frac{\partial}{\partial t} w(r, t) \\ &= -r[A(t)r + B(t)] + A(t)(r - \alpha)(r - \beta) + [A'(t)r + B'(t)] \\ &= [\alpha\beta A(t) + B'(t)] + r[-B(t) - (\alpha + \beta)A(t) + A'(t)] \end{aligned}$$

Importantly, the  $r^2$  terms cancel out. This explains why adding a quadratic term to the standard Ornstein-Uhlenbeck process substantially improves tractability. The PDF  $Q = 0$  is satisfied for all  $r$  if and only if:

$$\begin{aligned} \alpha\beta A(t) + B'(t) &= 0 \\ -B(t) - (\alpha + \beta)A(t) + A'(t) &= 0 \end{aligned} \quad (42)$$

Differentiating the last equation, we get:

$$-B'(t) - (\alpha + \beta)A'(t) + A''(t) = 0$$

which implies, via  $\alpha\beta A(t) + B'(t) = 0$ :

$$\alpha\beta A(t) - (\alpha + \beta)A'(t) + A''(t) = 0,$$

which is satisfied by a function  $e^{\xi t}$  if and only if:

$$0 = \alpha\beta - (\alpha + \beta)\xi + \xi^2 = (\xi - \alpha)(\xi - \beta)$$

so that the roots are  $\alpha$  and  $\beta$ . Therefore, for some constants  $c$  and  $C$ ,  $A(t) = ce^{\alpha t} + Ce^{\beta t}$ . The equations for  $c$  and  $C$  are given by  $A(T) = 0, B(T) = 1$  and (42). We get:

$$W_t = A(t)r_t + B(t) = \frac{\beta - r_t}{\beta - \alpha}e^{\alpha(t-T)} + \frac{r_t - \alpha}{\beta - \alpha}e^{\beta(t-T)}$$

The above shows that  $W_t$  satisfies  $-r_t W_t + E[dW_t]/dt = 0$ , and  $W_T = 1$ , but not yet that it is the equal to the bond price. To conclude the proof, we define  $L_t = \exp\left(-\int_0^t r_s ds\right) W_t$ . As  $E_t[dW_t - r_t W_t dt] = 0$ , we have  $E_t[dL_t] = E_t\left[\exp\left(-\int_0^t r_s ds\right)(dW_t - r_t W_t dt)\right] = 0$ . Hence  $L_t$  is a martingale, which implies,  $L_0 = E_0[L_T]$ , i.e.  $W_0 = E_0\left[\exp\left(-\int_0^T r_s ds\right) W_T\right] = E_0\left[\exp\left(-\int_0^T r_s ds\right)\right]$ , i.e.

$$\frac{\beta - r_0}{\beta - \alpha}e^{-\alpha T} + \frac{r_0 - \alpha}{\beta - \alpha}e^{-\beta T} = E_0\left[\exp\left(-\int_0^T r_s ds\right)\right]$$

which is the statement of the Theorem.

## 7.2 Proof of Proposition 3

The Proposition is a direct re-expression of (3), applied to the process  $r_t = R - g_t$  which follows, with  $r_* = R - g_*$ :  $dr_t = -\phi(r_* - r)dt + (r_t - r_*)^2 dt - dM_t$ .

We also present an alternative, arguably more direct derivation. We look for a formula:  $V_t$  of the type

$$V_t = v(D_t, g_t) = D_t(a + b(g_t - g_*))$$

which implies:

$$\begin{aligned} \frac{1}{D_t}E[dV_t]/dt &= E\left[\frac{dD_t}{D_t}\right]/dt + bE[dg_t]/dt \\ &= g_t(a + b(g_t - g_*)) + b\left[-\phi(g_t - g_*) - (g_t - g_*)^2\right]. \end{aligned}$$

Expressing the above in terms of  $(g_t - g_*)$  rather than  $g_t$ , the PDE is  $D_t - RV_t + E[dV_t]/dt = 0$ :

$$\begin{aligned} \bar{0} &= \bar{1} - \bar{R}(a + b(g_t - g_*)) + [(g_t - g_*) + g_*][(a + b(g_t - g_*))] + b\left[\phi(g_t - g_*) - (g_t - g_*)^2\right] \\ &= [1 - Ra + g_*a] + (g_t - g_*)[-Rb + a + bg_* - b\phi] + (g_t - g_*)^2[b - b], \end{aligned}$$

which is satisfied if and only if:

$$\begin{aligned} 1 - Ra + g_*a &= 0 \\ -Rb + a + bg_* - b\phi &= 0. \end{aligned}$$

The term  $(g_t - g_*)^2$  cancels out, which justifies its inclusion in the definition of the quadratic process.

The last two equations give:

$$\begin{aligned} a &= 1/(R - g_*) \\ b &= a/(R - g_* + \phi), \end{aligned}$$

i.e.,

$$V_t = \frac{D_t}{R - g_*} \left( 1 + \frac{g_t - g_*}{R - g_* + \phi} \right).$$

### 7.3 Proof of Proposition 5

This is a simple corollary of Theorem 1 and Girsanov's theorem. To simplify notation, we normalize the initial time to 0,  $g = 0$ ,  $D_0 = 1$ . Define:

$$D_t = e^{N_t - [N]_t/2}$$

where  $[N]_t$  is the quadratic variation of  $N$  up to time  $t$ . Define

$$dM_t^Q = dM_t - d\langle M_t, N_t \rangle = dM_t - (ar_t + b) dt$$

then

$$\begin{aligned} dr_t &= (r - \lambda)(r - \mu) + dM_t \\ &= (r^2 - (\lambda + \mu - a)r + \lambda\mu + b) dt + dM_t^Q \\ dr_t &= (r - \lambda')(r - \mu') dt + dM_t^Q. \end{aligned} \tag{43}$$

Let  $Q$  denote the probability with Radon-Nikodym derivative  $dQ = e^{N_t - [N]_t/2} dP$ :

$$V = E \left[ e^{-\int_t^s r_u du} D_s \right] = E \left[ e^{-\int_t^s r_u du} e^{N_t - [N]_t/2} \right] = E^Q \left[ e^{-\int_t^s r_u du} \right].$$

By Girsanov's theorem,  $dM_t^Q$  is a martingale under  $Q$ , so Theorem 1, applied to process (43) gives:

$$V = e^{-\lambda'(s-t)} \frac{\mu' - r_t}{\mu' - \lambda'} + e^{-\mu'(s-t)} \frac{r_t - \lambda'}{\mu' - \lambda'}.$$

### 7.4 Proof of Theorem 2

It is sufficient to consider  $X_* = b = 0$ , by redefining  $X_t$  to be  $X_t - X_*$  if needed. Also, it is enough to consider  $r_* = 0$ , by redefining  $r_t$  to be  $r_t - r_*$ . We consider the  $Z(X_t, T)$  the price of zero-coupon bond of maturity  $T$ . The arbitrage equation for  $Z$  is:

$$0 = -r_t Z + E[dZ_t]/dt.$$

We guess the functional form  $Z(X_t, T) = 1 + \gamma'(T)X_t$ , for a function  $\gamma(T)$  with values in  $\mathbb{R}^n$  to be determined. With this functional form,  $E[dZ_t/dt] = \gamma'(T)(-AX_t + r_t X_t) - \partial_T Z(X_t, T)$ , so that

the PDE becomes:

$$\begin{aligned} 0 &= -r_t (1 + \gamma'(T) X_t) + \gamma'(T) (-AX_t + r_t X_t) - \frac{d}{dT} \gamma(T)' X_t \\ &= -r_t - \gamma'(T) A X_t - \frac{d}{dT} \gamma(T)' X_t. \end{aligned}$$

The  $r_t X_t$  terms cancel out, which justifies their inclusion in the quadratic process. Using  $r_t = \beta' X_t$ , we obtain:

$$\begin{aligned} 0 &= -\beta' - \gamma'(T) A - \frac{d}{dT} \gamma(T)' = 0 \\ \Leftrightarrow \frac{d}{dT} \gamma(T)' &= -\gamma'(T) A - \beta'. \end{aligned}$$

One integrates this matrix equation like a scalar equation, using the boundary condition  $\gamma(0) = 0$ , which gives:

$$\gamma(T)' = \beta' \frac{e^{-AT} - 1}{A}$$

and finally the earlier expression for the bond price.

## 7.5 Proof of Proposition 7

We integrate (22):

$$\begin{aligned} P_t &= \int_0^\infty Z_t(T) dT = \int_0^\infty e^{-r_* T} dT + \int_0^\infty e^{-r_* T} \beta' \frac{e^{-AT} - 1}{A} (X_t - X_*) dT \\ &= \frac{1}{r_*} + \beta' \left( \int_0^\infty e^{-(r_* + A)T} - e^{-r_* T} dT \right) \frac{1}{A} (X_t - X_*) \\ &= \frac{1}{r_*} + \beta' \left( \frac{1}{r_* + A} - \frac{1}{r_*} \right) \frac{1}{A} (X_t - X_*) = \frac{1}{r_*} - \beta' \frac{1}{r_* (r_* + A)} (X_t - X_*). \end{aligned}$$

## 7.6 Proof of Theorem 3

We use the convention  $\delta_0 = 0$  and  $\delta_{n+1} = \delta_n$ . We define  $\alpha_i = 1/\delta_{i+1}$  and

$$\psi_j^{(i)} = \frac{\delta_j (\delta_{j+1} - \delta_{j-1})}{\delta_i \delta_{i+1}} 1_{j \leq i}.$$

We check the conditions of Lemma 2.  $\psi_j^{(i)} \geq 0$ , and

$$\sum_j \psi_j^{(i)} = \sum_{j=1}^i \frac{\delta_j \delta_{j+1} - \delta_{j-1} \delta_j}{\delta_i \delta_{i+1}} = \frac{\delta_i \delta_{i+1} - \delta_0 \delta_1}{\delta_i \delta_{i+1}} = 1.$$

We next calculate each side of (30), evaluated at the  $k$ -th coordinate. We obtain:

$$\begin{aligned} LHS_k - 1 &= \left( \alpha_i \delta + (1 - \alpha_i A) \Gamma^{(i)} \right)_k - 1 = \frac{\delta_k}{\delta_{i+1}} + \left( 1 - \frac{\delta_k}{\delta_{i+1}} \right) \frac{\delta_{i \wedge k}}{\delta_i} - 1 \\ &= \left( \frac{\delta_k}{\delta_{i+1}} - 1 \right) \left( 1 - \frac{\delta_{i \wedge k}}{\delta_i} \right) \end{aligned}$$

and, using  $\sum_j \psi_j^{(i)} = 1$ ,

$$\begin{aligned} RHS_k - 1 &= \sum_j \psi_j^{(i)} \Gamma_k^{(j)} - \sum_j \psi_j^{(i)} = \sum_j \psi_j^{(i)} \left( \Gamma_k^{(j)} - 1 \right) \\ &= \frac{1}{\delta_i \delta_{i+1}} \sum_j \delta_j (\delta_{j+1} - \delta_{j-1}) 1_{j \leq i} \left( \frac{\delta_{j \wedge k}}{\delta_j} - 1 \right). \end{aligned}$$

If  $k \geq i$ ,  $RHS_k - 1 = 0 = LHS_k - 1$ . If  $k < i$ ,

$$\begin{aligned} RHS_k - 1 &= \frac{1}{\delta_i \delta_{i+1}} \sum_j \delta_j (\delta_{j+1} - \delta_{j-1}) 1_{k < j \leq i} \left( \frac{\delta_k}{\delta_j} - 1 \right) \\ &= \frac{1}{\delta_i \delta_{i+1}} \sum_{k+1 \leq j \leq i} (\delta_{j+1} - \delta_{j-1}) (\delta_k - \delta_j) \\ &= \frac{1}{\delta_i \delta_{i+1}} \left[ \left( \sum_{k+1 \leq j \leq i} (\delta_{j+1} - \delta_{j-1}) \right) \delta_k + \sum_{k+1 \leq j \leq i} -\delta_{j+1} \delta_j + \delta_j \delta_{j-1} \right] \\ &= \frac{1}{\delta_i \delta_{i+1}} [(\delta_{i+1} + \delta_i - \delta_{k+1} - \delta_k) \delta_k - \delta_{i+1} \delta_i + \delta_{k+1} \delta_k] \\ &= \frac{1}{\delta_i \delta_{i+1}} [(\delta_{i+1} + \delta_i - \delta_k) \delta_k - \delta_{i+1} \delta_i] = \frac{1}{\delta_i \delta_{i+1}} (\delta_{i+1} - \delta_k) (\delta_k - \delta_i) \\ &= LHS_k - 1. \end{aligned}$$

So, we get  $\forall k, RHS_k - 1 = LHS_k - 1$ , i.e., (30) holds.

## 7.7 Proof of Proposition 8

We have:

$$\begin{aligned} Z_t(T+1) &= E_t[m_{t+1} \dots m_{t+T+2}] = E_t[m_{t+1} \dots m_{t+T+1} E_{t+T+1}[m_{t+T+2}]] \\ &= E_t \left[ m_{t+1} \dots m_{t+T+1} \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{ab m_{t+T+1}} \right) \right] \\ &= \left( \frac{1}{a} + \frac{1}{b} \right) E_t[m_{t+1} \dots m_{t+T+1}] - \frac{1}{ab} E_t[m_{t+1} \dots m_{t+T}] \\ Z_t(T+1) &= \left( \frac{1}{a} + \frac{1}{b} \right) Z_t(T) - \frac{1}{ab} Z_t(T-1). \end{aligned}$$



The characteristic equation is:  $\xi^2 = (\frac{1}{a} + \frac{1}{b})\xi - \frac{1}{ab}$ , which has roots  $1/a$  and  $1/b$ . Hence, the solution is of the type:  $Z_t(T) = Aa^{-T} + Bb^{-T}$ , for some constants  $A$  and  $B$ . Using the values  $Z_t(0) = 1$ , and  $Z_t(1) = E_t[m_{t+1}] = 1/a + 1/b - 1/(abm_t)$ , we get the announced formula.

## 7.8 Proof of Theorem 2

$$\begin{aligned} Z_t(T) &= E_t[m_{t+1} \dots m_{t+T}] = E_t[m_{t+1} \dots m_{t+T-1} R_*^{-1} (1 - \beta' X_{t+T})] \\ &= R_*^{-1} E_t[m_{t+1} \dots m_{t+T-1}] - R_*^{-1} \beta' E_t[m_{t+1} \dots m_{t+T-1} X_{t+T}] \\ &= R_*^{-1} Z_t(T-1) - R_*^{-1} \beta' \left( \frac{\Gamma}{R_*} \right)^T \frac{X_t}{m_t} \end{aligned}$$

as the previous Lemma, applied to  $Y_t = X_t$  and  $C = \Gamma/R_*$ , gives  $E_t[m_{t+1} \dots m_{t+T-1} X_{t+T}] = \left( \frac{\Gamma}{R_*} \right)^{-T} \frac{X_t}{m_t}$ . Thus,

$$R_*^T Z_t(T) = R_*^{T-1} Z_t(T-1) - R_*^{-1} \beta' \Gamma^T \frac{X_t}{m_t}$$

and so, summing from 1 to  $T$ ,

$$R_*^T Z_t(T) = 1 - R_*^{-1} \beta' \sum_{s=1}^T \Gamma^s \frac{X_t}{m_t} = 1 - R_*^{-1} \beta' \Gamma \frac{I_n - \Gamma^T}{I_n - \Gamma} \frac{X_t}{m_t}.$$

## Appendix C. A class of processes admitting a closed form for perpetuities

The following Proposition shows a way to generate processes that have closed form for perpetuities. However, typically that functional process does not yield a closed form for finite-maturity claims, unlike quadratic processes.

**Proposition 11** (*Class of processes admitting a closed form for perpetuities*) Suppose that there is a process  $\psi_t$ , and constants  $\alpha$  and  $\beta$  such that

$$d\psi_t = (r_t \psi_t + \alpha r_t - \beta) dt + dM_t,$$

where  $dM_t$  is an adapted martingale, and is essentially arbitrary except for technical conditions. Then:

$$V_t = \frac{\psi_t + \alpha}{\beta} \tag{44}$$

is a solution of the perpetuity PDE:  $1 - r_t V_t + E[dV_t]/dt = 0$ . If  $dM_t$  satisfies regularity conditions that make the process well-defined, then  $V_t$  is the price of a perpetuity,  $V_t = E_t \left[ \int_t^\infty e^{-\int_t^s r_u du} ds \right]$ .

**Proof.** Call  $V_t = (\psi_t + \alpha)/\beta$ .

$$1 - r_t V_t + E[dV_t]/dt = 1 - r_t \frac{\psi_t + \alpha}{\beta} + \frac{1}{\beta} (r_t \psi_t + \alpha r_t - \beta) = 0.$$

■

A first example is the quadratic process:  $\psi(r) = r$ ,  $\alpha = -(\lambda + \mu)$ ,  $b = -\lambda\mu$ . Eq. 44 gives  $V_t = (\lambda + \mu - r_t) / (\lambda\mu)$ , the formula 3 for perpetuities.

A second example is

$$d(1/r_t) = \phi(r_t - r^*)dt + dM_t$$

where  $dM_t$  ensures  $r_t > 0$ , i.e.,  $r_t$  mean-reverts to a central value  $r^*$ . This yields a price for perpetuities equal to:

$$P_t = \left( \frac{1}{r_t} + \frac{\phi}{r^*} \right) / (1 + \phi) \quad (45)$$

by applying Proposition 11 to  $\psi(r) = 1/r$ ,  $\alpha = \phi$ ,  $b = 1 + \phi r^*$ . The formula (45) admits the following interpretation. If  $r_t$  remains at its current value, the perpetuity price would be  $1/r_t$ . If  $r_t$  was at its long run central value  $r^*$ , the perpetuity price would be  $1/r^*$ . In general, the perpetuity price (45) is a simple weighted sum of those two perpetuity values.  $\phi$  indicates the speed of mean reversion, and gives the weight between the “current value”  $1/r_t$  and the “long run value”  $1/r^*$ .

## References

- Ait-Sahalia, Yacine, “Testing Continuous-Time Models of the Spot Interest Rate,” *Review of Financial Studies*, 1996, 9, 385-426.
- Bansal, Ravi and Amir Yaron (2004). “Risks for the long run: A potential resolution of asset pricing puzzles,” *Journal of Finance* 59, 1481-1509.
- Barro, Robert, 2006, “Rare disasters and asset markets in the twentieth century”, *Quarterly Journal of Economics*, 121.
- Brigo, Damiano and Fabio Mercurio, *Interest rate models, Theory and practice*. Springer (Berlin, New York), 2001.
- Brown, Stephen J., William N. Goetzmann, Stephen A. Ross, “Survival”, *Journal of Finance*, 50, 1995.
- Campbell, John Y. and John Cochrane “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior”, *Journal of Political Economy*, 1999.
- Campbell, John Y. and Robert J. Shiller, 1988, “The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors,” *Review of Financial Studies* 1, 195-228.
- Cochrane, John, 2006, “The Dog That Did Not Bark: A Defense of Return Predictability,” NBER WP.
- Collin-Dufresne, Pierre and Robert S. Goldstein, 2002, “Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility,” *Journal of Finance* 57.
- Cox, John C, Ingersoll, Jonathan E, Jr, and Ross, Stephen A, 1985. “A Theory of the Term Structure of Interest Rates,” *Econometrica*, vol. 53(2), pages 385-407, March.

- Croce, Massimiliano, Martin Lettau and Sidney Ludvigson, "Investor Information, Long-Run Risk, and the Duration of Risky Cash Flows," NYU WP, 2006.
- Driessen, Joost, Pascal Maenhout and Grigory Vilkov, "Option - Implied Correlations and the Price of Correlation Risk," Insead Working Paper, 2005.
- Duffie, Darrell, *Dynamic Asset Pricing Theory*, Princeton University Press, 2001.
- Duffie, Darrell and Rui Kan, "A Yield-Factor Model of Interest Rates," *Mathematical Finance*, vol. 6 (1996), no. 4, 379-406.
- Duffie, Darrell, Jun Pan and Kenneth Singleton "Transform Analysis and Asset Pricing for Affine Jump Diffusions," *Econometrica*, Vol. 68 (2000), pp. 1343-1376
- Fama, Eugene and Kenneth R. French, "Multifactor Explanations of Asset Pricing Anomalies," *The Journal of Finance*, 1996.
- Gabaix, Xavier and David Laibson, "The 6D Bias and the Equity Premium Puzzle," *NBER Macroeconomics Annual*, MIT Press, 2002, Ben Bernanke and Ken Rogoff ed., vol. 16, pp. 257-312.
- Gabaix, Xavier; Gopikrishnan, Parameswaran; Plerou, Vasiliki and Stanley, H. Eugene. "A Theory of Power Law Distributions in Financial Market Fluctuations." *Nature*, 2003, 423(6937), pp. 267-230.
- Gabaix, Xavier; Gopikrishnan, Parameswaran; Plerou, Vasiliki and Stanley, H. Eugene. "Institutional Investors and Stock Market Volatility." *Quarterly Journal of Economics*, 1999, 121 (2), pp. 461-504.
- Hansen, Lars, John C. Heaton and Nan Li "Consumption Strikes Back?: Measuring Long Run Risk," WP, 2005.
- Hansen, Lars, and José Scheinkman, "Long Term Risk: an Operator Approach," WP, 2005.
- Heath, David, Robert Jarrow, Andrew Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, Vol. 60, No. 1 (Jan., 1992) , pp. 77-105
- Julliard, Christian, and Jonathan Parker, "Consumption Risk and the Cross-Section of Expected Returns," *Journal of Political Economy*, 113(1), February 2005.
- Karlin, Samuel and Howard M. Taylor, 1981, *A second course in stochastic processes*, Academic Press.
- Litterman, Robert and José Scheinkman, "Common factors affecting bond returns," *Journal of Fixed Income*, June, 54-61, 1991.
- Menzly, Lior, Tano Santos and Pietro Veronesi, "Understanding Predictability," *Journal of Political Economy*, February 2004, pp. 1-47
- Parker, Jonathan, "The Consumption Risk of the Stock Market," *Brookings Papers on Economic Activity*, Vol 2, (2001) 279-348.

Pastor, Lubos and Pietro Veronesi, "Rational IPO Waves," *Journal of Finance*, 60, August 2005, 1713 – 1757.

Vasicek, Oldrich "An equilibrium characterization of the term structure," *Journal of Financial Economics* 5 (1977) 177-188



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