MIT Sloan School of Management

MIT Sloan School Working Paper 4703-08
5/9/2008

Behavior Modes, Pathways and Overall Trajectories: Eigenvector and Eigenvalue Analysis of Dynamic Systems

Paulo Gonçalves

Behavior Modes, Pathways and Overall Trajectories: Eigenvector and Eigenvalue Analysis of Dynamic Systems

Paulo Gonçalves*

Abstract

One of the most fundamental principles in system dynamics is the premise that the structure of the system will generate its behavior. Such philosophical position has fostered the development of a number of formal methods aimed at understanding the causes of model behavior. To most in the field of system dynamics, behavior is commonly understood as *modes of behavior* (e.g., exponential growth, exponential decay, and oscillation) because of their direct association with the feedback loops (e.g., reinforcing, balancing, and balancing with delays, respectively) that generate them. Hence, traditional research on formal model analysis has emphasized which loops cause a particular "mode" of behavior, with *eigenvalues* representing the most important link between structure and behavior. The main contribution of this work arises from a choice to focus our analysis in the overall trajectory of a state variable – a broader definition of behavior than that of a specific behavior mode. When we consider overall behavior trajectories, contributions from *eigenvectors* are just as central as those from *eigenvalues*. Our approach to understanding model behavior derives an equation describing overall behavior trajectories in terms of both eigenvalues and eigenvectors. We then use the derivatives of both eigenvalues and eigenvectors with respect to link (or loop) gains to measure how they affect overall behavior trajectories over time. The direct consequence of focusing on behavior trajectories is that system dynamics researchers' reliance on eigenvalue elasticities can be seen as too-narrow a focus on model behavior – a focus that has excluded the short term impact of a change in loop (or link) gain in its analysis.

Keywords: Formal model analysis; overall trajectories; behavior modes; eigenvectors; eigenvalues; eigenvalue elasticity; loop dominance; behavior contribution.

Paulo Gonçalves is a Visiting Assistant Professor at the MIT Sloan School of Management, 30 Wadsworth St. E53-339, Cambridge, MA, 02142. Phone (617) 253-3886. (Email: paulog@mit.edu). I am grateful to Jim Hines and Christian Kampmann for their valuable comments and insightful suggestions on earlier versions of the paper. I also thank the helpful comments of two anonymous reviewers and participants of the 2006 International System Dynamics Conference.

1. Introduction

The premise that structure generates behavior is one of the fundamental principles in system dynamics, second only to the concept of information feedback.[i] The importance of the connection between structure and behavior is easily seen in Forrester's introduction to Industrial Dynamics (1961) and subsequent interpretations:

- "Information-feedback systems… owe their behavior to three characteristics – structure, delays, and amplification." (Forrester 1961, p15);
- "The system dynamics approach … takes the philosophical position that feedback structures are responsible for the changes we experience over time. The premise is that *dynamic behavior is a consequence of system structure.*" (Richardson and Pugh 1981, p15, emphasis in original); and
- "A fundamental principle of system dynamics states that the structure of the system will give rise to its behavior." (Sterman 2000, p 28).

While "solving complex problems in [feedback] systems require understandings of the relationships between feedback structure and the problematic behavior observed" (Richardson and Pugh 1981, p12), researchers interested in formal model analysis have traditionally interpreted "behavior" in a very restrictive sense. Conventionally, "behavior" has been readily associated with "behavior modes," that is, modes such as exponential growth, exponential decay or oscillations directly associated with reinforcing loops, balancing loops or higher order balancing loops and the eigenvalues they generate (positive real, negative real and positive/negative complex). This narrow interpretation of behavior has its roots both in the system dynamics modeling approach as well as the historical process of model analysis.

Consider first the role of the system dynamics modeling approach. Richardson and Pugh (1981, p19) state that to begin the system dynamics process "one defines problems dynamically, that is, in terms of graphs of variables over time." Then, one must formulate a dynamic hypothesis, where "[t]he *dynamic hypothesis* is a statement of feedback structures that are conjectured to have the power to create or at least contribute to problem behavior." (Richardson and Pugh 1981, p63, emphasis in original). Because there are only two types of feedback processes (balancing and reinforcing), generating characteristic modes of behavior (exponential growth, decay or oscillations), the dynamic hypotheses can easily focus on theories that can potentially generate the problematic behavior over time.

Incidentally, the process by which we build our dynamic theories directly influences how we develop intuition about model behavior, that is, the process by which we analyze models (the second component of our interpretation of behavior). “By exploring the behavior generated by individual feedback loops … the modeler learns about structure and behavior…Simulation experiments isolating and combining [feedback loops] … can precisely pinpoint the structure responsible.” (Richardson and Pugh 1981, p268). The feedback loop is defined as the structural unit of analysis because it provides a more adequate way of characterizing the cause of behavior. For instance, is it more appropriate to consider the growth in population as caused by an increase in births, or an increase in births as caused by population growth? Since both population and births are causally linked, the reinforcing population feedback loop is a better unit of analysis to explain the growth in births (and population) behavior.

Due to the focus on feedback loops as the unit of analysis for causes of behavior, the discussion developed into how different loops and how shifts in loop dominance could influence observed behavior. Nathan Forrester (1982) discusses two traditional methods used to indentify dominant feedback loops. “The first method involves disconnecting unimportant loops and showing that the remaining, isolated loops produce behavior similar to that of the whole model. The second approach involves making small changes in model behavior. Loops containing influential parameters are assumed to be dominant.” (N. Forrester 1982, p178). Shortly after, Richardson (1984) provides rigorous definitions for the important building blocks for loop analysis, such as loop polarity, loop dominance and shifts in loop dominance. The useful notion of dominant feedback loops as drivers of behavior is common today. According to Sterman (2000, p897) “several methods exist to identify the dominant loops at any point in a simulation, quantify the contribution of any parameter or loop to a given mode [of behavior], and show how nonlinearities change the dominant feedback structure.”

The main contribution of this work arises from a choice to focus our analysis in a broader definition of behavior, which differs from the definition adopted by prior research in formal model analysis. To many in the field of system dynamics behavior is commonly understood as “modes of behavior” (e.g., exponential growth, exponential decay and oscillations) because of their direct association with the feedback loops that generate them. Hence, traditional research on formal model analysis emphasizes which loops cause a particular “mode” of behavior. In such context, eigenvalues are the most important link between structure and behavior in model analysis; and, considerations about eigenvectors and their contributions are largely irrelevant.

However, when we consider behavior more broadly in terms of overall behavior trajectories, contributions from *eigenvectors* are just as central as those from *eigenvalues*. Our approach to understanding model behavior derives an equation describing overall behavior trajectories in terms of both eigenvalues and eigenvectors. We then use the derivatives of both eigenvalues and eigenvectors with respect to link (or loop) gains to measure how they affect overall behavior trajectories over time. The direct consequence of focusing on behavior trajectories is that system dynamics researchers' reliance on eigenvalue elasticities can be seen as too-narrow a focus on model behavior – a focus that has excluded the short term impact of a change in loop (or link) gain in its analysis.

2. Literature review

Formal model analysis remains an important and challenging area in system dynamics. Several methods aimed at understanding the causes of model behavior have been proposed in recent years (Kampmann 1996; Mojtahedzadeh 1997; Gonçalves, Lertpattarapong and Hines 2000; Saleh and Davidsen 2001; Saleh 2002; Mojtahedzadeh, Richardson and Andersen 2004; Oliva 2004; Oliva and Mojtahedzadeh 2004; Güneralp 2005; Hines 2005; Kampmann and Oliva 2005; Saleh, Davidsen and Bayoumi 2005). These methods trace back two threads in model analysis: the loop dominance work of Richardson (1984) and eigenvalue elasticity work of Forrester (1982).

Mojtahedzadeh (1997) and Mojtahedzadeh, Richardson and Andersen (2004) extend the loop dominance work first proposed by Richardson (1984). The research proposes pathway participation metrics (PPM) to find the structure that most influences the time path of a given variable. The PPM method provides a local assessment of how changes in a state variable of interest influence the net change of the same variable ($d\dot{x}_k/dx_k$). Furthermore, the ratio $d\dot{x}_k/dx_k$ can be transformed into a ratio between $d\dot{x}_k/dt$ and dx_k/dt, i.e., a ratio between the curvature and slope of state x_k at time *t*. Because the method captures information on both the curvature and slope of the behavior of state x_k at time *t*, it has valuable information about the local behavior of state variable x_k. The quantity $d\dot{x}_k/dx_k$ is called the Total Participation Metric and can be partitioned among pathways that contribute to the a net-flow influencing state variable x_k. Since several pathways will affect the state, the PPM method computes which pathways are most influential, defined as the pathway "whose participation is the largest in magnitude and has the same sign as the total changes in the net-flow X when it is disturbed by a infinitesimal change in the state variable at the tail of the pathway." (Mojtahedzadeh, Richardson and Andersen 2004). The method has the advantage of being computationally simple. More important, while the Total Participation Metric is obtained from slopes and curvatures computed at a specific time *t*, researchers applying the PPM method are interested in the overall trajectory of a state variable.

Most of the remaining research traces back to eigenvalue elasticity theory originally proposed by Perez–Arriaga (1981) and introduced to the system dynamics field by Nathan Forrester (1982). The method calls for the computation of eigenvalues and then explores how the eigenvalues change as link gains change, that is, link gain elasticities. Forrester showed that a complete description of *link* elasticities allows one in principle to calculate *loop* elasticities. This suggestion though never implemented in software, promised to provide an answer to how model structure, that is a set of feedback loops, determines model behavior. The particular calculation that Forrester suggested is actually not feasible. As he realized later, Forrester's suggested approach results in a system of equations that is over-determined – an effect of the fact that the number of loops increases much faster than the number links. Kampmann discovered that a small subset of loops is sufficient to uniquely describe eigenvalues (i.e. the behavior) of a system dynamics model (Kampmann 1996). Using an Independent Loop Set (ILS) produces a smaller system of equations, a system that can be solved. The Independent loop set (ILS) method has the important advantage of allowing us to calculate loop gains from link gains, where the number of links in a model is often small. However, it has the disadvantage of relying on an *ad hoc* procedure to select the independent loop set (ILS).

Gonçalves, Lertpattarapong and Hines (2000) use Mason's rule to express the characteristic equation and its solutions (eigenvalues) in terms of loop gains (instead of link gains), which allows them to obtain *loop* gain elasticities directly. The method has the advantage of sidestepping the problem associated with an arbitrary selection of loops, however, it has the shortcoming of requiring the computation of all loop gains and cycle compositions in the model to obtain the characteristic polynomial. While the maximum number of loops rise quickly even for moderately sized models, it is unlikely that the rise will exceed current computational power.

Oliva (2004) provides an extension to the method selecting first the shortest loops. The shortest independent loop set (SILS) provides a systematic representation of the feedback complexity in its simplest components and it is the most granular description of the structure in a cycle partition. Oliva and Mojtahedzadeh (2004) compare the results obtained with the SILS approach to that of PPM and find that the loops generating the main dynamics are often included in the SILS. More recently, Kampmann and Oliva (2006) explore the application of loop eigenvalue elasticity to three models to assess the potential of the method and find that the insights depend on the character and dynamics of the model. The work of Saleh, Davidsen and Bayoumi (2005) is most akin to ours in its interest in understanding the contribution of both eigenvalues and eigenvectors on model behavior. While we focus on the analytical computation of the influence of eigenvalues and eigenvectors on model behavior, Saleh et al. (2005) provide a computational method (implemented in Matlab) to calculate such influence. Automated approaches that allow researchers to understand how changes in the structure of their models affect overall behavior are fundamental to policy design. Our work provides a mathematical framework for future research and automated engines using the contribution of both eigenvectors and eigenvalues for formal model analysis.

3. How Links Influence Overall State Trajectories

A linear system dynamics model with a vector of state variables $\mathbf{x}$(t), where $\mathbf{x}$(t) = ($x_1, x_2, \ldots, x_n$)', a vector of first time derivatives of the state variables $\dot{\mathbf{x}}$(t), where $\dot{\mathbf{x}}$(t) = ($\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$)', and a gain matrix $\mathbf{A}$ capturing the partial derivatives of the net change of a state variable with respect to another ($\mathbf{A}_{\mathbf{n\,x\,n}} = \partial\dot{\mathbf{x}}/\partial\mathbf{x}$), can be represented compactly in the following way:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1}$$

The linear system above can be solved if $\boldsymbol{A}$ is not degenerate (see Appendix A for details of this derivation), leading to:

$$\mathbf{x}(t) = \mathbf{R}\mathbf{z}(t) \tag{2}$$

where **R** is the matrix of right eigenvectors and $\mathbf{z}(t) = (z_1(t), z_2(t), \ldots\ z_n(t))'$ is a column vector.

Expanding equation (2) to write the individual eigenvectors and components of $\mathbf{z}(t)$ yields:

$$\mathbf{x}(t) = e^{\lambda_1 t} z_1(0)\mathbf{r_1} + e^{\lambda_2 t} z_2(0)\mathbf{r_2} + \ldots + e^{\lambda_n t} z_n(0)\mathbf{r_n} \tag{3}$$

The behavior of each state $x_i(t)$ in the system can be described by:

$$x_i(t) = r_{1i} e^{\lambda_1 t} z_1(0) + r_{2i} e^{\lambda_2 t} z_2(0) + \ldots + r_{ni} e^{\lambda_n t} z_n(0) \tag{4}$$

where r_{1i} is the i-th component of the first eigenvector.

Equation 4 highlights that the overall behavior trajectory of state variable $x_i(t)$ is determined by the linear combination of the product of eigenvector components (r_{ji}), behavior mode ($e^{\lambda j t}$) generated by eigenvalue (λ_j) and initial condition ($z_j(0)$).

Also, we could easily rewrite equation (4) in matrix form, to obtain:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \ldots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & \ldots & r_{n1} \\ r_{12} & r_{22} & \ldots & r_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ r_{1n} & r_{2n} & \ldots & r_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \ldots \\ e^{\lambda_n t} z_n(0) \end{bmatrix} \tag{5}$$

Note that in the traditional focus on behavior modes, model analysis might emphasize on understanding why state $x_i(t)$ oscillates or grows exponentially according to a behavior mode that is best characterized by a specific eigenvalue (λ_j). Researchers will traditionally characterize eigenvalue λ_j as the dominant behavior mode and will search for clues that inform which parameters might influence the strength of such eigenvalue.

If instead we are interested in the overall behavior trajectory of the state variable $x_i(t)$, we observe that it will be determined not only by the mode of behavior ($e^{\lambda_j t}$) due generated by eigenvalue (λ_j), but also by the influence of each j-th component of each eigenvector (r_{ji}). The equations also highlight that the behavior of each state variable $x_i(t)$ is influenced both by eigenvalues (λ_j) and eigenvector components (r_{ji}). In addition, both eigenvalues (λ_j) and eigenvector components (r_{ji}) depend on the values of link gains (i.e., parameters in the model), because eigenvalues are solutions to the characteristic polynomial ($P(\lambda)$), where $P(\lambda) = |\lambda I_n - \mathrm{A}| = 0$ and the entries of the ***A*** *matrix* are parameters (i.e., the partial derivatives or the link gains (a_{kl})) in a system dynamics model. Furthermore, we compute eigenvectors by solving a system of equations ($\boldsymbol{Ar_i}=\lambda_i\boldsymbol{r_i}$) that depend on the value of eigenvalues. Therefore, a change in the gain of an arbitrary link (a_{kl}) results in a new ***A*** *matrix* and different values for both eigenvalues (λ_i) and eigenvector components (r_{ji}).

To understand the nature of the impact of changes in link gains on overall system behavior, we take the partial derivative of each state variable $x_i(t)$ in the system with respect to its link gains. From equation (4), we obtain the change in overall behavior of each state variable $x_i(t)$ due to changes in link gain (a_{kl}) as:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \frac{\partial}{\partial a_{kl}}\left[r_{1i}e^{\lambda_1 t}z_1(0) + \ldots + r_{ni}e^{\lambda_n t}z_n(0)\right] \tag{6}$$

and taking the derivative of individual components, we obtain:[ii]

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \frac{\partial r_{1i}}{\partial a_{kl}}e^{\lambda_1 t}z_1(0) + r_{1i}\frac{\partial e^{\lambda_1 t}}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial a_{kl}}z_1(0) + \ldots + \frac{\partial r_{ni}}{\partial a_{kl}}e^{\lambda_n t}z_n(0) + r_{ni}\frac{\partial e^{\lambda_n t}}{\partial \lambda_n}\frac{\partial \lambda_n}{\partial a_{kl}}z_n(0) \tag{7}$$

Rewriting equation (7) in a more compact way, we get:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n}\left(\frac{\partial r_{ji}}{\partial a_{kl}}e^{\lambda_j t} + r_{ji}\frac{\partial e^{\lambda_j t}}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial a_{kl}}\right)z_j(0) \tag{8}$$

Because the eigenvalues and eigenvectors in liner systems are constant, the derivative of the exponential of the j-th behavior mode ($e^{\lambda jt}$) with respect to its eigenvalue (λ_j) yields a term that depends on time ($te^{\lambda jt}$). Therefore, we can rewrite equation (8) to yield:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n}\left(\frac{\partial r_{ji}}{\partial a_{kl}} + r_{ji}\frac{\partial \lambda_j}{\partial a_{kl}}t\right)e^{\lambda_j t}z_j(0) \tag{9}$$

Equation (9) suggests that a change in behavior of state $x_i(t)$ due to a change in link gain (a_{kl}) will be composed by two terms for each behavior mode ($e^{\lambda jt}$) contributing to the overall behavior trajectory of state variable $x_i(t)$. Each of the terms corresponds to:

1. The derivative of eigenvector component (r_{ji}) with respect to link gain (a_{kl}); and
2. The product of eigenvector component (r_{ji}), the derivative of eigenvalue (λ_i) with respect to link gain (a_{kl}), and time (t).

The first term captures the change in weight in behavior mode ($e^{\lambda jt}$) due to the partial derivative of eigenvector component (r_{ji}) with respect to link gain (a_{kl}). The second term captures a more complicated change in weight in behavior mode ($e^{\lambda jt}$). The weight changes with time, eigenvector component (r_{ji}) and the partial derivative of eigenvalue (λ_i) with respect to link gain (a_{kl}). Note that, if eigenvalues (λ) and eigenvectors ($\boldsymbol{r}$) are complex their derivatives will also be complex. In such cases, the exponentials will be multiplied by complex values which will influence not only the amplitude of the behavior mode, but will also lead to a phase shift (see derivation in appendix B).

The equation above suggests that early in time ($t \cong 0$),behavior mode ($e^{\lambda jt}$) will be mainly influenced by the first term, i.e., the derivative of the eigenvector with respect to the link gain; and later on (as $t \to \infty$), behavior mode ($e^{\lambda jt}$) will be more influenced by the second term, i.e., the derivative of the eigenvalue with respect to the link gain. In a linear system, the weight of behavior mode ($e^{\lambda jt}$) will be highly determined by the second term at high values of time (t) – determined by the value of $r_{ji}\dfrac{\partial \lambda_j}{\partial a_{kl}}$. Since most research in model analysis has dealt with eigenvalue elasticity – closely associated with the derivative of the eigenvalue with respect to link gains $\left(\partial \lambda_j / \partial a_{kl}\right)$ – we have focused myopically at the long term behavior impact of a link change. That is, we have focused on how changes in links (or loops) affect the long term behavior mode of a state variable. However, since we care deeply about transient behavior it is important to characterize the likely impact of short term behavior due to link (or loop) changes.

3.1. Interpreting the Impact on Behavior Modes

To understand and interpret the impact that a change in a link gain has on each behavior mode composing the overall trajectory of a state variable, it is useful to consider the ratio between the changed weight in the behavior mode due to the change in link gain and the original weight. Note that the ratio can be a complex number. The real part of the ratio determines a factor that multiplies the behavior mode, either amplifying or dampening it. The complex part determines a phase gain to the behavior mode. To obtain the behavior mode impact, we must divide each component in equation (9) by the corresponding component in equation (4):

$$\frac{\partial x_{ij}(t)/\partial a_{kl}}{x_{ij}(t)} = \frac{\left(\frac{\partial r_{ji}}{\partial a_{kl}} + r_{ji}\frac{\partial \lambda_j}{\partial a_{kl}}t\right)e^{\lambda_j t}z_j(0)}{r_{ji}e^{\lambda_j t}z_j(0)} = \frac{1}{r_{ji}}\frac{\partial r_{ji}}{\partial a_{kl}} + \frac{\partial \lambda_j}{\partial a_{kl}}t \tag{10}$$

Equation (10) reemphasizes the role that the first time derivatives of both eigenvector and eigenvalue with respect to the link gain have on each behavior mode ($e^{\lambda_j t}$) influencing the overall trajectory of state $x_i(t)$. Since the ultimate goal of formal model analysis is inform policy, it is important to compute the overall impact of changes by a link (or loop) gain to the overall behavior trajectory of specific states. This overall impact requires addition of the individual impacts of different modes. Since the overall trajectory is composed by a mix of behavior modes (oscillatory, exponential growth and decay) and their weights change with time, automated implementation of the method will provide a mechanism to visualize the result from changes in link gains, and subsequent policy design, by selecting links (or loops) to change to obtain the desired behavior.

3.2. System Behavior: Link Eigenvalue and Link Eigenvector Sensitivities

In equation (9), the partial derivatives of eigenvalue (λ_i) and eigenvector component (r_{ji}) with respect to a link gain (a_{kl}), respectively $\frac{\partial \lambda_j}{\partial a_{kl}}$ and $\frac{\partial r_{ji}}{\partial a_{kl}}$, can be understood in the context of previous work on link gain eigenvalue elasticity (N. Forrester 1982, 1983). According to Nathan Forrester (1982, 1983), $\frac{\partial \lambda_j}{\partial a_{kl}}$ measures the sensitivity of eigenvalue (λ_i) with respect to link (a_{kl}), which allows us to understand how the strength of a link (a_{kl}) can impact behavior mode ($e^{\lambda_j t}$).

$$S_{\lambda_i kl} = \frac{\partial \lambda_i}{\partial a_{kl}} \tag{11}$$

It is possible to normalize the sensitivity measure defined above (11) to isolate the effect of the change in link gain from the magnitude of the eigenvalue and link gain. This normalization can be obtained multiplying the sensitivity by the ratio of the magnitude of the link gain (a_{kl}) to the magnitude of the eigenvalue (λ_i). Nathan Forrester (1983) defined this measure *eigenvalue elasticity with respect to link gain* or *link gain (eigenvalue) elasticity*.

$$E_{ikl} = \frac{\partial \lambda_i}{\partial a_{kl}} \frac{|a_{kl}|}{\|\lambda_i\|} \tag{12}$$

where $|a_{kl}|$ is the absolute value of the link gain and $\|\lambda_i\|$ is the Euclidean norm of a potentially complex eigenvalue (λ_i).

Note that the partial derivative of the eigenvalue (λ_i) with respect to that link gain (a_{kl}) is present in the second term of equation (9) characterizing how a change in a link gain would affect behavior mode ($e^{\lambda_j t}$).

While it has been suggested that eigenvector elasticity would be required to understand how structure ultimately influences behavior, there is little research implementing it (a welcome exception is Saleh et al. 2005). To incorporate eigenvector elasticity in formal model analysis, we must first define it. Let the elasticity eigenvector component (r_{ji}) with respect to a link gain (a_{kl}) be defined in a similar way as the link gain eigenvalue elasticity. First, let $\frac{\partial r_{ji}}{\partial a_{kl}}$ define the sensitivity of an eigenvector component (r_{ji}) with respect to a specific link (a_{kl}). The eigenvector component (r_{ji}) sensitivity provides a measure of how the strength of link gain (a_{kl}) impacts eigenvector component (r_{ji}).

$$S_{r_{ij}kl} = \frac{\partial r_{ij}}{\partial a_{kl}} \tag{13}$$

Next, it is possible to normalize the eigenvector component sensitivity measure to isolate the effect of the change in link gain from the magnitude of the eigenvector component and link gain. This normalization can be obtained by multiplying the sensitivity by the ratio of the magnitude of the link gain (a_{kl}) to the magnitude of the eigenvector component (r_{ij}). Finally, define this measure as the *eigenvector component* (r_{ji}) *elasticity with respect to link gain* or *link gain eigenvector component elasticity*.

$$E_{r_{\mathrm{ij}}kl} = \frac{\partial\, r_{\mathrm{ij}}}{\partial\, a_{kl}} \frac{|a_{kl}|}{\|r_{\mathrm{ij}}\|} \tag{14}$$

where $|a_{kl}|$ is the absolute value of the link gain and $\|r_{\mathbf{ij}}\|$ is the Euclidean norm of the eigenvector component (r_{ij}).

Note that the partial derivative of the eigenvector component (r_{ij}) with respect to the link gain (a_{kl}) is present in the first term of equation (9) characterizing how a change in link gain (a_{kl}) affects the weight of behavior mode ($e^{\lambda_j t}$).

While the notion of link gain eigenvalue and eigenvector component elasticities are useful, equation (9) provides an integrated way to assess how eigenvalue and eigenvector component *sensitivities* work together to influence the weight of behavior mode ($e^{\lambda_j t}$). Rewriting equation (9) using eigenvalue and eigenvector component sensitivities, we obtain:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n} \left(S_{r_{ij}kl} + r_{ji} S_{\lambda_j kl} t \right) e^{\lambda_j t} z_j(0) \qquad (15)$$

- Eigenvector component sensitivity $S_{r_{ij}kl} = \frac{\partial r_{ji}}{\partial a_{kl}}$ captures the change in weight in behavior mode ($e^{\lambda_j t}$) due to a change in a link gain (a_{kl});
- Eigenvalue sensitivity $S_{\lambda_j kl} = \frac{\partial \lambda_j}{\partial a_{kl}}$ captures the change in weight in the behavior mode ($e^{\lambda_j t}$) due to a change in the link gain (a_{kl}).
- The contribution of the eigenvalue sensitivity to the weight changes with time and it becomes the main determinant of weight of behavior mode ($e^{\lambda_j t}$) as time grows.

4. Behavior in Nonlinear Dynamic Systems

The method of analysis described above applies only to linear systems. However, most system dynamics models are nonlinear. Hence, we cannot apply it right away and instead we must consider ways to apply the results derived for linear systems to nonlinear ones. One possibility to apply the method to nonlinear systems is to linearize the system. The local linearization option is limited, however, because linearized solutions are good approximations of nonlinear systems solutions only close to the operating point. By linearizing the nonlinear system at every point in time (in practice, every time step in the simulation), however, the analysis can be generalized to the rest of the system, providing insight into how change in link gains influence the behavior trajectory of interest. Considering how the overall trajectory, $x_i(t)$, of a linearized system, might be affected by a change in link gain (a_{kl}) at the linearization time (t_0) yields:

$$\frac{\partial x_i(t)}{\partial a_{kl}} = \sum_{j=1}^{n} \left(S_{r_{ij}kl} + r_{ji} S_{\lambda_j kl} t \right) e^{\lambda_j (t - t_0)} z_j(t_0) \tag{16}$$

where each $z_j(t_0)$ refers to the position of the system at the linearization time (t_0).

Since the linearized system provides a good approximation to the nonlinear system only close to the operating point, we only care about solutions to equation (16) that happen early in time ($t \cong t_0$). The result of equation (16) at later times ($t \to \infty$) departs too far from where the system is a close approximation to the nonlinear system. Hence, for nonlinear systems that are linearized at every point in time, the impact of a change in link gain on overall system behavior can be simplified by substituting $t \cong t_0$ in equation (16). Despite the additional complexity of nonlinear systems, by linearizing the system at every point in time and then considering the impact of the link gains, we arrive at a general solution that is similar to that of a linear system. Equation (16) suggests that *eigenvector component* sensitivity also plays an important role in determining the impact that a change in structure has on model behavior in nonlinear systems. The equation above also provides a framework to include *eigenvector component* sensitivity in the formal model analysis research.

5. Application to a Linear System: The Inventory-Workforce Oscillator

We illustrate the concepts above with a version of the familiar inventory–workforce model. The model captures a simple production system that attempts to maintain inventory at the desired level by adjusting production through hiring and firing workers. More precisely, inventory integrates the difference between production and shipments. Shipments are determined by demand reduced by stock-outs, should inventory fall too low. Production depends on the available workforce and its productivity. Workforce level is “anchored” to the level necessary to meet expected demand with normal productivity. The workforce is adjusted above (below) this anchor when inventory is below (above) desired inventory. Expected demand is given by a first order exponential smooth of actual demand.

A stock and flow diagram of the model is shown below. The model is composed of three state variables (inventory, workforce, and expected demand), four flows (producing, shipments, hiring/firing rate, and change in demand), three auxiliary variables (desired workforce, desired producing, and inventory correction), six constants (desired inventory, correction time, hire/fire time, time to change in expectations, minimum processing time, and productivity), and one exogenous variable (demand).

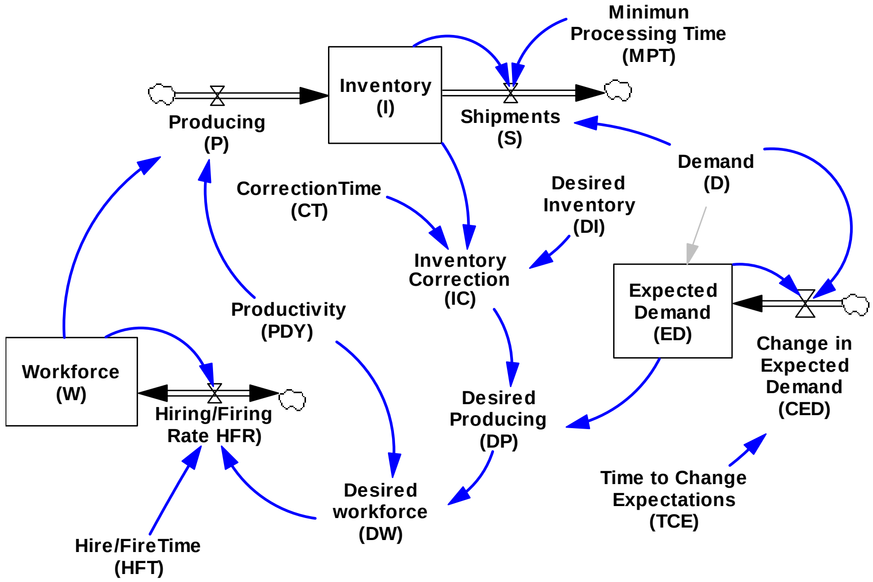

Figure 1 – Diagram of the linear inventory-workforce system dynamics model.

$$\dot{I} = P - S = PDY \cdot W - D \qquad IC = (DI - I)/CT$$

$$\dot{W} = HFR = (DW - W)/HFT \qquad DP = IC + ED$$

$$\dot{ED} = CED = (D - ED)/TCE \qquad DW = DP/PDY$$

The ***A*** matrix of the system above leads to the following relation:

$$\mathrm{J} = \begin{bmatrix} 0 & PDY & 0 \\ -1/HFT \cdot PDY \cdot CT & -1/HFT & 1/HFT \cdot PDY \\ 0 & 0 & -1/TCE \end{bmatrix}$$

Alternatively, we could have written the ***A*** matrix of the system in terms of loop gains. This system has three loops:

1. Workforce adjustment: a minor balancing loop adjusting workforce (*W*), with a loop gain of $g_1 = -1/HFT$.
2. Demand adjustment: a minor balancing loop adjusting demand (*ED*), with a loop gain of $g_2 = -1/TCE$.

3. Inventory–workforce: a major balancing loop linking inventory and workforce (*W*), with a loop gain of $g_3 = -1/(CT^{*}HFT)$.

We can rewrite the ***A*** matrix in terms of the loop gains to obtain:[iii]

$$\mathbf{A} = \begin{bmatrix} 0 & PDY & 0 \\ g_3/PDY & g_1 & -g_1/PDY \\ 0 & 0 & g_2 \end{bmatrix}$$

We find the characteristic polynomial ($P(\lambda)$) of the ***A*** matrix in terms of the loop gains by computing the determinant of ($\lambda\boldsymbol{I}$-$\boldsymbol{A}$):

$$P(\lambda) = \lambda^3 + (-g_1 - g_2)\lambda^2 + (g_1 g_2 - g_3)\lambda + g_2 g_3$$

We find the eigenvalues of the ***A*** matrix, by computing the roots of the characteristic polynomial ($P(\lambda) = |\lambda\boldsymbol{I}\text{-}\boldsymbol{A}| = 0$):

$$\lambda_1 = g_2\,,\ \lambda_2 = \frac{g_1}{2} - \frac{1}{2}\sqrt{g_1^{\,2} + 4g_3}\ \text{, and }\ \lambda_3 = \frac{g_1}{2} + \frac{1}{2}\sqrt{g_1^{\,2} + 4g_3}$$

Next, we compute the eigenvectors of the system solving the system of equations $\boldsymbol{A}\boldsymbol{r_i} = \lambda_i \boldsymbol{r_i}$:

$$\mathbf{r_1} = \begin{bmatrix} \dfrac{g_1}{(g_1 - g_2)g_2 + g_3} \\ \dfrac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} \\ 1 \end{bmatrix};\ \mathbf{r_2} = \begin{bmatrix} -\dfrac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \\ 1 \\ 0 \end{bmatrix};\ \mathbf{r_3} = \begin{bmatrix} \dfrac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \\ 1 \\ 0 \end{bmatrix}$$

With the results for eigenvalues and eigenvectors we can write the equations for the behavior of each state $x_i(t)$ in the system according to the result in equation (4):

$$I(t) = \frac{g_1}{(g_1 - g_2)g_2 + g_3} e^{g_2 t} z_1(0) - \frac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} e^{\frac{1}{2}\left(g_1 - \sqrt{g_1^2 + 4g_3}\right)t} z_2(0) + \frac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} e^{\frac{1}{2}\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)t} z_3(0)$$

$$W(t) = \frac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} e^{g_2 t} z_1(0) + e^{\frac{1}{2}\left(g_1 - \sqrt{g_1^2 + 4g_3}\right)t} z_2(0) + e^{\frac{1}{2}\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)t} z_3(0)$$

$$ED(t) = e^{g_2 t} z_1(0)$$

To understand how the state variables are impacted by changes in the loop gains, we need to compute both the derivatives of eigenvalues and eigenvectors with respect to the loop gains. The two tables below present the necessary derivatives for eigenvalues and eigenvectors.

Table 1 – Derivatives of eigenvalues wrt loop gains for Inventory-Workforce model.

	Eigenvalue 1 $\lambda_1 = g_2$	Eigenvalue 2 $\lambda_2 = \frac{g_1}{2} - \frac{1}{2}\sqrt{g_1^{\,2} + 4g_3}$	Eigenvalue 3 $\lambda_3 = \frac{g_1}{2} + \frac{1}{2}\sqrt{g_1^{\,2} + 4g_3}$
Loop 1 – Workforce Adjustment (g_1)	$\frac{\partial\lambda_1}{\partial g_1} = 0$	$\frac{\partial\lambda_2}{\partial g_1} = \frac{1}{2}\left(1 - \frac{g_1}{\sqrt{g_1^{\,2} + 4g_3}}\right)$	$\frac{\partial\lambda_3}{\partial g_1} = \frac{1}{2}\left(1 + \frac{g_1}{\sqrt{g_1^{\,2} + 4g_3}}\right)$
Loop 2 – Demand Adjustment (g_2)	$\frac{\partial\lambda_1}{\partial g_2} = 1$	$\frac{\partial\lambda_2}{\partial g_2} = 0$	$\frac{\partial\lambda_3}{\partial g_2} = 0$
Loop 3 - Inventory – Workforce (g_3)	$\frac{\partial\lambda_1}{\partial g_3} = 0$	$\frac{\partial\lambda_2}{\partial g_3} = -\frac{1}{\sqrt{g_1^{\,2} + 4g_3}}$	$\frac{\partial\lambda_3}{\partial g_3} = +\frac{1}{\sqrt{g_1^{\,2} + 4g_3}}$

First, note that the derivative of the eigenvalues 2 and 3 are not influenced by loop gain 2 (the derivatives are equal to zero). Note also that loop 3 does not affect the real part of the complex eigenvalues (λ_2 and λ_3) and that increasing the gain of loop 1 (g_1) increases the dampening and decreases the frequency (f), i.e., increases the period (T), of oscillation. Note that frequency and period are inversely related ($f = 1/T$). Also, the complex part in the derivative has a different sign than the sign of the eigenvalue's complex part (b).[iv] Therefore**,** a change in g_1 decreases the complex part of the eigenvalue and since $f = 2\pi b$ (or $T = 2\pi/b$) a lower value of b leads to lower frequency (or, a longer period.) Analogously, increasing g_3 increases the frequency of oscillation, since the complex part of the derivative has the same sign as the sign of the eigenvalue's complex part (b).

Table 2 – Derivatives of eigenvectors wrt loop gains for Inventory-Workforce model.

	Eigenvector 1 $\mathbf{r_1} = \left[\frac{g_1}{(g_1 - g_2)g_2 + g_3} \quad \frac{g_1 g_2}{((g_1 - g_2)g_2 + g_3)PDY} \quad 1\right]$	Eigenvector 2 $\mathbf{r_2} = \left[-\frac{\left(g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \quad 1 \quad 0\right]$	Eigenvector 3 $\mathbf{r_3} = \left[\frac{\left(-g_1 + \sqrt{g_1^2 + 4g_3}\right)PDY}{2g_3} \quad 1 \quad 0\right]$
Loop 1 Workforce (g_1)	$\frac{\partial \mathbf{r_1}}{\partial g_1} = \left[\frac{-g_2^2 + g_3}{((g_1 - g_2)g_2 + g_3)^2} \quad \frac{(-g_2^2 + g_3)g_2}{((g_1 - g_2)g_2 + g_3)^2 PDY} \quad 0\right]$	$\frac{\partial \mathbf{r_2}}{\partial g_1} = \left[-\frac{PDY}{2g_3}\left(1 + \frac{g_1}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$	$\frac{\partial \mathbf{r_3}}{\partial g_1} = \left[\frac{PDY}{2g_3}\left(-1 + \frac{g_1}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$
Loop 2 Demand Adj. (g_2)	$\frac{\partial \mathbf{r_1}}{\partial g_2} = \left[\frac{-g_1(g_1 - 2g_2)}{((g_1 - g_2)g_2 + g_3)^2} \quad \frac{g_1(g_2^2 + g_3)}{((g_1 - g_2)g_2 + g_3)^2 PDY} \quad 0\right]$	$\frac{\partial \mathbf{r_2}}{\partial g_2} = [0 \quad 0 \quad 0]$	$\frac{\partial \mathbf{r_3}}{\partial g_2} = [0 \quad 0 \quad 0]$
Loop 3 Inventory-wkforce (g_3)	$\frac{\partial \mathbf{r_1}}{\partial g_3} = \left[\frac{-g_1}{((g_1 - g_2)g_2 + g_3)^2} \quad \frac{-g_1 g_2}{((g_1 - g_2)g_2 + g_3)^2 PDY} \quad 0\right]$	$\frac{\partial \mathbf{r_2}}{dg_3} = \left[\frac{PDY}{2g_3^2}\left(g_1 + \frac{g_1^2 + 2g_3}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$	$\frac{\partial \mathbf{r_3}}{dg_3} = \left[\frac{PDY}{2g_3^2}\left(g_1 - \frac{g_1^2 + 2g_3}{\sqrt{g_1^2 + 4g_3}}\right) \quad 0 \quad 0\right]$

Consider the impact of the changes of loop gains in the eigenvectors (table 2). Focusing mainly on the oscillatory eigenvalues let us consider the derivative of r_{21} with respect to g_1. First, the real part suggests that every incremental change in g_1 causes a multiplication of (-PDY/2g_3). The complex part of the derivative suggests a reduction in the complex value b, reducing the phase lag that it could have on the system behavior. Since the real and complex parts have different signs the inverse tangent that defines the phase lag would lead to a negative phase lag. Loop 3 has a positive impact on the phase lag. Incorporating the results from tables 1 and 2 in equation (9) provides an integrated way to assess how the partial derivatives of the states with respect to a loop gain influence system behavior.

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_1} \\ \frac{\partial W(t)}{\partial g_1} \\ \frac{\partial ED(t)}{\partial g_1} \end{bmatrix} = \begin{bmatrix} \frac{-g_2^2+g_3}{((g_1-g_2)g_2+g_3)^2} & -\frac{PDY}{2g_3\sqrt{g_1^2+4g_3}}\left(g_1+\sqrt{g_1^2+4g_3}+2g_3t\right) & \frac{PDY}{2g_3\sqrt{g_1^2+4g_3}}\left(g_1-\sqrt{g_1^2+4g_3}+2g_3t\right) \\ \frac{\left(-g_2^2+g_3\right)g_2}{((g_1-g_2)g_2+g_3)^2 PDY} & \frac{1}{2}\left(1-\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t & \frac{1}{2}\left(1+\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2t}z_1(0) \\ e^{\frac{1}{2}\left(g_1-\sqrt{g_1^2+4g_3}\right)t}z_2(0) \\ e^{\frac{1}{2}\left(g_1+\sqrt{g_1^2+4g_3}\right)t}z_3(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_2} \\ \frac{\partial W(t)}{\partial g_2} \\ \frac{\partial ED(t)}{\partial g_2} \end{bmatrix} = \begin{bmatrix} \left(\frac{-g_1(g_1-2g_2)}{((g_1-g_2)g_2+g_3)^2}+\frac{g_1}{(g_1-g_2)g_2+g_3}t\right) & 0 & 0 \\ \left(\frac{g_1\left(g_2^2+g_3\right)}{((g_1-g_2)g_2+g_3)^2 PDY}+\frac{g_1g_2}{((g_1-g_2)g_2+g_3)PDY}t\right) & 0 & 0 \\ t & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2t}z_1(0) \\ e^{\frac{1}{2}\left(g_1-\sqrt{g_1^2+4g_3}\right)t}z_2(0) \\ e^{\frac{1}{2}\left(g_1+\sqrt{g_1^2+4g_3}\right)t}z_3(0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_3} \\ \frac{\partial W(t)}{\partial g_3} \\ \frac{\partial ED(t)}{\partial g_3} \end{bmatrix} = \begin{bmatrix} \frac{-g_1}{((g_1-g_2)g_2+g_3)^2} & \frac{PDY}{2g_3}\left[\left(\frac{g_1}{g_3}+\frac{g_1^2+2g_3}{g_3\sqrt{g_1^2+4g_3}}\right)+\left(1+\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t\right] & \frac{PDY}{2g_3}\left[\left(\frac{g_1}{g_3}-\frac{g_1^2+2g_3}{g_3\sqrt{g_1^2+4g_3}}\right)+\left(1-\frac{g_1}{\sqrt{g_1^2+4g_3}}\right)t\right] \\ \frac{-g_1g_2}{((g_1-g_2)g_2+g_3)^2 PDY} & -\frac{1}{\sqrt{g_1^2+4g_3}}t & \frac{1}{\sqrt{g_1^2+4g_3}}t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{g_2t}z_1(0) \\ e^{\frac{1}{2}\left(g_1-\sqrt{g_1^2+4g_3}\right)t}z_2(0) \\ e^{\frac{1}{2}\left(g_1+\sqrt{g_1^2+4g_3}\right)t}z_3(0) \end{bmatrix}$$

Each mode of behavior ($e^{\lambda_j t}$) is multiplied by a (potentially complex) factor $\left(\frac{\partial r_{ji}}{\partial g_k}+r_{ji}\frac{\partial \lambda_j}{\partial g_k}t\right)$, influencing the weight of the original behavior mode and potentially the phase lag. Interpreting the set of matrices above, we note that changes in g_2 do not affect the oscillatory mode of behavior, as seen in the zeros in the second and third columns of the $\partial x_i(t)/\partial g_2$ equations. This result makes intuitive sense because loop 2, a minor balancing loop associated with Expected Demand (*ED*), does not contribute to the generation of the oscillatory mode, as can be seen from the equations for λ_2 and λ_3. Nevertheless, a change in g_2 impacts all states in the system, increasing the rate associated with the exponential decay. Note also that the weight of the impact depends on time, resulting from our previous results. The equations above also suggest that changes in g_1 and g_3 do not impact the behavior of expected demand (*ED*), which can be seen by the last row of zeros in the matrices capturing the derivatives of states with respect to g_1 and g_3.

Further results may be easier to derive after we substitute values for each of the loop gains. With this purpose, we allow the time constants for inventory correction time (*CT*), hire-fire time (*HFT*), and change demand expectations (*TCE*) to equal (e.g. 2 months), we obtain that $g_1 = -1/HFT = -1/2$, $g_2 = -1/TCE = -1/2$, $g_3 = -1/(CT^{*}HFT) = -1/4$, and $PDY = 10$, providing us with the following eigenvectors:

$$\mathbf{r_1} = \begin{bmatrix} 2 \\ 0.1 \\ 1 \end{bmatrix}; \mathbf{r_2} = \begin{bmatrix} -10 + i10\sqrt{3} \\ 1 \\ 0 \end{bmatrix}; \mathbf{r_3} = \begin{bmatrix} -10 - i10\sqrt{3} \\ 1 \\ 0 \end{bmatrix}$$

With the numerical results for eigenvalues and eigenvectors we can write the equations for the behavior of each state $x_i(t)$ in the system as well as interpret them:

$$\begin{bmatrix} I(t) \\ W(t) \\ ED(t) \end{bmatrix} = \begin{bmatrix} 2 & -10\left(1-i\sqrt{3}\right) & -10\left(1+i\sqrt{3}\right) \\ -0.1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix}$$

The set of equations suggest that the behavior of state $ED(t)$ follows an exponential decay with rate $g_2 (= -1/2)$ – only loop 2 (with gain g_2) influences the behavior of $ED(t)$. In addition, the behavior of states $I(t)$ and $W(t)$ are composed by a linear combination of two modes of behavior: an exponential decay and a decaying oscillation. Overall states $I(t)$ and $W(t)$ will follow decaying oscillatory exponentials.

Having the description of the original behavior provides a reference to interpret the impact introduced by changes in the loop gains. Such comparison can be made by comparing the cells of the original system behavior with cells from each of the three matrices below:

$$\begin{bmatrix} \frac{\partial I(t)}{\partial g_1} \\ \frac{\partial W(t)}{\partial g_1} \\ \frac{\partial ED(t)}{\partial g_1} \end{bmatrix} = \begin{bmatrix} -8 & \left(20+i\frac{20\sqrt{3}}{3}\right)+i\left(\frac{20\sqrt{3}}{3}\right)t & \left(20-i\frac{20\sqrt{3}}{3}\right)-i\left(\frac{20\sqrt{3}}{3}\right)t \\ 0.4 & \frac{1}{2}\left(1-i\frac{\sqrt{3}}{3}\right)t & \frac{1}{2}\left(1+i\frac{\sqrt{3}}{3}\right)t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial I(t)}{\partial g_2} \\ \dfrac{\partial W(t)}{\partial g_2} \\ \dfrac{\partial ED(t)}{\partial g_2} \end{bmatrix} = \begin{bmatrix} (4+2t) & 0 & 0 \\ (-0.1t) & 0 & 0 \\ t & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix};$$

$$\begin{bmatrix} \dfrac{\partial I(t)}{\partial g_3} \\ \dfrac{\partial W(t)}{\partial g_3} \\ \dfrac{\partial ED(t)}{\partial g_3} \end{bmatrix} = \begin{bmatrix} 8 & \left(-40+i\dfrac{40\sqrt{3}}{3}\right)-\left(20+i\dfrac{20\sqrt{3}}{3}\right)t & \left(-40-i\dfrac{40\sqrt{3}}{3}\right)-\left(20-i\dfrac{20\sqrt{3}}{3}\right)t \\ -0.4 & i\dfrac{2\sqrt{3}}{3}t & -i\dfrac{2\sqrt{3}}{3}t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} z_1(0) \\ e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t} z_2(0) \\ e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t} z_3(0) \end{bmatrix}$$

Note that a change in g_1 multiplies the weight of the original exponential decay ($e^{-\frac{1}{2}t}$) mode by a factor of four while also changing its sign. Perhaps more difficult to understand is the impact on the weight of the oscillatory mode of behavior for inventory, state $I(t)$, as seen in the coefficients for both $e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t}$ and $e^{-\frac{1}{4}\left(1-i\sqrt{3}\right)t}$. Again, the real part of the ratio (of the changed state behavior to the original one) determines a factor that multiplies the original weight of this complex behavior mode; and, the complex part of the ratio determines a phase lag to the original behavior mode. Consider first the impact of a change in g_1 on inventory's behavior mode $e^{-\frac{1}{4}\left(1+i\sqrt{3}\right)t}$: the ratio between changed and original state is $\frac{t}{2}-i\left(\frac{2\sqrt{3}}{3}+\frac{\sqrt{3}}{6}t\right)$. The result suggests that the weight multiplying this behavior mode depends on time. The complex coefficient contributes to the amplification with the square root of the sum of squares of the real and complex parts ($\sqrt{\left(\frac{t}{2}\right)^2+\left(-\frac{2\sqrt{3}}{3}-\frac{\sqrt{3}}{6}t\right)^2}$) and to the phase shift by the inverse tangent of the ratio of the real by the complex parts ($tan^{-1}\left(-\left(\frac{2\sqrt{3}}{3}+\frac{\sqrt{3}}{6}t\right)\Big/\left(\frac{t}{2}\right)\right)$). When time is close to zero ($t \cong 0$), the amplification to the oscillatory mode is given by a factor of $\frac{2\sqrt{3}}{3}$ and the phase shift is of $-\frac{\pi}{2}$. To compute the impact on the inventory (I) behavior at a specific time t, it would be required to substitute the adequate value of time. For instance, at $t = 4$ the change in g_1 causes an amplification to the oscillatory mode by a factor of 3.05 (since $\sqrt{\left(\frac{4}{2}\right)^2+\left(-\frac{4\sqrt{3}}{3}\right)^2}=\sqrt{\frac{28}{3}}=3.05$) and a phase shift of approximately -49^{o} (since $tan^{-1}\left(-2\sqrt{3}/3\right) \approx -49^{o}$). It is necessary to proceed in a similar way to compute the impact on different behavior modes.

The discussion above suggests that while there are some insights that are readily available from this type of analysis, deeper analyses will require further visualization, interpretation and measures of contribution of changed weights after changes in loop (or link) gains. Since the overall trajectory of any state is a linear combination of different behavior modes, graphs or metrics that can provide a clear visualization of the contribution of individual modes of behavior to the overall trajectory will likely be useful tools for the design of improved policies. Given that the Pathway Participation Method allows us to visualize and draw inferences from pathways that contribute most to the Total Participation Metric, it seems that we can readily apply a similar approach to visualize and interpret how the weights of different behavior modes affect overall behavior trajectories.

5. Discussion

The main contribution of this paper arises from a broader definition of behavior as the overall trajectory of a state variable, instead of the traditional definition associating behavior with *behavior modes* (e.g., exponential growth, exponential decay, and oscillation). When we consider overall behavior trajectories, influences from *eigenvectors* as well as *eigenvalues* are central to understanding how the structure of the system generates the observed behavior. The paper provides a mathematical framework to understand the contribution that changes in link (or loop) gains have on the time path behavior of state variables in linear dynamic systems. Our approach to understanding model behavior uses the derivatives of both eigenvalues and eigenvectors with respect to link (or loop) gains, following closely the research tradition established by Forrester (1982). In particular, we derive an equation that characterizes the relative contribution of both eigenvalues and eigenvectors to changes in overall behavior over time. The direct consequence of focusing on behavior trajectories is that previous focus on behavior modes and the use of eigenvalue elasticities has led to a myopic attention on long-term impact of a change in loop (or link) gain in its analysis.

The paper develops an analytical framework to understand how eigenvectors can be incorporated to the analysis of overall behavior trajectories of linear systems. The approach is precise, reproducible, and provides a standard way to analyze linear dynamic systems. In addition, the method provides a direct measure of the impact of different loops on the behavior response of the system. By capturing both the short-term and long-term impact of a change in loop (or link) gain in the overall trajectory, the method also contributes to our understanding of transient analysis instead of simply steady state analysis of linear systems. Finally, by linearizing a nonlinear system at every point in time, we arrive at a general solution that provides a good approximation of the impact of a change in link gains on overall behavior trajectories of state x_i.

The method offers new opportunities for formal model analysis, but also has its own limitations. First, the main derivations apply to the impact of a change in structure to the overall behavior trajectory of states in a linear system, consecutive system linearization at every point in time extends the application to nonlinear systems. While this result is stated, no example is provided. Second, further research implementing the computation of eigenvalues, eigenvectors and the results of the main equation derived here to different nonlinear models is required to assess the usefulness of the proposed method. In addition, it is likely that the method can benefit from visualization tools showing how different behavior modes contribute to the overall trajectory and within a specific behavior mode how the first and second term contribute to the total weight of the behavior mode. Computationally, the application to nonlinear systems requires linearization of the system at every time step of the simulation, calculation of the A matrix, numerical evaluation of eigenvalues, eigenvectors, equation results, and overall trajectory contribution data as well as visualization of such data for adequate analysis and policy design.

As the simple linear example suggests, interpreting the results of the method poses challenges in terms of evaluating the specific impact of eigenvector and eigenvalue contribution to behavior modes. Evaluation of impact of a change in a link gain on overall system behavior has to be done by inspection and requires tedious processing case-by-case. Policy design has also to be done manually based on inferences about which links (or loops) cause most impact on the desired system trajectory. Despite current challenges and limitations, we are hopeful that the method provides a useful step on the analysis of how structure influences behavior as well as a new direction for future research on the analysis of nonlinear dynamic systems.

Appendix A – Behavior in Linear Dynamic Systems

The formal structure of a linear system dynamics model with a vector of state variables **x**(t), where **x**(t) = ($x_1, x_2, \ldots, x_n$)', a vector of first time derivatives of the state variables $\dot{\mathbf{x}}$(t), where $\dot{\mathbf{x}}$(t) = ($\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$)', a gain matrix **A** capturing the partial derivatives of the net change of a state variable with respect to another (the matrix $\mathbf{A}_{\mathbf{n\,x\,n}} = \partial\dot{\mathbf{x}}/\partial\mathbf{x}$ is commonly known as the *A matrix*), and a constant vector **b,** can be represented compactly in the following way:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{A1}$$

Consider now the solution to the homogeneous system. A standard result in linear systems theory is that the eigenvalues (λ) of matrix ***A*** describe the behavior modes inherent in the model and are the solutions of the characteristic polynomial ($P(\lambda)$), where ($P(\lambda) = |\lambda I_n - \mathbf{A}| = 0$). Assume for simplicity that the system matrix $\boldsymbol{A}_{nxn}$ has a complete set of n linearly independent eigenvectors ($\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_n}$) with corresponding eigenvalues ($\lambda_1, \lambda_2, \ldots, \lambda_n$), where eigenvalues may or may not be distinct. Since the eigenvectors are linearly independent, they span the n dimensional space, therefore an arbitrary value of the state $\boldsymbol{x}(t)$ can be expressed by the linear combination of the eigenvectors:

$$\mathbf{x}(t) = z_1(t)\mathbf{r_1} + z_2(t)\mathbf{r_2} + \ldots + z_n(t)\mathbf{r_n} \tag{A2}$$

where $z_i(t)$, $i=1, 2, \ldots, n$ are scalars.

Using the fact that by definition multiplication of the system matrix by their eigenvectors results in the product of the eigenvectors by eigenvalues ($\boldsymbol{A}\boldsymbol{r_i} = \lambda_i \boldsymbol{r_i}$), we can rewrite equation (A2) by multiplying it by the system matrix $\boldsymbol{A}_{nxn}$.

$$\mathbf{A}\mathbf{x}(t) = \dot{\mathbf{x}}(t) = z_1(t)\mathbf{A}\mathbf{r_1} + z_2(t)\mathbf{A}\mathbf{r_2} + \ldots + z_n(t)\mathbf{A}\mathbf{r_n}$$

$$\dot{\mathbf{x}}(t) = z_1(t)\lambda_1\mathbf{r_1} + z_2(t)\lambda_2\mathbf{r_2} + \ldots + z_n(t)\lambda_n\mathbf{r_n} \tag{A3}$$

Since equation (A2) defines the state vector $\boldsymbol{x}(t)$, we can take its first time derivative. In addition, using the fact that eigenvalues and eigenvectors are constant in linear systems, we can rewrite (A2) to get:

$$\dot{\mathbf{x}}(t) = \dot{z}_1(t)\mathbf{r_1} + \dot{z}_2(t)\mathbf{r_2} + \ldots + \dot{z}_n(t)\mathbf{r_n} \tag{A4}$$

Comparing the right hand side of (A4) and (A3), we obtain:

$$\dot{z}_1(t)\mathbf{r_1} + \dot{z}_2(t)\mathbf{r_2} + \ldots + \dot{z}_n(t)\mathbf{r_n} = z_1(t)\lambda_1\mathbf{r_1} + z_2(t)\lambda_2\mathbf{r_2} + \ldots + z_n(t)\lambda_n\mathbf{r_n} \qquad \text{(A5)}$$

And since the eigenvectors are linearly independent, the equality can only hold if:

$$\dot{z}_i(t) = z_i(t)\lambda_i \qquad \text{(A6)}$$

The system above can be represented in matrix form as: [v]

$$\begin{bmatrix} \dot{\mathrm{z}}_1(\mathrm{t}) \\ \dot{\mathrm{z}}_2(\mathrm{t}) \\ \ldots \\ \dot{\mathrm{z}}_\mathrm{n}(\mathrm{t}) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathrm{z}_1(\mathrm{t}) \\ \mathrm{z}_2(\mathrm{t}) \\ \ldots \\ \mathrm{z}_\mathrm{n}(\mathrm{t}) \end{bmatrix} \qquad \text{(A7)}$$

The solution of the homogeneous system of decoupled equations presented above is known:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \ldots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{\lambda_n} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ \ldots \\ z_n(0) \end{bmatrix} \text{ or } z_i(t) = e^{\lambda_i t} z_i(0) \qquad \text{(A8)}$$

Substituting the result in (A8) in our original equation (A2) yields:[vi]

$$\mathbf{x}(t) = e^{\lambda_1 t} z_1(0)\mathbf{r_1} + e^{\lambda_2 t} z_2(0)\mathbf{r_2} + \ldots + e^{\lambda_n t} z_n(0)\mathbf{r_n} \qquad \text{(A9)}$$

Appendix B – The Product of a Complex Number by Complex Exponentials

To understand the implication of multiplying a complex exponential by a complex number, consider the following example: $(a+b\mathrm{i})e^{(c+d\mathrm{i})}$

we can rewrite the exponential as: $e^{(c+d\mathrm{i})} = e^{c}e^{d\mathrm{i}}$

and by definition $e^{d\mathrm{i}} = cos(d) + \mathrm{i}\, sin(d)$

so we can rewrite the equation above as:

$$e^{c}(a+b\mathrm{i})(cos(d)+\mathrm{i}\,sin(d))$$
$$(ae^{c})(cos(d)+\mathrm{i}\,sin(d)) + (be^{c})(\mathrm{i}\,cos(d) - sin(d))$$
$$e^{c}[(a\,cos(d) - b\,sin(d)) + \mathrm{i}(b\,cos(d) + a\,sin(d))]$$

Multiplying by 1($\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$) and defining $\frac{b}{a} = tan(\phi)$, we observe that $\frac{a}{\sqrt{a^2+b^2}} = cos(\phi)$ and $\frac{b}{\sqrt{a^2+b^2}} = sin(\phi)$ we can rewrite the equation above as:

$$\left(\sqrt{a^2+b^2}\right)e^{c}\left[\left(\frac{a}{\sqrt{a^2+b^2}}cos(d) - \frac{b}{\sqrt{a^2+b^2}}sin(d)\right) + \mathrm{i}\left(\frac{b}{\sqrt{a^2+b^2}}cos(d) + \frac{a}{\sqrt{a^2+b^2}}sin(d)\right)\right]$$
$$\left(\sqrt{a^2+b^2}\right)e^{c}[(cos(\phi)cos(d) - sin(\phi)sin(d)) + \mathrm{i}(sin(\phi)cos(d) + cos(\phi)sin(d))]$$

Since $cos(d+\phi) = (cos(\phi)cos(d) - sin(\phi)sin(d))$ and $sin(d+\phi) = (sin(\phi)cos(d) + cos(\phi)sin(d))$, we obtain:

$$\left(\sqrt{a^2+b^2}\right)\, e^{c}[cos(d+\phi) + \mathrm{i}\,sin(d+\phi)]$$
$$\left(\sqrt{a^2+b^2}\right)\, e^{c+\mathrm{i}(d+\phi)}$$
$$\left(\sqrt{a^2+b^2}\right)\, e^{c+\mathrm{i}\left(d+tan^{-1}\left(\frac{b}{a}\right)\right)}$$

Therefore, the complex number multiplying the exponential contributes to the amplification with the square root of the sum of squares of the real and complex parts, and to the phase shift by the inverse tangent of the ratio of the complex by the real parts. The inverse tangent of (x) is defined in the interval $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. The inverse tangent takes a value of zero when x is zero; and it takes a positive (negative) value when x is positive (negative).

Appendix C - How loops influence system behavior?

To understand how changes in loop gains (i.e., the strength of a feedback loop) influence system behavior, we follow a derivation analogous to the one in section 3. The behavior of each state in the system $x_i(t)$ is described by equation (4), which demonstrates that the behavior of each state is influenced both by eigenvalues (λ_i) and eigenvector components (r_{ji}).

$$x_i(t) = r_{1i} e^{\lambda_1 t} z_1(0) + r_{2i} e^{\lambda_2 t} z_2(0) + \ldots + r_{ni} e^{\lambda_n t} z_n(0)$$

While it is more common to write the characteristic polynomial ($P(\lambda)$) and eigenvalues in terms of the link gains (a_{kl}), it is also possible to write them in terms of loop gains (g_k), as shown in the example provided in section 5. Loops, and their gains, may be a more comprehensive (better) way to describe structure, since modelers often decide to include (or exclude) loops based on the dynamic hypotheses that they believe are important in a system. Since we are ultimately interested in how structure drives behavior, understanding how changes in loop gains influence system behavior may be more appropriate than looking at how changes in links influence behavior. To capture how loops influence system behavior, we take the partial derivative of each state in the system $x_i(t)$ with respect to an arbitrary loop gain (g_k). Therefore we take a partial derivative of equation (4), characterizing the behavior of state $x_i(t)$, with respect to a loop gain (g_k).

$$\frac{\partial x_i(t)}{\partial g_k} = \frac{\partial}{\partial g_k}\left[r_{1i} e^{\lambda_1 t} z_1(0) + \ldots + r_{ni} e^{\lambda_n t} z_n(0)\right] \qquad \text{(B1)}$$

Which for linear systems, we can rewrite as:

$$\frac{\partial x_i(t)}{\partial g_k} = \sum_{j=1}^{n}\left(\frac{\partial r_{ji}}{\partial g_k} + r_{ji}\frac{\partial \lambda_j}{\partial g_k} t\right) e^{\lambda_j t} z_j(0) \qquad \text{(B2)}$$

Equation (B2) suggests that a change in behavior of state $x_i(t)$ due to a change in loop gain (g_k) will be composed by two terms for each behavior mode ($e^{\lambda j t}$) contributing to the overall behavior trajectory of state variable $x_i(t)$. Each of the terms corresponds to:

1. The derivative of eigenvector component (r_{ji}) with respect to loop gain (g_k); and
2. The product of eigenvector component (r_{ji}), the derivative of eigenvalue (λ_i) with respect to loop gain (g_k), and time (t).

With his suggestion of finding the characteristic polynomial in terms of the loop gains, Forrester (1983) extended the results of link sensitivity and link elasticity to loop sensitivity and loop elasticity.

$$S_{\lambda_i k} = \frac{\partial \lambda_i}{\partial g_k} \text{ and } E_{ik} = \frac{\partial \lambda_i}{\partial g_k} \frac{|g_k|}{\|\lambda_i\|} \tag{B3}$$

In addition, we can extend the concept of link eigenvector component sensitivity and elasticity introduced in section 3.2 to loop *eigenvector component sensitivity* and eigenvector component elasticity with respect to loop gain or *loop gain eigenvector component elasticity*.

$$S_{r_{ij} k} = \frac{\partial r_{ij}}{\partial g_k} \text{ and } E_{r_{ij} k} = \frac{\partial r_{ij}}{\partial g_k} \frac{|g_k|}{\|r_{ij}\|} \tag{B4}$$

Equation (B2) provides an integrated way to assess how loop eigenvalue and eigenvector sensitivity (i.e., the partial derivatives with respect to a loop gain) work together to influence system behavior. In particular, we can rewrite equation (B2) as:

$$\frac{\partial x_i(t)}{\partial g_k} = \sum_{j=1}^{n} \left(S_{r_{ij} k} + r_{ji} S_{\lambda_j k} t \right) e^{\lambda_j t} z_j(0) \tag{B5}$$

- Loop eigenvector component sensitivity $S_{r_{ij} k} = \frac{\partial r_{ij}}{\partial g_k}$ captures a change in weight in behavior mode ($e^{\lambda_j t}$) due to a change in loop gain (g_k);
- Loop eigenvalue sensitivity $S_{\lambda_i k} = \frac{\partial \lambda_i}{\partial g_k}$ captures the change in weight in the behavior mode ($e^{\lambda_j t}$) due to a change in the loop gain (g_k); and
- The contribution of the eigenvalue sensitivity to the weight changes with time and it becomes the main determinant of weight of behavior mode ($e^{\lambda_j t}$) as time grows.

References

Eberlein, R.L. 1989. Simplification and Understanding of Models. *System Dynamics Review*. **5**(1).

Forrester, N. 1982. A Dynamic Synthesis of Basic Macroeconomic Policy: Implications for Stabilization Policy Analysis. Unpublished Ph.D. Thesis, M.I.T., Cambridge, MA.

Forrester, N. 1983. Eigenvalue Analysis of Dominant Feedback Analysis. *Proceeding of the 1983 International System Dynamics Conference, Plenary Session Papers.* System Dynamics Society: Albany, NY. pp178-202.

Gonçalves, P, Lertpattarapong C, Hines J. 2000. Implementing formal model analysis. *Proceedings of the 2000 International System Dynamics Conference, Bergen, Norway*. System Dynamics Society, Albany, NY.

Güneralp, B. 2005. Towards Coherent Loop Dominance Analysis: Progress in Eigenvalue Elasticity Analysis. *Proceedings of the 2005 International System Dynamics Conference.* Boston.

Hines, J. 2005. How to Visit a Great Model Like Yours. *Proceedings of the 2005 International System Dynamics Conference.* Boston.

Kampmann, C.E. 1996. Feedback Loop Gains and System Behavior. *Proceeding of the 1996 International System Dynamics Conference Boston*. System Dynamics Society: Albany, NY. pp. 260-263.

Kampmann, C.E. and R. Oliva. 2006. Loop eigenvalue elasticity analysis: Three case studies. *System Dynamics Review* (forthcoming).

Mojtahedzadeh, M.T. 1996. A Path Taken: Computer-Assisted Heuristics for Understanding Dynamic Systems*.* Unpublished Ph.D. Dissertation, Rockefeller College of Public Affairs and Policy, State University of New York at Albany, 1996, Albany NY.

Mojtahedzadeh, M., G. Richardson and D. Andersen. 2004. Using Digest to implement the pathway participation method for detecting influential system structure. *System Dynamics Review.* **20**(1):1-20.

Oliva, R. 2004. Model structure analysis through graph theory: Partition heuristics and feedback structure decomposition. *System Dynamics Review.* **20**(4):313-336.

Oliva, R. and M. Mojtahedzadeh. 2004. Keep it Simple: Dominance Assessment of Short Feedback Loops. *Proceedings of the 2004 International System Dynamics Conference*. Oxford, UK.

Perez-Arriaga. 1981. Selective Modal analysis with Applications to Electric Power Systems. Unpublished Ph.D. dissertation. Department of Electrical Engineering, MIT.

Reinschke, KJ. *Multivariate Control: A Graph Theoretical Approach. Lecture Notes in Control and Information Sciences*. 1988. Berlin: Springer-Verlag.

Richardson, GP. 1984. "Loop Polarity, Loop Dominance, and the Concept of Dominant Polarity." In *Proceeding of the 1984 International System Dynamics Conference*. Oslo, Norway.
Reprinted as Richardson GP. 1995. Loop polarity, loop dominance, and the concept of dominant polarity. *System Dynamics Review* 11(1): 67-88.

Richardson, GP. 1986. Dominant Structure. *System Dynamics Review* 2(1): 68-75.

Richardson, GP, Pugh III, AL. 1981. *Introduction to System Dynamics Modeling with Dynamo*. MIT Press: Cambridge, MA. (Now available from Pegasus Communications, Waltham, MA.)

Saleh, M. and P. Davidsen. 2000. An Eigenvalue Approach to Feedback Loop Dominance Analysis in Non-linear Dynamic Models. *Proceedings of the 2000 International System Dynamics Conference*. Bergen, Norway.

Saleh, M. and P. Davidsen. 2001. The Origins of Business Cycles. *Proceedings of the 2001 International System Dynamics Conference*. Atlanta.

Saleh, M. 2002. *The Characterization of Model Behavior and its Causal Foundation*. PhD Dissertation, University of Bergen, Bergen, Norway.

Saleh, M., P. Davidsen and K. Bayoumi. 2005. A Comprehensive Eigenvalue Analysis of System Dynamics Models. *Proceedings of the 2005 International System Dynamics Conference.* Boston.

Sterman 2000. *Business Dynamics. Systems Thinking and Modeling for a ComplexWorld*. Irwin McGraw-Hill: Boston, MA.

---

[i] "The first and most important foundation for [system] dynamics is the concept of servo-mechanisms (or information-feedback systems)." (Forrester 1961, p14).

[ii] Note that the computation of the partial derivative of each term $r_{ji}e^{\lambda_j t}z_j(0)$ assumes that the initial state $z_j(0)$ does not depend on the link gain. State $z_j(0)$ is a new state variable – obtained after the change of variables – given by ($\mathbf{z}(0)=\mathbf{R}^{-1}\mathbf{x}(0)$) where $\mathbf{z}(0)$ is the initial position vector of the new state variables and $\mathbf{x}(0)$ is the initial position vector of the original state variables. The inverse of the matrix of eigenvectors ($\mathbf{R}^{-1}$) depends on the value of all eigenvectors and thus varies with changes in the link gain. However, we abstract away from those changes because we are interested in deriving an expressions that hold no matter what the initial conditions are.

[iii] Gonçalves, Hines, Lertpattarapong (2000) provide a derivation of the characteristic polynomial of the inventory-workforce model in terms of the loop gains. To compute the eigenvalues in terms of loop gains readers are also directed to Forrester (1983), Kampmann (1996) and Kampmann and Oliva (2006).

[iv] While the table results shows the same sign, note that the complex number is in the denominator and will need to be multiplied by its conjugate to arrive at the correct sign of the complex number.

[v] Note that we rewrite the results above more compactly in matrix form defining $\boldsymbol{R}$ as the *nxn* matrix whose *n* columns are the eigenvectors of $\boldsymbol{A}$ and defining the column vector $\boldsymbol{z}$(t) with components ($z_1(t), z_2(t), \ldots\ z_n(t)$). Defining $\boldsymbol{R}$ that way allows us to write equation (A2) as $\mathbf{x}(t)=\mathbf{R}\mathbf{z}(t)$. We can interpret the new equation as a change in variable and use it to rewrite the dynamic system, which yields: $\mathbf{R}\dot{\mathbf{z}}(t)=\mathbf{A}\mathbf{R}\mathbf{z}(t)$ or simply: $\dot{\mathbf{z}}(t)=\mathbf{R}^{-1}\mathbf{A}\mathbf{R}\mathbf{z}(t)$, where the computation of the inverse of the matrix of eigenvectors ($\boldsymbol{R}^{-1}$) depends on the value of all the system eigenvectors. The new system ($\dot{\mathbf{z}}(t)$) is related to the original one ($\dot{\mathbf{x}}(t)$) by a change of variable. The new system matrix ($\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}$) corresponds to the system governing the $\boldsymbol{z}(t)$ state equations, where the change in each state

($\dot{z}_i(t)$) depends only on the product of the associated eigenvalue (λ_i) and the own state ($z_i(t)$). Accordingly, we can write $\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}=\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is the diagonal matrix with the eigenvalues of $\boldsymbol{A}$ in the diagonal.

[vi] The initial values of $z(0)$ can be obtained in terms of $x(0)$ from the change in variable definition: $\mathbf{z}(0)=\mathbf{R}^{-1}\mathbf{x}(0)$.