Orbifold points on Teichmüller curves and Jacobians with complex multiplication

by

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Submitted to the Department of Mathematics
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Abstract

For each integer $D \geq 5$ with $D \equiv 0$ or $1 \pmod{4}$, the Weierstrass curve W_D is an algebraic curve and a finite volume hyperbolic orbifold which admits an algebraic and isometric immersion into the moduli space of genus two Riemann surfaces. The Weierstrass curves are the main examples of Teichmüller curves in genus two.

The primary goal of this thesis is to determine the number and type of orbifold points on each component of W_D . Our enumeration of the orbifold points, together with [Ba] and [Mc3], completes the determination of the homeomorphism type of W_D and gives a formula for the genus of its components. We use our formula to give bounds on the genus of W_D and determine the Weierstrass curves of genus zero. We will also give several explicit descriptions of each surface labeled by an orbifold point on W_D .

Thesis Supervisor: Curtis T. McMullen
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Chapter 1

Introduction

Let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces. The space $\mathcal{M}_g$ can be viewed as both a complex orbifold and an algebraic variety and carries a complete Teichmüller metric. A *Teichmüller curve* is an algebraic and isometric immersion of a finite volume hyperbolic Riemann surface:

$$f : C = \mathbb{H}/\Gamma \rightarrow \mathcal{M}_g.$$

The modular curve $\mathcal{M}_1$ is the first example of a Teichmüller curve. Other examples emerge from the study of polygonal billiards [Ve, MT] and square-tiled surfaces. While the Teichmüller curves in $\mathcal{M}_2$ have been classified [Mc5], much less is known about Teichmüller curves in $\mathcal{M}_g$ for $g > 2$ [BaM, BM2].

For each integer $D \geq 5$ with $D \equiv 0$ or $1 \pmod{4}$, the *Weierstrass curve* W_D is the moduli space of Riemann surfaces whose Jacobians have real multiplication by the quadratic order $\mathcal{O}_D = \mathbb{Z} \left[\frac{D+\sqrt{D}}{2} \right]$ stabilizing a holomorphic one form with double zero up to scale. The curve W_D is a finite volume hyperbolic orbifold and the natural immersion:

$$W_D \rightarrow \mathcal{M}_2,$$

is algebraic and isometric and has degree one onto its image [Ca, Mc1]. The curve W_D is a Teichmüller curve unless $D > 9$ with $D \equiv 1 \pmod{8}$ in which case $W_D = W_D^0 \sqcup W_D^1$ is a disjoint union of two Teichmüller curves distinguished by a spin invariant in

$D \bmod 16$	$e_2(W_D)$
1, 5, 9, or 13	$\frac{1}{2}\tilde{h}(-4D)$
0	$\frac{1}{2}(\tilde{h}(-D) + 2\tilde{h}(-D/4))$
4	0
8	$\frac{1}{2}\tilde{h}(-D)$
12	$\frac{1}{2}(\tilde{h}(-D) + 3\tilde{h}(-D/4))$

Table 1.1: For $D > 8$, the number of orbifold points of order two on W_D is given by a weighted sum of class numbers. The function $\tilde{h}(-D)$ is defined below.

$\mathbb{Z}/2\mathbb{Z}$ [Mc3]. A major challenge is to describe W_D as an algebraic curve and as a hyperbolic orbifold. To date, this has been accomplished only for certain small D [BM1, Mc1, Lo].

The purpose of this thesis is to study the orbifold points on W_D . Such points label surfaces with automorphisms commuting with $\mathcal{O}_D$. The first two Weierstrass curves W_5 and W_8 were studied by Veech [Ve] and are isomorphic to the $(2, 5, \infty)$ - and $(4, \infty, \infty)$ -orbifolds. The surfaces with automorphisms labeled by the three orbifold points are drawn in Figure 1-1.

Our primary goal is to give a formula for the number and type of orbifold points on W_D . Together with [Mc3] and [Ba], our formula completes the determination of the homeomorphism type of W_D and gives a formula for the genus of W_D . We will use our formula to give bounds for the genera of W_D and W_D^ϵ and list the components of $\bigcup_D W_D$ of genus zero. We will also give several explicit descriptions of the surfaces labeled by orbifold points on W_D , giving the first examples of algebraic curves labeled by points of W_D for most D .

Main results. Our main theorem determines the number and type of orbifold points on W_D :

Theorem 1.1. *For $D > 8$, the orbifold points on W_D all have order two, and the number of such points $e_2(W_D)$ is the weighted sum of class numbers of imaginary quadratic orders shown in Table 1.1.*

We also give a formula for the number of orbifold points on each spin component:

Theorem 1.2. *Fix $D \geq 9$ with $D \equiv 1 \pmod{8}$. If $D = f^2$ is a perfect square, then all of the orbifold points on W_D lie on the component with spin $(f + 1)/2 \pmod{2}$:*

$$e_2 \left(W_D^{(f+1)/2} \right) = \frac{1}{2} \tilde{h}(-4D) \text{ and } e_2 \left(W_D^{(f-1)/2} \right) = 0.$$

Otherwise, $e_2(W_D^0) = e_2(W_D^1) = \frac{1}{4} \tilde{h}(-4D)$.

When D is not a square and W_D is reducible, the spin components of W_D have algebraic models defined over $\mathbb{Q}(\sqrt{D})$ and are Galois conjugate [BM1]. Theorem 1.2 confirms that the spin components have the same number and type of orbifold points.

The class number $h(-D)$ is the order of the ideal class group $H(-D)$ for $\mathcal{O}_{-D}$ and counts the number of elliptic curves with complex multiplication by $\mathcal{O}_{-D}$ up to isomorphism. The weighted class number

$$\tilde{h}(-D) = 2h(-D) / |\mathcal{O}_{-D}^\times|$$

appearing in Table 1.1 is the number of elliptic curves with complex multiplication weighted by their orbifold order in $\mathcal{M}_1$. Note that $\tilde{h}(-D) = h(-D)$ unless $D = 3$ or 4 . When D is odd, the orbifold points on W_D are labeled by elements of the group $H(-4D)/[P]$ where $[P]$ is the ideal class in $\mathcal{O}_{-4D}$ representing the prime ideal with norm two.

The orbifold Euler characteristics of W_D and W_D^ϵ were computed in [Ba] and the cusps on W_D were enumerated and sorted by component in [Mc3]. Theorems 1.1 and 1.2 complete the determination of the homeomorphism type of W_D and give a formula for the genera of W_D and its components.

Corollary 1.3. *For any $\epsilon > 0$, there are constants C_ϵ and N_ϵ such that:*

$$C_\epsilon D^{3/2+\epsilon} > g(V) > D^{3/2}/650,$$

whenever V is a component of W_D and $D \geq N_\epsilon$.

Figure 1-1: The first two Weierstrass curves W_5 and W_8 are isomorphic to the $(2, 5, \infty)$ and $(4, \infty, \infty)$ -orbifolds. The point of order two is related to billiards on the L -shaped table (left) corresponding to the golden mean $\gamma = \frac{1+\sqrt{5}}{2}$. The points of order five (center) and four (right) are related to billiards on the regular pentagon and octagon.

Modular curves of genus zero play an important role in number theory [Ti]. We also determine the components of Weierstrass curves of genus zero.

Corollary 1.4. *The genus zero components of $\bigcup_D W_D$ are the 23 components of $\bigcup_{D \leq 41} W_D$ and the curves W_{49}^0 , W_{49}^1 and W_{81}^1 .*

We include a table listing the homeomorphism type of W_D for $D < 250$ in §A.2.

Orbifold points on Hilbert modular surfaces. Theorem 1.1 is closely related to the classification of orbifold points on Hilbert modular surfaces we prove in §3. The *Hilbert modular surface* X_D is the moduli space of principally polarized abelian varieties with real multiplication by $\mathcal{O}_D$. The period map sending a Riemann surface to its Jacobian embeds W_D in X_D .

Central to the story of the orbifold points on X_D and W_D are the moduli spaces $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$ of genus two surfaces with actions of the dihedral groups of orders 8 and 12:

$$D_8 = \langle r, J : r^2 = (Jr)^2 = J^4 = 1 \rangle \text{ and } D_{12} = \langle r, Z : r^2 = (Zr)^2 = Z^6 = 1 \rangle.$$

The surfaces in $\mathcal{M}_2(D_8)$ (respectively $\mathcal{M}_2(D_{12})$) whose Jacobians have complex mul-

multiplication have real multiplication commuting with J (respectively Z). The complex multiplication points on $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$ give most of the orbifold points on $\bigcup_D X_D$:

Theorem 1.5. *The orbifold points on $\bigcup_D X_D$ which are not products of elliptic curves are the two points of order five on X_5 and the complex multiplication points on $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$.*

Since the Z -eigenforms on D_{12} -surfaces have simple zeros and the J -eigenforms on D_8 -surfaces have double zeros (cf. Proposition 3.2), we have:

Corollary 1.6. *The orbifold points on $\bigcup_D W_D$ are the point of order five on W_5 and the complex multiplication points on $\mathcal{M}_2(D_8)$.*

Corollary 1.6 explains the appearance of class numbers in the formula for $e_2(W_D)$. The involutions r and Jr on a D_8 -surface X have genus one quotients X/r and X/Jr whose Jacobians are related by a degree two isogeny and the family $\mathcal{M}_2(D_8)$ is birational to the modular curve $Y_0(2)$. The Jacobian $\text{Jac}(X)$ has complex multiplication by an order in $\mathbb{Q}(\sqrt{D}, i)$ if and only if $\text{Jac}(X/r)$ has complex multiplication by an order in $\mathbb{Q}(\sqrt{-D})$. The formula for $e_2(W_D)$ follows by sorting the $3\tilde{h}(-D)$ surfaces with D_8 -action covering elliptic curves with complex multiplication by $\mathcal{O}_D$ by their orders for real multiplication.

The product locus P_D . A recurring theme in the study of the Weierstrass curves is the close relationship between W_D and the product locus $P_D \subset X_D$. The *product locus* P_D consists of products of elliptic curves with real multiplication by $\mathcal{O}_D$.

The cusps on W_D were first enumerated and sorted by spin in [Mc3] and, for non-square D , are in bijection with the cusps on P_D (cf. §7). The Hilbert modular surface X_D has a meromorphic modular form with a simple pole along P_D and a simple zero along W_D . This modular form can be used to give a formula for the Euler characteristic of W_D and, for non-square D , the Euler characteristics of W_D , X_D and P_D satisfy ([Ba], Cor. 10.4):

$$\chi(W_D) = \chi(P_D) - 2\chi(X_D).$$

Our classification of the orbifold points on X_D and W_D in Theorem 1.5 and Corollary 1.6 show that all of the orbifold points of order two on X_D lie on W_D or P_D .

Theorem 1.7. *For non-square D , the homeomorphism type of W_D is determined by the homeomorphism types of X_D and P_D and $D \bmod 8$.*

The D_8 -family. A secondary goal of our analysis is to give several explicit descriptions of D_8 -surfaces and to characterize those with complex multiplication. We will outline a similar discussion for $\mathcal{M}_2(D_{12})$ in §A.1. For a genus two surface $X \in \mathcal{M}_2$, the following are equivalent:

1. *Automorphisms.* The automorphism group $\text{Aut}(X)$ admits an injective homomorphism $\rho : D_8 \rightarrow \text{Aut}(X)$.
2. *Algebraic curves.* The field of meromorphic functions $\mathbb{C}(X)$ is isomorphic to:

$$K_a = \mathbb{C}(z, x) \text{ with } z^2 = (x^2 - 1)(x^4 - ax^2 + 1),$$

for some $a \in \mathbb{C} \setminus \{\pm 2\}$.

3. *Jacobians.* There is a number $\tau \in \mathbb{H}$ such that the Jacobian $\text{Jac}(X)$ is isomorphic to the principally polarized abelian variety:

$$A_\tau = \mathbb{C}^2 / \Lambda_\tau,$$

where $\Lambda_\tau = \mathbb{Z} \langle \begin{pmatrix} \tau \\ \tau+1 \end{pmatrix}, \begin{pmatrix} \tau \\ -\tau-1 \end{pmatrix}, \begin{pmatrix} \tau+1 \\ \tau \end{pmatrix}, \begin{pmatrix} \tau+1 \\ -\tau \end{pmatrix} \rangle$ and A_τ is polarized by the symplectic form $\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \rangle = \frac{\text{Im}(a\bar{c} + b\bar{d})}{2\text{Im}(\tau)}$.

4. *Pinwheels.* The surface X is isomorphic to the surface X_τ obtained from the polygonal *pinwheel* P_τ (Figure 1-2) for some τ in the domain:

$$U = \left\{ \tau \in \mathbb{H} : \tau \neq \frac{\pm 1 + i}{2}, |\tau|^2 \geq \frac{1}{2} \text{ and } |\text{Re } \tau| \leq \frac{1}{2} \right\}.$$

Figure 1-2: For τ in the shaded domain U , the pinwheel P_τ has vertices at $z = \frac{1 \pm i}{2}$, $\frac{-1 \pm i}{2}$, $\pm \tau$, and $\pm i\tau$. Gluing together opposite sides on P_τ by translation gives a genus two surface admitting an action of D_8 . The one form induced by dz is a J -eigenform and has a double zero.

5. *Parallelograms.* The surface X is isomorphic to the surface Y_w obtained from the parallelogram Q_w (Figure 1-3) for some w in the domain:

$$V = \{w \in \mathbb{H} : 0 \leq \operatorname{Re} w \leq 2 \text{ and } |w - 1| \geq 1\}.$$

It is straightforward to identify the action of D_8 in each of the descriptions above. The field K_a has automorphisms $r(z, x) = (z, -x)$ and $J(z, x) = (iz/x^3, 1/x)$. The linear transformations $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ preserve the polarized lattice Λ_τ . The action of D_8 on the surface Y_w is identified in Figure 1-3. The surface X_τ satisfies $\mathbb{C}(X_\tau) \cong K_a$ for some a because the order four automorphism obtained by rotating P_τ acts as a product of two disjoint transpositions on the Weierstrass points of X_τ (cf. Propositions 3.2 and 3.3).

The function relating the number τ determining the polygon P_τ and abelian variety A_τ to the number a determining the field K_a is the modular function:

$$a(\tau) = -2 + \frac{1}{\lambda(\tau)\lambda(\tau+1)}.$$

The function $\lambda(\tau)$ is modular for the group $\Gamma(2) = \ker(\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}))$ and descends to the isomorphism $\lambda : \mathbb{H}/\Gamma(2) \xrightarrow{\sim} \mathbb{C} \setminus \{0, 1\}$ sending the cusps $\Gamma(2) \cdot 0$, $\Gamma(2) \cdot 1$ and $\Gamma \cdot \infty$ to 0, 1 and ∞ respectively. The function relating τ and w is the Riemann mapping $w : U \rightarrow V$ whose extension to ∂U has $w((-1+i)/2) = 0$, $w((1+i)/2) = 2$ and $w(\infty) = \infty$. In Sections 4 and 5 we will prove:

Theorem 1.8. *Fix $\tau \in U$. The surface X_τ obtained from the polygon P_τ admits a faithful D_8 -action, is isomorphic to the surface obtained from the parallelogram $Q_{w(\tau)}$ and satisfies:*

$$\mathrm{Jac}(X_\tau) \cong A_\tau \text{ and } \mathbb{C}(X_\tau) \cong K_{a(\tau)}.$$

The Jacobian $\mathrm{Jac}(X_\tau)$ has complex multiplication if and only if τ is imaginary quadratic.

Teichmüller and Shimura curves. There are very few examples of Teichmüller curves parametrizing surfaces whose Jacobians lie on a Shimura curve [Mo]. The families $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$ are examples of Teichmüller-Shimura curves and are related to the known examples by a branched covering construction.

The families $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$ are Teichmüller curves arising from square-tiled surfaces (see Figures 1-3 and A-2). The Jacobians of D_8 - and D_{12} -surfaces lie on different components of the transverse self-intersection of the immersion of X_4 into the moduli space of principally polarized abelian varieties. The Jacobians of D_8 - and D_{12} -surfaces admit proper actions of the involutive rings

$$R_{D_8} = \mathbb{Z}[D_8]/(J^2 + 1) \text{ and } R_{D_{12}} = \mathbb{Z}[D_{12}]/(Z^3 + 1),$$

such that the actions of r and Jr in D_8 and r and Zr in D_{12} are self-adjoint. The Weierstrass curve W_D , by contrast, is *not* a Shimura curve ([Mc1], Cor. 10.2) be-

Figure 1-3: For w in the shaded domain V , the parallelogram Q_w has vertices $\{0, 1, 4w, 4w + 1\}$. Gluing the marked segments together as indicated gives a genus two surface Y_w with a degree four cyclic covering map to the double of a parallelogram. The surface Y_w has an action of D_8 generated by the Deck transformation and the involution covered by $z \mapsto -z + 2w + 1$. The quadratic differential q on Y_w induced by dz^2 has two double zeros and is a product of J -eigenforms.

cause of the condition on the $\mathcal{O}_D$ -eigenform. The abelian varieties in X_D which are Jacobians of surfaces in W_D cannot be distinguished by their endomorphism rings.

Algebraic models for D_8 -surfaces with complex multiplication. The function $a(\tau)$ is a transcendental function of τ and modular for the group $\Gamma_0(2) = \langle \Gamma(2), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$. Setting $q = e^{2\pi i\tau}$, we have:

$$a(\tau) = -2 - 256q - 6144q^2 - 76800q^3 - 671744q^4 + \dots$$

Usually, at most one of τ and $a(\tau)$ is algebraic. When τ is imaginary quadratic, the number $a(\tau)$ generates a finite abelian extension of $\mathbb{Q}(\tau)$. The D_8 -surfaces with complex multiplication give the first explicit examples of algebraic curves labeled by points on W_D for most D .

For instance, when $\tau = \frac{1+\sqrt{-19}}{2}$, X_τ is labeled by an orbifold point on W_{76} and $a(\tau)$ generates a degree three abelian extension of $\mathbb{Q}(\sqrt{-19})$. Computer experiment

indicates that the number $a(\tau) = -1.999710899\dots$ is the unique real root of:

$$x^3 + 3x^2 + 3459x + 6913 = 0.$$

Similarly, the point of order two on W_5 is obtained from the polygon $P_{\frac{1}{2}(1+\sqrt{-5})}$ and $a(\frac{1}{2}(1+\sqrt{-5})) = 34 - 16\sqrt{5}$. The point of order four on W_8 is obtained from the polygon $P_{\sqrt{-2}/2}$ and $a(\sqrt{-2}/2) = -6$.

Outline. We conclude this Introduction with an outline of the proofs of our main results.

1. An automorphism ϕ on a surface $X \in \mathcal{M}_2$ induces a permutation σ_ϕ on the Weierstrass points. A brief analysis of the possibilities for σ_ϕ shows that the most of the orbifold points on $\bigcup_D X_D$ which are not products of elliptic curves lie on $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$ and that most of the orbifold points on $\bigcup_D W_D$ lie on $\mathcal{M}_2(D_8)$ (§3).
2. In §4 we study the D_8 -family. For a genus two surface X with faithful D_8 -action ρ , the correspondence $X \rightarrow X/r \times X/Jr$ induces a degree two isogeny between elliptic curves:

$$c_\rho : \text{Jac}(X/r) \rightarrow \text{Jac}(X/Jr).$$

We compute $\mathbb{C}(X)$ and $\text{Jac}(X)$ in terms of the isogeny c_ρ and show that the map $(X, \rho) \mapsto c_\rho$ embeds $\mathcal{M}_2(D_8)$ as the complement the degree two endomorphism of the square torus in the modular curve $Y_0(2)$ parametrizing degree two isogenies between elliptic curves. We then prove Theorem 1.8 by computing the isogenies associated to the D_8 -actions on the surfaces X_τ and $Y_{w(\tau)}$. The outer automorphism σ of D_8 acts on $\mathcal{M}_2(D_8)$ and induces the Atkin-Lehner involution on $Y_0(2)$ sending an isogeny to its dual. Note that the domain U for pinwheels is a fundamental domain for the group $\left\langle \Gamma_0(2), \sigma = \begin{pmatrix} 0 & 1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \right\rangle$. The family $\mathcal{M}_2(D_{12})$ admits a similar analysis, which we outline in §A.1.

3. Let (X, ρ) be a D_8 -surface and let $E = \text{Jac}(X/r)$. In §5 we use the isogeny

$\text{Jac}(X) \rightarrow E \times E$ induced by the map $X \rightarrow X/r \times X/\eta r$ to embed $\text{End}(\text{Jac}(X))$ as an order in $M_2(\text{End}(E) \otimes \mathbb{Q})$. It follows that $\text{Jac}(X)$ has complex multiplication if and only if E has complex multiplication and that the complex multiplication points on $\mathcal{M}_2(D_8)$ give orbifold points on Weierstrass curves. We then sort the $3\tilde{h}(-D)$ surfaces with D_8 -action covering elliptic curves with complex multiplication by $\mathcal{O}_{-D}$ by their orders for real multiplication commuting with J giving the formula for $e_2(W_D)$ in Theorem 1.1.

4. In §6 we sort the orbifold points on W_D by spin component when $D \equiv 1 \pmod{8}$. For such discriminants, the orbifold points on W_D correspond to ideal classes for $\mathcal{O}_{-4D}$. These form a group $H(-4D)$ and there is a spin homomorphism:

$$\epsilon_0 : H(-4D) \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

Fix a proper ideal I for $\mathcal{O}_{-4D}$ and let (X, ρ) be the D_8 -surface with $X/r \cong \mathbb{C}/I$ and with real multiplication by $\mathcal{O}_D$. The spin-invariant for the corresponding orbifold point on W_D is given by:

$$\epsilon = \frac{f+1}{2} + \epsilon_0(I) \pmod{2},$$

where f is the conductor of $\mathcal{O}_D$, i.e. the index of $\mathcal{O}_D$ in the maximal order of $\mathbb{Q}(\sqrt{D})$. The spin homomorphism is the zero map iff D is a square giving the formula in Theorem 1.2.

5. In §7 we use the formula for the number and type of orbifold points on W_D to give bounds on the genera of the components of W_D .

Open problems. While the homeomorphism type of W_D is now understood, describing the components of W_D as Riemann surfaces remains a challenge.

Problem 1. *Describe W_D as a hyperbolic orbifold and as an algebraic curve.*

Our analysis of the orbifold points on W_D have given explicit descriptions of some complex multiplication points on W_D . By the André-Oort conjecture [KY], there are

only finitely many complex multiplication points on W_D and it would be interesting to find them.

Problem 2. *Describe the complex multiplication points on W_D .*

The complex multiplication points on $\mathcal{M}_2(D_8)$ generate Teichmüller curves and the complex multiplication points on $\mathcal{M}_2(D_{12})$ generate complex geodesics with infinitely generated fundamental group. It would be interesting to find other examples of Shimura varieties whose complex multiplication points lie on interesting complex geodesics.

Problem 3. *Find other Shimura varieties whose complex multiplication points generate Teichmüller curves.*

The divisors supported at cusps on modular curves generate a finite subgroup of the associated Jacobian [Ma]. It would be interesting to know if the same is true for Teichmüller curves. The first Weierstrass curve with genus one is W_{44} .

Problem 4. *Compute the subgroup of $\text{Jac}(W_{44})$ generated by divisors supported at the cusps and points of order two.*

Algebraic geometers and number theorists have been interested in exhibiting explicit examples of algebraic curves whose Jacobians have endomorphisms. A parallel goal is to exhibit Riemann surfaces whose Jacobians have endomorphisms as polygons in the plane glued together by translations as we did for the complex multiplication points on $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$.

Problem 5. *Exhibit surfaces whose Jacobians have complex multiplication as polygons in $\mathbb{C}$ glued together by translation.*

Notes and references. For a survey of results related to the Teichmüller geodesic flow, Teichmüller curves and relations to billiards see [KMS, MT, KZ, Zo]. Background about abelian varieties, Hilbert modular surfaces and Shimura varieties can be found in [vdG], [Sh2], [BL], and [Sh1]. The orbifold points on X_D are studied in [Pr] and the family $\mathcal{M}_2(D_8)$ was studied in [Si].

Chapter 2

Background

2.1 Abelian varieties

In this section we will collect background and definitions about abelian varieties, endomorphisms of abelian varieties, Shimura varieties, Hilbert modular surfaces and modular curves. For more background on abelian varieties, endomorphisms and Shimura varieties see [BL], for more background on Hilbert modular surfaces see [vdG] and for more background on modular curves see [DS].

Abelian varieties. A g -dimensional complex torus $B = V/\Lambda$ is the quotient of a g -dimensional $\mathbb{C}$ -vector space V by a cocompact lattice Λ . Any connected, compact complex Lie group is a complex torus. The vector space V is naturally isomorphic to the dual to the space of holomorphic one forms on B and the lattice Λ is naturally isomorphic to the first integral homology:

$$B \cong \Omega(B)^*/H_1(B, \mathbb{Z}).$$

A *polarization* on B is the first Chern class of an ample line bundle on A , or equivalently, an integral symplectic form $E_B(\cdot, \cdot)$ on $H_1(B, \mathbb{Z})$ with the property that $E_B(i\cdot, \cdot) + iE_B(\cdot, \cdot)$ is a positive definite Hermitian form on $\Omega(B)^* \cong H_1(B, \mathbb{Z}) \otimes \mathbb{R}$. A complex torus is an *abelian variety* if it admits a polarization. A polarization on

B is *principal* if the symplectic form is unimodular on $H_1(B, \mathbb{Z})$. We will denote by A_g the moduli space of principally polarized abelian varieties of dimension g .

Homomorphisms and endomorphisms of abelian varieties. A *homomorphism of abelian varieties* $f : B_1 \rightarrow B_2$ is a holomorphic map which is also a homomorphism. Any holomorphic map between abelian varieties is the composition of a homomorphism with a translation. A homomorphism of abelian varieties f is determined by its lift to the universal cover, which is a $\mathbb{C}$ -linear map $\tilde{f} : \Omega(B_1)^* \rightarrow \Omega(B_2)^*$ with $\tilde{f}(H_1(A, \mathbb{Z})) \subset H_1(B, \mathbb{Z})$. The map f is an *isogeny* if it has finite kernel and cokernel. Principal polarizations on B_1 and B_2 determine a dual map $f^* : B_2 \rightarrow B_1$ characterized by:

$$E_{B_2}(\tilde{f}x, y) = E_{B_1}(x, \tilde{f}^*y),$$

whenever $x \in H_1(B_1, \mathbb{Z})$ and $y \in H_1(B_2, \mathbb{Z})$.

Now fix a principally polarized abelian variety B . The *endomorphisms* of B are the holomorphic self-homomorphisms and form a $\mathbb{Z}$ -algebra $\text{End}(B)$ called the *endomorphism ring of B* . An endomorphism f of B is an *automorphism* if it preserves the polarization, i.e. $f^* = f^{-1}$. The *rational endomorphism ring of B* is the $\mathbb{Q}$ -algebra $\text{End}(B) \otimes \mathbb{Q}$. The ring $\text{End}(B)$ is a subring with unit of $\text{End}(B) \otimes \mathbb{Q}$ which is also a lattice, i.e. an *order* in $\text{End}(B) \otimes \mathbb{Q}$. The ring $\text{End}(B)$ has an *analytic representation* Φ_a on $\Omega(B)^*$, a *rational representation* Φ_r on $H_1(B, \mathbb{Q})$ and a representation $\Phi_{\mathbb{Q}}$ on the $\mathbb{Q}$ -vector space $\text{End}(B) \otimes \mathbb{Q}$. The anti-involution $f \mapsto f^*$ of $\text{End}(B)$ is called the *Rosati involution*. The Rosati involution is *positive* in the sense that $(f, g) \mapsto \text{Tr}(\Phi_{\mathbb{Q}}(fg^*))$ is a positive definite bilinear form.

Every abelian variety has a self-dual action of $\mathbb{Z}$ and a typical abelian variety has $\text{End}(B) = \mathbb{Z}$. The subgroup $B[n]$ of B is the kernel of the multiplication by n map and $B[n]^*$ is set of points of order n . The endomorphism ring of the product of elliptic

curves $E \times F$ is given by:

$$\text{End}(E \times F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a \in \text{End}(E), b \in \text{Hom}(F, E) \\ c \in \text{Hom}(E, F) \text{ and } d \in \text{End}(F) \end{array} \right\}.$$

The Rosati involution induced by the product polarization is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$.

Families of abelian varieties with endomorphisms. Let $\mathcal{O}$ be a $\mathbb{Z}$ -algebra with positive anti-involution $*$. An $\mathcal{O}$ -action on an abelian variety $B \in A_g$ is an injective homomorphism:

$$\iota : \mathcal{O} \rightarrow \text{End}(B) \text{ with } \iota(x)^* = \iota(x^*).$$

The $\mathcal{O}$ -action ι is *proper* if it does not extend to a larger ring in $\mathcal{O} \otimes \mathbb{Q}$. The family:

$$A_g(\mathcal{O}) = \{(B, \iota) : \iota \text{ is a proper } \mathcal{O}\text{-action on } B\} / \sim,$$

parametrizes principally polarized abelian varieties with an $\mathcal{O}$ -action. Two pairs (B_1, ι_1) and (B_2, ι_2) are equivalent, i.e. $(B_1, \iota_1) \sim (B_2, \iota_2)$, if there is an isomorphism of polarized abelian varieties $B_1 \rightarrow B_2$ intertwining ι_i .

The components of $A_g(\mathcal{O})$ can be distinguished by the rational representation of $\mathcal{O}$. A *unimodular $\mathcal{O}$ -module* is an $\mathcal{O}$ -module M with a unimodular symplectic form $E_M(\cdot, \cdot)$ which is positive, i.e. the trace of $(xx^*) \in \text{End}(M)$ is positive for each $x \in \mathcal{O}$, and which is compatible with $*$, i.e. $E_M(xm_1, m_2) = E_M(m_1, x^*m_2)$ for every $x \in \mathcal{O}$ and $m_i \in M$. A unimodular $\mathcal{O}$ -module M is an *ideal* if it is a submodule of $\mathcal{O} \otimes \mathbb{Q}$ and M is *proper* if the action of $\mathcal{O}$ does not extend to a larger ring in $\mathcal{O} \otimes \mathbb{Q}$. A proper, unimodular $\mathcal{O}$ -module M with $\mathbb{Z}$ -rank $2g$ determines a component of $A_g(\mathcal{O})$:

$$A_g(M) = \left\{ (B, \iota, \phi) : \begin{array}{l} (B, \iota) \in A_g(\mathcal{O}) \text{ and } \phi : M \rightarrow H_1(B, \mathbb{Z}) \\ \text{is a symplectic } \mathcal{O}\text{-module isomorphism} \end{array} \right\} / \sim.$$

The triple (B_1, ι_1, ϕ_1) is equivalent to (B_2, ι_2, ϕ_2) if there is an isomorphism $B_1 \rightarrow B_2$

intertwining ι_i and ϕ_i .

Shimura varieties. A *Shimura variety* is the quotient of a Hermitian symmetric domain by an arithmetic group. The family $A_g(M)$ is a prototypical example of a Shimura variety. Complex structures on $M \otimes_{\mathbb{Z}} \mathbb{R}$ compatible with E_M form a Hermitian symmetric domain $\mathcal{H}(M)$. A point $\tau \in \mathcal{H}(M)$ turns the real torus $M \otimes_{\mathbb{Z}} \mathbb{R}/M$ into an abelian variety B_τ with $\mathcal{O}$ -action ι_τ . Two complex structures in $\mathcal{H}(M)$ give the same point in $A_g(M)$ if they differ by an element of the arithmetic group $\mathrm{SL}(M)$ of symplectic $\mathcal{O}$ -module automorphisms M .

Let G be the subgroup of $\mathrm{SL}(M)$ which fixes every point in $\mathcal{H}(M)$. The group G acts by automorphisms commuting with $\mathcal{O}$ on each abelian variety in $A_g(M)$ and the group $\mathrm{PSL}(M) = \mathrm{SL}(M)/G$ acts faithfully on $\mathcal{H}(M)$. The space $A_g(M)$ is presented as a complex orbifold and Shimura variety by:

$$A_g(M) \cong \mathcal{H}(M)/\mathrm{PSL}(M).$$

Real and complex multiplication. Let L be a commutative $\mathbb{Q}$ -algebra with positive anti-involution $*$ and let L^0 denote the set of self-adjoint elements of L . Since $*$ is positive, L^0 is a direct sum of totally real fields and either $L = L^0$ or L is a degree two totally imaginary extension of L^0 (see [Sh1], Prop. 1).

When $L = L^0$ and $\mathcal{O}$ is an order in L , we will say that a $g = [L : \mathbb{Q}]$ dimensional abelian variety $B \in A_g$ has *real multiplication by $\mathcal{O}$* if B admits a proper $\mathcal{O}$ -action. For a proper unimodular module M over $\mathcal{O}$, the space $\mathcal{H}(M)$ is isomorphic to $(\mathbb{H})^g$ and the group $\mathrm{SL}(M)$ is commensurable to $\mathrm{SL}_2(\mathcal{O})$.

When $[L : L^0] = 2$, $[L : \mathbb{Q}] = 2g$ and $\mathcal{O}$ is an order in L , we will say $B \in A_g$ has *complex multiplication by the order $\mathcal{O}$* if B admits a proper $\mathcal{O}$ -action. For a proper unimodular module M over $\mathcal{O}$, the space $\mathcal{H}(M)$ is a point.

Real quadratic orders. Each integer $D \equiv 0$ or $1 \pmod{4}$ determines a quadratic ring:

$$\mathcal{O}_D = \frac{\mathbb{Z}[t]}{(t^2 - Dt + D(D-1)/4)}.$$

Every quadratic ring is isomorphic to $\mathcal{O}_D$ for some D . The integer D is called the *discriminant* of $\mathcal{O}_D$ and $\mathcal{O}_D$ is totally real whenever $D > 0$. Let $K_D = \mathcal{O}_D \otimes \mathbb{Q}$, let $\mathcal{O}_{K_D}$ be the maximal order in K_D and let $\sqrt{D}$ be the square root of D equal to $2t - D$. The *conductor* f of $\mathcal{O}_D$ is the index of $\mathcal{O}_D$ in $\mathcal{O}_{K_D}$. The *inverse different* $\mathcal{O}_D^\vee = \frac{1}{\sqrt{D}}\mathcal{O}_D$ is the fractional ideal dual to $\mathcal{O}_D$ under the trace pairing. When $D > 0$, let σ_+ and σ_- denote the two homomorphisms $K_D \rightarrow \mathbb{R}$ satisfying $\sigma_+(\sqrt{D}) > 0 > \sigma_-(\sqrt{D})$. We will also write σ_+ and σ_- for the induced homomorphisms $M_2(K_D) \rightarrow M_2(\mathbb{R})$ and $\mathrm{SL}_2(K_D) \rightarrow \mathrm{SL}_2(\mathbb{R})$.

Our analysis works equally well when D is a square, even though the ring $\mathcal{O}_D$ is no longer a domain and the homomorphisms $\sigma_\pm$ are no longer injective. When $D = f^2$, we have $K_D = \mathbb{Q} \times \mathbb{Q}$, $\mathcal{O}_{K_D} = \mathbb{Z} \times \mathbb{Z}$ and $\mathcal{O}_D$ is the subring of $\mathbb{Z} \times \mathbb{Z}$ consisting of pairs (a, b) with $a \equiv b \pmod{f}$.

Hilbert modular surfaces. The $\mathcal{O}_D$ -module $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$ has a unimodular symplectic form:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \mathrm{Tr}_{\mathbb{Q}}^K(x_1 y_2 - x_2 y_1).$$

Up to isomorphism of symplectic $\mathcal{O}_D$ -modules, $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$ is the unique unimodular and proper $\mathcal{O}_D$ -module isomorphic to $\mathbb{Z}^4$ as a $\mathbb{Z}$ -module. The Hermitian symmetric domain $\mathcal{H}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ is isomorphic to $\mathbb{H} \times \mathbb{H}$ with a point $\tau = (\tau_1, \tau_2)$ corresponding to the abelian variety:

$$B_\tau = \mathbb{C}^2 / \phi_\tau(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$$

where $\phi_\tau(x, y) = (\sigma_+(x) + \tau_1 \sigma_+(y), \sigma_-(x) + \tau_2 \sigma_-(y))$. The diagonal action $x \mapsto \begin{pmatrix} \sigma_+(x) & 0 \\ 0 & \sigma_-(x) \end{pmatrix}$ of $\mathcal{O}_D$ on $\mathbb{C}^2$ covers a proper $\mathcal{O}_D$ -action ι_τ on B_τ , giving a point $(B_\tau, \iota_\tau) \in A_2(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$. Two points in $\mathbb{H} \times \mathbb{H}$ determine the same point in $A_g(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ if and only if they differ by an element of the group:

$$\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} ad - bc = 1, a, d \in \mathcal{O}_D, \\ b \in \sqrt{D}\mathcal{O}_D \text{ and } c \in \mathcal{O}_D^\vee \end{array} \right\}.$$

The matrix $A \in \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ acts on $\mathbb{H} \times \mathbb{H}$ by $A \cdot (\tau_1, \tau_2) = (\sigma_+(A)\tau_1, \sigma_-(A)\tau_2)$ where the matrices $\sigma_\pm(A) \in \mathrm{SL}_2(\mathbb{R})$ act on $\mathbb{H}$ by linear fractional transformation. The group G fixing every point in $\mathbb{H} \times \mathbb{H}$ is generated by the elements of order two and G has order two unless $D = 1$ or 4 , in which case G has order four. The *Hilbert modular surface*:

$$X_D = (\mathbb{H} \times \mathbb{H}) / \mathrm{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee).$$

is in natural bijection with the points in the Shimura variety $A_2(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and presents $A_2(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ as a complex orbifold.

Quaternion algebras. Now fix two positive discriminants D and E . Let $\mathcal{O}$ be an order in the quaternion algebra:

$$\left(\frac{D, E}{\mathbb{Q}} \right) = \mathbb{Q} \langle a, b \rangle / (a^2 = D, b^2 = E, ab = -ba),$$

which is invariant under the anti-involution generated by $a^* = a$, $b^* = b$ and $(ab)^* = ba$ and has $\mathcal{O} \cap \mathbb{Q}(a) = \mathcal{O}_D$ and $\mathcal{O} \cap \mathbb{Q}(b) = \mathcal{O}_E$. Let M be a proper, unimodular $\mathcal{O}$ -module. The family $A_2(M)$ parametrizes a family of abelian varieties with real multiplication by $\mathcal{O}_D$ and $\mathcal{O}_E$ and the projection $A_2(M) \rightarrow A_2$ covers a component of the intersection of the image of X_D with the image of X_E . The Hermitian symmetric domain $\mathcal{H}(M)$ is isomorphic to $\mathbb{H}$ and the arithmetic group $\mathrm{SL}(M)$ is isomorphic to the group of units in $\mathcal{O}$.

Square discriminants. When D is a square, the Hilbert modular surface X_D parametrizes abelian varieties which are isogenous to a product of elliptic curves. The Hilbert modular surface X_1 is isomorphic to $\mathcal{M}_1 \times \mathcal{M}_1$ and parametrizes abelian varieties which are isomorphic to products of elliptic curves.

Elliptic transformations and orbifold points. An element $A \neq 1$ in $\mathrm{PSL}(M)$ is *elliptic* if it fixes a point in $\mathcal{H}(M)$. For $\tau \in \mathcal{H}(M)$, let $\mathrm{Stab}(\tau)$ denote the stablizer of τ in $\mathrm{SL}(M)$. The group $\mathrm{Stab}(\tau)$ acts by automorphisms commuting with $\iota_\tau(\mathcal{O})$ on the associated abelian variety B_τ . When $\mathrm{Stab}(\tau)$ is strictly larger than G , the point

$(B_\tau, \iota_\tau) \in A_g(M)$ is called an *orbifold point* and the cardinality of $\text{Stab}(\tau)/G$ is called the *orbifold order* of (B_τ, ι_τ) .

An elliptic transformation in $A \in \text{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ fixes either a unique point in $\mathbb{H} \times \mathbb{H}$ or every point on a complex geodesic of the form $\tau \times \mathbb{H}$ or $\mathbb{H} \times \tau$. It is easy to check that, when $D \geq 4$, the elliptic elements of $\text{PSL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ have unique fixed points. For such discriminants, the following are equivalent:

- A has finite order,
- $|\text{Tr}(\sigma_+(A))| < 2$ and $|\text{Tr}(\sigma_-(A))| < 2$,
- A fixes a point in $\mathbb{H} \times \mathbb{H}$, and
- A fixes a unique point in $\mathbb{H} \times \mathbb{H}$.

There is a natural bijection between the elements of order $n > 2$ in $\text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ up to conjugacy and abelian varieties with complex multiplication by orders $\mathcal{O}$ containing $\mathcal{O}_D[\zeta_n]$ with $\mathcal{O} \cap (\mathcal{O}_D \otimes \mathbb{Q}) = \mathcal{O}_D$ and ζ_n a primitive n th root of unity.

Modular curves. The modular curve $\mathcal{M}_1 \cong A_1$ is the moduli space of elliptic curves and is isomorphic to the hyperbolic Riemann surface $\mathbb{H}/\text{SL}_2(\mathbb{Z})$. A number $\tau \in \mathbb{H}$ corresponds to the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. Let $\Gamma(n)$ be the kernel of the natural homomorphism $\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/n\mathbb{Z})$, let $\Gamma_1(n) = \langle \Gamma(n), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$ and let $\Gamma_0(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{n} \}$. The quotients:

$$Y(n) = \mathbb{H}/\Gamma(n), Y_1(n) = \mathbb{H}/\Gamma_1(n) \text{ and } Y_0(n) = \mathbb{H}/\Gamma_0(n),$$

parametrize elliptic curves distinguished points of order n . Note that $\Gamma_0(2) = \Gamma_1(2)$.

We will be most concerned with the curve $Y_0(n)$ which parametrizes cyclic degree- n isogenies $i : E \rightarrow F$ between elliptic curves up to isomorphism (see Figure 4-1). The point $\tau \in \mathbb{H}$ corresponds to the isogeny i_τ on E_τ whose kernel is generated by the image T_τ of $1/n$ in $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. The modular curve $Y_0(n)$ has an *Atkin-Lehner involution* sending the isogeny i_τ to its dual $i_\tau^* = i_{-1/n\tau}$.

Class numbers of imaginary quadratic fields. Fix a negative discriminant $D < 0$. The following sets are in natural bijection:

- Elliptic curves with complex multiplication by $\mathcal{O}_D$ up to isomorphism,
- Proper $\mathcal{O}_D$ -ideal classes, and
- Triples of integers (a, b, c) with $D = b^2 - 4ac$, $\gcd(a, b, c) = 1$, $|b| \leq a \leq c$, and if $|b| = a$ or $a = c$ then $b \geq 0$.

The triple (a, b, c) corresponds to the ideal class of $I = a\mathbb{Z} \oplus \frac{-b+\sqrt{D}}{2}\mathbb{Z}$, which in turn corresponds to the elliptic curve $\mathbb{C}/I$. The conditions on (a, b, c) ensure that I is a proper $\mathcal{O}_D$ -ideal and the number $\tau = (-b+\sqrt{D})/(2a)$ is in the standard fundamental domain for the action of the group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

The *class number* $h(-D)$ is the cardinality the set of $\mathcal{O}_D$ -ideal classes. Using $|D| = 4ac - b^2 \geq 4a^2 - a^2$, we have that $|b|, a < \sqrt{D}/3$ and (cf. [Coh], pg. 232):

$$h(-D) < 2D/3.$$

2.2 Riemann surfaces, Jacobians and Automorphisms

In this section we will collect background and definitions about the moduli space of curves, Jacobians, the period map and the Teichmüller geodesic flow. References for this section include [BL] (especially §11) for Jacobians and the period map and [MT] and [Zo] for the Teichmüller geodesic flow.

Jacobians of Riemann surfaces. Let $\mathcal{M}_g$ denote the moduli space of genus g Riemann surfaces. The space $\mathcal{M}_g$ is both an algebraic variety and a complex orbifold. The space $\mathcal{M}_g$ has a *Deligne-Mumford compactification* $\overline{\mathcal{M}}_g$ by stable algebraic curves.

For $X \in \mathcal{M}_g$, let $\Omega(X)$ denote space of holomorphic one forms on X . The *Jacobian* of X is the g -dimensional principally polarized abelian variety:

$$\mathrm{Jac}(X) = \Omega(X)^*/H_1(X, \mathbb{Z}).$$

A holomorphic map $f : X \rightarrow Y$ between Riemann surfaces induces a holomorphic homomorphism $\text{Jac}(X) \rightarrow \text{Jac}(Y)$. The automorphisms of X give automorphisms of the polarized abelian variety $\text{Jac}(X)$ and when $g = 2$, the automorphism groups coincide $\text{Aut}(X) = \text{Aut}(\text{Jac}(X))$.

A point $P \in X$ determines a map $\theta_P : X \rightarrow \text{Jac}(X)$ and any holomorphic map from X to an abelian variety B factors through θ_P . The Jacobian $\text{Jac}(X)$ is isomorphic to the identity component $\text{Pic}^0(X)$ of the *Picard variety*. The abelian variety $\text{Pic}^0(X)$ parametrizes degree zero line bundles on X .

The period map. The *period map* $X \mapsto \text{Jac}(X)$ embeds $\mathcal{M}_g$ in A_g and extends to the moduli space $\widetilde{\mathcal{M}}_g \subset \overline{\mathcal{M}}_g$ of stable algebraic curves with compact Jacobians. The boundary of $\mathcal{M}_g$ in $\widetilde{\mathcal{M}}_g$ consists of polarized products of Jacobians of smooth curves of lower genus.

When $g \leq 2$, the period map is dominant and extends to an isomorphism of complex orbifolds on $\widetilde{\mathcal{M}}_g$. The space $\widetilde{\mathcal{M}}_2$ is the partial compactification of $\mathcal{M}_2$ by the moduli space of pairs of elliptic curves $E \vee F$ joined at a single simple node:

$$\widetilde{\mathcal{M}}_2 = \mathcal{M}_2 \cup \text{Sym}^2(\mathcal{M}_1).$$

The Jacobian of $E \vee F$ is the product $E \times F$ with the product polarization. The complement of $\mathcal{M}_2$ in A_2 is the image of the Hilbert modular surface X_1 :

$$\begin{array}{ccc} \mathcal{M}_2 & \xrightarrow{\sim} & A_2 \setminus X_1 \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_2 & \xrightarrow{\sim} & A_2 \end{array}$$

Surfaces with automorphisms. For a finite group G , the family of genus g surfaces with a faithful action of G is parametrized by the space:

$$\widetilde{\mathcal{M}}_g(G) = \left\{ (X, \rho) : \begin{array}{l} X \in \widetilde{\mathcal{M}}_g, \text{ and } \rho : G \rightarrow \text{Aut}(X) \\ \text{is injective.} \end{array} \right\} / \sim.$$

Two pairs (X_1, ρ_1) and (X_2, ρ_2) are equivalent, i.e. $(X_1, \rho_1) \sim (X_2, \rho_2)$, if there is an isomorphism $X_1 \rightarrow X_2$ intertwining the ρ_i . The set $\widetilde{\mathcal{M}}_g(G)$ has a natural topology and a unique holomorphic structure making the map $\widetilde{\mathcal{M}}_g(G) \rightarrow \mathcal{M}_g$ holomorphic.

We will denote by $\mathcal{M}_g(G)$ the subspace of $\widetilde{\mathcal{M}}_g(G)$ parametrizing the family of smooth surfaces with an action of G . The action of G on $\text{Jac}(X)$ extends to a (not necessarily faithful) action of the group ring $\mathbb{Z}[G]$. The Rosati-involution is generated by $g^* = g^{-1}$ for $g \in G$.

An injective group homomorphism $G_1 \rightarrow G_2$ gives a holomorphic map $\widetilde{\mathcal{M}}_2(G_2) \rightarrow \widetilde{\mathcal{M}}_2(G_1)$. The automorphism group of G acts on $\widetilde{\mathcal{M}}_2(G)$ and the inner automorphisms act trivially. This gives an action of the outer automorphism group $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ on $\widetilde{\mathcal{M}}_2(G)$. Note that $\text{Out}(D_8)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the non-trivial outer automorphism σ has $\sigma(r) = Jr$ and $\sigma(J) = J$.

Correspondences. Suppose $f : Y \rightarrow X_1$ and $g : Y \rightarrow X_2$ are holomorphic maps between Riemann surfaces and $f_* : \text{Jac}(Y) \rightarrow \text{Jac}(X_1)$ and $g_* : \text{Jac}(Y) \rightarrow \text{Jac}(X_2)$ are the maps induced on Jacobians. The map $Y \rightarrow X_1 \times X_2$ is called a *correspondence between X_1 and X_2* and induces a holomorphic homomorphism $g_*(f_*)^* : \text{Jac}(X_1) \rightarrow \text{Jac}(X_2)$.

Under an identification of $\text{Jac}(X_i)$ with $\text{Pic}^0(X_i)$, the holomorphic homomorphism $g_*(f_*)^*$ is obtained by taking a degree zero divisor on X_1 , pulling it back to Y and pushing it forward to X_2 :

$$(g_*)(f_*)^* : \sum_i (P_i - Q_i) \mapsto \sum_i g(f^{-1}(P_i)) - g(f^{-1}(Q_i)),$$

for any points P_i and Q_i are points in X_1 .

The hyperelliptic involution and Weierstrass points. Every genus two surface with compact Jacobian $X \in \widetilde{\mathcal{M}}_2$ has a *hyperelliptic involution* η acting by $\omega \mapsto -\omega$ on $\Omega(X)$.

When X is smooth, the quotient X/η is a sphere naturally isomorphic to $\mathbb{P}\Omega(X)^*$. The surface X has six distinguished *Weierstrass points* fixed by η and we will denote

the set of Weierstrass points by $X^W = \text{Fix}(\eta)$. For each $P \in X$, there is a holomorphic one form in $\Omega(X)$ with zeros at P and $\eta(P)$ and the Weierstrass points are the zeros of holomorphic one-forms with double zero.

When $X = E_1 \vee E_2$ is singular, η restricts to the elliptic involution on each component of X fixing the node. Taking the node to be the base point for E_1 and E_2 , the set of Weierstrass points $X^W = E_1[2]^* \sqcup E_2[2]^*$ consists of the points of order two. The node is not a Weierstrass point even though it is fixed by η .

For any $X \in \widetilde{\mathcal{M}}_2$, the hyperelliptic involution η is in the center of the automorphism group $\text{Aut}(X)$ and the group $\text{Aut}(X)$ acts naturally on both X^W and X/η . The group $\text{Sym}(X^W)$ is isomorphic to the symmetric group on six letters S_6 and the conjugacy classes in $\text{Sym}(X^W)$ are naturally labeled by partitions of six:

$$[n_1, \dots, n_k] \text{ with } \sum_i n_i = 6.$$

Involutions of genus two surfaces. Fix an order two automorphism ϕ on a genus two surface $X \in \widetilde{\mathcal{M}}_2$. There are three possibilities for ϕ , distinguished by the action of $\mathbb{Z}[\phi] \cong \mathcal{O}_4$ on $\text{Jac}(X)$:

- $\mathbb{Z}[\phi]$ is not faithful. The map ϕ acts by -1 on $\text{Jac}(X)$ and ϕ is the hyperelliptic involution.
- $\mathbb{Z}[\phi]$ is faithful but not proper. The action of $\mathbb{Z}[\phi]$ on $\text{Jac}(X)$ extends to real multiplication by the maximal order $\mathcal{O}_1 = \mathbb{Z} \left[\frac{1+\phi}{2} \right]$. The surface X is nodal and ϕ restricts to the identity on one component of X and the elliptic involution on the other. The eigenspaces for ϕ in $\Omega(X)$ are one dimensional and ϕ fixes the components of X .
- $\mathbb{Z}[\phi]$ is proper. The quotient X/ϕ has genus one. This is true for nodal X since ϕ interchanges the components of X because it does not commute with $\mathcal{O}_1$. For smooth X , ϕ is an involution different from the hyperelliptic involution, so the quotient must have genus one. The eigenspaces for ϕ on $\Omega(X)$ are one-dimensional and $\text{Jac}(X)$ has real multiplication by $\mathcal{O}_4 = \mathbb{Z}[\phi]$.

Fix a surface X with involution ϕ such that the induced action of $\mathbb{Z}[\phi]$ on $\text{Jac}(X)$ is proper. Since ϕ commutes with the hyperelliptic involution η , η induces an elliptic involution η_E on the quotient $E = X/\phi$. It is easy to check that $\text{Fix}(\eta\phi)$ forms a single ϕ -orbit and that η_E fixes its image. Setting this point to be the base point, i.e. $E[1] = \text{Fix}(\eta\phi)/\phi$, identifies E with its Jacobian via $\theta_{E[1]} : E \rightarrow \text{Jac}(E)$. The remaining points fixed by η_E are covered by the Weierstrass points, giving an identification:

$$E[2]^* = X^W/\phi.$$

Similarly, the quotient $F = X/\eta\phi$ has genus one, a natural elliptic involution η_F , base point $F[1] = \text{Fix}(\phi)/\eta\phi$ and identification $F[2]^* = X^W/\eta\phi$.

Abelian and quadratic differentials. For a Riemann surface $X \in \mathcal{M}_g$, the space of *holomorphic quadratic differentials* $Q(X)$ consists of sections of the square of the cotangent bundle on X . The space of holomorphic one-forms on X admits a quadratic map $\Omega(X) \rightarrow Q(X)$ and when $g \leq 3$, the space $\text{Sym}^2(\Omega(X))$ is naturally isomorphic to $Q(X)$. The bundle of abelian differentials $\Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$ parametrizes Riemann surfaces with a non-zero holomorphic one-form. The bundle of quadratic differentials $Q\mathcal{M}_g \rightarrow \mathcal{M}_g$ parametrizes Riemann surfaces with a non-zero quadratic differential and is isomorphic to the complement of the zero section in the cotangent bundle $T^*\mathcal{M}_g$.

A typical way to give a point in $(X, q) \in Q\mathcal{M}_g$ is to glue polygons P_i in $\mathbb{C}$ together along their sides by affine maps of the form $z \mapsto \pm z + c$ as in Figures 1-2 and 1-3. The quadratic differential $dz^2 \in Q(\mathbb{C})$ is invariant under such maps and gives a quadratic differential on the quotient:

$$(X, q) = \bigcup_i (P_i, dz^2) / \sim .$$

A quadratic differential (X, q) is the square of a one-form, i.e. $q = \omega^2$, if (X, q) can be written as a union of polygons glued together by affine maps of the form $z \mapsto z + c$.

Complex geodesics. The real geodesic flow on $\mathcal{M}_g$ gives $\mathbb{R}$ -actions on $Q\mathcal{M}_g$ and $\Omega\mathcal{M}_g$ which extend to $\mathrm{SL}_2(\mathbb{R})$ -actions. For the quadratic differential $(X, q) = \bigcup_i (P_i, dz^2)/\sim$ obtained from polygons $P_i \subset \mathbb{C}$, the matrix $A \in \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C}$ by real-linear transformations giving a new surface with quadratic differential $A \cdot (X, q) = \bigcup_i (A(P_i), dz^2)/\sim$.

The subgroup $\mathrm{SO}_2(\mathbb{R})$ acts on (X, q) by leaving X fixed and rotating q . The $\mathrm{SL}_2(\mathbb{R})$ -orbit $\mathrm{SL}_2(\mathbb{R}) \cdot (X, q)$ covers a holomorphic and isometric immersion of the hyperbolic plane called a *complex geodesic*:

$$f_{(X,q)} : \mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \rightarrow \mathcal{M}_g.$$

The map $f_{(X,q)}$ further factors through the quotient by the *Veech group* $\mathrm{SL}(X, q) = \mathrm{Stab}(X, q)$ in $\mathrm{SL}_2(\mathbb{R})$ and the projectivizations $\mathbb{P}Q\mathcal{M}_g = Q\mathcal{M}_g/\mathbb{C}^*$ and $\mathbb{P}\Omega\mathcal{M}_g = \Omega\mathcal{M}_g/\mathbb{C}^*$ are foliated by complex geodesics. The $\mathrm{SL}_2(\mathbb{R})$ -action preserves the quadratic differentials which are squares of holomorphic one-forms. The bundle $\Omega\mathcal{M}_g$ extends to the bundle $\widetilde{\Omega\mathcal{M}_g}$ whose fiber over the nodal surface with compact Jacobian X with components X_i is isomorphic to $\oplus_i \Omega(X_i)$. The $\mathrm{SL}_2(\mathbb{R})$ -action extends naturally to $\widetilde{\Omega\mathcal{M}_g}$. The bundle $Q^G\mathcal{M}_g(G) \rightarrow \mathcal{M}_g(G)$ parametrizing G -invariant quadratic differentials is closed under the foliation by complex geodesics.

Teichmüller curves. When $\mathrm{SL}(X, q)$ is a lattice, i.e. $\mathrm{SL}(X, q)$ has finite covolume in $\mathrm{SL}_2(\mathbb{R})$, the complex geodesic $f_{(X,q)}$ factors through an algebraic and isometric immersion of the quotient $C = \mathbb{H}/\mathrm{SL}(X, q)$. Such an immersion is called a *Teichmüller curve*. If P_i are polygons whose vertices lie in the lattice $\mathbb{Z}[i] \subset \mathbb{C}$, then the quadratic differential $(X, q) = \bigcup_i (P_i, dz^2)/\sim$ is *square-tiled*, $\mathrm{SL}(X, q)$ is a lattice in $\mathrm{SL}_2(\mathbb{Z})$ and (X, q) generates a Teichmüller curve.

Topological invariants of hyperbolic Riemann surfaces. The topological invariants of a two real dimensional finite volume hyperbolic orbifold V with real dimension two include the number of components $h^0(V)$, the genus $g(V) = \dim_{\mathbb{C}} \Omega(V)$, the number of cusps $C(V)$ and the number of orbifold points $e_n(V)$ of order n for

$n > 1$. These numbers are combined to give an *orbifold Euler characteristic* $\chi(V)$:

$$\chi(V) = 2h^0(V) - 2g(V) - C(V) - \sum_n e_n(V) \left(1 - \frac{1}{n}\right).$$

When V is connected (i.e. $h^0(V) = 1$) the numbers $g(V)$, $e_n(V)$ and $C(V)$ determine the homeomorphism type of V .

2.3 The Weierstrass curve

In this section we will collect background related to the Weierstrass curve and its components. References for this section include [Mc1, Mc4, Ba] for the definition of W_D , [Mc3] for the components of W_D and [At] for spin structures on Riemann surfaces.

Eigenforms and the Weierstrass curve. Fix a surface $X \in \widetilde{\mathcal{M}}_2$ and a proper $\mathcal{O}_D$ -action $\iota : \mathcal{O}_D \rightarrow \text{End}(\text{Jac}(X))$. The space of one-forms up to scale $\mathbb{P}\Omega(X)$ has a distinguished point corresponding to the line of eigenforms $[\omega]$ with:

$$\iota(x)\omega = \sigma_+(x)\omega,$$

where σ_+ is the homomorphism $\mathcal{O}_D \rightarrow \mathbb{R}$ with $\sigma_+(\sqrt{D}) > 0$. The map sending (X, ι) to its σ_+ -eigenform up to scale gives an embedding of the Hilbert modular surface X_D in $\mathbb{P}\Omega\widetilde{\mathcal{M}}_2$ (cf. [Ba], Section 4.3). The images of X_D and X_E in $\mathbb{P}\Omega\widetilde{\mathcal{M}}_2$ are disjoint whenever $D \neq E$.

The Weierstrass curve W_D parametrizes the family of surfaces with real multiplication by $\mathcal{O}_D$ whose eigenforms have a double zero:

$$W_D = \left\{ (X, \iota) : \begin{array}{l} X \in \mathcal{M}_2, \iota \text{ is a proper } \mathcal{O}_D\text{-action on} \\ \text{Jac}(X), \text{ and the } \sigma_+\text{-eigenform up to} \\ \text{scale } [\omega] \in \mathbb{P}\Omega(X) \text{ has a double zero} \end{array} \right\} / \sim.$$

The period map embeds W_D as a suborbifold of X_D .

The Hilbert modular surface X_D and the Weierstrass curve W_D are foliated by Teichmüller geodesics [Ca, Mc1]:

Theorem 2.1 (Calt, McMullen). *The images of W_D and X_D in $\mathbb{P}\Omega\widetilde{\mathcal{M}}_2$ are closed under the foliation by complex geodesics.*

Orbifolds points on W_D . Since W_D is a suborbifold of X_D , the orbifold points on W_D correspond to pairs (X, ι) where $\iota : \mathcal{O}_D \rightarrow \text{End}(\text{Jac}(X))$ extends to complex multiplication by an order containing $\mathcal{O}_D[\zeta_n]$ for some $n > 2$. Since the automorphisms of $\text{Jac}(X)$ coincide with automorphisms of X , (X, ι) is an orbifold point on W_D if and only if X has an automorphism commuting with $\iota(\mathcal{O}_D)$. The orbifold order of (X, ι) is the cardinality of the group $\text{Aut}(X)'/\eta$ of automorphisms commuting with $\iota(\mathcal{O}_D)$ modulo the hyperelliptic involution.

Components of W_D . The components of W_D were enumerated in [Mc3]. Usually W_D is irreducible; when W_D is reducible the components of W_D can be distinguished by a spin invariant which we describe below.

A *spin structure* on a topological surface X is a quadratic form

$$q : H_1(X, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$$

satisfying $q(x + y) = q(x) + q(y) + \langle x, y \rangle$ where $\langle, \rangle$ is the intersection pairing. The parity of q is given by the Arf-invariant:

$$\text{Arf}(q) = \sum_i q(A_i)q(B_i) \in \mathbb{Z}/2\mathbb{Z}$$

for a standard symplectic basis $\{A_i, B_i\}$ of $H_1(X, \mathbb{Z}/2\mathbb{Z})$. The number $\text{Arf}(q)$ is independent of basis.

If $X \in \mathcal{M}_2$ is a Riemann surface, a one-form $\omega \in \Omega(X)$ with double zero determines an odd spin structure q , i.e. $\text{Arf}(q) = 1$. Any $x \in H_1(X, \mathbb{Z}/2\mathbb{Z})$ is the image of the fundamental class of the circle S^1 under a smooth immersion $C : S^1 \rightarrow X$ that

avoids the zero $Z(\omega)$ of ω . The Gauss map is the map $S^1 \rightarrow S^1$ given by:

$$G_x : \theta \mapsto \frac{\omega(C'(\theta))}{|\omega(C'(\theta))|},$$

and the function $q(x) = \deg G_x + 1 \bmod 2$ does not depend on the immersion C and defines an odd spin structure on X .

Now fix $D > 1$ with $D \equiv 1 \bmod 8$ and a point $(X, \iota) \in W_D$ and let f be the conductor of $\mathcal{O}_D$. Let q be the spin structure defined by the σ_+ -eigenform up to scale $[\omega]$. The Jacobian endomorphism $(f + \sqrt{D})/2$ gives a linear transformation on $H_1(X, \mathbb{Z}/2\mathbb{Z})$ whose image V is a two dimensional subspace. The spin invariant of (X, ι) is defined to be the Arf-invariant of q restricted to V :

$$\epsilon(X, \omega) = \epsilon(X, \iota) = \text{Arf}(q|_V).$$

The spin component associated to $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ is given by:

$$W_D^\epsilon = \{(X, \iota) : (X, \iota) \in W_D \text{ and } \epsilon(X, \iota) = \epsilon\}.$$

McMullen showed that the discriminant for real multiplication and the spin invariant distinguish the irreducible components of $\bigcup_D W_D$ ([Mc3], Thm. 1.1):

Theorem 2.2 (McMullen). *The curves W_1 and W_4 are empty. For discriminants $D > 9$ with $D \equiv 1 \bmod 8$, the spin components W_D^0 and W_D^1 are non-empty and irreducible. For all other discriminants, W_D is irreducible.*

Bouw and Möller showed the spin components are Galois conjugate and homeomorphic ([BM1], Thm. 3.3):

Theorem 2.3 (Bouw-Möller). *Fix $D \equiv 1 \bmod 8$ with D not a square. The curves W_D^0 and W_D^1 have algebraic models defined over $\mathbb{Q}(\sqrt{D})$ and are Galois conjugate.*

When $D = f^2$ is an odd square, the spin invariant has a more elementary interpretation. For such discriminants, $(X, \iota) \in W_D$ is a degree f branched cover of an

elliptic curve E branched over a single point B , and the number $N = |f^{-1}(B) \cap X^W|$ determines the spin invariant $\epsilon(X, \iota) = \frac{N-1}{2}$.

Chapter 3

Orbifold points on Hilbert modular surfaces

In this section, we will show that most of the orbifold points on $\bigcup_D X_D$ are Jacobians of D_8 - and D_{12} -surfaces with complex multiplication. In §4, 5 and A.1 we will show that every complex multiplication point on $\mathcal{M}_2(D_8)$ and $\mathcal{M}_2(D_{12})$ gives orbifold point on $\bigcup_D X_D$ establishing Theorem 1.5.

Automorphisms of abelian surfaces. Let $\tilde{S}_4$ and $\tilde{D}_{12}$ be the automorphism groups of the polarized abelian varieties $A_{\sqrt{-2}/2}$ and $\tilde{A}_{\sqrt{-3}/3}$ defined in Sections 1 and A.1. The groups $\tilde{S}_4$ and $\tilde{D}_{12}$ are $\mathbb{Z}/2\mathbb{Z}$ -central extensions of the groups S_4 and D_{12} respectively. The following proposition is well known (cf. [BL], pg. 340):

Proposition 3.1. *The automorphism group of a two dimensional principally polarized abelian variety which is not a product of elliptic curves is isomorphic to one of the following: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/10\mathbb{Z}$, D_4 , D_8 , D_{12} , $\tilde{S}_4$ or $\tilde{D}_{12}$.*

Note that the group $\tilde{S}_4$ contains D_8 and the group $\tilde{D}_{12}$ contains both D_8 and D_{12} .

Orbifold points on Hilbert modular surfaces. We will prove the following variant of Proposition 3.1:

Proposition 3.2. *Fix a discriminant $D > 0$, an integer $n > 2$ and an abelian variety $B \in A_2$ with $\mathcal{O}_D[\zeta_n]$ -action ι . One of the following holds:*

- B is a product of elliptic curves,
- (B, ι) is a point of orbifold order five on X_5 ,
- An iterate of $\iota(\zeta_n)$ extends to a faithful action of D_8 on B by automorphisms,
or
- An iterate of $\iota(\zeta_n)$ extends to a faithful action of D_{12} on B by automorphisms.

Recall that the orbifold points of order n on X_D correspond to abelian varieties with complex multiplication by orders containing $\mathcal{O}_D[\zeta_{2n}]$ (§2.1). Proposition 3.2 shows that the orbifold points on $\bigcup_D X_D$ which are not products of elliptic curves are the points of order five on X_5 and Jacobians of D_8 - and D_{12} -surfaces with complex multiplication.

Proof. Every two dimensional principally polarized abelian variety is a product of elliptic curves or the Jacobian of a smooth surface (cf. 2.2). Suppose B is the Jacobian of $X \in \mathcal{M}_2$. Let ϕ be the automorphism of X inducing $\iota(\zeta_n)$ on $\text{Jac}(X)$, and let σ_ϕ be the induced permutation of the Weierstrass points $X^W = \text{Fix}(\eta)$. Since ϕ and each of its iterates can fix at most three points on the sphere X/η , the conjugacy class of σ_ϕ is one of the following:

- $[2,2,2]$: There is a coordinate $x : X/\eta \xrightarrow{\sim} \widehat{\mathbb{C}}$ with $x \circ \phi = -x$ and $x(X^W) = \{\pm 1, \pm a, \pm b\}$ for some a and $b \in \mathbb{C}^*$. The field of functions on X satisfies:

$$\mathbb{C}(X) \cong \frac{\mathbb{C}(z)[x]}{(z^2 - (x^2 - 1)(x^2 - a^2)(x^2 - b^2))},$$

and $\phi(z, x) = (z, -x)$ or $(-z, -x)$. This is impossible since $\phi^2 \neq 1$.

- $[2,2,1,1]$: Let P_1, P_2, Q_1 and Q_2 be Weierstrass points so $\phi(P_i) = P_i$ and $X^W = \{P_1, P_2, Q_1, \phi(Q_1), Q_2, \phi(Q_2)\}$. There is a unique coordinate $x : X/\eta \rightarrow \widehat{\mathbb{C}}$ with $x(P_1) = 1$, $x(P_2) = -1$ and $x(Q_1) = -x(Q_2)$. This coordinate satisfies $x \circ \phi = 1/x$ and $x(X^W) = \{\pm 1, \pm x(Q_1), \pm x(Q_1)^{-1}\}$. The field of functions on X satisfies:

$$\mathbb{C}(X) \cong \frac{\mathbb{C}(z)[x]}{(z^2 - (x^2 - 1)(x^4 - ax^2 + 1))},$$

where $a = x(Q_1)^2 + x(Q_1)^{-2}$. The surface X has a faithful D_8 -action generated by $r(z, x) = (z, -x)$ and $J = (iz/x^3, 1/x)$. Since $x \circ \phi = 1/x$, $\phi = J$ or J^3 and $\iota(\zeta_n)$ extends to a faithful action of D_8 on B .

- $[3, 3]$: Let P and Q be points in X^W in different ϕ -orbits. There is a coordinate $x : X/\eta \xrightarrow{\sim} \widehat{\mathbb{C}}$ with $x \circ \phi = \zeta_3 x$ and $x(P) = x(Q)^{-1}$. Setting $a = x(P)$, we have $x(X^W) = \{a, \zeta_3 a, \zeta_3^2 a, a^{-1}, \zeta_3 a^{-1}, \zeta_3^2 a^{-1}\}$ and the field of functions on X satisfies:

$$\mathbb{C}(X) \cong \frac{\mathbb{C}(z)[x]}{(z^2 - (x^3 - a^3)(x^3 - a^{-3}))}.$$

The surface X has a faithful D_{12} -action generated by $r(z, x) = (z/x^3, 1/x)$ and $Z(z, x) = (-z, \zeta_3 x)$. Since $x \circ \phi = \zeta_3 x$, ϕ is in the subgroup of D_{12} generated by Z and $\iota(\zeta_n)$ extends to an action of D_{12} on B .

- $[4, 1, 1]$ or $[4, 2]$: The automorphism ϕ^2 acts on X^W by a permutation in the conjugacy class $[2, 2, 1, 1]$ and B has a faithful D_8 -action by automorphisms extending $\iota(\zeta_n^2)$. The surface X is the unique genus two surface whose automorphism group is $\tilde{S}_4$:

$$\mathbb{C}(X) \cong \frac{\mathbb{C}(z)[x]}{(z^2 - (x^2 - 1)(x^4 + 6x^2 + 1))}.$$

- $[5, 1]$: There is a coordinate $x : X/\eta \xrightarrow{\sim} \widehat{\mathbb{C}}$ with $x \circ \phi = \zeta_5 x$ and $x(X^W) = \{1, \zeta_5^{\pm 1}, \zeta_5^{\pm 2}, \infty\}$. The surface X is the unique genus two surface with order five automorphism and satisfies:

$$\mathbb{C}(X) \cong \frac{\mathbb{C}(z)[x]}{(z^2 - (x^5 - 1))}.$$

The order for real multiplication $\mathcal{O}_D$ is $\mathcal{O}_5 = \mathbb{Z}[\zeta_5] \cap \mathbb{R}$.

- $[6]$: The automorphism ϕ^2 induces a permutation of X^W in the conjugacy class $[3, 3]$ and B has a faithful D_{12} -action extending $\iota(\zeta_n^2)$. The surface X is the

unique genus two surface whose automorphism group is $\tilde{D}_{12}$ and X satisfies:

$$\mathbb{C}(X) \cong \frac{\mathbb{C}(z)[x]}{(z^2 - (x^6 + 1))}.$$

□

We record the following consequence of the proof of Proposition 3.2 which will be useful in our analysis of the D_8 -family:

Proposition 3.3. *Suppose $X \in \mathcal{M}_2$ has an order four automorphism ϕ . The induced permutation $\sigma_\phi \in \text{Sym}(X^W)$ is in the conjugacy class $[2, 2, 1, 1]$ and ϕ extends to an action of D_8 .*

Proof. Since the kernel of $\text{Aut}(X) \rightarrow \text{Sym}(X^W)$ is generated by the hyperelliptic involution, the permutation σ_ϕ has order two or four. Since the automorphisms inducing permutations in the conjugacy class $[2, 2, 2]$ have order two and those inducing permutations in $[4, 1, 1]$ and $[4, 2]$ have order eight, the permutation σ_ϕ must be in $[2, 2, 1, 1]$. □

Orbifold points on W_D . Our characterization of the orbifold points on X_D which lie on W_D follows readily from Proposition 3.2.

Proposition 3.4. *Fix $X \in \mathcal{M}_2$ with $\mathcal{O}_D[\zeta_n]$ -action ι on $\text{Jac}(X)$ and suppose an eigenform for ι has a double zero. Either:*

- *X is the genus two surface with order five automorphism, or*
- *X admits a faithful action of D_8 extending an iterate of $\iota(\zeta_n)$.*

Proof. If ϕ is the automorphism inducing $\iota(\zeta_n)$ on $\text{Jac}(X)$ and σ_ϕ is the induced permutation on X^W , the conjugacy class of σ_ϕ in $\text{Sym}(X^W)$ is $[2, 2, 1, 1]$, $[4, 1, 1]$ or $[5, 1]$ since ϕ fixes at least one point in X^W , namely the zero of the eigenform. □

Remark. The orbifold points on X_D which are products of elliptic curves are also easy to characterize. For $D = 1$, the complex orbifold X_1 is isomorphic to the product

$\mathcal{M}_1 \times \mathcal{M}_1$ and the product $E \times F$ with $\mathcal{O}_1$ generated by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an orbifold point if and only if E or F is an orbifold point on $\mathcal{M}_1$. For $D \geq 4$, if the product $E \times F$ is labeled by an orbifold point on X_D , then E and F are isomorphic and have complex multiplication by $\mathbb{Z}[\zeta_3]$, $\mathbb{Z}[i]$ or an order in $\mathbb{Q}(\sqrt{-D})$.

Chapter 4

The D_8 -family

The family $\widetilde{\mathcal{M}}_2(D_8)$ parametrizes pairs (X, ρ) where X is in $\widetilde{\mathcal{M}}_2$ and $\rho : D_8 \rightarrow \text{Aut}(X)$ is an injective homomorphism (cf. §2.2). In this section, we will identify the components of $\widetilde{\mathcal{M}}_2(D_8)$ and study the component whose general member is smooth. For $(X, \rho) \in \mathcal{M}_2(D_8)$ we will show that the correspondence $X \rightarrow X/r \times X/Jr$ induces a degree two isogeny between elliptic curves:

$$c_\rho : \text{Jac}(X/r) \rightarrow \text{Jac}(X/Jr).$$

We will compute $\text{Jac}(X)$ and $\mathbb{C}(X)$ in terms of c_ρ to show:

Proposition 4.1. *The map $(X, \rho) \mapsto c_\rho$ gives an embedding:*

$$\mathcal{M}_2(D_8) \rightarrow Y_0(2),$$

whose image is the complement of the degree two endomorphism of the square torus. The action of the non-trivial outer automorphism σ of D_8 on $\mathcal{M}_2(D_8)$ is induced by the Atkin-Lehner involution on $Y_0(2)$.

In §1, we defined the domain U , the field K_a , the abelian variety A_τ , the surfaces X_τ and Y_w and the functions $a(\tau)$ and $w(\tau)$. We will also show that the surfaces X_τ and $Y_{w(\tau)}$ admit faithful D_8 -actions inducing the isogeny i_τ on $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ whose kernel is generated by the image T_τ of $1/2$ in $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ (see Figure 4-1), establishing:

Figure 4-1: The domain V obtained from U and the image of U under $\tau \mapsto -1/2\tau$ is a fundamental domain for the group $\Gamma_0(2)$. A point $\tau \in V$ determines an elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ and a degree two isogeny i_τ whose kernel is generated by the point of order two T_τ . The isogeny $i_{-1/2\tau}$ is dual to i_τ since $i_{-1/2\tau} \circ i_\tau$ is the multiplication by two map on E_τ .

Proposition 4.2. *Fix $\tau \in U$. The surface X_τ admits a faithful D_8 -action ρ with $c_\rho = i_\tau$, is isomorphic to the surface $Y_{w(\tau)}$ and satisfies:*

$$\mathbb{C}(X_\tau) \cong K_{a(\tau)} \text{ and } \text{Jac}(X_\tau) \cong A_\tau.$$

Components of $\widetilde{\mathcal{M}}_2(D_8)$. Fix a surface $X \in \widetilde{\mathcal{M}}_2$ and let $\rho : D_8 \rightarrow \text{Aut}(X)$ be a faithful representation of D_8 . We will call ρ *proper* if the induced actions of $\mathbb{Z}[r]$ and $\mathbb{Z}[Jr]$ on $\text{Jac}(X)$ are proper. We will call ρ *improper* if it is not proper. We will denote the corresponding components of $\widetilde{\mathcal{M}}_2(D_8)$ by $\widetilde{\mathcal{M}}_2(D_8)^{pr}$ and $\widetilde{\mathcal{M}}_2(D_8)^{im}$.

A faithful representation on a smooth surface is automatically proper and we will show that $\widetilde{\mathcal{M}}_2(D_8)^{pr}$ is the closure of $\mathcal{M}_2(D_8)$. For a nodal surface $X = E \vee F$, an action of D_8 is proper if and only if both r and Jr interchange the components of X . For any genus one surface $E \in \mathcal{M}_1$, the singular surface $E \vee E$ has a faithful D_8 -action which is improper.

Nodal surfaces. We now characterize the nodal surfaces in $\widetilde{\mathcal{M}}_2(D_8)$.

Proposition 4.3. *The nodal surface $X = E \vee F$ admits an improper D_8 -action iff E is isomorphic to F . The nodal surface $X = E \vee F$ admits a proper D_8 -action if and*

only if both E and F are isomorphic to the square torus.

Proof. It is easy to show that $E \vee E$ has an improper action of D_8 for any $E \in \mathcal{M}_1$. It is also easy to check that when $E = \mathbb{C}/\mathbb{Z}[i]$, the automorphism group of $E \vee E$ contains a proper action of D_8 , which is unique up to conjugation in $\text{Aut}(E \vee E)$.

For the converse, suppose $\rho : D_8 \rightarrow \text{Aut}(X)$ is injective. Since the subgroup of $\text{Aut}(X)$ fixing the components of X is the commutative group $\text{Aut}(E) \times \text{Aut}(F)$, either r or Jr interchanges the components of X and E is isomorphic to F . Now suppose ρ is also proper. Both r and Jr interchange the components of X and the composition $J = (Jr)(r)$ fixes the components of X . Since $J^4 = 1$, both E and F are isomorphic to the square torus. $\square$

Remark. Proposition 4.3 can be used to show that $\widetilde{\mathcal{M}}_2(D_8)^{im}$ has two components both of which are isomorphic to $\mathcal{M}_1$ (the components are distinguished by whether r or Jr generates a proper $\mathcal{O}_4$ -action). Also, the proper D_8 -action on $E \vee E$ when $E = \mathbb{C}/\mathbb{Z}[i]$ is unique up to $\sim$ since it is unique up to conjugation in $\text{Aut}(E \vee E)$.

Weierstrass points. Let $\phi : D_8 \rightarrow S_6$ be the representation into the symmetric group on six letters given by:

$$\phi(r) = (12)(34)(56) \text{ and } \phi(J) = (13)(24).$$

For a proper D_8 -surface, the action D_8 induces on the Weierstrass points is isomorphic to ϕ .

Proposition 4.4. *Fix a surface $X \in \widetilde{\mathcal{M}}_2$ with a proper D_8 -action ρ . There is a bijection $X^W \rightarrow S_6/S_5$ inducing a homomorphism $f : \text{Sym}(X^W) \rightarrow S_6$ so the following diagram commutes:*

$$\begin{array}{ccc} D_8 & \xrightarrow{\rho} & \text{Sym}(X^W) \\ \parallel & & \downarrow f \\ D_8 & \xrightarrow{\phi} & S_6 \end{array}$$

Proof. We will show that J^2 is the hyperelliptic involution and that the involutions r and Jr fix no point in X^W . As a consequence, the action D_8 induces on X^W factors through the commutative group $D_4 \cong D_8/J^2$ and the proposition follows easily.

Since $\mathbb{Z}[r]$ and $\mathbb{Z}[Jr]$ act faithfully on $\text{Jac}(X)$, the eigenspaces for r and Jr in $\Omega(X)$ are one-dimensional. Since r and Jr do not commute, their eigenspaces are distinct. The automorphism J^2 commutes with both r and Jr and fixes every line in $\Omega(X)$. The automorphism J^2 is the hyperelliptic involution.

Since $\mathbb{Z}[r]$ and $\mathbb{Z}[Jr]$ are proper, the quotients X/r and X/Jr have genus one. For smooth X , the fixed points of r and Jr are the simple zeros of the one-forms pulled back from X/r and X/Jr , so they are disjoint from X^W . For nodal X , r and Jr interchange the components of X and again the fixed points of r and Jr are disjoint from X^W . $\square$

We will denote by $\text{Fix}_W(J)$ the set of Weierstrass points fixed by J . We will need the following fact in our discussion of surfaces obtained from polygons:

Proposition 4.5. *Two proper D_8 -surfaces (X_1, ρ_1) and (X_2, ρ_2) have an isomorphism $\psi : X_1 \rightarrow X_2$ intertwining $\rho_i(J)$ if and only if:*

$$(X_1, \rho_1) \sim (X_2, \rho_2) \text{ or } (X_1, \rho_1) \sim (X_2, \rho_2 \circ \sigma).$$

Proof. Let t be the automorphism of X_1 pulled back from $\rho_2(r)$ under ψ . By Proposition 4.4, the action of t on X_1^W coincides with that of r or Jr and, as a consequence, t is one of r , Jr , ηr or ηJr . If t is r or Jr , ϕ gives the desired equivalence. Otherwise, $\phi \circ \rho_1(J)$ gives the desired equivalence. $\square$

Proper D_8 -actions, correspondences and isogeny. We will now show that a proper D_8 -surface $(X, \rho) \in \widetilde{\mathcal{M}}_2(D_8)^{pr}$ determines a degree two isogeny c_ρ between the genus one surfaces $\text{Jac}(X/r)$ and $\text{Jac}(X/Jr)$. The non-trivial outer automorphism σ of D_8 has $\sigma(r) = Jr$ and $\sigma(J) = J$. We will also show that the surface $(X, \rho \circ \sigma)$ determines the dual isogeny c_ρ^* .

Recall that the quotients $E = X/r$ and $F = X/Jr$ have genus one, elliptic involutions η_E and η_F induced by η , natural base points $E[1] = \text{Fix}(\eta r)/r$ and $F[1] = \text{Fix}(r)/\eta r$ and identifications $E[2]^* = X^W/r$ and $F[2]^* = X^W/Jr$ (see §2.2).

Proposition 4.6. *Suppose $X \in \widetilde{\mathcal{M}}_2$ has a proper D_8 -action ρ . The correspondence $X \rightarrow X/r \times X/Jr$ induces an isogeny:*

$$c_\rho : \text{Jac}(X/r) \rightarrow \text{Jac}(X/Jr)$$

whose kernel is generated by the point of order two $[\text{Fix}_W(J)/r - \text{Fix}(\eta r)/r]$. The isogeny c_ρ is dual to the isogeny $c_{\rho\sigma}$.

Proof. We will show that $\ker(c_\rho)$ contains the point of order two labeled by $\text{Fix}_W(J)/r$ and that $c_\rho \circ c_{\rho\sigma}$ is the multiplication by two map on $\text{Jac}(X/r)$. It follows that both c_ρ and $c_{\rho\sigma}$ have degree two and $c_\rho^* = c_{\rho\sigma}$.

The isogeny c_ρ is given by summing the images of the fibers of the map $X \rightarrow X/r$ under the map $X \rightarrow X/Jr$. For the point in X/r labeled by the r -orbit $\{Q, rQ\}$ we have:

$$c_\rho(\{Q, rQ\}) = \{Q, JrQ\} +_F \{rQ, JQ\}.$$

The point $\text{Fix}_W(J)/r$ is in $\ker(c_\rho)$ since $\{rQ, JQ\} = \eta_F(\{Q, JrQ\})$ whenever $J(Q) = Q$.

The non-trivial outer automorphism σ of D_8 is generated by $\sigma(J) = J$ and $\sigma(r) = Jr$. Using the formula above for c_ρ and the relations on D_8 , we have that $c_{\rho\sigma} \circ c_\rho(\{Q, rQ\})$ is equal to:

$$\{Q, rQ\} +_E \{JrQ, J^3Q\} +_E \{rQ, Q\} +_E \{JQ, rJQ\}.$$

Since $J^2 = \eta$, we have $\{JrQ, J^3Q\} = \eta_E(\{JQ, rJQ\})$ and the second and fourth terms cancel, giving $c_\rho \circ c_{\rho\sigma}(\{Q, rQ\}) = \{Q, rQ\} +_E \{Q, rQ\}$. $\square$

The unique nodal and proper D_8 -surface (X, ρ) has:

$$\text{Jac}(X/r) = \text{Jac}(X/Jr) = \mathbb{C}/\mathbb{Z}[i],$$

and $c_\rho = i_{(1+i)/2}$ is point of orbifold order two on the modular curve $Y_0(2)$ (see §2.1).

Jacobians. We now compute the Jacobian $\text{Jac}(X)$ for a proper D_8 -surface (X, ρ) in terms of the isogeny c_ρ .

Proposition 4.7. *Fix a D_8 -surface $(X, \rho) \in \widetilde{\mathcal{M}}_2(D_8)^{pr}$ and a number $\tau \in \mathbb{H}$ so $c_\rho = i_\tau$. The Jacobian of X satisfies:*

$$\text{Jac}(X) \cong A_\tau.$$

The polarized abelian variety A_τ was defined in §1 and the isogeny i_τ is the map on $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ whose kernel is generated by the image T_τ of $1/2$ in $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$.

Proof. Since the lattice Λ_τ contains the lattice $\mathbb{Z} \langle (\frac{2}{0}), (\frac{2\tau}{0}), (\frac{0}{2}), (\frac{0}{2\tau}) \rangle$, the abelian variety A_τ satisfies:

$$A_\tau \cong (E_\tau \times E_\tau) / \Gamma^{T_\tau},$$

where $\Gamma^{T_\tau} \subset E_\tau[2]^* \times E_\tau[2]^*$ is the graph of the transposition fixing T_τ . Twice the product polarization on $E_\tau \times E_\tau$ is pulled back from the principal polarization on A_τ under the isogeny $E_\tau \times E_\tau \rightarrow A_\tau$.

Let $E = \text{Jac}(X/r)$ and $\widetilde{E} = \text{Jac}(X/\eta r)$. Since $c_\rho = i_\tau$, the elliptic curve E_τ is isomorphic to E . The automorphism J induces an isomorphism:

$$\phi : X/r \xrightarrow{\sim} X/\eta r,$$

since $J^{-1}rJ = \eta r$ so $\widetilde{E}$ is also isomorphic to E_τ . The equality $X^W/r = X^W/\eta r$ gives a bijection on the points of order two $\psi : E[2]^* \rightarrow \widetilde{E}[2]^*$ and the composition $\phi^{-1} \circ \psi$ is the action J induces on $X^W/r = E[2]^*$. We computed the action of ρ on X^W in Proposition 4.4 and $\phi^{-1} \circ \psi$ is the transposition fixing T_τ .

The map $X \rightarrow X/r \times X/\eta r$ factors through a degree four map on Jacobians $\text{Jac}(X) \rightarrow E \times \widetilde{E}$. The image of X^W in $E \times \widetilde{E} \cong E_\tau \times E_\tau$ is the graph Γ^{T_τ} of $\psi^{-1} \circ \phi$ and the image of $\text{Jac}(X)[2]$ in $E_\tau \times E_\tau$ is the subgroup generated by Γ^{T_τ} . The quotient

map:

$$\mathrm{Jac}(X) \rightarrow (E_\tau \times E_\tau)/\Gamma^{T_\tau}$$

has degree sixteen, vanishes on the two torsion $\mathrm{Jac}(X)[2]$ and factors through multiplication by two to give an isomorphism. The polarization is a product since the self-adjoint operator r has eigenspaces pulled back from $\Omega(E)$ and $\Omega(\tilde{E})$. Unimodularity implies the polarization is twice the product polarization. $\square$

Remark. More generally, if $Y \in \widetilde{\mathcal{M}}_2$ has an involution r generating a proper $\mathcal{O}_4$ -action on $\mathrm{Jac}(Y)$, the quotients $E_1 = Y/r$ and $E_2 = Y/\eta r$ have genus one, natural elliptic involutions and base points. The points of order two on E_1 and E_2 have an identification coming from Y^W :

$$E_1[2]^* = Y^W/r = Y^W/\eta r = E_2[2]^*.$$

The middle equality comes from the fact that η fixes Y^W pointwise. The Jacobian of Y satisfies:

$$\mathrm{Jac}(Y) \cong (E_1 \times E_2)/\Gamma^W,$$

where Γ^W is the graph of this identification and the right hand side is polarized by twice the product polarization. Proposition 4.7 is a special instance of this computation.

Since the Jacobian of a D_8 -surface can be computed from the isogeny c_ρ , we have established that the map $(X, \rho) \mapsto c_\rho$ embeds $\mathcal{M}_2(D_8)$ in $Y_0(2)$:

Proof of Proposition 4.1. We will show that the map $\widetilde{\mathcal{M}}_2(D_8)^{pr} \rightarrow Y_0(2)$ given by $(X, \rho) \mapsto c_\rho$ is an isomorphism. The remaining claims follow from Proposition 4.6 and our observation that the unique nodal and proper D_8 -surface maps to the isogeny $i_{(1+i)/2}$.

We will define an inverse for the map $(X, \rho) \mapsto c_\rho$. Fix an isogeny i_τ in $Y_0(2)$ and let X be the (possibly nodal) surface with Jacobian A_τ . Let $\rho_\tau : D_8 \rightarrow \mathrm{Aut}(X)$ be the action of D_8 inducing the Jacobian automorphisms $\rho_\tau(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\rho_\tau(J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The map $i_\tau \rightarrow (X_\tau, \rho_\tau)$ is the desired inverse by Proposition 4.7. $\square$

Remark. The family $\widetilde{\mathcal{M}}_2(D_8)^{pr}$ is connected and one dimensional. Since there is a unique nodal and proper D_8 -surface, $\widetilde{\mathcal{M}}_2(D_8)^{pr}$ is the closure of $\mathcal{M}_2(D_8)$ in $\widetilde{\mathcal{M}}_2(D_8)$.

Algebraic curves. The function $a(\tau) = -2 + \frac{1}{\lambda(\tau)\lambda(\tau+1)}$ defined in §1 is modular for $\Gamma_0(2)$ and covers an isomorphism $a : Y_0(2) \rightarrow \mathbb{C} \setminus \{-2, \infty\}$. Using $\lambda\left(\frac{-1+i}{2}\right) = \lambda\left(\frac{1+i}{2}\right) = \frac{1}{2}$, the isomorphism a is uniquely determined by:

$$a(\Gamma_0(2) \cdot 0) = \infty, a(\Gamma_0(2) \cdot \infty) = -2, \text{ and } a(\Gamma_0(2) \cdot (1+i)/2) = 2.$$

The number $a(\tau)$ is characterized by:

$$\mathbb{C}(E_\tau) \cong \frac{\mathbb{C}(z)[w]}{(z^2 - (w-1)(w^2 - a(\tau)w + 1))} \text{ with } T_\tau = [w^{-1}(1) - w^{-1}(\infty)],$$

where as usual $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ and $T_\tau = \frac{1}{2} + \mathbb{Z} \oplus \tau\mathbb{Z}$.

Proposition 4.8. *Fix a smooth D_8 -surface (X, ρ) and $\tau \in \mathbb{H}$ with $c_\rho = i_\tau$. The field of functions on X satisfies:*

$$\mathbb{C}(X) \cong K_{a(\tau)}.$$

Proof. Let Y be the genus two surface with:

$$\mathbb{C}(Y) \cong K_{a(\tau)} = \mathbb{C}(z, x) \text{ with } z^2 = (x^2 - 1)(x^4 - a(\tau)x^2 + 1),$$

and let ψ be the D_8 -action on Y with $\psi(r)(z, x) = (z, -x)$ and $\psi(J)(z, x) = (iz/x^3, 1/x)$. The quotient Y/r has $\mathbb{C}(Y/r) \cong \mathbb{C}(z, x^2)$, a field isomorphic to $\mathbb{C}(E_\tau)$. The action $\psi(J)$ induces on the points of order two $E_\tau[2]^*$ is the transposition fixing $[w^{-1}(1) - w^{-1}(\infty)]$. The isogeny c_ψ is isomorphic to i_τ and Proposition 4.1 implies that $(Y, \psi) \sim (X, \rho)$. $\square$

Remark. The field $K_{(-2a+12)/(a+2)}$ is isomorphic to K_a . Equipping K_a with the D_8 -actions $\rho_a(z, x) = (z, -x)$ and $J(z, x) = (iz/x^3, 1/x)$, the D_8 -surfaces associated to (K_a, ρ_a) and $(K_{(-2a+12)/(a+2)}, \rho_{(-2a+12)/(a+2)})$ differ by the outer automorphism σ of D_8 .

Cusps on $\mathcal{M}_2(D_8)/\sigma$. There are two cusps on $\mathcal{M}_2(D_8)/\sigma$, one of which maps to the cusp of the $(2, 4, \infty)$ -orbifold $\widetilde{\mathcal{M}}_2(D_8)^{pr}/\sigma$. For $(X, \rho) \in \mathcal{M}_2(D_8)$, let $|q_\rho|$ be the flat metric with singularities induced by a quadratic differential on $X/\rho(J)$ with simple poles at $X^W/\rho(J)$ and unit area $\int_{X/\rho(J)} |q_\rho| = 1$.

Proposition 4.9. *For a sequence of D_8 -surfaces $(X_i, \rho_i) \in \mathcal{M}_2(D_8)$, the following are equivalent:*

1. *The D_8 -surfaces (X_i, ρ_i) diverge in $\widetilde{\mathcal{M}}_2(D_8)^{pr}/\sigma$.*
2. *The quotients $X_i/\rho_i(r)$ diverge in $\mathcal{M}_1$.*
3. *The surfaces X_i tend to a stable limit with geometric genus zero.*
4. *The diameter of $\text{Fix}_W(\rho_i(J))$ in the metric space $(X_i/\rho_i(J), |q_{\rho_i}|)$ tends to zero.*

Proof. Condition (1) and (2) are equivalent since the two maps $\widetilde{\mathcal{M}}_2(D_8)^{pr} \cong Y_0(2) \rightarrow \mathcal{M}_1$ given by $(X, \rho) \mapsto X/r$ and $(X, \rho) \mapsto X/Jr$ are proper. Now suppose (2) holds and X_i/r diverge in $\mathcal{M}_1$. This happens if and only if there is a sequence of numbers a_i tending to -2 so $\mathbb{C}(X_i) \cong K_{a_i}$. The stable limit X_∞ has:

$$\mathbb{C}(X_\infty) \cong K_2 = \mathbb{C}(z, x) \text{ with } z^2 = (x^2 - 1)(x^2 + 1)^2,$$

and the geometric genus of X_∞ is zero since $K_2 = \mathbb{C}\left(\frac{z}{(x^2+1)(x-1)}\right)$. Conversely, a surface with geometric genus zero does not have compact Jacobian so (2) and (3) are equivalent.

Finally, the fields $\mathbb{C}(X_i/J) \subset K_{a_i}$ are generated by the function $g = x + 1/x$ and $g(\text{Fix}_W(J)) = \{\pm 2\}$. If a_i tends to -2 , the remaining points in X^W tend to 0 under g and the diameter of $\text{Fix}_W(\rho_i(J))$ tends to zero in the metric space $(X_i/J, |q_{\rho_i}|)$. If a_i tends to 2, the remaining points in X^W tend to ± 2 under g and the diameter of $\text{Fix}_W(\rho_i(J))$ tends to ∞ in $(X_i/J, |q_{\rho_i}|)$. This shows that (2) and (4) are equivalent. $\square$

Figure 4-2: The surfaces X_τ and $X_{\tau+1}$ are isomorphic since the polygons P_τ and $P_{\tau+1}$ differ by a cut-and-paste operation.

Polygons. We now turn to the surfaces obtained from the polygons P_τ and Q_w . The following proposition, together with Propositions 4.7 and 4.8 completes the proof of Proposition 4.2:

Proposition 4.10. *Fix $\tau \in U$. The surfaces X_τ and $Y_{w(\tau)}$ have proper D_8 -actions ρ_X and ρ_Y with $c_{\rho_X} = c_{\rho_Y} = i_\tau$.*

Proof. Let J_τ denote the order four automorphism on X_τ obtained from counterclockwise rotation the polygon P_τ and let J_w denote the order four Deck transformation for the map on Y_w depicted in Fig. 1-3. We will show that the maps $\tau \rightarrow (X_\tau, J_\tau)$ and $w \mapsto (Y_w, J_w)$ give maps:

$$g_U : U / \sim_U \rightarrow \mathcal{M}_2(D_8) / \sigma \text{ and } g_V : V / \sim_V \rightarrow \mathcal{M}_2(D_8) / \sigma$$

with $\tau \sim_U \tau+1$, $\tau \sim_U -1/2\tau$, $w \sim_V w+2$ and $w \sim_V w/(w+1)$. Since $U / \sim_U$, $V / \sim_V$ and $\mathcal{M}_2(D_8) / \sigma$ are all isomorphic to the $(2, \infty, \infty)$ -orbifold, the maps g_U and g_V are isomorphisms. We will also show that $g_U(\tau)$ and $g_V(w)$ diverge in $\widetilde{\mathcal{M}}_2(D_8)^{pr} / \sigma$ as $\text{Im}(\tau)$ and $\text{Im}(w)$ tend to ∞ , determining the isomorphisms g_U and g_V and showing that X_τ and $Y_{w(\tau)}$ have D_8 -actions ρ_X and ρ_Y with $c_{\rho_X} = c_{\rho_Y} = i_\tau$.

By Proposition 3.3, the automorphisms J_τ and J_w extend to actions of D_8 on X_τ and Y_w . By Proposition 4.5, the maps $\tau \mapsto (X_\tau, J_\tau)$ and $w \mapsto (Y_w, J_w)$ give

holomorphic maps $g_U : U \rightarrow \mathcal{M}_2(D_8)/\sigma$ and $g_V : V \rightarrow M_2(D_8)/\sigma$.

The polygons P_τ and $P_{\tau+1}$ differ by a cut-and-paste operation (see Figure 4-2) and the polygons P_τ and $P_{-1/2\tau}$ differ by Euclidean similarity. The surfaces X_τ , $X_{\tau+1}$ and $X_{-1/2\tau}$ are isomorphic with isomorphisms intertwining the actions of J_τ , $J_{\tau+1}$ and $J_{-1/2\tau}$ and the map g_U descends to the quotient $U/\sim_U$. As $\text{Im } \tau \rightarrow \infty$, the stable limit of X_τ has geometric genus zero and $g_U(\tau)$ diverges in $\mathcal{M}_2(D_8)^{pr}/\sigma$.

It is easy to check that the Veech group of $(Y_i, q_i) = (Q_i, dz^2)/\sim$ contains $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The surfaces Y_w , Y_{w+2} and $Y_{w/(w+1)}$ are all isomorphic with isomorphisms intertwining J_w , J_{w+2} and $J_{w/(w+1)}$ and the map g_V descends to the quotient $V/\sim_V$. The flat metric $|q_\rho|$ on Y_w/J_w is induced by $|dz^2|/\text{Im}(w)$ (Fig. 1-3) and the diameter of $\text{Fix}_W(J_w)$ in $(Y_w/J_w, |q_\rho|)$ is $1/\sqrt{\text{Im}(w)}$ which tends to zero as $\text{Im}(w) \rightarrow \infty$. $\square$

Chapter 5

Endomorphisms

Fix a smooth D_8 -surface (X, ρ) and let $E = \text{Jac}(X/\tau)$. In this section we embed $\text{End}(\text{Jac}(X))$ as an order in $M_2(\text{End}(E) \otimes \mathbb{Q})$, characterize the proper D_8 -surfaces with complex multiplication and sort them by their orders for real multiplication to compute $e_2(W_D)$ and prove Theorem 1.1.

Theorem 1.8 describing the D_8 -surfaces with complex multiplication follows from Proposition 4.2 and:

Proposition 5.1. *The abelian variety A_τ has complex multiplication iff τ is imaginary quadratic.*

Whenever E has complex multiplication, $\rho(J)$ extends to complex multiplication by an order in $\text{End}(E) \otimes \mathbb{Q}[i]$ giving an orbifold point on $\bigcup_D W_D$.

Proposition 5.2. *Fix a D_8 -surface $(X, \rho) \in \mathcal{M}_2(D_8)$ and suppose $\text{Jac}(X)$ has complex multiplication. The action of J on $\text{Jac}(X)$ extends to complex multiplication ι by an order in $\mathbb{Q}(\sqrt{D}, i)$ and the pair (X, ι) gives an orbifold point on $\bigcup_D W_D$.*

Propositions 3.4 and 5.2 complete our characterization of the orbifold points on $\bigcup_D W_D$ asserted in Corollary 1.6. The analogous results for D_{12} -surfaces show that the complex multiplication points of $\mathcal{M}_2(D_{12})$ are labeled by orbifold points on $\bigcup_D X_D$ (see §A.1). Together with Proposition 3.2, this establishes our characterization the orbifold points on $\bigcup_D X_D$ in Theorem 1.5.

Isogeny and endomorphism. The endomorphism rings of isogenous abelian varieties are related by the following proposition.

Proposition 5.3. *Suppose $f : A \rightarrow B$ is an isogeny between principally polarized abelian varieties and n -times the polarization on A is the polarization pulled back from B . The ring $\text{End}(B)$ is isomorphic as an involutive algebra to the subring $R_f \subset \text{End}(A) \otimes \mathbb{Q}$ given by:*

$$R_f = \frac{1}{n} \{ \phi \in \text{End}(A) : \phi(\ker(nf)) \subset \ker(f) \}.$$

The ring R_f satisfies $\frac{1}{n} \text{End}(A) \supset R_f \supset n \text{End}(A)$. Proposition 5.3 shows that $\text{End}(A)$ is an order in $\text{End}(B) \otimes \mathbb{Q}$ and that the rational endomorphism ring is an isogeny invariant. Propositions 5.1 and 5.2 follow:

Proof of Proposition 5.1. The abelian variety A_τ has a degree four isogeny to $E_\tau \times E_\tau$ and twice the polarization on A_τ is pulled back from the product polarization. When τ is imaginary quadratic, the ring $\text{End}(A_\tau)$ is an order in $M_2(\mathbb{Q}(\tau))$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$ and A_τ has complex multiplication by an order in $\mathbb{Q}(\tau, i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. When τ is not imaginary quadratic, $\text{End}(A_\tau)$ is an order in $M_2(\mathbb{Q})$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and A_τ does not have complex multiplication since every commutative subalgebra of $M_2(\mathbb{Q})$ has rank two over $\mathbb{Q}$. $\square$

Proof of Proposition 5.2. Suppose (X, ρ) is a smooth D_8 -surface and $\text{Jac}(X)$ has complex multiplication. The center of $\text{End}(A_\tau)$ is an order in $\mathbb{Q}(\tau)$ and the order four automorphism $\rho(J)$ extends to complex multiplication ι by an order $\mathcal{O}$ in $\mathbb{Q}(\tau, i)$. The $\mathcal{O}$ -eigenforms have double zeros by Proposition 3.3, so the pair (X, ι) gives an orbifold point on $\bigcup_D W_D$. $\square$

Remark. The Jacobian $\text{Jac}(X)$ for a D_{12} -surface (X, ρ) is isogenous to $E \times E$ where $E = \text{Jac}(X/r)$ (§A.1). The same argument shows that the Z -eigenforms on D_{12} -surfaces with complex multiplication are σ -eigenforms for orbifold points on $\bigcup_D X_D$, establishing our characterization of the orbifold points on Hilbert modular surfaces of Theorem 1.5.

Proof of Proposition 5.3. We will show that the map:

$$\begin{aligned}\rho : \text{End}(B) &\rightarrow \text{End}(A) \otimes \mathbb{Q} \\ \phi &\mapsto \frac{1}{n} f^* \circ \phi \circ f\end{aligned}$$

is an injective homomorphism and has image $\rho(\text{End}(B)) = R_f$. The condition on the polarization implies that $f^* \circ f$ and $f \circ f^*$ are the multiplication by n maps on A and B respectively and it is easy to check that ρ is a homomorphism. The map ρ is injective since rationally it is an isomorphism. The inverse for ρ is given by $\rho^{-1}(\psi) = \frac{1}{n} f \circ \psi \circ f^*$.

The image of $\text{End}(B)$ is contained in R_f since:

$$\ker(nf) \xrightarrow{f} B[n] \xrightarrow{\phi} B[n] \xrightarrow{f^*} \ker(f)$$

for any $\phi \in \text{End}(B)$. To see that the image of ρ is all of R_f , fix $\psi \in \text{End}(A) \otimes \mathbb{Q}$ satisfying $\psi(\ker(nf)) \subset \ker(f)$. The endomorphism $\phi = \frac{1}{n^2} f \circ \psi \circ f^*$ is integral on B since:

$$B[n^2] \xrightarrow{f^*} \ker(nf) \xrightarrow{\psi} \ker(f) \xrightarrow{f} 0.$$

Since $\rho(\phi) = \psi$, the image of ρ is all of R_f .

The isomorphism between $\text{End}(B)$ and R_f has $\rho(\phi)^* = \rho(\phi^*)$ since $(f^* \phi f)^* = f^* \phi^* f$. $\square$

Invertible modules over finite rings. The Jacobian of X has a degree four isogeny $f : \text{Jac}(X) \rightarrow E \times E$ with $f \circ f^*$ acting by multiplication by two on $E \times E$. In light of Proposition 5.3, if E has complex multiplication by $\mathcal{O}_C$, the order for real multiplication on X commuting with J has discriminant $D = -C/4$, $-C$ or $-4C$. The actual order for real multiplication on $\text{Jac}(X)$ commuting with J depends only on $\text{End}(E)$, the $M_2(\text{End}(E))$ -module $E[4] \times E[4]$ and the $\mathbb{Z}$ -submodules $\ker(f)$ and $\ker(2f)$.

To sort the complex multiplication points on $\mathcal{M}_2(D_8)$ by their orders for real

multiplication we need to determine the possible $\mathcal{O}_C$ -modules $E[4]$:

Proposition 5.4. *Let $\mathcal{O}$ be a quadratic order and I be a proper $\mathcal{O}$ -ideal. The module I/nI is isomorphic to $\mathcal{O}/n\mathcal{O}$ as $\mathcal{O}$ -modules.*

For imaginary quadratic orders, Proposition 5.4 can be deduced from the fact that $\mathbb{C}/I$ and $\mathbb{C}/J$ have algebraic models which are Galois conjugate whenever I and J are proper $\mathcal{O}$ -ideals.

Proof. Since $\mathcal{O}$ is quadratic, the Galois conjugate I' is an inverse for I and $I \otimes I'$ is isomorphic to $\mathcal{O}$. The quotient I/nI is also invertible and the invertible modules over $R = \mathcal{O}/n\mathcal{O}$ form a group $\text{Pic}(R)$. We will show $\text{Pic}(R)$ is trivial.

The ring R is an Artin ring and by the structure theorem there are local Artin rings R_i with:

$$R = \prod_i R_i \text{ and } \text{Pic}(R) = \prod_i \text{Pic}(R_i).$$

The group $\text{Pic}(R_i)$ is trivial by Nakayama's lemma: if I_i is an invertible R_i module, and m_i is the maximal ideal, I_i/m_i is an invertible module over the field R_i/m_i . The R_i/m_i -vector space $I_i/m_i I_i$ is one dimensional and if $x \in I_i$ generates this vector space, $R_i x + m_i I_i = I_i$ which implies $I_i = R_i x$. $\square$

We are now ready to sort the complex multiplication points on $\mathcal{M}_2(D_8)$ by their orders for real multiplication.

Proof of Theorem 1.1. The only D_8 -surface whose automorphism group is larger than D_8 is the surface obtained from the regular octagon labeling the point of order four on W_8 (cf. Proposition 3.1). For discriminants $D > 8$, all of the orbifold points on W_D are labeled by complex multiplication points on $\mathcal{M}_2(D_8)$ and have order two. To complete our proof, we need to determine the number of D_8 -surfaces with real multiplication by $\mathcal{O}_D$ commuting with $\rho(J)$.

For each $C \equiv 0$ or $1 \pmod{4}$, set:

$$\mathcal{O}_C = \frac{\mathbb{Z}[x]}{x^2 - Cx + \frac{C(C-1)}{4}}.$$

$C \bmod 16$	D_1	D_2	D_3
0	$-4C$	$-C$	$-4C$
4	$-4C$	$-4C$	$-C$
8	$-4C$	$-C$	$-4C$
12	$-4C$	$-4C$	$-C/4$
1 or 9	$-4C$	$-4C$	$-4C$
5 or 13	$-4C$	$-4C$	$-4C$

Table 5.1: The elliptic curve $E = \mathbb{C}/\mathcal{O}_C$ is covered by three D_8 -surfaces (X_i, ρ_i) . The discriminant D_i of the order for real multiplication on $\text{Jac}(X_i)$ commuting with J is computed using Proposition 5.3.

The ring $\mathcal{O}_C/4\mathcal{O}_C$ depends only on $C \bmod 16$. Let $P_1 = 1$, $P_2 = x$ and $P_3 = x + 1$ be the points of order two in $\mathcal{O}_C/2\mathcal{O}_C$ and let (X_i, ρ_i) be the D_8 -surface in $\widetilde{\mathcal{M}}_2(D_8)^{pr}$ with $X_i/\rho_i(r) \cong \mathbb{C}/\mathcal{O}_C$ and $\ker(c_{\rho_i})$ generated by P_i . In Table 5.1 we compute the discriminant D_i of the order for real multiplication on $\text{Jac}(X_i)$ commuting with J using Proposition 5.3. Most of the entries in Table 5.1 are determined by the observation that $\text{Jac}(X_i)$ has complex multiplication by an order containing $\mathcal{O}_C[i]$ iff the $\mathcal{O}_C$ -module $P_i\mathcal{O}_C/2\mathcal{O}_C$ is a *proper* submodule of $\mathcal{O}_C/2\mathcal{O}_C$.

For any genus one surface E with complex multiplication by $\mathcal{O}_C$, we have $E[4] \cong \mathcal{O}_C/4$ by Proposition 5.4. With an appropriate ordering, the three D_8 -surfaces covering E have the same orders for real multiplication as the three D_8 -surfaces covering $\mathbb{C}/\mathcal{O}_C$ by Proposition 5.3. The formula for $e_2(W_D)$ in Table 1.1 follows easily. Note that the factor of two in the formula for $e_2(W_D)$ comes from the fact that the D_8 -surfaces (X, ρ) and $(X, \rho \circ \sigma)$ label the same orbifold point on $\bigcup_D W_D$. $\square$

Chapter 6

Spin

For a number $\tau \in U$, let $\omega_\tau \in \Omega(X_\tau)$ be the holomorphic one-form with double zero on the surface X_τ obtained by gluing together opposite sides on P_τ :

$$(X_\tau, \omega_\tau) = (P_\tau, dz) / \sim .$$

In this section we compute the spin structure $q : H_1(X_\tau, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ coming from the form ω_τ and then, for $D \equiv 1 \pmod{8}$, sort the orbifold points on W_D by their spin component.

Spin structure. Let a be the number $\frac{\tau-1+i}{2}$ and x_t be the homology class in $H_1(X_\tau, \mathbb{Z})$ with period $t \in \mathbb{Z}[i] \oplus a\mathbb{Z}[i]$ under the one-form ω_τ .

Proposition 6.1. *The spin structure q on X_τ obtained from the form with double zero ω_τ has:*

$$q(k_1x_1 + k_2x_i + k_3x_a + k_4x_{ai}) = k_1^2 + k_2^2 + k_1k_3 + k_2k_4 + k_3k_4.$$

Proof. Isotoping representatives away from the zero $Z(\omega_\tau)$ and computing the degree of the Gauss map shows that the spin structure q associated to ω_τ satisfies:

$$q(x_1) = q(x_i) = 1 \text{ and } q(x_a) = q(x_{ai}) = 0.$$

From the relation $q(x + y) = q(x) + q(y) + \langle x, y \rangle$, any basis x_n of $H_1(X_\tau, \mathbb{Z}/2\mathbb{Z})$ has:

$$q\left(\sum_n k_n x_n\right) = \sum_n k_n^2 q(x_n) + \sum_{l < n} k_l k_n \langle x_l, x_n \rangle.$$

Computing the intersection pairing on the basis $\{x_1, x_i, x_a, x_{ai}\}$ gives the stated formula for q . $\square$

Spin homomorphism. Now fix an integer $D \geq 9$ with $D \equiv 1 \pmod{8}$ and let f be the conductor of D . An ideal $I \subset \mathcal{O}_{-4D}$ has norm $\text{Nm}(I) = 2^k l$ with l odd. We define:

$$\epsilon_0(I) = \frac{l-1}{2} \pmod{2}.$$

Proposition 6.2. *The number $\epsilon_0(I)$ depends only on the ideal class of I and defines a spin homomorphism:*

$$\epsilon_0 : H(-4D) \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

The spin homomorphism ϵ_0 is the zero map iff D is a square.

Proof. Any $x \in \mathcal{O}_{-4D}$ has norm $\text{Nm}(x) = x_1^2 + x_2^2 D = 2^k l$ with $l \equiv 1 \pmod{4}$. If the ideals I and J are in the same ideal class, they satisfy $xI = yJ$ for some x and y in $\mathcal{O}_{-4D}$ and $\epsilon_0(I) = \epsilon_0(J)$. The map ϵ_0 is a homomorphism since the norm of ideals is a homomorphism.

Now suppose $D = f^2$ is a square. Any $\mathcal{O}_{-4D}$ -ideal class has a representative of the form $I = x\mathbb{Z} \oplus (fi - y)\mathbb{Z}$. Since I is an ideal, x divides $f^2 + y^2$ and since I is proper $\gcd(x, y, (f^2 + y^2)/x) = 1$. If an odd prime p divides $\text{Nm}(I)$, then p divides x^2 , $f^2 + y^2$, and y . Since $f^2 \equiv -y^2 \pmod{p}$ and p does not divide both f and y , -1 is a square mod p , $p \equiv 1 \pmod{4}$ and $\epsilon_0(I) = 0$.

If D is not a square, $D = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$ with p_l distinct odd primes and k_1 odd. By Dirichlet's theorem, there is a prime p with:

- $p \equiv 3 \pmod{4}$,
- $p \equiv 1 \pmod{p_l}$ for $l > 1$, and

- $\left(\frac{p}{p_1}\right) = -1$.

Quadratic reciprocity gives $\left(\frac{-D}{p}\right) = 1$. If $x^2 \equiv -D \pmod{p}$, then $I = p\mathbb{Z} \oplus (\sqrt{-D} - x)\mathbb{Z}$ is an $\mathcal{O}_{-4D}$ -ideal and has norm p and $\epsilon_0(I) = 1$. $\square$

Remark. When $D \equiv 1 \pmod{8}$, the ideal (2) ramifies in $\mathcal{O}_{-4D}$ and there is a prime ideal P with $P^2 = (2)$. Since $\text{Nm}(P) = 2$, we have $\epsilon_0(P) = 0$. The ideal classes represented by $I = \mathbb{Z} \oplus \tau\mathbb{Z}$ and $J = \mathbb{Z} \oplus -1/2\tau\mathbb{Z}$ satisfy $[I] = [PJ]$. This is related to the fact the polygonal stars P_τ and $P_{-1/2\tau}$ give the same point on W_D and so must have the same spin invariant.

Spin and orbifold points. As before, fix a discriminant $D \geq 9$ with $D \equiv 1 \pmod{8}$ and conductor f .

Proposition 6.3. *Suppose (X_τ, ω_τ) is an eigenform for real multiplication ι_τ by $\mathcal{O}_D$ obtained from the polygon P_τ and let $I = \mathbb{Z} \oplus \tau\mathbb{Z}$. The spin invariant for (X_τ, ι_τ) is given by:*

$$\epsilon(X_\tau, \iota_\tau) = \frac{f+1}{2} + \epsilon_0(I) \pmod{2}.$$

The formula for $e_2(W_D^\epsilon)$ stated in Theorem 1.2 follows from Propositions 6.2 and 6.3.

Proof. According to Table 5.1, $\text{Jac}(X_\tau)$ has real multiplication by $\mathcal{O}_D$ with eigenform ω_τ iff I is a proper $\mathcal{O}_{-4D}$ -ideal and $1 + \sqrt{-D} \in 2I$.

Since I is an ideal, $\sqrt{-D} = x\tau + y$ for some x and $y \in \mathbb{Z}$, x divides $D + y^2$ and I has the same class as $I_0 = x\mathbb{Z} \oplus (\sqrt{-D} - y)\mathbb{Z}$. Since I is proper, $\gcd(x, y, (D + y^2)/x) = 1$ and the norm of I_0 is x up to a factor of two. The condition $1 + \sqrt{-D} \in 2I$ ensures $x \equiv 2 \pmod{4}$ and $\epsilon_0(I) \equiv \frac{x-2}{4} \pmod{2}$.

To compute the spin invariant $\epsilon(X_\tau, \omega_\tau)$, we need to determine the subspace $V = \text{Im}\left(\frac{f+\sqrt{D}}{2}\right)$ of $H_1(X, \mathbb{Z}/2\mathbb{Z})$ and evaluate $\text{Arf}(q|_V)$. The subspace V is spanned by v and iv where:

$$v = \frac{f - i\sqrt{-D}}{2} \equiv \frac{2f+x}{4} + i\frac{x-2y}{4} + ai \pmod{2}.$$

Since $q(v) = q(iv)$, we have by Proposition 6.1:

$$\epsilon(X_\tau, \omega_\tau) = q(v)^2 = \frac{f+1}{2} + \epsilon_0(I) \bmod 2.$$

□

Remark. For square discriminants $D = f^2$, there is an elementary argument that shows $e_2(W_D^{(f-1)/2}) = 0$. When the surface X_τ is labeled by an orbifold point on W_D , the number τ is in $\mathbb{Q}(i)$ and rescaling P_τ , we can exhibit X_τ as the quotient of a polygon with vertices in $\mathbb{Z}[i]$ and area f . The area determines the number of points in X^W lying on the lattice $\mathbb{Z}[i]$ which in turn determines the spin invariant.

Chapter 7

Genus

Together with [Ba, Mc3], Theorems 1.1 and 1.2 complete the determination of the homeomorphism type of W_D . For non-square D , we will show that the homeomorphism type of W_D can be determined from the homeomorphism type of the Hilbert modular surface X_D and the product locus P_D :

Proposition 7.1. *For non-square $D > 8$, the homeomorphism type of W_D is determined by the orbifold Euler characteristic $\chi(X_D)$, the number of cusps $C(P_D)$ on P_D , the number of orbifold points of order two $e_2(X_D)$ and $e_2(P_D)$ and $D \bmod 8$.*

We will use our formula to give bounds for the genus of W_D and its components.

Proposition 7.2. *For any $\epsilon > 0$, there are positive constants C_ϵ and N_ϵ so:*

$$C_\epsilon D^{3/2+\epsilon} > g(V)$$

whenever V is a component of W_D and $D > N_\epsilon$.

We will also give effective lower bounds.

Proposition 7.3. *Suppose $D > 0$ is a discriminant and V is a component of W_D . If D is not a square, the genus of V satisfies:*

$$g(V) > D^{3/2}/600 - D/6 - D^{3/4}/2 - 75.$$

If D is a square, the genus of V satisfies:

$$g(V) > D^{3/2}/240 - D - D^{3/4}/2 - 75.$$

The components of $\bigcup_D W_D$ with genus $g \leq 4$ are all listed in §A.2.

Corollary 7.4. *The components of $\bigcup_D W_D$ with genus $g \leq 4$ all lie on $\bigcup_{D \leq 121} W_D$.*

Proof. The bounds in Proposition 7.3 show that $g(V) > 4$ whenever $D > 17500$ for non-square D and $D > 250^2$ for square D . The remaining discriminants were checked by computer. $\square$

Euler characteristic of X_D and W_D . The Hilbert modular surface X_D has a meromorphic modular form with a simple zero along W_D and simple pole along P_D . This gives a simple relationship between the orbifold Euler characteristics of W_D , P_D and X_D and a modular curve S_D in the boundary of X_D . The curve S_D is empty unless $D = f^2$ is a square, in which case $S_D \cong X_1(f)$.

Theorem 7.5 ([Ba] Cor. 10.4). *The Euler characteristic of W_D satisfies:*

$$\chi(W_D) = \chi(P_D) - 2\chi(X_D) - \chi(S_D).$$

For a discriminant D , define:

$$F(D) = \prod_{p \nmid f} \left(1 - \left(\frac{D_0}{p} \right) p^{-2} \right).$$

where f is the conductor of $\mathcal{O}_D$, $D_0 = D/f^2$ is the discriminant of the maximal order in $\mathbb{Q}(\sqrt{D})$ and the product is over primes dividing f . The number $F(D)$ satisfies $1 > F(D) > \zeta_{\mathbb{Q}}(2)^{-1} > 6/10$.

For square discriminants, $\chi(S_D) = -f^2 F(D)/12$ and the Euler characteristic of

W_D and its components are given by ([Ba] Thm. 1.4):

$$\chi(W_{f^2}) = -f^2(f-1)F(D)/16,$$

$$\chi(W_{f^2}^0) = -f^2(f-1)F(D)/32, \text{ and}$$

$$\chi(W_{f^2}^1) = -f^2(f-3)F(D)/32.$$

For non-square discriminants, $\chi(S_D) = 0$ and $\chi(P_D) = -\frac{5}{2}\chi(X_D)$ giving $\chi(W_D) = -\frac{9}{2}\chi(X_D)$. The Euler characteristic $\chi(X_D)$ can be computed from ([Ba] Thm. 2.12):

$$\chi(X_D) = 2f^3\zeta_{D_0}(-1)F(D).$$

Here ζ_{D_0} is the Dedekind-zeta function and can be computed from Siegel's formula ([Bru], Cor. 1.39):

$$\zeta_{D_0}(-1) = \frac{1}{60} \sum_{e^2 < D_0, e \equiv D_0 \pmod{2}} \sigma\left(\frac{D_0 - e^2}{4}\right),$$

where $\sigma(n)$ is the sum of the divisors of n . For reducible W_D , the spin components satisfy $\chi(W_D^0) = \chi(W_D^1) = \frac{1}{2}\chi(W_D)$ ([Ba] Thm. 1.3).

The well-known bound $\sigma(n) = o(n^{1+\epsilon})$ gives constants C_ϵ and N_ϵ so:

$$C_\epsilon D^{3/2+\epsilon} > \chi(X_D)$$

whenever $D > N_\epsilon$. Using $\sigma(n) > n + 1$ and $F(D) > 6/10$ gives:

$$\chi(X_D) > D^{3/2}/300.$$

We can now prove the upper bounds for the genus of W_D :

Proof of Proposition 7.2. For square discriminants $D = f^2$, we have $|\chi(W_D)| < f^3$. For non-square discriminants, the bounds for $\chi(X_D)$ and the formula $\chi(W_D) = -9\chi(X_D)/2$ gives $|\chi(W_D)| = O(D^{3/2+\epsilon})$. Since W_D has one or two components,

$$g(W_D) = O(|\chi(W_D)|).$$

□

The modular curve P_D . The modular curve P_D is isomorphic to:

$$\left(\bigsqcup_{(e,l,m)} Y_0(m) \right) / g,$$

where the union is over triples of integers (e, l, m) with:

$$D = e^2 + 4l^2m, \quad l, m > 0, \quad \text{and} \quad \gcd(e, l) = 1,$$

and g is the automorphism sending the degree m isogeny i on the component labeled by (e, l, m) to the isogeny i^* on the $(-e, l, m)$ -component (cf. [Mc3] Thm. 2.1). The isogeny $i : E \rightarrow F$ on the (e, l, m) -component corresponds to the abelian variety $B = E \times F$ with $\mathcal{O}_D$ generated by $\iota \left(\frac{e + \sqrt{D}}{2} \right) = \begin{pmatrix} e & i^* \\ i & 0 \end{pmatrix}$.

The components of P_D are labeled by triples (e, l, m) as above subject to the additional condition $e \geq 0$. We will need the following bound on the number of such triples:

Proposition 7.6. *The number of components of P_D satisfies:*

$$h_0(P_D) \leq D^{3/4} + 150.$$

Proof. Let $l(n)$ denote the largest integer whose square divides n and let $f(n) = d(l(n))$ be the number of divisors of $l(n)$. The function f is multiplicative and the number of triples (e, l, m) with e fixed is bounded above by $f \left(\frac{D - e^2}{4} \right)$. There is a finite set S of natural numbers n for which $f(n) > n^{1/4}$ (they are all divisors of $2^{12}3^65^47^211^2$) since $d(n)$ is $o(n^\epsilon)$ for any $\epsilon > 0$ and it is easy to check that $\sum_{n \in S} f(n) - n^{1/4} < 150$. The asserted bound on $h^0(P_D)$ follows from:

$$h_0(P_D) \leq \sum_{\substack{e \equiv D \pmod{2} \\ 0 \leq e < \sqrt{D}}} f \left(\frac{D - e^2}{4} \right) \leq 150 + \sum_e \left(\frac{D - e^2}{4} \right)^{1/4}.$$

□

Cusps on W_D and P_D . Let $C_1(W_D)$ and $C_2(W_D)$ be the number of one- and two-cylinder cusps on W_D respectively and $C(W_D) = C_1(W_D) + C_2(W_D)$ be the total number of cusps. The cusps on W_D were first enumerated and sorted by component in [Mc3]:

Proposition 7.7. *For non-square discriminants, the number of cusps on W_D is equal to the number of cusps on P_D :*

$$C(W_D) = C_2(W_D) = C(P_D),$$

and $C(W_D^0) = C(W_D^1)$ when W_D is reducible. For square discriminants $D = f^2$, the number of one- and two-cylinder cusps satisfy:

$$C_2(W_{f^2}) < C(P_{f^2}) \text{ and } C_1(W_{f^2}) < f^2/3.$$

When f is odd, $\left|C(W_{f^2}^1) - C(W_{f^2}^0)\right| < 7f^2/12$.

Proof. Except for the explicit bounds on $C_1(W_{f^2})$ and $\left|C(W_{f^2}^1) - C(W_{f^2}^0)\right|$, the claims in the proposition follow from the enumeration of cusps on P_D and W_D in [Ba], Section 3.1. We will sketch another argument for the relationship between $C(W_D)$ and $C(P_D)$. The cusps on W_D are labeled by splitting saddle connections for eigenforms ([Mc3], §3). For a point $(X, \iota) \in W_D$ near a cusp, there is a unique short splitting saddle connection and the splitting gives a point on P_D near a cusp. This gives a correspondence between the two-cylinder cusps on W_D and the cusps on P_D . One can show that this correspondence is always injective and is bijective whenever D is not a square by showing there is an inverse correspondence defined on a subset of the cusps of P_D .

We now turn to the bounds on $C_1(W_{f^2})$ and $\left|C(W_{f^2}^1) - C(W_{f^2}^0)\right|$. When D is not a square, there are no one-cylinder cusps and when $D = f^2$ is a square, the one-cylinder cusps are parametrized by cyclically ordered triples (a, b, c) with (cf. [Mc3] Theorem

A.1):

$$f = a + b + c, a, b, c > 0 \text{ and } \gcd(a, b, c) = 1.$$

Reordering (a, b, c) so $a < b$ and $a < c$ ensures that $a < f/3$ and $b < f$, giving $C_1(W_D) < f^2/3 = D/3$. The difference in the number of two cylinder cusps is given by (Theorem A.4 in [Mc3]):

$$C_2(W_D^0) - C_2(W_D^1) = \sum_{b+c=f, 0 < c < b} \phi(\gcd(b, c)),$$

which is smaller than $D/4$ using $c < f/2$ and $\phi(\gcd(b, c)) < f/2$. The bound asserted for $|C(W_D^0) - C(W_D^1)|$ follows. $\square$

Orbifold points on W_D . Our formula for $e_2(W_D)$ gives:

Theorem 7.8. *The number of points of order two on W_D satisfies:*

$$e_2(W_D) = e_2(X_D) - e_2(P_D) \text{ and } e_2(W_D) < \frac{4}{3}D.$$

For non-square D , Theorem 7.8 establishes Proposition 7.1 together with Proposition 7.7 and the enumeration of the irreducible components of W_D in [Mc3].

Proof. Theorem 1.5 and Corollary 1.6 say that every orbifold point of order two on X_D is an orbifold point on P_D or W_D , giving $e_2(W_D) = e_2(X_D) - e_2(P_D)$. The bound for $e_2(W_D)$ follows from the bound $h(-D) < 2D/3$ (§2) for discriminants $-D < 0$ and the formula in Theorem 1.1. $\square$

Lower bounds, non-square discriminants. Now suppose D is a non-square discriminant and $D > 8$ so all of the orbifold points on W_D have order two. Using the formula for $\chi(W_D)$, the equality $C(P_D) = C(W_D)$, and $\chi(S_D) = 0$ and ignoring several terms which contribute positively to the $g(W_D)$ gives:

$$g(W_D) > \chi(X_D) - h^0(P_D) - e_2(W_D)/4.$$

Combining the bound above with $\chi(X_D) > D^{3/2}/300$, $h_0(P_D) < D^{3/4} + 150$, $e_2(W_D) < \frac{4}{3}D$ and $g(V) \geq \frac{1}{2}g(W_D)$ whenever V is a component of W_D gives the bound in Proposition 7.3.

Lower bounds, square discriminants. Now suppose $D = f^2$. Using the formula for $\chi(W_D)$ in terms of $\chi(X_D)$, $\chi(P_D)$ and $\chi(S_D)$, the bound $C_2(W_D) < C(P_D)$ and ignoring some terms which contribute positively to $g(W_D)$ gives:

$$g(W_D) \geq \chi(X_D) - h_0(P_D) - e_2(W_D)/4 + \chi(S_D)/2 - C_1(W_D)/2.$$

As before we have $h_0(P_D) < D^{3/4} + 150$, $e_2(W_D) < \frac{4}{3}D$ and $C_1(W_D) < D/3$. By Theorem 2.12 and Proposition 10.5 of [Ba] and using $\zeta_{\mathbb{Q}}(2) > 6/10$, we have $\chi(X_D) + \chi(S_D)/2 > D^{3/2}/120 - D/40$ so long as $D > 36$, giving:

$$g(W_D) \geq D^{3/2}/120 - 3D/5 - D^{3/4} - 150.$$

Finally, to bound $g(V)$ when V is a component of W_D , we bound the difference:

$$|g(W_D^0) - g(W_D^1)| \leq \left| \frac{\chi(W_D^1) - \chi(W_D^0)}{2} \right| + \left| \frac{C(W_D^1) - C(W_D^0)}{2} \right| + e_2(W_D)/4$$

We have seen that $|C(W_D^1) - C(W_D^0)| < 7D/12$ and $e_2(W_D)/4 < D/3$. Theorem 1.4 of [Ba] gives $|\chi(W_D^0) - \chi(W_D^1)| < D/16$ and $|g(W_D^1) - g(W_D^0)| < 7D/10$. The bound asserted for $g(V)$ in Proposition 7.3 follows. $\square$

Appendix A

The D_{12} -family

A.1 The D_{12} -family

In this section we will describe the surfaces in $\mathcal{M}_2(D_{12})$. For a smooth surface $X \in \mathcal{M}_2$, the following are equivalent:

- *Automorphisms.* The automorphism group $\text{Aut}(X)$ admits an injective homomorphism $\rho : D_{12} \rightarrow \text{Aut}(X)$.
- *Algebraic curves.* The field of functions $\mathbb{C}(X)$ is isomorphic to the field:

$$\tilde{K}_a = \mathbb{C}(z, x) \text{ with } z^2 = x^6 - ax^3 + 1,$$

for some $a \in \mathbb{C} \setminus \{\pm 2\}$.

- *Jacobians.* The Jacobian $\text{Jac}(X)$ is isomorphic to the principally polarized abelian variety:

$$\tilde{A}_\tau = \mathbb{C}^2 / \tilde{\Lambda}_\tau,$$

where $\tilde{\Lambda}_\tau = \mathbb{Z} \left\langle \begin{pmatrix} 1 \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} \tau \\ \sqrt{3}\tau \end{pmatrix}, \begin{pmatrix} 1 \\ -1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} \tau \\ -\sqrt{3}\tau \end{pmatrix} \right\rangle$ and is polarized by the symplectic form $\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \rangle = \frac{\text{Im}(a\bar{c} + b\bar{d})}{2\text{Im}\tau}$.

- *Hexagonal pinwheels.* The surface X is isomorphic to the surface $\tilde{X}_\tau$ obtained by gluing the hexagonal pinwheel H_τ (Fig. A-1) to $-H_\tau$ for some τ in the

Figure A-1: For τ in the shaded domain $\tilde{U}$, the hexagonal pinwheel H_τ has vertices lying on two equilateral triangles. Gluing together sides on H_τ and $-H_\tau$ by translation gives a genus two surface admitting an action of D_{12} . The one form induced by dz is a Z -eigenform.

domain:

$$\tilde{U} = \left\{ \tau \in \mathbb{H} : \tau \neq \zeta_{12} \text{ or } \zeta_{12}^5, |\operatorname{Re} \tau| \leq \frac{1}{2} \text{ and } |\tau| \geq 1 \right\}.$$

- *Parallelograms.* The surface X is isomorphic the degree six cyclic cover $\tilde{Y}_w$ of the double of a parallelogram with vertices $\{0, 1, w/2, (w+1)/2\}$ depicted in Fig. A-2 for some w in the domain:

$$\tilde{V} = \left\{ w \in \mathbb{H} : |\operatorname{Re} \tau| \leq \frac{1}{2}, |\tau| \geq 1 \text{ and } |\tau - 1| \geq 1 \right\}.$$

It is straightforward to identify the action of D_{12} on the surfaces described above. The field $\tilde{K}_a$ has automorphisms $Z(z, x) = (-z, \zeta_3 x)$ and $r(z, x) = (z/x^3, 1/x)$. The polarized lattice $\tilde{\Lambda}_\tau$ is preserved by the linear transformations $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Z = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$. The surface obtained from $\tilde{X}_\tau$ satisfies $\mathbb{C}(\tilde{X}_\tau) \cong \tilde{K}_a$ for some a since $\tilde{X}_\tau$ has an order six automorphism acting on the Weierstrass points by a permutation the conjugacy class $[3, 3]$. The surface $\tilde{Y}_w$ has an order six Deck transformation for the map depicted in Fig. A-2 and an involution covered by $z \mapsto -z + 5w + 1$.

Figure A-2: For w in the shaded domain $\tilde{V}$, the parallelogram $\tilde{Q}_w$ has vertices $\{0, 1, 6w, 6w + 1\}$. Gluing together the marked segments as indicated gives a genus two surface $Y_w = Q_w / \sim$ with a degree six cyclic covering map to the double a parallelogram. The surface Y_w has an action of D_{12} generated by the Deck transformation and the involution obtained by rotating Q_w about $z = 3w + 1/2$.

The isogeny $c_\rho : \text{Jac}(X/r) \rightarrow \text{Jac}(X/Zr)$ induced by the correspondence $X \rightarrow X/r \times X/Zr$ is a degree three isogeny between elliptic curves and the map:

$$\tilde{\mathcal{M}}_2(D_{12}) \rightarrow X_0(3)$$

given by $(X, \rho) \mapsto c_\rho$ is an isomorphism. The non-trivial outer automorphism of D_{12} given by $\sigma(Z) = Z$ and $\sigma(r) = Zr$ induces the Atkin-Lehner involution on $\tilde{\mathcal{M}}_2(D_{12})$ and the unique nodal D_{12} -surface corresponds to the point of order three on $X_0(3)$. The function relating τ and a is modular and the function relating τ and w is a Riemann mapping between $\tilde{U}$ and $\tilde{V}$.

The surface obtained from H_τ has complex multiplication and is labeled by an orbifold point on a Hilbert modular surface iff τ is imaginary quadratic. When τ is in $\mathbb{Q}(\zeta_3)$, the vertices of H_τ lie in a lattice and the Teichmüller geodesic generated by the Z -eigenform for $\tilde{X}_\tau$ is a Teichmüller curve associated to a square tiled surface. When τ is imaginary quadratic but not in $\mathbb{Q}(\zeta_3)$, the complex geodesic in $\mathcal{M}_2$ generated by the Z -eigenform on $\tilde{X}_\tau$ is an example of the type studied in [Mc2] and has infinitely

generated fundamental group.

A.2 Homeomorphism type of W_D

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$	D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
5	0	1	1	$-\frac{3}{10}$	52	1	0	15	-15
8	0	0	2	$-\frac{3}{4}$	53	2	3	7	$-\frac{21}{2}$
9	0	1	2	$-\frac{1}{2}$	56	3	2	10	-15
12	0	1	3	$-\frac{3}{2}$	57	$\{1, 1\}$	$\{1, 1\}$	$\{10, 10\}$	$\{-\frac{21}{2}, -\frac{21}{2}\}$
13	0	1	3	$-\frac{3}{2}$	60	3	4	12	-18
16	0	1	3	$-\frac{3}{2}$	61	2	3	13	$-\frac{33}{2}$
17	$\{0, 0\}$	$\{1, 1\}$	$\{3, 3\}$	$\{-\frac{3}{2}, -\frac{3}{2}\}$	64	1	2	17	-18
20	0	0	5	-3	65	$\{1, 1\}$	$\{2, 2\}$	$\{11, 11\}$	$\{-12, -12\}$
21	0	2	4	-3	68	3	0	14	-18
24	0	1	6	$-\frac{9}{2}$	69	4	4	10	-18
25	$\{0, 0\}$	$\{0, 1\}$	$\{5, 3\}$	$\{-3, -\frac{3}{2}\}$	72	4	1	16	$-\frac{45}{2}$
28	0	2	7	-6	73	$\{1, 1\}$	$\{1, 1\}$	$\{16, 16\}$	$\{-\frac{33}{2}, -\frac{33}{2}\}$
29	0	3	5	$-\frac{9}{2}$	76	4	3	21	$-\frac{57}{2}$
32	0	2	7	-6	77	5	4	8	-18
33	$\{0, 0\}$	$\{1, 1\}$	$\{6, 6\}$	$\{-\frac{9}{2}, -\frac{9}{2}\}$	80	4	4	16	-24
36	0	0	8	-6	81	$\{2, 0\}$	$\{0, 3\}$	$\{16, 14\}$	$\{-18, -\frac{27}{2}\}$
37	0	1	9	$-\frac{15}{2}$	84	7	0	18	-30
40	0	1	12	$-\frac{21}{2}$	85	6	2	16	-27
41	$\{0, 0\}$	$\{2, 2\}$	$\{7, 7\}$	$\{-6, -6\}$	88	7	1	22	$-\frac{69}{2}$
44	1	3	9	$-\frac{21}{2}$	89	$\{3, 3\}$	$\{3, 3\}$	$\{14, 14\}$	$\{-\frac{39}{2}, -\frac{39}{2}\}$
45	1	2	8	-9	92	8	6	13	-30
48	1	2	11	-12	93	8	2	12	-27
49	$\{0, 0\}$	$\{2, 0\}$	$\{10, 8\}$	$\{-9, -6\}$	96	8	4	20	-36

Table A.1: The Weierstrass curve W_D is a finite volume hyperbolic orbifold and for $D > 8$ its homeomorphism type is determined by the genus $g(W_D)$, the number of orbifold points of order two $e_2(W_D)$, the number of cusps $C(W_D)$ and the Euler characteristic $\chi(W_D)$. The values of these topological invariants are listed for each curve W_D with $D < 250$ as well as several larger discriminants. When $D > 9$ with $D \equiv 1 \pmod{8}$, the curve W_D is reducible and the invariants are listed for both spin components with the invariants for W_D^0 appearing first.

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
97	$\{4, 4\}$	$\{1, 1\}$	$\{19, 19\}$	$\{-\frac{51}{2}, -\frac{51}{2}\}$
100	4	0	30	-36
101	6	7	15	$-\frac{57}{2}$
104	9	3	20	$-\frac{75}{2}$
105	$\{6, 6\}$	$\{2, 2\}$	$\{16, 16\}$	$\{-27, -27\}$
108	10	3	21	$-\frac{81}{2}$
109	8	3	25	$-\frac{81}{2}$
112	10	2	29	-48
113	$\{6, 6\}$	$\{2, 2\}$	$\{16, 16\}$	$\{-27, -27\}$
116	11	0	25	-45
117	10	4	16	-36
120	16	2	20	-51
121	$\{6, 3\}$	$\{3, 0\}$	$\{26, 26\}$	$\{-\frac{75}{2}, -30\}$
124	15	6	29	-60
125	11	5	15	$-\frac{75}{2}$
128	13	4	22	-48
129	$\{8, 8\}$	$\{3, 3\}$	$\{22, 22\}$	$\{-\frac{75}{2}, -\frac{75}{2}\}$
132	15	0	26	-54
133	15	2	22	-51
136	17	2	36	-69
137	$\{9, 9\}$	$\{2, 2\}$	$\{19, 19\}$	$\{-36, -36\}$
140	19	6	18	-57
141	18	4	18	-54
144	11	4	38	-60
145	$\{10, 10\}$	$\{2, 2\}$	$\{29, 29\}$	$\{-48, -48\}$
148	20	0	37	-75
149	16	7	19	$-\frac{105}{2}$

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
152	22	3	18	$-\frac{123}{2}$
153	$\{10, 10\}$	$\{2, 2\}$	$\{26, 26\}$	$\{-45, -45\}$
156	25	8	26	-78
157	20	3	25	$-\frac{129}{2}$
160	22	4	40	-84
161	$\{14, 14\}$	$\{4, 4\}$	$\{20, 20\}$	$\{-48, -48\}$
164	20	0	34	-72
165	24	4	18	-66
168	29	2	24	-81
169	$\{14, 7\}$	$\{0, 3\}$	$\{37, 39\}$	$\{-63, -\frac{105}{2}\}$
172	29	3	37	$-\frac{189}{2}$
173	22	7	13	$-\frac{117}{2}$
176	27	6	29	-84
177	$\{17, 17\}$	$\{1, 1\}$	$\{26, 26\}$	$\{-\frac{117}{2}, -\frac{117}{2}\}$
180	28	0	36	-90
181	26	5	33	$-\frac{171}{2}$
184	37	2	38	-111
185	$\{17, 17\}$	$\{4, 4\}$	$\{23, 23\}$	$\{-57, -57\}$
188	31	10	19	-84
189	27	6	26	-81
192	31	4	34	-96
193	$\{19, 19\}$	$\{1, 1\}$	$\{37, 37\}$	$\{-\frac{147}{2}, -\frac{147}{2}\}$
196	25	0	60	-108
197	26	5	21	$-\frac{147}{2}$
200	31	3	36	$-\frac{195}{2}$
201	$\{20, 20\}$	$\{3, 3\}$	$\{34, 34\}$	$\{-\frac{147}{2}, -\frac{147}{2}\}$
204	38	6	40	-117

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
213	36	4	18	-90
216	38	3	46	-2432
217	{25, 25}	{2, 2}	{38, 38}	{-87, -87}
220	46	8	44	-138
221	32	8	30	-96
224	42	8	34	-120
225	{21, 16}	{4, 0}	{42, 42}	{-84, -72}
228	43	0	42	-126
229	42	5	37	$-\frac{243}{2}$
232	49	1	52	$-\frac{297}{2}$
233	{27, 27}	{3, 3}	{26, 26}	$\{-\frac{159}{2}, -\frac{159}{2}\}$
236	45	9	35	$-\frac{255}{2}$
237	42	6	20	-105
240	52	4	40	-144
241	{31, 31}	{3, 3}	{45, 45}	$\{-\frac{213}{2}, -\frac{213}{2}\}$
244	53	0	61	-165
245	40	6	24	-105
248	52	4	22	-126
249	{32, 32}	{3, 3}	{40, 40}	$\{-\frac{207}{2}, -\frac{207}{2}\}$
41376	164821	112	1552	-331248
41377	{113276, 113276}	{28, 28}	{1442, 1442}	{-228006, -228006}
41380	178100	0	3154	-359352
41381	119380	89	665	$-\frac{478935}{2}$
41384	145957	68	884	-292830
41385	{107869, 107869}	{24, 24}	{1284, 1284}	{-217032, -217032}
41388	155386	54	1188	-311985
41389	146346	81	1475	$-\frac{588411}{2}$
41392	173203	48	2340	-348768

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