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SIAM J. Discrete Math.
Vol. 25, No. 3, pp. 1349–1364

LAPLACIANS FOR FLOW NETWORKS*

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Abstract. We define a class of Laplacians for multicommodity, undirected flow networks, and bound their smallest nonzero eigenvalues with a generalization of the sparsest cut.

Key words. spectral graph theory, multicommodity flow network, Cheeger inequality, sparsest cut

AMS subject classifications. 05C21, 05C50, 90B10

DOI. 10.1137/100787726

1. Introduction.

Spectral graph theory [6] offers powerful tools for analysis and design of systems that are well modeled by graphs. However, many systems have important features not captured by purely graphical descriptions. Flow networks [1, 22] describe a wide variety of such systems, for example, electric power grids and communication networks, yet have a minimal level of detail additional to the underlying graph. In this work, we apply spectral graph theory to flow networks. We formulate a class of Laplacian matrix pencils for undirected, multicommodity flow networks and a Cheeger-like parameter that generalizes the sparsest cut [25] and relate them with bounds similar to the Cheeger inequality [5, 6, 8, 11]. When there are many commodities, finding the correct eigenvalue entails solving a combinatorial optimization problem, for which we formulate a semidefinite relaxation using the methodologies of [26] and [23].

2. Background.

2.1. The Laplacian of a graph and the Cheeger constant.

We are given an undirected, connected graph G with vertices $V(G)$, edges $E(G)$, and corresponding adjacency matrix A. The Laplacian of G is defined as $L = D - A$, where D is a diagonal matrix with $D_{vv} = d_v = \sum_u A_{uv}$. The normalized Laplacian is $\mathcal{L} = D^{-1/2} L D^{-1/2}$, and its eigenvalues can be written $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2$. The eigenvalues of $\mathcal{L}$ are equivalent to those of the generalized eigensystem $Lx - \lambda Dx = 0$, which is referred to as the pencil (L, D); for convenience we use this notation [16].

Let f be a function assigning a complex value $f(v)$ to each vertex v, where the notation f denotes the vector of these values. The Rayleigh quotient of (L, D) is

$$\frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v},$$

where the sum subscript $u \sim v$ denotes summation over all pairs of vertices connected by edges. The first nonzero eigenvalue, often called the algebraic connectivity [11], satisfies

*Received by the editors March 5, 2010; accepted for publication (in revised form) June 30, 2011; published electronically September 13, 2011. This work was supported by the Office of Naval Research, through grant N00014-02-1-0623, monitored by Dr. T. Ericsen, and grant N00014-09-1-0700, monitored by Dr. M. Steinberg.
http://www.siam.org/journals/sidma/25-3/78772.html

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$$\lambda_1 = \inf_{f \perp D\mathbf{1}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d_v}.$$

The Cheeger constant [5, 6, 8, 11], sometimes referred to as the conductance, is a measure of the level of bottlenecking in a graph. It is defined as

$$h = \min_X \frac{|C(X, \bar{X})|}{\min(\mathrm{vol}(X), \mathrm{vol}(\bar{X}))},$$

where $C(X, \bar{X})$ is the set of edges with only one vertex in X, $|C(X, \bar{X})| = \sum_{u \in X, v \in \bar{X}} A_{uv}$, and $\mathrm{vol}(X) = \sum_{v \in X} d_v$. h is related to the algebraic connectivity by what is known as the Cheeger inequality

$$2h \geq \lambda_1 > \frac{h^2}{2}.$$

2.2. Flow networks. A flow network is a weighted graph on which flows travel between vertices by way of the edges [1, 22]. In this work we consider only undirected flow networks. Suppose further that we have a multicommodity flow network with m different types of flows or commodities [20] and that we are given a supply and demand vector for each commodity i, p^i, which satisfies $\sum_v p^i_v = 0$.

We denote the flow of commodity i from u to v by $g_i(u, v)$, and the weight of the edge between u and v by $c(u, v)$, which we refer to as a capacity. In this work, we equate capacities with edge weights such that the capacity of an edge, $c(u, v)$, is identical to its weight in the graph adjacency matrix, A_{uv}. We say that a flow network is feasible if there exists a flow $g_i \colon V(G) \times V(G) \to \mathbb{R}^+$ satisfying $\sum_{i=1}^m g_i(u, v) \leq c(u, v)$, and $\sum_v g_i(u, v) - g_i(v, u) = p^i_u$.

For many purposes, a network with multiple sources and sinks can be reduced to one with a single source and sink by introducing a supersource and supersink [22], for example, maximizing the flow through a network. As will be seen in the next section, this simplification is not compatible with our development, and so we allow as many vertices as are in the network to be sources or sinks provided that the total flow is conserved.

3. A flow-based Cheeger constant. We identify a quantity that measures bottlenecking of flows rather than graphical structure and is, in fact, a generalization of the sparsest cut [25]. We begin with the single commodity version. Define

$$q = \min_X \frac{|C(X, \bar{X})|}{|\sum_{v \in X} p_v|}.$$

The denominator is the flow that would be sent from X to $\bar{X}$ in the absence of edge capacities. By the max-flow min-cut theorem, the actual flow from X to $\bar{X}$ can be no greater than $|C(X, \bar{X})|$ [9, 21]. In fact, it is well known that $q \geq 1$ is also a sufficient condition for the existence of a feasible flow [13, 18]; an implication is that q is not NP-hard when there is only one commodity.

Before discussing the multicommodity case, we give a brief example for which the introduction of a supersource changes the value of q. Consider a three vertex line graph

with $p = [1, -2, 1]^T$ and $c(1,2) = 3$ and $c(2,3) = 2$. Simple calculation gives $q = 2$ and $C(X, \bar{X}) = c(2,3)$ for this network. Now append a supersource with $p_s = 2$, connected to vertices one and three by edges of unit capacity, and set p_1 and p_3 to zero. The optimal q for the modified network is $q = 1$, and furthermore the optimal cut has changed so that vertices one and three are now on the same side. Hence the usual simplification of multisource, multisink problems to single-source, single-sink problems is not applicable here.

We now generalize q to multicommodity flow networks. Let $\kappa \in \{-1, 1\}^m$ and $p_v^\kappa = \sum_{i=1}^m \kappa^i p_v^i$, where κ^i is element i of κ, and define

$$S(X) = \max_\kappa \sum_{v \in X} p_v^\kappa.$$

Because the objective is linear in κ, it is equivalent to the continuous linear optimization problem in which $\kappa \in [-1, 1]^m$. The purpose of the maximization is merely to ensure that the net demand that would leave set X of each commodity has the same sign. We then define the multicommodity version to be

$$q = \min_X \frac{|C(X, \bar{X})|}{S(X)}, \tag{3.1}$$

which in matrix form is given by

$$q = \min_{x \in \{0,1\}^n, \kappa} \frac{x^T L x}{|x^T P_\kappa x|}, \tag{3.2}$$

where P_κ is a matrix with p^κ on the main diagonal and zeros elsewhere.

q also has the minimax network flow formulation

$$\begin{aligned} q &= \min_\kappa \max_{g,\tau} \tau \\ &\sum_{u:\, u \sim v} g_{uv} - g_{vu} = \tau p_v^\kappa \quad \forall\ v \\ &0 \le g_{uv} \le A_{uv} \quad \forall\ u \sim v \\ &-1 \le \kappa^i \le 1 \quad \forall\ i. \end{aligned} \tag{3.3}$$

Intuitively, we are optimally consolidating the supplies and demands into a single commodity, the maximum flow of which is equal to the minimum cut by the max-flow min-cut theorem and the result of [13, 18].

4. Laplacians for flow networks. We now derive Cheeger-like inequalities for eigenvalues of flow normalized Laplacians. Note that we have not assumed feasibility, rather only that $\sum_v p_v^i = 0$ for each commodity i. The pencil (L, P_κ) is a natural starting point because its smallest magnitude eigenvalue is a continuous relaxation of q. However, it is defective, which is to say that an eigenfunction is missing. It has two zero eigenvalues corresponding to the constant eigenfunction; in the simplest case of a two vertex network, the eigenvalues provide no meaningful information.

4.1. Variational formulation. We can see why (L, P_κ) is defective by considering the quotient

$$\frac{f^T P_\kappa f}{f^T L f}.$$

It is undefined at $f = \mathbf{1}$, but approaches infinity as f approaches $\mathbf{1}$ from any direction. Now consider the perturbed pencil $(P_\kappa, L + a\mathbf{1}\mathbf{1}^T)$ for $a > 0$, the eigenvalues of which are one over those of $(L + a\mathbf{1}\mathbf{1}^T, P_\kappa)$: It is similar to $\sqrt{L + a\mathbf{1}\mathbf{1}^T} P_\kappa \sqrt{L + a\mathbf{1}\mathbf{1}^T}$, which can be real symmetric because $L + a\mathbf{1}\mathbf{1}^T$ is positive definite. By the Rayleigh–Ritz theorem [19], the largest positive and negative eigenvalues satisfy

$$\sup_f \frac{f^T P_\kappa f}{f^T (L + a\mathbf{1}\mathbf{1}^T) f} \quad \text{and} \quad \inf_f \frac{f^T P_\kappa f}{f^T (L + a\mathbf{1}\mathbf{1}^T) f}.$$

As a approaches zero, the two eigenvalues will approach positive and negative infinity. This is distinctly a consequence of $\mathbf{1}$ being in the null space of L and the fact that $\mathbf{1}^T P_\kappa \mathbf{1} = 0$; were the latter not true, only one of the eigenvectors could converge to $\mathbf{1}$ and not cause the quotient to switch signs.

The zero eigenvalue of (L, P) does have a generalized eigenfunction, as guaranteed by the Jordan canonical form theorem [19]. Solving the equation $(L - 0P_\kappa)x = P_\kappa \mathbf{1}$ yields $x = L^\dagger p^\kappa$, where $L^\dagger$ is the Moore–Penrose pseudoinverse of L.

We rectify (L, P_κ) by adding an infinite rank-one perturbation. Consider either of the pencils $\lim_{b\to\infty}(L + brr^T, P_\kappa)$ and $\lim_{b\to\infty}(L, P_\kappa + brr^T)$, where $r \in \mathbb{R}^n$ is not orthogonal to $\mathbf{1}$. They will, respectively, have an infinite and a zero eigenvalue, both corresponding to the eigenfunction r, and will share the remaining eigenvalues and eigenfunctions.

Because both matrices of the pencil are real symmetric and the left matrix is positive definite, the eigenvalues and eigenfunctions are real and admit a variational characterization. The magnitude of the smallest, which we denote μ_κ^r, has the variational characterization

$$\begin{aligned} \mu_\kappa^r &= \lim_{b\to\infty} \inf_f \left| \frac{f^T (L + brr^T) f}{f^T P_\kappa f} \right| \\ &= \inf_{f \perp r} \left| \frac{f^T L f}{f^T P_\kappa f} \right|. \end{aligned} \tag{4.1}$$

Define

$$\mu^r = \min_\kappa \mu_\kappa^r. \tag{4.2}$$

Even for the simple case in which r is not a function of κ, a continuous relaxation of (4.2) is not guaranteed to have a unique global minimum. This is evident from the reciprocal

$$(\mu^r)^{-1} = \max_\kappa \sup_{f \perp r} \left| \frac{f^T P_\kappa f}{f^T L f} \right|,$$

which is the maximum of the pointwise supremum of a family of linear functions of $\boldsymbol{\kappa}$, and hence a convex maximization problem [4]. A consequence is that there is no easy way of computing μ^r when there are many commodities; however, when the number of commodities is small, it may be straightforward to guess the optimal $\boldsymbol{\kappa}$, or simply try all of the likely ones. Furthermore, convexity does guarantee that the optimal $\boldsymbol{\kappa}$ is at a corner, and thus the continuous relaxation is equivalent to the binary formulation.

4.2. Bounds on $\boldsymbol{\mu^r}$. We have the following Cheeger-like inequality.

THEOREM 4.1.

$$\frac{q|\sum_v r_v|}{|\sum_{v\in X} r_v - \sum_{v\in\bar{X}} r_v|} \geq \mu^r > \frac{qh|\sum_v r_v|}{2\sqrt{\sum_v d_v \sum_v |r_v^2|/d_v}},$$

where X is the vertex set associated with q and $r \in \mathbb{R}^n$.

Proof. The structure of our proof for the most part follows that of the Cheeger inequality given in [6]. Although we assume unit capacities, the proof straightforwardly extends to networks with nonnegative capacities by generalizing the definition of the Laplacian to allow for weighted graphs.

We begin with the upper bound. Define the function

$$f(v) = \begin{cases} \sum\limits_{u\in X} r_u & \text{if } v \in \bar{X}, \\ -\sum\limits_{u\in\bar{X}} r_u & \text{if } v \in X, \end{cases}$$

where X is the optimal vertex set associated with q. Let $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ be optimal for (3.1) and (4.2), respectively. Substituting f into (4.1) gives

$$\begin{aligned} \mu^r &\leq \mu^r_{\boldsymbol{\kappa}_1} \\ &\leq \frac{|C(X,\bar{X})||(\sum_{v\in X} r_v + \sum_{v\in\bar{X}} r_v)^2|}{S(X)|(\sum_{v\in X} r_v)^2 - (\sum_{v\in\bar{X}} r_v)^2|} \\ &= \frac{q|\sum_v r_v|}{|\sum_{v\in X} r_v - \sum_{v\in\bar{X}} r_v|}. \end{aligned}$$

We now prove the lower bound. Let f be the eigenfunction of $\lim_{b\to\infty}(L + brr^T, P_{\boldsymbol{\kappa}_2})$ associated with $\mu^r_{\boldsymbol{\kappa}_2}$. Order the vertices in $V(G)$ so that $|f(v_i)| \leq |f(v_{i+1})|$ for $i = 1, \ldots, n-1$. For each i define the cut $D_i = \{\{j,k\} \in E(G) \,|\, 1 \leq j \leq i < k \leq n\}$, and set

$$\alpha = \min_{1\leq i<n} \frac{|D_i|}{|\sum_{j\leq i} p^{\boldsymbol{\kappa}_2}_j|}.$$

By definition, $\alpha \geq q$ regardless of whether $\kappa_1 = \kappa_2$. We have

$$\begin{aligned}
\mu^r &= \left| \frac{\sum_{u\sim v}(f(v)-f(u))^2 \sum_{u\sim v}(f(v)+f(u))^2}{\sum_v f(v)^2 p_v^{\kappa_2} \sum_{u\sim v}(f(u)+f(v))^2} \right| \\
&\geq \left| \frac{(\sum_{u\sim v}|f(u)^2 - f(v)^2|)^2}{2\sum_v f(v)^2 p_v^{\kappa_2} \sum_v f(v)^2 d_v} \right| \quad \text{by Cauchy–Schwarz} \\
&= \left| \frac{(\sum_i |f(v_i)^2 - f(v_{i+1})^2||D_i|)^2}{2\sum_v f(v)^2 p_v^{\kappa_2} \sum_v f(v)^2 d_v} \right| \quad \text{by counting} \\
&\geq \left| \frac{(\sum_i (f(v_i)^2 - f(v_{i+1})^2)\alpha|\sum_{j\leq i} p_{v_j}^{\kappa_2}|)^2}{2\sum_v f(v)^2 p_v^{\kappa_2} \sum_v f(v)^2 d_v} \right| \quad \text{by the definition of } \alpha \\
&\geq \left| \frac{\alpha^2(\sum_i (f(v_i)^2 - f(v_{i+1})^2)\sum_{j\leq i} p_{v_j}^{\kappa_2})^2}{2\sum_v f(v)^2 p_v^{\kappa_2} \sum_v f(v)^2 d_v} \right| \quad \text{by the triangle inequality} \\
&\geq \left| \frac{q^2(\sum_i f(v_i)^2(\sum_{j\leq i} p_{v_j}^{\kappa_2} - \sum_{j\leq i-1} p_{v_j}^{\kappa_2}))^2}{2\sum_v f(v)^2 p_v^{\kappa_2} \sum_v f(v)^2 d_v} \right| \\
&= \left| \frac{q^2(\sum_i f(v_i)^2 p_{v_i}^{\kappa_2})^2}{2\sum_v f(v)^2 p_v^{\kappa_2} \sum_v f(v)^2 d_v} \right| \\
&= \left| \frac{q^2 \sum_v f(v)^2 p_v^{\kappa_2}}{2\sum_v f(v)^2 d_v} \right|.
\end{aligned}$$

Switching to matrix notation and noting that $\mu^r f^T P_{\kappa_2} f = -f^T L f$, we simplify further so that

$$\mu^r \geq \frac{q^2 f^T L f}{2\mu^r f^T D f}.$$

Multiplying through by μ^r and taking the positive square root, we have that

$$\mu^r \geq q\sqrt{\frac{f^T L f}{2 f^T D f}} \geq q\sqrt{\frac{\lambda_r}{2}},$$

where λ_r is the smallest eigenvalue of the pencil $\lim_{b\to\infty}(L + brr^T, D)$.

λ_r may not be an intuitive quantity in some cases, so we also derive a slightly looser but more revealing lower bound, which is a function of λ_1 and thus h, by the Cheeger inequality. Using similarity and the substitution $l = D^{1/2} f$, we have

$$\mu^r \geq q\sqrt{\frac{l^T \mathcal{L} l}{2 l^T l}}.$$

Because $D^{1/2}\mathbf{1}$ is in the null space of $\mathcal{L}$, l in the numerator can be replaced with its projection onto the orthogonal complement of $D^{1/2}\mathbf{1}$, which we denote $\text{proj}_{D^{1/2}\mathbf{1}^\perp}(l)$. The minimum possible ratio of their lengths is given by

$$\beta = \min_{c \perp D^{-1/2} r} \frac{\|\text{proj}_{D^{1/2}\mathbf{1}^\perp}(c)\|}{\|c\|} = \min_{c \perp D^{-1/2} r} \frac{\|c - \text{proj}_{D^{1/2}\mathbf{1}}(c)\|}{\|c\|}. \tag{4.3}$$

The minimizing c is

$$\bar{c} = \text{proj}_{D^{-1/2}r^{\perp}}(D^{1/2}\mathbf{1}) = D^{1/2}\mathbf{1} - \text{proj}_{D^{-1/2}r}(D^{1/2}\mathbf{1}).$$

Substituting $\bar{c}$ into (4.3), after some algebra, yields

$$\beta = \frac{|\mathbf{1}^T r|}{\|D^{1/2}\mathbf{1}\|\|D^{-1/2}r\|} = \frac{|\sum_v r_v|}{\sqrt{\sum_v d_v \sum_v r_v^2/d_v}}.$$

Let k be a vector the same length as l and parallel to $\text{proj}_{D^{1/2}\mathbf{1}^{\perp}}(l)$. We then have

$$\begin{aligned}\mu^r &\geq q\beta\sqrt{\frac{k^T\mathcal{L}k}{2l^Tl}}\\ &\geq q\beta\sqrt{\frac{\lambda_1\|k\|^2}{2\|l\|^2}}\\ &= q\beta\sqrt{\frac{\lambda_1}{2}}\\ &> \frac{qh\beta}{2}. \qquad \square\end{aligned}$$

4.3. Orthogonality constraints. It is important that the upper bound stays finite for all networks of interest; for some r, there are certain networks that will cause the denominator to be zero, constituting an effective blind spot in μ^r. For analysis of a single network, one might heuristically construct an r for which it is clear that this cannot happen. Design and optimization, however, require that the upper bound remains finite for all possible networks, or else an algorithm may simply seek out networks for which the upper bound is infinite. We now examine several choices of r.

(i) The maximum possible lower bound is $qh/2$, which is attained by $r = D\mathbf{1}$. Unfortunately, the upper bound then becomes

$$\mu^{D\mathbf{1}} \leq \frac{q\text{vol}(V(G))}{|\text{vol}(X) - \text{vol}(\bar{X})|},$$

which is infinite if the sums of the degrees on either side of $C(X, \bar{X})$ are equal. This is reflected in $\mu^{D\mathbf{1}}$ as well: Consider a symmetric "dumbbell" network in which two identical halves are connected by a single edge, and assume that all vertices in one half are unit sources and in the other half unit sinks of a single commodity. As the size of the halves is increased, it can be observed that $\mu^{D\mathbf{1}}$ grows despite q_1 decreasing as one over the number of vertices.

(ii) q has a number of interpretations in which being larger is better, so we are interested in choices for which it is the only nonconstant factor in the upper bound. Let $Y \subseteq X$ or $Y \subseteq \bar{X}$. If we choose r to be

$$\delta_v^Y = \begin{cases} d_v & \text{if } v \in Y,\\ 0 & \text{if } v \in \bar{Y}, \end{cases}$$

the bound becomes

$$q \geq \mu^{\delta^Y} > \frac{qh}{2}\sqrt{\frac{\mathrm{vol}(Y)}{\mathrm{vol}(V(G))}}.$$

This may be somewhat impractical for most choices of Y, particularly in contexts in which the edges and hence X can change. However, if Y is the singleton z, X need not be known, and $\mathrm{vol}(Y)$ is simply replaced by d_z. The formulation is simple in this case, but the dependence on the vertex z and the potential $1/\sqrt{n}$ factor in the lower bound may be undesirable.

(iii) Rather than using a single orthogonality constraint, taking the minimum of two eigenvalues can result in an upper bound that is always finite. For a vertex set N, let N^+ (N^-) denote the subset for which $p_v^{\kappa_1} > 0$ ($p_v^{\kappa_1} < 0$), $v \in N$, where κ_1 is optimal for (3.1). Consider

$$s_v^+ = \begin{cases} p_v^\kappa & v \in V(G)^+, \\ 0 & v \in V(G)^- \end{cases} \quad \text{and} \quad s_v^- = \begin{cases} p_v^\kappa & v \in V(G)^-, \\ 0 & v \in V(G)^+, \end{cases}$$

and set $\mu^s = \min\{\mu^{s^+}, \mu^{s^-}\}$. The upper bound of the minimum of the two resulting eigenvalues is

$$\mu^s \leq \min\left\{\frac{q\sum_{v\in V(G)^+} p_v^{\kappa_1}}{|\sum_{v\in X^+} p_v^{\kappa_1} - \sum_{v\in \bar{X}^+} p_v^{\kappa_1}|}, \frac{-q\sum_{v\in V(G)^-} p_v^{\kappa_1}}{|\sum_{v\in X^-} p_v^{\kappa_1} - \sum_{v\in \bar{X}^-} p_v^{\kappa_1}|}\right\}.$$

Observe that

$$\begin{aligned} 2S(X) &= \left|\sum_{v\in X} p_v^{\kappa_1} - \sum_{v\in \bar{X}} p_v^{\kappa_1}\right| \\ &\leq \left|\sum_{v\in X^+} p_v^{\kappa_1} - \sum_{v\in \bar{X}^+} p_v^{\kappa_1}\right| + \left|\sum_{v\in X^-} p_v^{\kappa_1} - \sum_{v\in \bar{X}^-} p_v^{\kappa_1}\right|. \end{aligned}$$

Because the numerators are equal, we have

$$\mu^s \leq \frac{q\sum_v |p_v^{\kappa_1}|}{2S(X)},$$

which is finite because $S(X)$ is always greater than zero. Under certain conditions, μ^s is bounded above by q, as shown in the computational example in section 7.1 and by the following lemma.

LEMMA 4.2. *Suppose $V(G)^+ \subseteq X$ or $V(G)^- \subseteq X$. Then $\mu^s \leq q$.*

Proof. Let P^+ (P^-) be a matrix with s^+ (s^-) on the main diagonal and zeros elsewhere, and let x be the minimizer of (3.2). We have

$$
\begin{aligned}
q &= \frac{x^T L x}{|x^T P_{\kappa_1} x|} \\
&= \frac{x^T L x}{|x^T P^+ x + x^T P^- x|} \\
&\geq \min\left\{\frac{x^T L x}{x^T P^+ x}, \frac{x^T L x}{|x^T P^- x|}\right\} \quad \text{by the definiteness of } P^+ \text{ and } P^- \\
&\geq \min\left\{\min_{x\in\{0,1\}, x\perp s^-} \frac{x^T L x}{x^T P^+ x}, \min_{x\in\{0,1\}, x\perp s^+} \frac{x^T L x}{|x^T P^- x|}\right\} \quad \text{by the assumption} \\
&= \min\left\{\min_{x\in\{0,1\}, x\perp s^-} \frac{x^T L x}{|x^T P_{\kappa_1} x|}, \min_{x\in\{0,1\}, x\perp s^+} \frac{x^T L x}{|x^T P_{\kappa_1} x|}\right\} \\
&\geq \min\left\{\min_{\kappa, x\perp s^-} \frac{x^T L x}{|x^T P_{\kappa} x|}, \min_{\kappa, x\perp s^+} \frac{x^T L x}{|x^T P_{\kappa} x|}\right\} \\
&= \min\{\mu^{s^-}, \mu^{s^+}\}. \qquad \square
\end{aligned}
$$

(iv) Last we mention a complex orthogonality condition, for which the theory of the preceding section does not hold. Define

$$
t_v = \begin{cases} p_v^\kappa & \text{if } p_v^\kappa \geq 0, \\ i p_v^\kappa & \text{if } p_v^\kappa < 0. \end{cases}
$$

Based on observation, we conjecture the following bound:

$$
\frac{q \sum_v |p_v^{\kappa_1}|}{2S(X)} \geq \mu^t \geq \mu^s.
$$

4.4. Calculation via orthogonal transformation. One method by which to numerically compute μ^r using standard eigenvalue solvers is to approximate the limit of the pencil with a large number in place of b. This can be unsatisfactory because if the number is not large enough, the approximation is poor, while if it is too large numerical inaccuracies may arise, particularly for large networks. An orthogonal transformation can instead be used to obtain the exact answer.

Let R be an orthonormal matrix with first column equal to $r/\|r\|$. Because eigenvalues are invariant under orthogonal transformation, those of $\lim_{c\to\infty}(R^T L R + bR^T r r^T R, R^T P_\kappa R)$ are identical to those of $\lim_{c\to\infty}(L + brr^T, P_\kappa)$. Let L' and P' be the respective bottom right $n-1$ by $n-1$ submatrices of $R^T L R$ and $R^T P_\kappa R$. The eigenfunction with the infinite eigenvalue is the last column of $R^T L R$, and hence the remaining eigenvalues (among which is μ^r) are given by the reduced pencil (L', P'), which can be solved by any generalized eigenvalue algorithm. We remark that when $r = \delta^z$, this amounts to simply removing row and column z from L and P_κ.

5. An alternate relaxation of q. We now examine a slightly different formulation of q and arrive at a quantity similar to μ_κ^r, but which scales as capacity over flow squared. This has relevance in certain scenarios such as electrical current flow, which is conserved in networks, yet is proportional to the square root of power.

Consider the minimization

$$
\min_{x\in\{0,1\}^n} \frac{x^T L x}{|x^T p^\kappa|}.
$$

If $\kappa = \kappa_1$, the minimum is q. Relaxing x to take on continuous values and introducing the constraint $|x^T p^\kappa| = 1$ yields

$$\gamma_\kappa = \min_{|x^T p^\kappa|=1} \frac{x^T L x}{|x^T p^\kappa|} \tag{5.1}$$

$$= \frac{1}{p^{\kappa T} L^\dagger p^\kappa}. \tag{5.2}$$

We first make two observations: γ_κ is the sole finite eigenvalue of the pencil $(L, p^\kappa p^{\kappa T})$ and the optimal x associated with γ_κ, $L^\dagger p^\kappa / \gamma_\kappa$, is proportional to the generalized eigenvector of the zero eigenvalue of (L, P_κ).

Let

$$\gamma = \min_\kappa \gamma_\kappa, \tag{5.3}$$

and let κ_3 be optimal. A continuous relaxation of κ is again of little value here: Because $L^\dagger$ is positive semidefinite, it becomes a concave minimization problem, for which there will likely be multiple local minima.

We bound γ from above using q in the same fashion as μ^r.

LEMMA 5.1. *Let X be the set that minimizes q. Then*

$$\gamma \leq \frac{q}{S(X)}.$$

Proof. Let

$$x(v) = \begin{cases} \frac{1}{S(X)} & \text{if } v \in X, \\ 0 & \text{if } v \in \bar{X}. \end{cases}$$

Substituting x into (5.1), we have

$$\begin{aligned} \gamma &\leq \gamma_{\kappa_1} \\ &\leq \frac{|C(X, \bar{X})|}{S(X)^2} \\ &= \frac{q}{S(X)}. \end{aligned}$$

□

We can draw further comparison with current flow by considering resistive power loss in a direct current electrical network. Let p be the vector of currents entering and exiting the network through the vertices. Define the admittance Laplacian L_A to be the Laplacian with admittance (one over resistance) edge weights. The total power dissipated is

$$p^T L_A^\dagger p,$$

which is exactly $1/\gamma$ for the single commodity case.

6. Semidefinite relaxations. When there are many commodities, the minimizations over κ in (4.2) and (5.3) can pose intractable combinatorial optimization problems. Since its application to the max-cut problem [15], semidefinite programming [4, 29] has seen wide usage in developing relaxations for NP-hard problems, the most pertinent example here being the sparsest cut problem [2, 3, 14]. We apply the methodology of [26] and [23] to (4.2) and (5.3) and obtain simple semidefinite relaxations of μ^r and γ. As applied to the sparsest cut problem, these relaxations are less accurate than those based on geometric formulations [2, 3], but constitute a new approach in approximating the more general q.

6.1. μ^r. As shown in the previous sections, the orthogonality condition can be chosen so that μ^r is a lower bound for q. We formulate a semidefinite programming relaxation of (4.2), which consequently is also a relaxation of q.

μ^r can be expressed in terms of semidefinite programming as the following minimax problem:

$$\begin{aligned} &\min_{\kappa} \max_{\xi} \xi \\ &\xi P_\kappa \preccurlyeq \lim_{b\to\infty} L + brr^T \\ &-1 \le \kappa^i \le 1 \quad \forall\ i. \end{aligned} \tag{6.1}$$

As long as the graph is connected, h, q, and therefore μ^r are greater than zero, and so (6.1) is strictly feasible. We can thus replace the inner maximization with its dual and obtain the equivalent bilinear semidefinite program

$$\begin{aligned} &\min_{\kappa, Z} \mathbf{Tr}\, LZ \\ &\mathbf{Tr}\, rr^T Z = 0 \\ &\mathbf{Tr}\, P_\kappa Z = 1 \\ &0 \preccurlyeq Z \\ &-1 \le \kappa^i \le 1 \quad \forall\ i, \end{aligned} \tag{6.2}$$

where **Tr** denotes the trace operator.

If the first constraint is dropped, we obtain the result of applying the original max-cut relaxation of [14] to (3.2); this modification is, of course, always zero, as it corresponds to the defective pencil (L, P_κ). The first constraint makes the relaxation nontrivial and can be designed according to section 4.

The second constraint of (6.2) is bilinear and hence nonconvex. We proceed to formulate the simplest nontrivial relaxation within the framework of [23, 26]. Let P^i be a diagonal matrix with p^i, the vector of commodity i's supplies and demands, on its main diagonal. Note that $P_\kappa = \sum_{i=1}^m \kappa^i P^i$. Introduce a matrix W^i for each commodity i, and substitute W_i for each instance of the product $\kappa^i Z$. An additional constraint on each W_i is constructed by taking the product of the last two constraints. The resulting semidefinite relaxation is given by

$$
\begin{aligned}
&\min_{Z, W^i} \mathbf{Tr}\, LZ \\
&\mathbf{Tr}\; rr^T Z = 0 \\
&\mathbf{Tr}\, rr^T W^i = 0 \quad \forall\; i \\
&\mathbf{Tr} \sum_{i=1}^{m} P^i W^i = 1 \\
&0 \preccurlyeq Z \\
&-Z \preccurlyeq W^i \preccurlyeq Z \quad \forall\; i.
\end{aligned} \tag{6.3}
$$

As stated, (6.3) is an unwieldy relaxation due to the large number of new variables introduced. When r is an indicator vector, we can reduce the size by recognizing that often most vertices are not sources or sinks of most commodities, for example, in the sparsest cut problem, in which each commodity is attached to only two vertices.

Let $r = \delta^1$ (without loss of generality one can relabel the vertices so that any vertex, e.g., that of maximum degree, is the first). Define M^i to be the set of vertices v for which p^i_v is nonzero. Let $Z(M^i)$ denote the $|M^i|$ by $|M^i|$ principal submatrix of Z induced by the set M^i, and likewise let $P^i(M^i)$ denote the corresponding $|M^i|$ by $|M^i|$ submatrix of P^i. Note that the condition $Z \succcurlyeq 0$ implies that every principal submatrix of Z is positive semidefinite. Again substituting a matrix W^i wherever $\kappa^i Z(M^i)$ appears, we have the equivalent semidefinite program

$$
\begin{aligned}
&\min_{Z, W^i} \mathbf{Tr}\, LZ \\
&Z_{11} = 0 \\
&W^i_{11} = 0 \quad \forall\; i{:}1 \in M^i \\
&\mathbf{Tr} \sum_{i=1}^{m} P^i(M^i) W^i = 1 \\
&0 \preccurlyeq Z \\
&-Z(M^i) \preccurlyeq W^i \preccurlyeq Z(M^i) \quad \forall\; i.
\end{aligned} \tag{6.4}
$$

If, given an optimal solution $\tilde{Z}$ and $\tilde{W}^i$ to (6.3) or (6.4), there exists $\tilde{\kappa}^i \in [-1, 1]$ such that $\tilde{\kappa}^i \tilde{Z} = \tilde{W}^i$ (respectively, $\tilde{\kappa}_i \tilde{Z}(M^i) = \tilde{W}^i$) for each i, $\tilde{Z}$ and $\tilde{\kappa}^i$ are optimal for (6.2). In general, however, the relaxation is not tight, so we suggest the following rounding heuristic: If $\mathbf{Tr}\, \tilde{W}^i > 0$, set $\tilde{\kappa}^i = 1$; otherwise set $\tilde{\kappa}^i = -1$ for each i. Once $\tilde{\kappa}$ is known, the corresponding approximation to q is equal to the optimum of the linear program obtained by removing the outer minimization of (3.3) and setting $\kappa = \tilde{\kappa}$.

A natural question is whether linear relaxations can be directly formulated from (3.3). Repeating the steps used to obtain (6.3) from (6.1), one can apply the relaxation of [26] to the resulting bilinear program. We observed that a "second order" linear relaxation was uniformly zero; while higher order relaxations are possible, they are cumbersome in size, and furthermore it has been shown that the corresponding semidefinite relaxations of [23] are more efficient and numerically superior [24].

6.2. γ. The reciprocal of (5.3) can be written as

$$\max_{\kappa\in\{-1,1\}^m} p^{\kappa\,T} L^\dagger p^\kappa = \max_{\kappa\in\{-1,1\}^m} \sum_{i=1}^m \sum_{j=1}^m \kappa^i \kappa^j p^{i\,T} L^\dagger p^j.$$

A simple semidefinite relaxation in the fashion of [14] is

$$\begin{aligned} &\max_K \sum_{i=1}^m \sum_{j=1}^m K_{ij} p^{i\,T} L^\dagger p^j \\ &K_{ii} = 1 \quad \forall\ i \\ &K \succcurlyeq 0. \end{aligned} \tag{6.5}$$

7. Computational results.

7.1. One commodity. Although a substantial fraction of spectral graph theory applications deal directly or indirectly with NP-hard combinatorial optimization problems, we first focus on the single commodity case, which for the most part falls within the scope of linear programming and faster algorithms [1]. Our motivation comes from the amenability of eigenvalues to certain techniques not shared by linear and semidefinite programming, e.g., perturbation theory [27, 30].

We study the proximity of q to three variations of μ^r from section 4.3 as functions of size and edge density. The three eigenvalues considered are μ^s, μ^{δ^z}, and μ^{δ^X}, where z is the vertex of largest degree and X is the minimizer of q. Relative error, defined $e(x) = |1 - x/q|$, is averaged over 1,000 randomly generated, 100-vertex flow networks with unit capacities, which are generated as follows. An Erdös–Rényi random graph with edge formation probability p_{ER} is sampled [10]; since p_{ER} directly determines the expected number of edges, we use it as a parametrization of edge density. If the graph is disconnected, a new one is drawn, since q and μ^r are trivially zero in this case. For each graph, a random vector p of supplies and demands is drawn from the normal distribution $\mathcal{N}(\mathbf{0}, I)$, and then $\sum_v p_v/n$ is subtracted from each element so that $\sum_v p_v = 0$.

Tables 7.1 and 7.2 summarize the results. $e(\mu^{\delta^z})$ increases gradually, and $e(\mu^s)$ and $e(\mu^{\delta^X})$ tend toward the same value, approximately approaching 0.12 from above and below, respectively. As p_{ER} is increased, $e(\mu^{\delta^z})$ increases, but $e(\mu^s)$ and $e(\mu^{\delta^X})$ decrease. We can see why this is so for $e(\mu^{\delta^X})$ by applying a basic result from spectral graph theory. On a complete graph, $\lambda_1 = n/(n-1)$, and Theorem 4.1 reduces to $q \geq \mu^{\delta^X} \geq q\sqrt{\text{vol}(X)/2(n-1)^2}$. It is common in this case for X to contain all but a few vertices; when $|X| = n - 1$, the lower bound is $q/\sqrt{2}$. μ^s, which did not exceed q in any trial, also exhibits error decreasing with p_{ER}.

7.2. Multiple commodities. We now examine the quality of the relaxation (6.4) as a function of the number of commodities m and the number of vertices per commodity n_c. In each case, 100 30-vertex flow networks were randomly sampled as in the previous example with $p_{\text{ER}} = 1/2$, and the mean relative error was computed for μ^{δ^z}, its semidefinite relaxation, $\tilde{\mu}^{\delta^z}$, and the corresponding rounding approximation to q, $\tilde{q}$. Semidefinite programs were solved using the convex optimization tool CVX [17] and solver SeDuMi [28].

TABLE 7.1
Mean relative errors of each eigenvalue on single-commodity networks with $p_{\mathrm{ER}} = 10/n$ as a function of n.

n	100	200	300
$e(\mu^s)$	0.14	0.13	0.12
$e(\mu^{\delta^z})$	0.54	0.58	0.60
$e(\mu^{\delta^X})$	0.11	0.12	0.12

TABLE 7.2
Mean relative errors of each eigenvalue on single-commodity, 100-vertex networks as a function of p_{ER}.

p_{ER}	1/10	1/2	9/10
$e(\mu^s)$	0.14	0.085	0.084
$e(\mu^{\delta^z})$	0.54	0.70	0.73
$e(\mu^{\delta^X})$	0.11	0.013	0.0016

TABLE 7.3
Mean relative errors on five-commodity per node, 30-vertex networks as a function of m.

m	4	8	12
$e(\tilde{q})$	0.023	0.045	0.089
$e(\tilde{\mu}^{\delta^z})$	0.46	0.47	0.46
$e(\mu^{\delta^z})$	0.46	0.47	0.45

TABLE 7.4
Mean relative errors on 10-commodity, 30-vertex networks as a function of n_c.

n_c	10	20	30
$e(\tilde{q})$	0.066	0.12	0.19
$e(\mu^{\delta^z})$	0.46	0.42	0.39
$e(\tilde{\mu}^{\delta^z})$	0.47	0.43	0.40

Tables 7.3 and 7.4 show that as both m and n_c are increased, the tightness of the eigenvalue and semidefinite bounds do not change significantly, but the ultimate approximation error of $\tilde{q}$ increases.

8. Conclusions and future work. We have defined a class of Laplacian matrix pencils and a new cut parameter for undirected, multicommodity flow networks. The parameter, which is a generalization of the sparsest cut, bounds the smallest magnitude eigenvalue of each pencil via a Cheeger-like inequality. The eigenvalue is used to formulate semidefinite relaxations, the quality of which is assessed in computational examples.

There are a number of potential venues for further development. The most obvious is the extension to directed flow networks; a Laplacian and a Cheeger inequality exists for irreversible Markov chains [7, 12], but the formulation implicitly normalizes edge

weights, precluding an edge capacity interpretation. Better relaxation constraints, which do not necessarily correspond to eigenvector orthogonality conditions, are likely to exist. To this end, flow network versions of other spectral graph theory results, e.g., Poincaré and Sobolev inequalities [6, 8], may be useful in devising and perhaps bounding them. The distributions of μ^r and γ under random sources and sinks have many applications, for example, an electric power grid with intermittent wind or solar generation [31].

Acknowledgments. The authors would like to thank Professors Alan Edelman, Gilbert Strang, and Daniel Stroock for helpful discussions.

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