

MIT Open Access Articles

Bilayer quantum Hall phase transitions and the orbifold non-Abelian fractional quantum Hall states

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

Citation: Barkeshli, Maissam, and Xiao-Gang Wen. "Bilayer Quantum Hall Phase Transitions and the Orbifold non-Abelian Fractional Quantum Hall States." *Physical Review B* 84.11 (2011): n. pag. Web. 17 Feb. 2012. © 2011 American Physical Society

As Published: <http://dx.doi.org/10.1103/PhysRevB.84.115121>

Publisher: American Physical Society (APS)

Persistent URL: <http://hdl.handle.net/1721.1/69138>

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

Terms of Use: Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.



Bilayer quantum Hall phase transitions and the orbifold non-Abelian fractional quantum Hall states

Maissam Barkeshli* and Xiao-Gang Wen

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 3 December 2010; published 20 September 2011)

We study continuous quantum phase transitions that can occur in bilayer fractional quantum Hall (FQH) systems as the interlayer tunneling and interlayer repulsion are tuned. We introduce a slave-particle gauge theory description of a series of continuous transitions from the (ppq) Abelian bilayer states to a set of non-Abelian FQH states, which we dub orbifold FQH states, of which the Z_4 parafermion (Read-Rezayi) state is a special case. This provides an example in which Z_2 electron fractionalization leads to non-Abelian topological phases. The naive “ideal” wave functions and ideal Hamiltonians associated with these orbifold states do not in general correspond to incompressible phases but, instead, lie at a nearby critical point. We discuss this unusual situation from the perspective of the pattern-of-zeros/vertex algebra frameworks and discuss implications for the conceptual foundations of these approaches. Due to the proximity in the phase diagram of these non-Abelian states to the (ppq) bilayer states, they may be experimentally relevant, both as candidates for describing the plateaus in single-layer systems at filling fractions $8/3$ and $12/5$ and as a way to tune to non-Abelian states in double-layer or wide quantum wells.

DOI: [10.1103/PhysRevB.84.115121](https://doi.org/10.1103/PhysRevB.84.115121)

PACS number(s): 73.43.-f, 11.15.-q, 11.25.Hf, 03.65.Vf

I. INTRODUCTION

The discovery of topologically ordered phases over the past three decades has revolutionized our fundamental understanding of the possible quantum states of matter.¹ For a long time, it was believed that different states can be fully classified by their patterns of symmetry breaking, and transitions between states of different symmetry can be described through the concept of local order parameters and the associated Ginzburg-Landau theory of symmetry breaking. However, the discovery of the quantum Hall effect showed that even when we break all symmetries of a system explicitly, there can still be distinct quantum states of matter that cannot be connected to each other without passing through a phase transition. These different states are distinguished not by symmetry-breaking order, but by a totally different kind of order, called topological order. Understanding the topological phase of a system at equilibrium is, from one perspective, the coarsest and most basic question that can be asked of a quantum many-body system, because the result is independent of any particular symmetry of the problem. In this sense, almost all known conventional states of matter—superfluids, crystals, magnets, insulators, etc.—are topologically trivial states: if all symmetries of the system are broken, most known systems would not have any phase transition as the system parameters are tuned.

To have a fully developed theory of topological order, we need to understand how to characterize physically the possible topological states of quantum many-body systems, we need a mathematical framework that describes their properties, and we need to understand how to describe transitions between different topological states. While we know the mathematical framework for bosonic systems—tensor category theory^{2,3,53}—we do not completely know how to characterize them physically or how to describe their phase transitions. This is particularly true in the context of non-Abelian topological phases. Attempts to develop a systematic physical classification of non-Abelian topological orders in fractional quantum Hall (FQH) states has appeared recently

in the context of the pattern-of-zeros and vertex algebra approaches to classifying ideal FQH wave functions.^{4–10} Non-Abelian states are currently the subject of major experimental and theoretical focus, largely because of the possibility of utilizing them for robust quantum information storage and processing.^{11–13}

This paper presents three main conceptual advances. First, we develop a theory of a set of continuous quantum phase transitions in bilayer quantum Hall systems between well-known Abelian states—the (ppq) states¹⁴—and a set of non-Abelian topological phases that we call the orbifold FQH states. This generalizes the discovery in Ref. 15 regarding transitions between the $(p, p, p-3)$ states and the non-Abelian Z_4 parafermion states. These are all transitions at the same filling fraction and can be driven by tuning interlayer tunneling and/or interlayer repulsion. These results are theoretically significant because aside from this series of transitions, there is only one other set of transitions involving non-Abelian FQH states that is theoretically understood; this is the transition between the $(p, p, p-2)$ bilayer states and the Moore-Read Pfaffian states.^{16,17} The transitions presented here have experimental consequences: we see that there is a possibility of obtaining a wide array of possible non-Abelian states in bilayer or wide quantum wells by starting with well-known states such as the (330) state, and tuning the interlayer tunneling and/or interlayer repulsion. Furthermore, the non-Abelian states that we present here may also be relevant in explaining the single-layer plateaus seen in the second Landau level, such as at $\nu = 8/3$ and $\nu = 12/5$.

The second major advance relates to the implications of the orbifold FQH states for the pattern-of-zeros/vertex algebra classification. Currently, the pattern-of-zeros/vertex algebra approaches have a shortcoming: some patterns of zeros (called “sick” patterns of zeros) cannot be used to uniquely fix the ground state and/or quasiparticle wave functions of FQH states. These sick patterns of zeros may correspond to gapless states for the ideal Hamiltonian, leaving open the question of how such solutions may be relevant in describing

gapped, incompressible phases. The orbifold FQH states that we study here are significant for the theoretical foundations of the pattern-of-zeros/vertex algebra approach because the orbifold states are closely related to such sick pattern-of-zeros solutions. While we use effective field theory and slave-particle gauge theory techniques to demonstrate the existence of these phases, the “ideal wave functions” associated with most of these orbifold FQH states correspond to the sick pattern-of-zeros solutions and are therefore gapless. In this paper we discuss how to properly understand these sick pattern-of-zeros solutions and how they are actually relevant to describing gapped FQH states.

Finally, the study reported here shows how non-Abelian states can be obtained from a theory of Z_2 fractionalization, in which the transitions can be viewed as the condensation of a Z_2 charged field, while the non-Abelian excitations correspond to the Z_2 vortices. This suggests possible generalizations to transitions between Abelian and non-Abelian states based on other discrete gauge groups.

The results presented in this paper rely on a diverse, often complementary, array of techniques: Chern-Simons (CS) theory, slave-particle methods, conformal field theory (CFT), and vertex algebra. Each technique by itself is not powerful enough, but the confluence of all them allows us to see the underlying structure and to establish our results.

We begin in Sec. II by briefly reviewing the results of an analysis of a particular topological field theory: the $U(1) \times U(1) \rtimes Z_2$ CS theory, which suggests the possible existence of a class of non-Abelian FQH states: the orbifold states. However, the topological field theory alone does not imply that there is a possible FQH state of bosons or fermions with such topological properties. In Sec. III, we develop a slave-particle gauge theory of Z_2 fractionalization that shows that, in principle, there can be FQH states whose low-energy effective field theories are the $U(1) \times U(1) \rtimes Z_2$ CS theories. This slave-particle construction will yield projected trial wave functions for the orbifold FQH states. In Sec. IV, we study the edge theory of these orbifold FQH states and we develop a prescription for computing all topological quantum numbers of these phases. We present the results of this prescription in Sec. V.

In Sec. VI, we study the phase transition between the bilayer Abelian (ppq) states and the orbifold states. We find that the transition is continuous and in the three-dimensional (3D) Ising universality class; the critical theory is a Z_2 -gauged Ginzburg-Landau theory. These results give a physical manifestation of recent mathematical ideas of boson condensation in tensor category theory.¹⁸

In Sec. VII we study the consequences of our results for the pattern-of-zeros/vertex algebra approaches to classifying FQH states. Ideal wave functions are wave functions that can be obtained through correlation functions of vertex operators in a CFT; the naive ones for the orbifold FQH states are, in general, gapless and correspond to various sick pattern-of-zeros solutions. We discuss how to interpret this situation in the vertex algebra framework. The results show how the sick pattern-of-zeros/vertex algebra solutions should generally be viewed and how they are relevant to describing gapped FQH states even when their associated ideal Hamiltonians are gapless.

In Sec. VIII we briefly discuss some experimental consequences of this work and we conclude in Sec. IX.

II. $U(1) \times U(1) \rtimes Z_2$ CS THEORY AND ORBIFOLD FQH STATES

The $U(1) \times U(1) \rtimes Z_2$ CS theory was introduced in Ref. 19 and many of its topological properties were explicitly calculated. Here we give a brief review of the main results and pose the main questions that emerge. The Lagrangian is given by

$$\mathcal{L} = \frac{p}{4\pi}(a\partial a + \tilde{a}\partial\tilde{a}) + \frac{q}{4\pi}(a\partial\tilde{a} + \tilde{a}\partial a), \quad (1)$$

where a and \tilde{a} are two $U(1)$ gauge fields defined in $2+1$ dimensions, $a\partial a \equiv \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$, and there is an additional Z_2 gauge symmetry associated with interchanging the two gauge fields. The semidirect product \rtimes highlights the fact that the Z_2 transformation of interchanging the two $U(1)$ gauge fields does not commute with the individual $U(1) \times U(1)$ gauge transformations.

In the absence of the Z_2 gauge symmetry, Eq. (1) is a $U(1) \times U(1)$ CS theory and is the low-energy effective field theory for a bilayer (ppq) FQH state,^{20,21} where the currents in the two layers are given by

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda, \quad \tilde{j}^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \tilde{a}_\lambda. \quad (2)$$

In the presence of the Z_2 gauge symmetry and for $|p - q| > 1$, this theory describes a non-Abelian topological phase where Z_2 vortices are the fundamental non-Abelian excitations. Note that we use the same Lagrangian for both the $U(1) \times U(1)$ and the $U(1) \times U(1) \rtimes Z_2$ CS theories, even though they have different gauge structures and are therefore different topological theories.

When $p - q = 3$, all of the topological properties of the $U(1) \times U(1) \rtimes Z_2$ CS theory that we can compute agree precisely with those of the Z_4 parafermion FQH states at filling fraction $\nu = 2/(2q + 3)$. This, in conjunction with a number of other results, led us to suggest that for $p - q = 3$, this is the correct effective field theory for the Z_4 parafermion FQH states.

This leads us to ask whether, for more general choices of $p - q$, this theory also describes a valid, physically realistic, topological phase. In other words, does it describe a topological phase that can be realized, for some range of material parameters, in a physical system with realistic interactions? It is not clear because, aside from $p - q = 3$, there are no known trial wave functions or trial Hamiltonians that capture the properties of such a topological phase. In fact, the naive trial wave functions that are suggested from a projective construction analysis of these states are believed to be gapless for $p - q > 3$, which casts doubt on whether the phases described by the field theory are physical. A topological field theory by itself is not enough to know that it can be obtained from a physical system of interacting fermions or bosons. In this paper we remedy this problem.

To develop the theory for the topological phases that are described by $U(1) \times U(1) \rtimes Z_2$ CS theory, in the following we recall some of the topological properties of such a CS theory.

A. Topological properties of $U(1) \times U(1) \rtimes Z_2$ CS theory

The number of topologically distinct quasiparticles of the $U(1) \times U(1) \rtimes Z_2$ theory is given by the ground-state degeneracy on a torus, which was calculated to be¹⁹

$$\text{No. of quasiparticles} = (N + 7)|p + q|/2, \quad (3)$$

where $N \equiv |p - q|$. On genus g surfaces, the number of degenerate ground states was calculated to be

$$S_g(p, q) = |p + q| 2^{g-1} [N^g + 1 + (2^{2g} - 1)(N^{g-1} + 1)]. \quad (4)$$

Using $S_g(p, q)$, we can read off an important set of topological quantum numbers of the phase: the quantum dimensions of all the quasiparticles. The quantum dimension d_i of a quasiparticle of type i has the following meaning. In the presence of m quasiparticles of type i at fixed locations, the dimension of the Hilbert space grows like $\propto d_i^m$. Abelian quasiparticles have quantum dimension $d = 1$, while non-Abelian quasiparticles have quantum dimension $d > 1$.

The ground-state degeneracy on genus g surfaces is related to the quantum dimensions through the formula^{8,22}

$$S_g = D^{2(g-1)} \sum_{i=0}^{N_{\text{qp}}-1} d_i^{-2(g-1)}, \quad (5)$$

where N_{qp} is the number of quasiparticles, d_i is the quantum dimension of the i th quasiparticle, and $D = \sqrt{\sum_i d_i^2}$ is the total quantum dimension. Using Eqs. (4) and (5), we can calculate the quantum dimension d_i for each quasiparticle by studying the $g \rightarrow \infty$ limit. The results are as follows. The total quantum dimension is

$$D^2 = 4N|p + q|. \quad (6)$$

There are $2|p + q|$ quasiparticles of quantum dimension 1, $2|p + q|$ quasiparticles of quantum dimension \sqrt{N} , and $(N - 1)|p + q|/2$ quasiparticles of quantum dimension 2.

The fundamental non-Abelian excitations in the $U(1) \times U(1) \rtimes Z_2$ CS theory are Z_2 vortices, which are topological defects around which the two gauge fields transform into each other. We can understand the fact that the Z_2 vortices are non-Abelian by seeing that there should be a degeneracy of states associated with a number of Z_2 vortices at fixed locations. To see that there should be a degeneracy, see Fig. 1. The configurations of the two gauge fields a and \tilde{a} on a sphere with Z_2 vortices can be reinterpreted as though there is a single gauge field on a doubled space with genus $g = n - 1$, where n is the number of pairs of Z_2 vortices.

In Ref. 19, we studied the number of degenerate ground states in the presence of n pairs of Z_2 vortices at fixed locations on a sphere. The result for the number of such states is

$$\alpha_n = \begin{cases} (N^{n-1} + 2^{n-1})/2 & \text{for } N \text{ even,} \\ (N^{n-1} + 1)/2 & \text{for } N \text{ odd.} \end{cases} \quad (7)$$

This shows that the quantum dimension of the Z_2 vortices is $d = \sqrt{N}$. We can also compute the number of states that are odd under the Z_2 gauge transformation. The number of these Z_2 noninvariant states also turns out to be an important quantity, because it yields important information

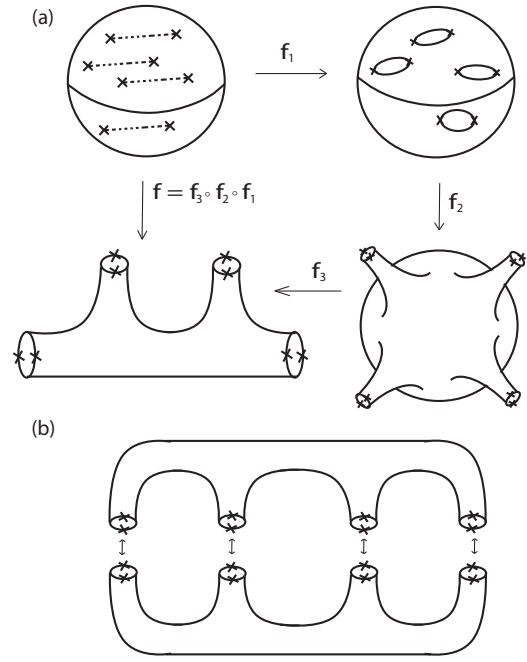


FIG. 1. How to see that Z_2 vortices at fixed locations come with a degeneracy of states and should therefore be non-Abelian. (a) The space is deformed arbitrarily; (b) we consider a doubled space with genus $g = n - 1$, where n is the number of pairs of Z_2 vortices. Because of the boundary conditions of the two gauge fields a and \tilde{a} , we can define a single continuous gauge field on the doubled space, which leaves us with $U(1)$ CS theory on a genus $g = n - 1$ surface. Thus the number of states in the presence of n pairs of Z_2 vortices grows exponentially in n . In this sense, Z_2 vortices are like “genons:” they effectively change the genus of the manifold.

about the fusion rules of the quasiparticles. The number of Z_2 noninvariant states yields the number of ways for n pairs of Z_2 vortices to fuse to an Abelian quasiparticle that carries Z_2 gauge charge. The ground-state degeneracy of Z_2 noninvariant states in the presence of n pairs of Z_2 vortices at fixed locations on a sphere was computed to be

$$\beta_n = \begin{cases} (N^{n-1} - 2^{n-1})/2 & \text{for } N \text{ even,} \\ (N^{n-1} - 1)/2 & \text{for } N \text{ odd.} \end{cases} \quad (8)$$

Thus if γ labels a Z_2 vortex, these calculations reveal the following fusion rules for γ and its conjugate $\bar{\gamma}$:

$$(\gamma \times \bar{\gamma})^n = \alpha_n \mathbb{I} + \beta_n j + \dots, \quad (9)$$

where j is a topologically nontrivial excitation that carries the Z_2 gauge charge. The ellipsis (\dots) represents additional quasiparticles that may appear in the fusion.

In what follows, we focus on the case $p - q > 0$, because these are the cases that are relevant for the bilayer (ppq) states.

III. SLAVE-PARTICLE GAUGE THEORY AND Z_2 FRACTIONALIZATION

The $U(1) \times U(1) \rtimes Z_2$ CS theory presented above defines a topological field theory, however, for $N \neq 3$ it is unclear whether it can arise as the low-energy effective field theory of a physical system with local interactions. In this section,

we show how the $U(1) \times U(1) \rtimes Z_2$ CS theory can arise from a slave-particle formulation, which adds strong evidence to the possibility of these states being realized in physical systems with local interactions. The slave-particle formulation provides us with candidate many-body wave functions that capture the topological properties of these phases. It also provides a UV completion, or lattice regularization, of the $U(1) \times U(1) \rtimes Z_2$ CS theory. This is useful for computing certain topological properties, such as the electric charge of the Z_2 vortices, which we were unable to calculate directly from the $U(1) \times U(1) \rtimes Z_2$ CS theory alone. Finally, this slave-particle formulation provides us with an example in which Z_2 electron fractionalization may lead to non-Abelian topological phases.

Consider a bilayer quantum Hall system, and suppose that the electrons move on a lattice. Let $\Psi_{i\sigma}$ denote the electron annihilation operator at site i ; $\sigma = \uparrow, \downarrow$ refers to the two layers. Now consider the positive and negative combinations:

$$\Psi_{i\pm} = \frac{1}{\sqrt{2}}(\Psi_{i\uparrow} \pm \Psi_{i\downarrow}). \quad (10)$$

We use a slave-particle decomposition to rewrite Ψ_{\pm} in terms of new bosonic and fermionic degrees of freedom, including appropriate constraints so as not to unphysically enlarge the Hilbert space. Such slave-particle decompositions allow us to access novel fractionalized phases. In the following section, we introduce a slave Ising construction that interpolates between the bilayer Abelian (ppq) states and the states described by the $U(1) \times U(1) \rtimes Z_2$ CS theory. In Appendix C, we introduce a slave rotor construction, which can describe these two phases with the advantage of including a larger set of fluctuations about the slave-particle mean-field states.

A. Slave Ising

We introduce two new fields at each lattice site i : an Ising field, $s_i^z = \pm 1$, and a fermionic field, c_{i-} , and we rewrite Ψ_{i-} as

$$\Psi_{i+} \equiv c_{i+}, \quad \Psi_{i-} = s_i^z c_{i-}. \quad (11)$$

This introduces a local Z_2 gauge symmetry, associated with the transformations

$$s_i^z \rightarrow -s_i^z, \quad c_{-i} \rightarrow -c_{-i}. \quad (12)$$

The electron operators are neutral under this Z_2 gauge symmetry, and therefore the physical Hilbert space at each site is the gauge-invariant set of states at each site:

$$\begin{aligned} (|\uparrow\rangle + |\downarrow\rangle) \otimes |n_{c_-} = 0\rangle, \\ (|\uparrow\rangle - |\downarrow\rangle) \otimes |n_{c_-} = 1\rangle, \end{aligned} \quad (13)$$

where $|\uparrow\rangle$ ($|\downarrow\rangle$) is the state with $s^z = +1$ (-1), respectively. In other words, the physical states at each site are those that satisfy

$$(s_i^x + 1)/2 + n_{c_{i-}} = 1. \quad (14)$$

If we imagine that the fermions $c_{i\pm}$ form some gapped state, then we would generally expect two distinct phases:²³ the deconfined/ Z_2 unbroken phase, where

$$\langle s_i^z \rangle = 0; \quad (15)$$

and the confined/Higgs phase, where upon fixing a gauge, we have

$$\langle s_i^z \rangle \neq 0. \quad (16)$$

We seek a mean-field theory where the deconfined phase corresponds to the orbifold FQH states, and the confined/Higgs phase corresponds to the bilayer (ppq) states. To do this, observe that in the Higgs phase, we have

$$\Psi_{i\pm} = c_{i\pm}, \quad (17)$$

since we may set $s_i^z = 1$ in this phase. In such a situation, we can use the parton construction^{24,25} to obtain the (ppq) states. For example, to obtain the (330) states, we rewrite the electron operators in each layer in terms of three partons:

$$\Psi_{i\uparrow} = \psi_{1i} \psi_{2i} \psi_{3i}, \quad \Psi_{i\downarrow} = \psi_{4i} \psi_{5i} \psi_{6i}, \quad (18)$$

where ψ_a carries electric charge $e/3$. We can then rewrite the theory in terms of the original electrons in terms of a theory of these partons, with the added constraint that

$$n_{1i} = n_{2i} = n_{3i}, \quad n_{4i} = n_{5i} = n_{6i}, \quad (19)$$

where $n_{ai} = \psi_{ai}^\dagger \psi_{ai}$, in order to preserve the electron anti-commutation relations and to avoid unphysically enlarging the Hilbert space at each site. The (330) state corresponds to the case where each parton forms a $\nu = 1$ integer quantum Hall state.

Therefore, to interpolate between the Z_4 parafermion state and the (330) state at $\nu = 2/3$, we write

$$\begin{aligned} \Psi_{i+} &= \psi_{1i} \psi_{2i} \psi_{3i} + \psi_{4i} \psi_{5i} \psi_{6i}, \\ \Psi_{i-} &= s_i^z (\psi_{1i} \psi_{2i} \psi_{3i} - \psi_{4i} \psi_{5i} \psi_{6i}). \end{aligned} \quad (20)$$

More generally, to describe the (ppq) states and the orbifold FQH states, we set

$$\begin{aligned} \Psi_{i+} &= c_{i+}, \quad \Psi_{i-} = s_i^z c_{i-}, \\ c_{i\pm} &= \left(\prod_{a=1}^N \psi_{ai} \pm \prod_{a=N+1}^{2N} \psi_{ai} \right) \prod_{b=2N+1}^{2N+q} \psi_{bi}, \end{aligned} \quad (21)$$

where $N \equiv p - q$ (note that we assume $p > q$). Furthermore, we assume that the interactions are such that the partons each form a $\nu = 1$ IQH state.

1. Topological properties of the Z_2 confined and deconfined phases

In what follows, let us focus on the case $q = 0$. When $\langle s_i^z \rangle = 1$, we can write

$$\Psi_{i\pm} = \prod_{a=1}^N \psi_{ai} \pm \prod_{a=N+1}^{2N} \psi_{ai}. \quad (22)$$

The low-energy theory will thus be a theory of $2N$ partons, each with electric charge e/N , and coupled to an $SU(N) \times SU(N)$ gauge field:

$$\mathcal{L} = i \psi^\dagger \partial_0 \psi + \frac{1}{2m} \psi^\dagger (\partial - i A_i Q)^2 \psi + \text{Tr} (j^\mu a_\mu) + \dots, \quad (23)$$

where a is an $SU(N) \times SU(N)$ gauge field, $\psi^\dagger = (\psi_1^\dagger, \dots, \psi_{2N}^\dagger)$, $(Q)_{ab} = \delta_{ab}e/N$, A is the external electromagnetic gauge field, and $j_{ab}^\mu = \psi_a^\dagger \partial^\mu \psi_b$. If the partons form a $\nu = 1$ IQH state, then we can integrate out the partons to obtain a $SU(N)_1 \times SU(N)_1$ CS theory as the long-wavelength, low-energy field theory. This $SU(N)_1 \times SU(N)_1$ CS theory reproduces all of the correct ground-state properties, such as the ground-state degeneracy on genus g surfaces, and the fusion rules of the quasiparticles. The quasiparticle excitations are related to holes in the parton integer quantum Hall states. The $SU(N)_1 \times SU(N)_1$ CS theory needs to be supplemented with additional information about the fermionic character of an odd number of holes to completely capture all of the topological quantum numbers. This can be done by using the $U(1)_N \times U(1)_N$ CS theory instead, which is known to be the correct low-energy effective field theory of the bilayer $(NN0)$ states.

Now consider the Z_2 deconfined phase, where $\langle s_i^z \rangle = 0$. What is the low-energy effective field theory? Since the partons still each form a $\nu = 1$ IQH state and are coupled to an $SU(N) \times SU(N)$ gauge field, integrating them out will yield a $SU(N)_1 \times SU(N)_1$ CS theory, and using the arguments outlined above, we are left with a $U(1)_N \times U(1)_N$ CS theory. Suppose that we also sum over the Ising spins $\{s_i^z\}$. Since there are no gapless modes associated with phases of the Ising spins, we expect a local action involving the Z_2 gauge field coupled to the $U(1)$ gauge fields. We do not know how to explicitly write this action down, because the CS terms are difficult to properly define on a lattice, while the discrete gauge fields require a lattice for their action. Nevertheless, we consider the theory on general grounds: observe that the Z_2 gauge symmetry interchanges ψ_a and ψ_{a+N} ; thus in the low-energy theory involving only the gauge fields, the Z_2 gauge symmetry interchanges the current densities associated with the two $U(1)$ gauge fields. This is precisely the content of the $U(1) \times U(1) \rtimes Z_2$ CS theory. Thus, we may think of the Z_2 deconfined phase of this slave Ising construction as providing a UV completion of the $U(1) \times U(1) \rtimes Z_2$ CS theory. This slave-particle gauge theory can be taken to be the complete definition of the $U(1) \times U(1) \rtimes Z_2$ CS theory. (An alternative, mathematical definition of CS theory for disconnected gauge groups is given in Ref. 30.)

Now let us further study the low-energy excitations of this Z_2 fractionalized phase. In this phase, the Ising spin s^z can propagate freely and is deconfined from the partons. This is an electrically neutral excitation that is charged under the Z_2 gauge symmetry and that fuses with itself to the identity. The phases described by the $U(1) \times U(1) \rtimes Z_2$ CS theory all have precisely such a Z_2 charged excitation; Eqs. (8) and (9) yield the number of ways for n pairs of Z_2 vortices to fuse to precisely this Z_2 charged excitation, which was denoted j .

The other novel topologically nontrivial excitation in the Z_2 deconfined phase is the Z_2 vortex. Since the Z_2 gauge field is coupled to the partons, the Z_2 vortex is non-Abelian. This is not an obvious result: in the low-energy $U(1) \times U(1) \rtimes Z_2$ CS theory the Z_2 vortex corresponds to a topological defect around which the two $U(1)$ gauge fields transform into each other. A detailed study of the Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ theory shows that there is a topological degeneracy

associated with the presence of n pairs of Z_2 vortices at fixed locations, which reveals that the Z_2 vortices are non-Abelian quasiparticles (see Fig. 1).

2. Electric charge of Z_2 vortices

Can we understand the allowed values of the electric charge carried by the Z_2 vortices? We believe that the $U(1) \times U(1) \rtimes Z_2$ CS theory, for certain choice of coupling constants, describes the Z_4 parafermion state. The Z_4 parafermion state has a fundamental non-Abelian excitation that carries a fractionalized electric charge; at $\nu = 2/3$, for example, the electric charge of the fundamental non-Abelian excitation comes in odd multiples of $e/6$. Since we believe that the Z_2 vortices in this theory correspond to the fundamental non-Abelian excitations, an important check on this slave Ising description will be whether it can account for these values of the fractionalized electric charge.

To calculate the electric charge, let us define the following parton operators, which are superpositions of the parton operators ψ_a :

$$\psi_{a\pm} = \frac{1}{\sqrt{2}}(\psi_a \pm \psi_{a+N}), \quad a = 1, \dots, N. \quad (24)$$

The local Z_2 gauge symmetry corresponds to the transformation:

$$s_i^z \rightarrow -s_i^z, \quad \psi_a \leftrightarrow \psi_{a+N}, \quad a = 1, \dots, N. \quad (25)$$

Thus, ψ_{a+} is Z_2 neutral, while ψ_{a-} is Z_2 charged. Furthermore, since the ψ_a each form a $\nu = 1$ IQH state, then ψ_{a+} and ψ_{a-} also each form $\nu = 1$ IQH states. The particle/hole excitations of the states formed by ψ_{a-} carry electric charge e/N [recall that we set $q = 0$ in Eq. (21)]. The Z_2 vortex acts as a π flux for ψ_{a-} . Thus in the low-energy field theory, the interaction between the excitations of the ψ_{a-} IQH state and the external electromagnetic gauge field A_μ and the Z_2 vortices is described by

$$\mathcal{L}_{\text{int};-} = \sum_{a=1}^N \left(\frac{e}{N} A_\mu + b_\mu \right) j_{a-}^\mu, \quad (26)$$

where a Z_2 vortex is associated with π flux of the $U(1)$ gauge field b_μ . j_{a-}^μ is the current density associated with the ψ_{a-} partons. Integrating out the partons, which are in a $\nu = 1$ IQH state, will generate a Chern-Simons term:

$$\begin{aligned} \mathcal{L}_{\text{int};-} &= \sum_{a=1}^N \frac{1}{4\pi} \left(\frac{e}{N} A + b \right) \partial \left(\frac{e}{N} A + b \right), \\ &= \frac{1}{4\pi} \frac{e^2}{N} A \partial A + \frac{N}{4\pi} b \partial b + \frac{e}{2\pi} A \partial b. \end{aligned} \quad (27)$$

Notice that the interaction between the ψ_{a+} current and the external electromagnetic gauge field will contribute another term, $\frac{1}{4\pi} \frac{e^2}{N} A \partial A$, to the action, from which we see that the filling fraction is $\nu = 2/N$. Furthermore, because of the coupling of b to the external gauge field A , we see that a π flux of the b_μ gauge field will carry charge $e/2$. Therefore, depending on

how many holes, m , of the parton integer quantum Hall states are attached to the Z_2 vortices, the Z_2 vortices can have an electric charge of

$$Q_{Z_2 \text{ vortex}} = e(2m + N)/2N. \quad (28)$$

When $N = 3$, this result agrees exactly with properties of the Z_4 parafermion state, which is that the electric charge of the fundamental non-Abelian quasiparticles comes in odd multiples of $e/6$. More generally, when N is odd(even), we see that the Z_2 vortices can only carry an electric charge in odd(even) integer multiples of $e/2N$. In Sec. V A, we again see precisely these results, through a totally different description of this phase!

B. Slave Ising projected wave functions

The slave-particle approach naturally suggests trial wave functions that capture the essential long-wavelength properties of the phase. First, we have the mean-field state of the partons and the Ising spins:

$$|\Phi_{\text{mf}}\rangle = |\{s_i^z\}\rangle |\{\psi_a\}\rangle, \quad (29)$$

where the partons ψ_a form a $\nu = 1$ IQH state. The Z_2 confined/Higgs phase, which describes the Abelian (ppq) states, will be associated with an ordered state of the Ising spins. The Z_2 deconfined phase will be described by an unordered, paramagnetic state of the Ising spins. The quantum state of the electrons will be given by a projection onto the physical Hilbert space:

$$|\Psi\rangle = \mathcal{P}|\Phi_{\text{mf}}\rangle, \quad (30)$$

where

$$\mathcal{P} = \prod_i \mathcal{P}_i, \quad \mathcal{P}_i = \mathcal{P}_i^{\text{Ising}} \mathcal{P}_i^{\text{parton}}. \quad (31)$$

The projection operator for the Ising sector is [see Eq. (14)]:

$$\mathcal{P}_i^{\text{Ising}} = \frac{1}{2} [1 - (-1)^{((s_i^z + 1)/2 + n_{c_i-})}], \quad (32)$$

where $n_{c_i-} = c_{i-}^\dagger c_{i-}$ is written in terms of the partons as

$$n_{c_i-} = \frac{1}{2}(n_{\uparrow i} + n_{\downarrow i}) - \frac{1}{2}[(\psi_{1i} \cdots \psi_{Ni})^\dagger (\psi_{N+1i} \cdots \psi_{2Ni}) + \text{H.c.}]. \quad (33)$$

$n_{i\uparrow}$ and $n_{i\downarrow}$ are the number of electrons in the top and bottom layer, respectively, at site i . The projection operator for the parton sector is

$$\mathcal{P}_i^{\text{parton}} = \prod_{a=1}^N [1 - (n_{i\uparrow} - n_{ai})^2] \prod_{a=N+1}^{2N} [1 - (n_{i\downarrow} - n_{ai})^2], \quad (34)$$

which implements the constraint $n_{1i} = \cdots = n_{Ni} = n_{i\uparrow}$ and $n_{N+1i} = \cdots = n_{2Ni} = n_{i\downarrow}$.

Alternatively, we can work with the spatial wave function. The amplitude of the electron wave function to have N_\uparrow electrons in one layer and N_\downarrow electrons in the second layer is given by

$$\Psi(\{\mathbf{r}_i\}, \{\mathbf{r}'_i\}) = \langle 0 | \prod_{i=1}^{N_\uparrow} \Psi_{\uparrow \mathbf{r}_i} \prod_{i=1}^{N_\downarrow} \Psi_{\downarrow \mathbf{r}'_i} | \Phi_{\text{mf}} \rangle, \quad (35)$$

where $\Psi_{\sigma \mathbf{r}}$ is given in terms of the partons and the Ising spins through Eqs. (10) and (21). Here, $|0\rangle = |0\rangle_{\text{parton}} |\{s_i^x = 1\}\rangle$ is the state with no partons and an eigenstate of \hat{s}_i^x with eigenvalue 1.

This wave function is important because currently it is the only wave function we have for these non-Abelian FQH states (for $N > 3$). As we discuss later, there is currently no corresponding ideal wave function for these states. The projected wave functions presented here can, in principle, be used for numerical studies to determine which phases are most likely under realistic physical conditions.

IV. EDGE THEORY OF THE ORBIFOLD FQH STATES

One use of the $U(1) \times U(1) \rtimes Z_2$ CS theory is that it can be used to study the edge theory of the associated topological phases. It is known that the $U(1)$ CS description of the Abelian quantum Hall liquids leads to the chiral Luttinger liquid edge theory.²⁶ More specifically, an n -component Abelian quantum Hall liquid can be described by a CS theory involving n $U(1)$ gauge fields:²⁷

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I \partial a_J + \frac{1}{2\pi} A \partial a_I, \quad (36)$$

where K is an $n \times n$ symmetric invertible matrix and A is the external electromagnetic gauge field. As a result the edge theory is described by n chiral free bosons.²⁶

$$\mathcal{L}_{\text{edge}} = K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J, \quad (37)$$

where V_{IJ} is a positive definite matrix that dictates the velocity of the edge modes and depends on microscopic properties of the edge.

We therefore expect that the edge of the phases described by $U(1) \times U(1) \rtimes Z_2$ CS theory will be described by two free chiral bosons, ϕ_1 and ϕ_2 , with the Lagrangian given above, and with an additional Z_2 gauge symmetry associated with the transformations

$$[\phi_1(z), \phi_2(z)] \sim [\phi_2(z), \phi_1(z)] \quad (38)$$

at each space-time point. Such a CFT is called an orbifold CFT, because the symmetry $U(1) \times U(1)$ of the original free boson theory is gauged by a discrete Z_2 symmetry. Thus we refer to this theory as the $[U(1) \times U(1)]/Z_2$ orbifold CFT. That the $U(1) \times U(1) \rtimes Z_2$ CS theory should correspond to this edge CFT may be expected in light of Witten's CS/CFT correspondence.²⁸⁻³⁰

As a check, we may perform a simple counting of the operator content of such a chiral CFT by following the considerations of Ref. 31. In that reference, it was argued that the number of primary operators (primary with respect to the orbifold chiral algebra) in a G/Z_k orbifold CFT is related to the number of primary operators in the unorbifolded CFT, with symmetry group G , by the formula

$$\text{No. of operators} = nk^2 + m. \quad (39)$$

Here, m is the number of groups of k operators in the original unorbifolded theory that are cyclically permuted by the Z_k action; together, they lead to m operators that are Z_k invariant. n is the number of operators in the original unorbifolded theory that are fixed under the Z_k action.

In the case of the orbifold states with $p - q = N$ and $q = 0$, the primary operators are labeled as

$$V_{ab}(z) = e^{ia/\sqrt{N}\phi_1(z)+ib/\sqrt{N}\phi_2(z)}. \quad (40)$$

The Z_2 action exchanges a and b , so we have $n = N$ and $m = N(N - 1)/2$. This leads to $N(N + 7)/2$ primary operators, which agrees exactly with the number of quasiparticles expected from the analysis of the torus ground-state degeneracy of the $U(1) \times U(1) \rtimes Z_2$ CS theory. Carrying out the calculation for general $q \neq 0$ yields

$$\text{No. of operators} = (N + 7)|p + q|/2, \quad (41)$$

again agreeing with the analysis from the $U(1) \times U(1) \rtimes Z_2$ CS theory [see Eq. (3)]. This highly nontrivial consistency check suggests that this is indeed the correct edge theory.

To obtain the full topological properties of these FQH states using the edge theory, we would need to obtain the scaling dimensions of each of the primary operators and their fusion rules. This can be done by first computing the characters of the chiral CFT, which are given by

$$\chi_i(\tau) = \text{Tr}_{[\mathcal{O}_i]} q^{L_0 - c/24}. \quad (42)$$

The trace is over states in the module labeled $[\mathcal{O}_i]$, where \mathcal{O}_i is a primary field of the chiral algebra. For FQH states, the \mathcal{O}_i label different quasiparticle sectors. L_0 is the generator of scale transformations, $q = e^{2\pi i\tau}$, and c is the central charge. The scaling dimensions and fusion rules of the primary fields can be obtained by studying the transformation rules of the characters under the modular transformations $S : \tau \rightarrow -1/\tau$ and $T : \tau \rightarrow \tau + 1$.³²

Using the chiral characters, we would also be able to obtain the full edge spectrum for the FQH states on a disk. The spectrum of edge states at each angular momentum in the topological sector labeled by \mathcal{O}_i are given by the coefficients $a_n^{(i)}$ in the expansion

$$\chi_i(\tau) = \sum_{n=0}^{\infty} a_n^{(i)} q^n. \quad (43)$$

Here, $a_n^{(i)}$ is the number of edge excitations with energy $E_n \propto n$ on a disk with the quasiparticle created by \mathcal{O}_i at the center of the disk.

Unfortunately, obtaining the characters is a highly nontrivial task. In known examples of orbifold CFTs, one common way to proceed is to first compute the torus partition function of the nonchiral theory, which includes both holomorphic and antiholomorphic sectors of the CFT. The partition function is related to the chiral characters through

$$Z(\tau, \bar{\tau}) = \sum_{ij} \chi_i(q) M_{ij} \bar{\chi}_j(\bar{q}), \quad (44)$$

where M is a matrix that specifies how to glue together the holomorphic and antiholomorphic sectors. Sometimes it is possible to take the “holomorphic square root” and guess the chiral characters $\chi_i(\tau)$ from the partition function $Z(\tau, \bar{\tau})$. This is done, for example, for the $U(1)/Z_2$ orbifold CFTs at $c = 1$.^{31,32} In the case of FQH states, $Z(\tau, \bar{\tau})$ may be used to compute the edge spectrum on a cylinder by expanding in powers of q and \bar{q} .

In the case of the $[U(1) \times U(1)]/Z_2$ orbifold CFT, it is possible to compute $Z(\tau, \bar{\tau})$, but we do not know at present how to take the holomorphic square root and thus derive the scaling dimensions and fusion rules of the operators in the edge theory. Despite this shortcoming, we can develop a prescription for computing the scaling dimensions and fusion rules of the operators in this CFT. We perform many highly nontrivial checks, both with the slave-particle gauge theory and with results of the $U(1) \times U(1) \rtimes Z_2$ CS theory, to confirm that the prescription given yields correct results. This prescription is necessary because it is currently the only way we have of computing all of the topological quantum numbers of the orbifold states. While the slave Ising and associated $U(1) \times U(1) \rtimes Z_2$ CS theory descriptions are powerful and can be used to calculate many highly nontrivial topological properties, we do not currently know how to use them to compute all topological properties of the orbifold states, such as the spin of the Z_2 vortices or the full set of fusion rules.

First, observe that if we consider the following combination of the chiral scalar fields,

$$\varphi_{\pm} = \frac{1}{\sqrt{2}}(\varphi_1 \pm \varphi_2), \quad (45)$$

then the action becomes equivalent to the action of a free chiral scalar field, φ_+ , and that of the $U(1)/Z_2$ orbifold, described by φ_- . However, the edge theory is not simply a direct product of these two independent theories. The reason is that the fields φ_1 and φ_2 are compactified: in the case $q = 0$, we have

$$\varphi_i \sim \varphi_i + 2\pi R. \quad (46)$$

The compactification radius R is related to N through $R^2 = N$.

The spectrum of compactified bosons includes winding sectors; on a torus with spatial length L , the bosons can wind:

$$\varphi_i(x + L, t) = \varphi_i(x, t) + 2\pi R. \quad (47)$$

As a result, the fields φ_+ and φ_- are not independent and instead are tied together by their boundary conditions. We may think of such a theory, which is equivalent to $[U(1) \times U(1)]/Z_2$, as a theory denoted $U(1) \otimes U(1)/Z_2$. The \otimes indicates the nontrivial gluing together of the $U(1)$ theory and the $U(1)/Z_2$ orbifold theory. Let us consider the gluing together of these two theories from the point of view of the chiral operator algebra.

Observe that the edge theory for the $(NN0)$ states is generated by the electron operators

$$\Psi_{e1}(z) = e^{i\sqrt{N}\phi_1(z)}, \quad \Psi_{e2}(z) = e^{i\sqrt{N}\phi_2(z)}, \quad (48)$$

where ϕ_1 and ϕ_2 are free scalar bosons in a 1 + 1D chiral CFT. In terms of φ_{\pm} , we have

$$\Psi_{e1} = e^{i\sqrt{N/2}(\varphi_+ + \varphi_-)}, \quad \Psi_{e2} = e^{i\sqrt{N/2}(\varphi_+ - \varphi_-)}. \quad (49)$$

φ_+ describes the electrically charged sector of the edge theory, while φ_- describes the neutral sector of the edge theory. More generally, for the (ppq) states, the electron operators in the top and bottom layers are

$$\Psi_{e1} = e^{i\sqrt{N/2}\varphi_-} e^{i\sqrt{\frac{p+q}{2}}\varphi_+}, \quad \Psi_{e2} = e^{-i\sqrt{N/2}\varphi_-} e^{i\sqrt{\frac{p+q}{2}}\varphi_+}, \quad (50)$$

where recall that $N = p - q > 0$.

The chiral algebra of the $[U(1) \times U(1)]/Z_2$ theory should be the Z_2 invariant subalgebra of the $U(1) \times U(1)$ chiral algebra. Therefore we expect it to be generated by

$$\Psi_{e+} \propto \Psi_{e1} + \Psi_{e2} \propto \cos(\sqrt{N/2}\varphi_-) e^{i\sqrt{\frac{p+q}{2}}\phi_+}. \quad (51)$$

Studying the chiral algebra of Ψ_{e+} should yield the spectrum of edge states; representations of this chiral algebra should yield the topologically distinct sectors in the edge theory and should correspond to the topologically inequivalent quasiparticles in the bulk. The operator product expansion (OPE) $\Psi_{e+}^\dagger(z)\Psi_{e+}(w)$ contains only operators even in φ_- . In particular, it contains the operator $\cos(\sqrt{2N}\varphi)$, which is known to generate the chiral algebra of the $U(1)_{2N}/Z_2$ orbifold CFT.³¹ Note that the level $2N$ is related to the compactification radius of the boson; see Appendix A for a review. The chiral algebra of this orbifold CFT is denoted \mathcal{A}_N/Z_2 , where \mathcal{A}_N is the chiral algebra of the $U(1)_{2N}$ Gaussian theory. \mathcal{A}_N is generated by the operators $\{e^{\pm i\sqrt{2N}\varphi}\}$, and \mathcal{A}_N/Z_2 is the Z_2 invariant subalgebra of \mathcal{A}_N , which is generated by $\cos(\sqrt{2N}\varphi)$. Focusing on the neutral sector of these FQH edge theories, we see that the electron operators at the edge of the (ppq) states can generate the algebra \mathcal{A}_N , while the operator Ψ_{e+} can only generate the algebra \mathcal{A}_N/Z_2 .

The operator $\cos(\sqrt{N/2}\varphi)$ is difficult to work with for our purposes, but it is very closely related to the primary field ϕ_N^1 in the $U(1)/Z_2$ orbifold CFT [see Appendix A for a detailed discussion of the operator content in the $U(1)/Z_2$ CFT], which motivates us to use the following operator as the electron operator:

$$\Psi_e(z) = \phi_N^1(z) e^{i\sqrt{(p+q)/2}\phi_c(z)}. \quad (52)$$

This describes an FQH state at filling fraction $\nu = 2/(p+q)$. ϕ_N^1 is a primary field of the Z_2 orbifold chiral algebra with scaling dimension $N/4$, and its fusion rules with other primary fields are known, so it is more convenient to work with ϕ_N^1 than with $\cos(\sqrt{N/2}\varphi)$. We expect that both operators could, in principle, be used to generate the same edge spectrum. The chiral algebra of the electron operator is referred to as \mathcal{A}_e ; note that it contains the entire orbifold chiral algebra as a subalgebra, $\mathcal{A}_N/Z_2 \subset \mathcal{A}_e$.

Now we make the following conjecture for the edge theory. The properties of the chiral operators in the $[U(1) \times U(1)]/Z_2$ theory can be obtained by studying operators in the $U(1)/Z_2 \times U(1)$ CFT that are local, that is, have a single-valued OPE, with respect to the electron operator, Eq. (52). Two operators are topologically equivalent if they can be related by an operator in the electron chiral algebra. Practically, this means that the topologically distinct quasiparticle operators V_γ are of the form

$$V_\gamma = \mathcal{O}_\gamma e^{iQ_\gamma \sqrt{v^{-1}}\phi_c}, \quad (53)$$

where \mathcal{O}_γ is a chiral primary operator from the $U(1)_{2N}/Z_2$ orbifold CFT and determines the non-Abelian properties of the quasiparticle, and Q_γ determines the electric charge of the quasiparticle.

The quasiparticle operators in the edge theory yield all the topological properties of the bulk excitations. The scaling dimensions $h_\gamma = h_{\mathcal{O}_\gamma} + Q_\gamma^2/2\nu$ of the quasiparticle operators in the CFT are related to an important topological

quantum number of the bulk excitations: the quasiparticle twist, $\theta_\gamma = e^{2\pi i h_\gamma}$, which specifies the phase accumulated as a quasiparticle is rotated by 2π . The fusion rules of the quasiparticles in the bulk are identical to the fusion rules of the quasiparticle operators in the edge theory.

To summarize, the conjecture is that the properties of the chiral primary fields in the $[U(1) \times U(1)]/Z_2$ CFT can be obtained by instead considering the electron operator, Eq. (52), and embedding the electron chiral algebra \mathcal{A}_e into the chiral algebra of the $U(1)_{2N}/Z_2 \times U(1)$ CFT. This allows us to study representations of \mathcal{A}_e in terms of primary fields in the $U(1)_{2N}/Z_2 \times U(1)$ CFT.

V. QUASIPARTICLE CONTENT AND TOPOLOGICAL QUANTUM NUMBERS OF ORBIFOLD FQH STATES

Using the above prescription for finding the topologically inequivalent quasiparticle operators in CFT, we obtain the complete topological quantum numbers that such an edge theory describes. Remarkably, the topological properties obtained through this CFT prescription agree exactly with all the properties that we can compute from the $U(1) \times U(1) \rtimes Z_2$ CS theory and the slave Ising theory through completely different methods. Below, we first illustrate a simple way of understanding the results obtained from this edge theory in terms of the $U(1) \times U(1) \rtimes Z_2$ CS theory and in terms of the quasiparticle content of the (ppq) states. We then proceed to study some specific examples in more detail.

A. General properties

To illustrate the main ideas, we set $q = 0$. When $q = 0$, the orbifold FQH states have $N(N+7)/2$ topologically inequivalent quasiparticles [see Eq. (3)]. $2N$ quasiparticles have quantum dimension 1, $2N$ have quantum dimension \sqrt{N} , and $N(N-1)/2$ have quantum dimension 2.

Label the $d = 1$ and $d = \sqrt{N}$ quasiparticles A_l and B_l , respectively, for $l = 0, \dots, 2N-1$. Let us label the $N(N-1)/2$ quasiparticles with $d = 2$ as C_{mn} , where $m, n = 0, \dots, N-1$ and $m > n$. These quasiparticles have the properties listed in Table I.

We find that when N is even, the non-Abelian quasiparticles B_l have charge l/N , and when N is odd, the non-Abelian quasiparticles B_l have charge $(2l+1)/2N$.

TABLE I. General properties of quasiparticles in the orbifold FQH states for $q = 0$. Quasiparticles are labeled A_l and B_l for $l = 0, \dots, 2N-1$, and C_{mn} for $m, n = 0, \dots, N-1$ and $m > n$. A_l and C_{mn} quasiparticles are closely related to the Abelian quasiparticles of the $(NN0)$ states, while B_l quasiparticles are the Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ CS theory.

	Charge	Scaling dimension	Quantum dimension, d
A_l	$2l/N$	l^2/N	1
B_l	$l/N, N \text{ even}$ $(2l+1)/2N, N \text{ odd}$		\sqrt{N}
C_{mn}	$(m+n)/N$	$(m^2+n^2)/2N$	2

Now consider the bilayer ($NN0$) states, which have N^2 Abelian quasiparticles that can be labeled by two integers (m, n) , and where $(m, n) \sim (m + N, n) \sim (m, n + N)$ all refer to topologically equivalent quasiparticles. The electric charge of these quasiparticles is given by $(m + n)/N$ and the scaling dimension is given by $(m^2 + n^2)/2N$.

The quasiparticle content of the orbifold FQH states can now be interpreted in the following way. A_l for $l = 0, \dots, N - 1$ is the same as the quasiparticles (l, l) from the ($NN0$) states: they are all Abelian, and A_l carries the same charge and statistics as (l, l) . Furthermore, the orbifold FQH states have an additional neutral, Abelian boson that squares to the identity. In terms of the $U(1) \times U(1) \rtimes Z_2$ CS theory, it can be interpreted as the quasiparticle that carries Z_2 gauge charge. The Z_2 charged quasiparticle can fuse with the A_l for $l = 0, \dots, N - 1$ to yield the A_l for $l = N, \dots, 2N - 1$. The quasiparticles C_{mn} correspond to the Z_2 invariant combinations of (m, n) : $C_{mn} \sim (m, n) + (n, m)$, for $m \neq n$. This is clear in the edge theory, where these quasiparticle operators take the form $\cos(l\varphi_-/\sqrt{2N})e^{iQ\sqrt{v^{-1}}\phi_c}$.

Finally, the quasiparticles B_l correspond to Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ CS description. Alternatively, in the edge orbifold theory, they correspond to twist operators. There is a fundamental Z_2 vortex, say B_0 , and the other B_l can be obtained by fusing with the A or C quasiparticles. Note that when N is odd, the minimal quasiparticle charge in the orbifold states is carried by a Z_2 vortex and is given by $1/2N$. This is half of the minimal quasiparticle charge in the corresponding ($NN0$) states.

B. Examples

One of the simplest examples of the above properties is shown in Table III, which describes the quasiparticle content for $(N, q) = (3, 0)$. When $N = 3$, the orbifold FQH states are the same as the Z_4 parafermion FQH states at filling fraction $\nu = 2/(2q + 3)$. In this example, we clearly see three different families of quasiparticles, and each family forms a representation of a magnetic translation algebra.⁸ Notice that the quasiparticle $j \sim \partial\varphi$ is odd under the $\varphi \sim -\varphi$ transformation of the orbifold CFT, which is one way of seeing that this quasiparticle should carry Z_2 gauge charge in the $U(1) \times U(1) \rtimes Z_2$ CS description.

In Tables II and IV we list the quasiparticle content for the cases $(N, q) = (2, 0)$ and $(4, 0)$. These states are slightly more complicated than the $N = 3$ case because there are more than three representations of a magnetic translation algebra and there is not a one-to-one correspondence between the pattern-of-zeros sequences $\{n_l\}$ and topologically inequivalent quasiparticles. We study these features further in Section VII.

In Tables II–IV, we have also listed the occupation number sequences $\{n_l\}$ of each quasiparticle, which are defined as follows. If Ψ_e is the electron operator and V_γ is a quasiparticle operator, we obtain a sequence of integers $\{l_{\gamma;a}\}$ from the following OPEs:

$$\Psi_e(z)V_{\gamma;a}(w) \sim (z - w)^{l_{\gamma;a+1}} V_{\gamma;a+1} + \dots, \quad (54)$$

TABLE II. Quasiparticle operators of the $(N, q) = (2, 0)$ orbifold states, with filling fraction $\nu = 1$. The different representations of the magnetic translation algebra⁸ are separated by spaces. Q is the electric charge and h^{sc} and h^{ga} are the scaling dimensions of the orbifold primary field and the $U(1)$ vertex operator $e^{i\alpha\varphi_c}$, respectively. φ_c is a free scalar boson that describes the charge sector. $\{n_l\}$ is the occupation number sequence associated with the quasiparticle pattern of zeros.

Quasiparticles for $N = 2$ orbifold FQH state ($\nu = 1$)				
	Z_2 orbifold label	$\{n_l\}$	$h^{\text{sc}} + h^{\text{ga}}$	Quantum dimension
0	I	2 0	0 + 0	1
1	ϕ_N^1	0 2	1/2 + 0	1
2	j	0 2	1 + 0	1
3	ϕ_N^2	2 0	1/2 + 0 = 1/2	1
4	$\sigma_1 e^{i1/2\sqrt{v^{-1}}\phi_c}$	1 1	1/16 + 1/8	$\sqrt{2}$
5	$\tau_1 e^{i1/2\sqrt{v^{-1}}\phi_c}$	1 1	9/16 + 1/8	$\sqrt{2}$
6	σ_2	2 0	1/16 + 0	$\sqrt{2}$
7	τ_2	0 2	9/16 + 0	$\sqrt{2}$
8	$\cos\left(\frac{\varphi}{\sqrt{2}}\right) e^{i1/2\sqrt{v^{-1}}\phi_c}$	1 1	1/8 + 1/8	2

TABLE III. Quasiparticles in the $(N, q) = (3, 0)$ orbifold FQH state, at $\nu = 2/3$. This state is also called the Z_4 parafermion FQH state. The different representations of the magnetic translation algebra⁸ are separated by spaces. Q is the electric charge and h^{sc} and h^{ga} are the scaling dimensions of the orbifold primary field and the $U(1)$ vertex operator $e^{i\alpha\varphi_c}$, respectively. φ_c is a free scalar boson that describes the charge sector. $\{n_l\}$ is the occupation number sequence associated with the quasiparticle pattern of zeros.

Quasiparticles for $N = 3$ orbifold state ($\nu = 2/3$)				
	Z_2 orbifold label	$\{n_l\}$	$h^{\text{sc}} + h^{\text{ga}}$	Quantum dimension
0	$\mathbb{I} \sim \phi_N^1 e^{i\sqrt{3/2}\varphi_c}$	1 1 1 0 0 1	0 + 0	1
1	$e^{i2/3\sqrt{3/2}\varphi_c}$	1 1 1 1 0 0	0 + $\frac{1}{3}$	1
2	$e^{i4/3\sqrt{3/2}\varphi_c}$	0 1 1 1 1 0	0 + $\frac{4}{3}$	1
3	$j_r \sim \partial\varphi_r$	0 0 1 1 1 1	1 + 0	1
4	$j_r e^{i2/3\sqrt{3/2}\varphi_c}$	1 0 0 1 1 1	1 + $\frac{1}{3}$	1
5	$j_r e^{i4/3\sqrt{3/2}\varphi_c}$	1 1 0 0 1 1	1 + $\frac{4}{3}$	1
6	$\sigma_1 e^{i1/6\sqrt{3/2}\varphi_c}$	1 1 0 1 0 1	$\frac{1}{16} + \frac{1}{48}$	$\sqrt{3}$
7	$\sigma_1 e^{i5/6\sqrt{3/2}\varphi_c}$	1 1 1 0 1 0	$\frac{1}{16} + \frac{25}{48}$	$\sqrt{3}$
8	$\sigma_1 e^{i9/6\sqrt{3/2}\varphi_c}$	0 1 1 1 0 1	$\frac{1}{16} + \frac{27}{16}$	$\sqrt{3}$
9	$\tau_1 e^{i1/6\sqrt{3/2}\varphi_c}$	1 0 1 1 1 0	$\frac{9}{16} + \frac{1}{48}$	$\sqrt{3}$
10	$\tau_1 e^{i5/6\sqrt{3/2}\varphi_c}$	0 1 0 1 1 1	$\frac{9}{16} + \frac{25}{48}$	$\sqrt{3}$
11	$\tau_1 e^{i9/6\sqrt{3/2}\varphi_c}$	1 0 1 0 1 1	$\frac{9}{16} + \frac{27}{16}$	$\sqrt{3}$
12	$\cos\left(\frac{\varphi_r}{\sqrt{6}}\right) e^{i1/3\sqrt{3/2}\varphi_c}$	1 0 1 1 0 1	$\frac{1}{12} + \frac{1}{12}$	2
13	$\cos\left(\frac{\varphi_r}{\sqrt{6}}\right) e^{i\sqrt{3/2}\varphi_c}$	1 1 0 1 1 0	$\frac{1}{12} + \frac{3}{4}$	2
14	$\cos\left(\frac{\varphi_r}{\sqrt{6}}\right) e^{i5/3\sqrt{3/2}\varphi_c}$	0 1 1 0 1 1	$\frac{1}{12} + \frac{25}{12}$	2

TABLE IV. Quasiparticles for the $(N, q) = (4, 0)$ orbifold FQH states, at $\nu = 1/2$. The different representations of the magnetic translation algebra⁸ are separated by spaces. Q is the electric charge, and h^{sc} and h^{ga} are the scaling dimensions of the orbifold primary field and the $U(1)$ vertex operator $e^{i\alpha\varphi_c}$, respectively. φ_c is a free scalar boson that describes the charge sector. $\{n_l\}$ is the occupation number sequence associated with the quasiparticle pattern of zeros.

Quasiparticles for $N = 4$ orbifold FQH state ($\nu = 1/2$)				
	CFT label	$\{n_{\gamma;l}\}$	$h^{\text{sc}} + h^{\text{ga}}$	Quantum dimension
0	\mathbb{I}	2 0 0 0	0 + 0	1
1	$e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	0 2 0 0	0 + 1/4	1
2	ϕ_N^1	0 0 2 0	1 + 0	1
3	$\phi_N^1 e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	0 0 0 2	1 + 1/4	1
4	j	0 1 0 1	1 + 0	1
5	$j e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	1 0 1 0	1 + 1/4	1
6	ϕ_N^2	0 1 0 1	1 + 0	1
7	$\phi_N^2 e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	1 0 1 0	1 + 1/4	1
8	σ_1	0 1 0 1	1/16 + 0	2
9	$\sigma_1 e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	1 0 1 0	1/16 + 1/4	2
10	τ_1	0 1 0 1	9/16 + 0	2
11	$\tau_1 e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	1 0 1 0	9/16 + 1/4	2
12	$\sigma_2 e^{i1/4\sqrt{\nu^{-1}}\varphi_c}$	1 1 0 0	1/16 + 1/16	2
13	$\sigma_2 e^{i3/4\sqrt{\nu^{-1}}\varphi_c}$	0 1 1 0	1/16 + 9/16	2
14	$\tau_2 e^{i1/4\sqrt{\nu^{-1}}\varphi_c}$	0 0 1 1	9/16 + 1/16	2
15	$\tau_2 e^{i3/4\sqrt{\nu^{-1}}\varphi_c}$	1 0 0 1	9/16 + 9/16	2
16	$\cos(\frac{\varphi}{\sqrt{8}}) e^{i1/4\sqrt{\nu^{-1}}\varphi_c}$	1 1 0 0	1/16 + 1/16	2
17	$\cos(\frac{\varphi}{\sqrt{8}}) e^{i3/4\sqrt{\nu^{-1}}\varphi_c}$	0 1 1 0	1/16 + 9/16	2
18	$\cos(\frac{3\varphi}{\sqrt{8}}) e^{i1/4\sqrt{\nu^{-1}}\varphi_c}$	0 0 1 1	9/16 + 1/16	2
19	$\cos(\frac{3\varphi}{\sqrt{8}}) e^{i3/4\sqrt{\nu^{-1}}\varphi_c}$	1 0 0 1	9/16 + 9/16	2
20	$\cos(\frac{2\varphi}{\sqrt{8}})$	0 1 0 1	1/4 + 0	2
21	$\cos(\frac{2\varphi}{\sqrt{8}}) e^{i1/2\sqrt{\nu^{-1}}\varphi_c}$	1 0 1 0	1/4 + 1/4	2

where $V_{\gamma;a} = \Psi_e^a V_\gamma$ is a bound state of a electrons and a quasiparticle. The ellipsis (\dots) indicates terms of order $\mathcal{O}[(z - w)^{l_{\gamma;a}+1}]$. The integer $n_{\gamma;l}$ is defined as the number of a such that $l_{\gamma;a} = l$. In the limit of large l , $n_{\gamma;l}$ is periodic and it is the unit cell that characterizes a quasiparticle. For single-component states, these occupation number sequences have been studied from many points of view and have proven to be an important way of understanding the topological order of FQH states.^{6-9,33-35}

For $N \equiv p - q = 1$, the orbifold FQH phase is an Abelian phase. The Z_2 vortices, which are non-Abelian excitations for $N > 1$, have unit quantum dimension when $N = 1$ [see Eq. (7)]. The ground-state degeneracy on genus g surfaces is $[4(2p - 1)]^g$, which shows that in fact all quasiparticles have

unit quantum dimension. Moreover, the $U(1)_2/Z_2$ orbifold CFT is actually equivalent to the $U(1)_8$ CFT,³¹ which contains only Abelian quasiparticles. Since this is an Abelian phase, it must exist within the K -matrix classification of Abelian FQH phases.²⁷ What is the K matrix of the $N = 1$ orbifold states? The K matrix and charge vector \mathbf{q} are

$$K = \begin{pmatrix} 1+q & q-1 \\ q-1 & 5+q \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (55)$$

In Sec. VII we explain how to arrive at this result. Notice that this phase is actually a two-component bilayer state, so we expect that the edge theory would contain two electron operators, while in Eq. (52) we list only one electron operator. We explain this situation further in Sec. VII as well.

These $N = 1$ Abelian states are interesting because two-component Abelian states are all described by $U(1) \times U(1)$ CS theories:

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I \partial a^J + \frac{e}{2\pi} \mathbf{q}_I A \partial a_I. \quad (56)$$

Therefore, for the K matrix in Eq. (55), we have found that there is a different yet equivalent CS theory that describes the same phase. This other CS theory is the $U(1) \times U(1) \rtimes Z_2$ CS theory with the Lagrangian in Eq. (1) and with $p - q = 1$.

VI. PHASE TRANSITION FROM ORBIFOLD FQH STATES TO (ppq) BILAYER STATES

The phases described by the $U(1) \times U(1) \times Z_2$ and $U(1) \times U(1)$ CS theories differ by an extra Z_2 gauge symmetry, which suggests that the transition between these two phases is described by a Z_2 “gauge symmetry-breaking” transition. In this section we further elucidate this idea.

First, consider the slave Ising construction presented in Sec. III. There, we found that the difference between the Z_2 confined and the Z_2 deconfined phases is associated with the condensation of a Z_2 charged scalar field, s^z . When $\langle s_i^z \rangle = 0$, the system is in the Z_2 deconfined phase and the low-energy theory is the $U(1) \times U(1) \rtimes Z_2$ CS theory. When $\langle s_i^z \rangle \neq 0$, the low-energy theory is the $U(1) \times U(1)$ CS theory. This analysis suggests that these two phases are separated by a continuous phase transition and that the critical theory is simply a theory of the Ising field s_i^z coupled to a Z_2 gauge field. This transition has been well studied³⁶ and is known to be in the 3D Ising universality class.

Now let us arrive at the above conclusion through totally different arguments as well. From the CFT prescription for computing the quasiparticle operators, we observe that the orbifold FQH states always contain an electrically neutral, topologically nontrivial Abelian quasiparticle with integer scaling dimension. In the edge CFT, this quasiparticle is denoted $j \sim \partial\varphi_-$. j has trivial braiding properties with respect to itself because of its integer scaling dimension and is therefore a boson. It is another way of viewing the deconfined Ising spin s^z , so we expect it to carry Z_2 gauge charge. What happens when j condenses? The condensation of j drives a topological phase transition to a state with different topological order. Based on general principles,¹⁸ we can deduce that the

topological order of the resulting phase is precisely that of the (ppq) states. This works as follows.

Upon condensation, j becomes identified with the identity sector of the topological phase. Any topologically inequivalent quasiparticles that differed from each other by fusion with j will become topologically equivalent to each other after condensation. Furthermore, quasiparticles that were not local with respect to j will not be present in the low-energy spectrum of the theory after condensation. They become “confined.” Finally, if, before condensation, a quasiparticle γ fused with its conjugate both to the identity and to j , then after condensation γ splits into multiple topologically inequivalent quasiparticles. Otherwise, since j is identified with the identity after condensation, there would be two ways for γ to fuse with its conjugate to the identity, which is assumed not to be possible in a topological phase.

Applying these principles, we can see that the condensation of j yields the (ppq) states. As a simple example, consider the cases where $q = 0$. Some of the topological properties of the $q = 0$ orbifold FQH states were described in Sec. V A. When j condenses, we find that A_l becomes topologically identified with A_{l+N} . The quasiparticles labeled B_l become confined, because the OPE of the operator j with the operators B_l in the edge theory always have a branch cut and so the B_l are nonlocal with respect to j . Finally, the quasiparticles C_{mn} each split into two distinct quasiparticles. This leaves a total of N^2 Abelian quasiparticles whose topological properties all agree exactly with those of the $(NN0)$ states.

From the results of the $U(1) \times U(1) \rtimes Z_2$ CS theory, we find that j carries Z_2 gauge charge, proving that it is indeed associated with s^z in the slave Ising description. We arrive at this result by first studying the number of Z_2 noninvariant states in the presence of n pairs Z_2 vortices at fixed locations on a sphere [see Eq. (8)]. We observe that this number coincides exactly with the number of ways for n pairs of the fundamental non-Abelian quasiparticles and their conjugates to fuse to j . That is, we can use the CFT prescription to calculate the fusion rules

$$(B_l \times \bar{B}_l)^n = a_n \mathbb{I} + b_n j + \dots, \quad (57)$$

and we observe that b_n agrees exactly with the β_n in Eq. (8). This shows that j carries Z_2 gauge charge. This makes sense from the perspective of the low-energy theory, because the condensation of j yields a Higgs phase of the Z_2 sector and leaves us with the $U(1) \times U(1)$ CS theory, which describes the (ppq) states. Moreover, in the edge theory, $j \sim \partial\varphi_-$ is odd under the Z_2 transformation $\varphi_- \rightarrow -\varphi_-$, which is consistent with the fact that j carries the Z_2 gauge charge in the bulk; the Z_2 in the orbifold sector of the edge theory is the “same” Z_2 as the Z_2 gauge transformation that interchanges the two $U(1)$ gauge fields in the $U(1) \times U(1) \rtimes Z_2$ CS theory.

As the energy gap to creating excitations of j is reduced to 0, the low-energy theory near the transition must be that of a Z_2 gauged Ginzburg-Landau theory and the transition is therefore in the 3D Ising universality class.^{15,36}

This close connection between the topological properties of the orbifold FQH states and those of the bilayer (ppq) states provides additional strong evidence for why the CFT prescription presented in Sec. IV is correct and describes the same topological theory as the $U(1) \times U(1) \rtimes Z_2$ CS theory.

From the $U(1) \times U(1) \rtimes Z_2$ CS theory, we know that there must be a Z_2 Higgs transition to the (ppq) states, and so the topological quantum numbers of the orbifold phase must agree with the condensate-induced transition to the (ppq) states. The CFT prescription presented in Sec. IV provides us with such a consistent set of topological quantum numbers.

We note that while the edge between the orbifold FQH states and a topologically trivial phase will have protected edge modes, we do not expect protected edge modes at the edge between the orbifold states and the corresponding (ppq) states, because they differ by a Z_2 transition. As a simple check, note that the edge CFT for both states has central charge $c = 2$, so the edge between these two states would have zero thermal Hall conductance.

A. Anyon condensation and transition from (ppq) states to orbifold FQH states

The above discussion shows that we may understand the transition from the orbifold FQH states to the (ppq) states through the condensation of an electrically neutral boson, ultimately leading us to conclude that the transition is continuous and is in the 3D Ising universality class.

An interesting, though currently unresolved, question relates to how we should understand this phase transition from the other direction: starting from the (ppq) states and ending with the orbifold FQH states. Perhaps this transition can be understood as the condensation of the Abelian anyons of the (ppq) state into some collective state and driving a phase transition to a more complicated topological phase.

Starting from the (ppq) state, which can exist in the absence of interlayer tunneling, we expect that the orbifold FQH state can be stabilized for some range of interlayer tunneling. There are two reasons for this expectation. First, in the slave Ising construction we see that the fluctuations of the Z_2 charged boson are accompanied by interlayer density fluctuations. We can see this relation by noticing that the relative density difference between layers is (see Sec. III)

$$\begin{aligned} n_r &= \Psi_{e\uparrow}^\dagger \Psi_{e\uparrow} - \Psi_{e\downarrow}^\dagger \Psi_{e\downarrow} \\ &= \Psi_{e+}^\dagger \Psi_{e-} + \Psi_{e-}^\dagger \Psi_{e+} = s^z (c_+^\dagger c_- + c_-^\dagger c_+). \end{aligned} \quad (58)$$

In the slave-particle gauge theory, the states associated with the c fermions do not change as we tune through the transition, while the fluctuations of s^z become critical at the transition, thus leading to interlayer density fluctuations as well. Since the interlayer density fluctuations can physically be tuned by the interlayer tunneling, we expect that interlayer tunneling will be one of the material parameters that will help tune through this transition.

A second, closely related, consideration that suggests interlayer tunneling can tune through this transition is the following. In the absence of electron tunneling, we have a bilayer state with two electron operators, $\Psi_{e\uparrow}$ and $\Psi_{e\downarrow}$, for the two layers. As the interlayer tunneling is increased, there will be a single-particle gap between the symmetric and the antisymmetric orbitals that also increases. In the limit of large interlayer tunneling, we expect all of the electrons to occupy the symmetric orbitals; thus, the electron operator that we need to be concerned with in the limit of large interlayer tunneling

is $\Psi_{e+} \propto \Psi_{e\uparrow} + \Psi_{e\downarrow}$. Now, the $(NN0)$ states can be obtained from a parton construction by setting

$$\Psi_{e\uparrow} = \psi_1 \cdots \psi_N, \quad \Psi_{e\downarrow} = \psi_{N+1} \cdots \psi_{2N}, \quad (59)$$

where ψ_i are charged e/N fermions that form $\nu = 1$ IQH states. The gauge group here is $SU(N) \times SU(N)$, and integrating out the partons gives the $SU(N)_1 \times SU(N)_1$ CS theory. In the phase where the interlayer tunneling is high, so that we effectively have a single electron operator $\Psi_e = \Psi_{e+} = \frac{1}{\sqrt{2}}(\psi_1 \cdots \psi_N + \psi_{N+1} \cdots \psi_{2N})$, and we ignore the other electron operator completely, the gauge group for $N > 2$ is $SU(N) \times SU(N) \times Z_2$, which, after integrating out the partons, is equivalent to the $U(1) \times U(1) \rtimes Z_2$ CS theory for the orbifold states.

The above two considerations suggest that interlayer tunneling will allow us to tune through the phase transition. Since the important dimensionless parameters are t/V_{inter} and $V_{\text{inter}}/V_{\text{intra}}$, where $V_{\text{inter/intra}}$ are the inter-/intralayer Coulomb repulsions and t is the interlayer tunneling, we expect that tuning the Coulomb repulsions should also be able to stabilize the orbifold state. Now, as discussed in Refs. 15 and 16, observe that bilayer FQH states have a particular kind of neutral excitation called a fractional exciton (f exciton), which is the bound state of a quasiparticle in one layer and a quasihole in the other layer. The f exciton carries fractional statistics; when the interlayer Coulomb repulsion is increased, the gap to the f exciton can be decreased to 0. This means that interlayer repulsion can drive a phase transition between two FQH states at the same filling fraction. This leads us to suspect that this anyon condensation can be related to the formation of the orbifold FQH states, because the orbifold states can also be obtained by tuning the Coulomb repulsions in the presence of interlayer tunneling. Of course, anyons can condense in different ways, and likewise we expect that there will be other microscopic interactions that will determine precisely which phases will be obtained when the interlayer tunneling/repulsion is tuned.

VII. IDEAL WAVE FUNCTIONS AND THE VERTEX ALGEBRA APPROACH TO THE ORBIFOLD FQH STATES

In the sections above, we have introduced and developed a theory for a novel series of non-Abelian FQH states: the orbifold FQH states. These are parameterized by two integers, (N, q) . They occur at filling fraction $\nu = 2/(N + 2q)$ and are separated from the (ppq) states (where $N = p - q$) by a continuous Z_2 phase transition.

For $N = 3$, these states are equivalent to the Z_4 parafermion states, which have an ideal wave function description.⁵ In other words, if we take the electron operator in Eq. (52) and evaluate the correlator,

$$\Psi(\{z_i\}) \sim \langle V_e(z_1) \cdots V_e(z_N) \rangle, \quad (60)$$

we will obtain a wave function that describes an incompressible FQH state. However, carrying out this construction for $N \neq 3$ will not yield a wave function that describes an incompressible FQH state. In fact, for $N > 3$, the pattern of zeros of the electron operator V_e corresponds to certain problematic,

or sick, pattern-of-zeros solutions: pattern-of-zeros solutions whose relevance to describing gapped topological phases had been uncertain because their associated ideal wave functions always appear to be gapless.¹⁰

In the following, we study the orbifold FQH states from the pattern-of-zeros and ideal wave function point of view. The main conclusion to draw is that the sick pattern-of-zeros solutions are still relevant to quantitatively characterizing topological order in FQH states, even when naively it appears as though the corresponding ideal wave function is gapless! In the analysis below, we will see how the orbifold FQH states provide profound lessons for the conceptual foundation of the pattern-of-zeros/vertex algebra approach to constructing ideal wave functions.

A. Review of the vertex algebra/conformal field theory approach

A wide class of FQH states can be described by ideal wave functions that are exact zero-energy ground states of Hamiltonians with interactions that are either δ functions or derivatives of δ functions. Such ideal Hamiltonians select for certain properties of the ground state wave functions, such as the order of the zeros in the wave function as various numbers of particles approach each other.

The ideal wave functions that we currently understand can all be written in terms of a correlation function of vertex operators:

$$\Psi(\{z_i\}) \sim \langle V_e(z_1) \cdots V_e(z_N) \rangle, \quad (61)$$

where V_e is a certain operator in a 2D chiral CFT, called the electron operator. The wave function $\Psi(\{z_i\})$ can be specified by simply specifying the operator algebra generated by the electron operator:

$$\begin{aligned} V_e(z)V_e(w) &\sim C_{ee\mathcal{O}_1}(z-w)^{h_{\mathcal{O}_1}-2h_e}\mathcal{O}_1 + \cdots \\ V_e(z)\mathcal{O}_1(w) &\sim C_{e\mathcal{O}_1\mathcal{O}_2}(z-w)^{h_{\mathcal{O}_2}-h_e-h_{\mathcal{O}_1}}\mathcal{O}_2 + \cdots \\ &\vdots \end{aligned} \quad (62)$$

This operator algebra is called a vertex algebra. Using this vertex algebra, the correlation function (61) can be evaluated by successively replacing products of two neighboring operators by a sum of single operators. For the result to be independent of the order in which these successive fusions are evaluated, there need to be various consistency conditions on the vertex algebra. In some cases, specifying the scaling dimension h_e and the filling fraction ν is enough to completely specify the vertex algebra, because the structure constants C_{abc} can be obtained through the various consistency conditions.³⁷ In these cases, an ideal Hamiltonian that selects for the pattern of zeros is believed to have a unique zero-energy wave function of highest density. Otherwise, one needs to find a way to use the Hamiltonian to select also for a certain choice of structure constants C_{abc} .

The quasiparticle wave functions can also be written as correlators:

$$\Psi_\gamma(\eta; \{z_i\}) \sim \langle V_\gamma(\eta)V_e(z_1) \cdots V_e(z_n) \rangle, \quad (63)$$

where V_γ is a ‘‘quasiparticle’’ operator and η is the position of the quasiparticle. To evaluate these wave functions, we need

to specify the operator algebra involving the quasiparticle operators. For the quasiparticle wave function, Eq. (63), to be single-valued in the electron coordinates, the allowed quasiparticle operators must be local with respect to the electron operators: their operator product expansion with the electron operator must not contain any branch cuts. Two quasiparticle operators are topologically equivalent if they are related by electron operators. By solving the consistency conditions on the vertex algebra, we can obtain the constraints on the allowed quasiparticles. In the vertex algebra approach to FQH states, we take all solutions of the consistency conditions to be valid quasiparticle operators, so there can be a finite number of quasiparticles only if the number of solutions to the consistency conditions is finite. This is equivalent to the expectation that ideal Hamiltonians cannot selectively pick some of the quasiparticle vertex algebra solutions as allowed zero-energy states and not others. This expectation is fulfilled in all known FQH states that can be described by ideal Hamiltonians and ideal wave functions.

When the ideal Hamiltonian can uniquely select for the zero-energy wave function of highest density, and there are a finite number of solutions to the quasiparticle consistency conditions, and the vertex algebra is unitary³⁸ we believe that these model wave functions belong to an incompressible FQH phase. Its topological properties are dictated by the properties of the quasiparticle operators in the CFT. Such is the case for the Read-Rezayi states and some of their generalizations.^{4–6,39–41} Remarkably, it is also the case that the edge CFT is the same as the CFT whose correlation function yields the ideal wave function.

For some other choices of vertex algebra, there are an infinite number of solutions to the associativity conditions for a quasiparticle.¹⁰ This situation means that the corresponding ideal wave function likely does not describe a gapped phase.

The orbifold FQH states are interesting because if we try to use their edge CFT to construct single-component ideal wave functions, we find that for $N > 3$, the corresponding ideal wave function is gapless. The vertex algebra of the electron operator allows for an infinite number of quasiparticle solutions, indicating the gapless nature of the ideal wave function.¹⁰ The case $N = 3$ is special: the pattern of zeros of the electron operator uniquely fixes the ground-state wave function and there are a finite number of quasiparticle solutions for the vertex algebra: this corresponds to the Z_4 parafermion FQH states and it possesses a well-behaved single-layer ideal wave function. For $N = 1, 2$, we find that the single-layer wave function is gapped but does not have the topological properties of the orbifold states; to have a description of these states in terms of ideal wave functions, we are forced to view the orbifold FQH states as double-layer states.

To shed light on the pattern-of-zeros/vertex algebra approach to constructing FQH states, we study the orbifold FQH states from this point of view. The analysis below suggests that while some of the apparently sick pattern-of-zeros/vertex algebra solutions may not describe gapped FQH phases, they lie near a critical point and can be driven to a nearby incompressible phase—the orbifold FQH state—by applying certain perturbations to the ideal Hamiltonian. In the vertex algebra framework, this corresponds to enlarging the vertex algebra by incorporating additional local operators.

B. Orbifold FQH states viewed through vertex algebra

In Sec. IV we explained that the electron operator in the orbifold FQH edge theory is given by the operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{v-1}\phi_c}, \quad (64)$$

where ϕ_N^1 is an operator from the $U(1)_{2N}/Z_2$ CFT and has scaling dimension $h_{\phi_N^1} = N/4$. When N is even, we have the following fusion rule:

$$\phi_N^1 \times \phi_N^1 = \mathbb{I}. \quad (65)$$

When N is odd, we have

$$\phi_N^1 \times \phi_N^1 = j, \quad j \times j = \mathbb{I}. \quad (66)$$

These fusion rules denote fusion between representations of the orbifold chiral algebra \mathcal{A}_N/Z_2 . The identity representation still contains an infinite set of Virasoro representations, labeled by the Virasoro primary fields $\cos(l\sqrt{2N}\varphi)$, for integer l .

Our task is to study the pattern of zeros of these electron operators, V_e , compare with results from the pattern-of-zeros approach^{6–9} and with the vertex algebra approach,¹⁰ and try to make sense of any discrepancies. Since the discussion depends on the choice of N , we study various choices of N individually. We note that the pattern of zeros that we calculate from the electron operator, using the prescription in Sec. IV, assumes that the highest weight field appears in the OPEs if they are allowed by the fusion rules. In other words, the structure constants involving the highest weight fields are assumed to be nonzero. This is consistent with cases in which the Z_2 orbifold vertex algebra is known (e.g., for $N = 3$, because of the relation to Z_4 parafermion CFT) and can perhaps be viewed as a consequence of the naturality theorem for rational CFTs.⁴²

1. $N = 1$

Here the electron operator is given by

$$V_e(z) = \phi_N^1 e^{i\sqrt{1/2+q}\phi_c(z)}, \quad (67)$$

and ϕ_N^1 has scaling dimension $1/4$. The pattern of zeros associated with $V_e(z)$ is the pattern of zeros of the Laughlin $\nu = 1/(q+1)$ wave function, which describes a state with a different topological order than the $N = 1$ orbifold FQH states. An ideal Hamiltonian that selects for such a pattern of zeros will actually yield the Laughlin $\nu = 1/(q+1)$ state and not the $N = 1$ orbifold FQH state.

To obtain an ideal wave function for the orbifold FQH state, we need to reinterpret the system as a bilayer system. This means that we need to specify a second electron operator. The second electron operator will resolve the difference between $e^{i\sqrt{q+1}\phi_c(z)}$ —whose correlation function yields the Laughlin states—and $V_e(z) = \phi_N^1 e^{i\sqrt{1/2+q}\phi_c(z)}$, because these two operators will have a different pattern of zeros as viewed by the second electron operator. Another way to think about this is in terms of the ideal Hamiltonian. When the Hilbert space is enlarged to that of a double-layer system, the original ideal Hamiltonian—which only selects for the way wave functions go to 0 when one flavor of particles comes together—will be gapless. It can be modified by adding additional terms that

also select for the pattern of zeros involving the other flavor of particles. This modified Hamiltonian may then be gapped.

Returning to the vertex algebra language, notice that it suffices to add an electrically neutral bosonic operator V_o to the chiral algebra; then the composite operator $V_e V_o$ will be considered the second electron operator. To do this, it is helpful to observe that the $U(1)_2/Z_2$ CFT is actually dual to the $U(1)_8$ CFT, whose chiral algebra is generated by the operators $e^{\pm i\sqrt{8}\phi_n(z)}$, where ϕ_n is a free chiral boson. The operators ϕ_N^1 and ϕ_N^2 are then equivalent to the operators $e^{\pm i\phi_n/\sqrt{2}}$, both of which have scaling dimension $1/4$. This suggests that we should seek an operator of the form

$$V_o = e^{il\sqrt{2}\phi_n}, \quad (68)$$

because for any integer l it is local with respect to $e^{\pm i\phi_n/\sqrt{2}} e^{i\sqrt{1/2+q}\phi_c}$ and it is bosonic.

For each l , we can design an ideal bilayer Hamiltonian so that the bilayer wave function

$$\Psi(\{z_i\}, \{w_i\}) \sim \left\langle \prod_i V_{e1}(z_i) V_{e2}(w_i) \right\rangle \quad (69)$$

is an exact zero-energy ground state and the unique one of highest density. Here, $V_{e1} = e^{i\frac{1}{\sqrt{2}}\phi_n(z)} e^{i\sqrt{1/2+q}\phi_c(z)}$ and $V_{e2} = V_{e1} V_o = e^{i\frac{2l+1}{\sqrt{2}}\phi_n(z)} e^{i\sqrt{1/2+q}\phi_c(z)}$. These states correspond to bilayer Abelian states with K matrix and charge vector

$$K = \begin{pmatrix} q+1 & q+l+1 \\ q+l+1 & q+1+2l+l^2 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (70)$$

The case $l=1$ corresponds to the (ppq) states, while the case $l=2$ corresponds to the orbifold FQH states with $N=1$. We comment on other choices of l in Appendix B.

The $l=1$ (ppq) state and the $l=2$ orbifold state with $N=1$ are connected by a continuous phase transition. In the $l=2$ orbifold state, the operator $e^{i\sqrt{2}\phi_n}$ is a topologically nontrivial neutral boson that squares to V_o , which lives in the identity sector. When this neutral boson $e^{i\sqrt{2}\phi_n}$ condenses, it is added to the identity sector and we obtain the (ppq) states.

Therefore, we see that the original gapless ideal Hamiltonian can be perturbed to many different incompressible phases. The critical point contains many different bosons that can be condensed; condensing a particular one will yield a particular gapped FQH state. From the vertex algebra point of view, there are many different bosonic operators that can be added to the vertex algebra. One particular choice ($l=2$) will yield the $N=1$ orbifold states, while another choice ($l=1$) will yield the (ppq) states.

2. $N=2$

The case $N=2$ is similar to the case $N=1$ in that these orbifold states also need to be interpreted as bilayer states for the ideal Hamiltonian to yield the orbifold FQH phase. If we take the electron operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{q+1}\phi(z)} \quad (71)$$

for $N=2$, then we see that it has the same pattern of zeros as the Pfaffian ground-state wave function at $\nu=1/(q+1)$ (see, e.g., Table II). To construct an ideal wave function for the orbifold FQH phase, we need to reinterpret the system as a two-component phase, which again means adding a second electron operator to the chiral algebra. We leave a detailed analysis of this for future work.

3. $N=3$

For $N=3$, we find that the electron operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{3/2+q}\phi(z)} \quad (72)$$

has the same pattern of zeros as the Z_4 parafermion wave functions, which are known to be exact ground states of single-layer ideal Hamiltonians. The topological order of the Z_4 parafermion states is that of the orbifold states with $N=3$. Thus for $N=3$, the ideal wave functions and ideal Hamiltonians do properly describe the orbifold phases.

4. $N=4$

The $N=4$ case is the first highly nontrivial example that we encounter. The pattern of zeros of the electron operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{2+q}\phi(z)} \quad (73)$$

corresponds to the pattern of zeros associated with multiplying a $\nu=1$ Pfaffian wave function by a $\nu=1/(q+1)$ Pfaffian wave function. For $q=0$, the pattern of zeros is simply that of the square of the $\nu=1$ Pfaffian wave function, which is called the Haffnian wave function:

$$\begin{aligned} \Phi_{\text{Haffnian}} &= \left[Pf \left(\frac{1}{z_i - z_j} \right) \right]^2 \prod_{i<j} (z_i - z_j)^2 \\ &= \mathcal{S} \left(\frac{1}{(z_i - z_j)^2} \right) \prod_{i<j} (z_i - z_j)^2. \end{aligned} \quad (74)$$

Here, \mathcal{S} denotes symmetrization: $\mathcal{S}(M_{ij}) = \sum_P M_{P(1)P(2)} \cdots M_{P(N_e-1)P(N_e)}$, where \sum_P is the sum over all permutations of N_e elements. This pattern of zeros was studied in detail through the vertex algebra framework in Ref. 10; the vertex algebra there was named $Z_2|Z_2$ vertex algebra. It was found that the structure constants for one class of quasiparticles come with a free continuous parameter, indicating that the ideal wave function is likely gapless (see also Ref. 38 for a similar discussion). This conclusion of gaplessness is corroborated from a totally different analysis:⁴³ the Haffnian wave function corresponds to a critical point of d -wave paired composite fermions.

However, the $N=4$ orbifold FQH states indeed exist as gapped FQH states, and in particular, in Table IV we see that many of the quasiparticle patterns of zeros are repeated: there is not a one-to-one correspondence between the pattern of zeros and the topologically distinct quasiparticles. In the following, we describe how to understand these results through the vertex algebra framework.

From Ref. 10 we learn that one set of sequences $\{n_{\gamma,j}\}$ is associated with operators whose structure constants can take on any continuous parameter, α . For certain discrete values of α , the associated quasiparticle is a boson, that is, it is local with respect to itself, and may also be local with respect to

the electron operator. In such a case, this operator can and should be added to the chiral vertex algebra and treated as a second electron operator. From the point of view of the ideal Hamiltonian, this is like adding a perturbation so that the system is driven away from the critical point and into a nearby incompressible phase. The perturbation should be viewed as driving the condensation of a bosonic operator, which adds a second component to the FQH state.

Since this nearby incompressible phase should now be viewed as a two-component state, it thus should have an ideal wave function description in terms of a double-layer state, described by the enlarged chiral algebra. Note that there may be several different, mutually exclusive choices for which operator to add to the chiral algebra; equivalently, there may be several different directions in which to perturb the ideal Hamiltonian, each of which leads to a different incompressible phase. When the chiral algebra is enlarged in such a way, then the continuous set of quasiparticles described by α will not all be allowed, and instead only a finite, discrete set of them will be distinct and local with respect both to the original electron operator and the second electron operator. From this point of view we can now understand why the pattern of zeros for the other quasiparticles may appear multiple times, even when there was a unique solution for the structure constants: there is now an additional electron operator that may resolve the difference between two quasiparticles that the first electron operator could not distinguish between.

Let us study this more concretely for the case $q = 0$ using the vertex algebra framework. The vertex algebra generated by the electron operator, Eq. (73), contains two pieces, the “charge” part, described by the $U(1)$ vertex operator $e^{i\sqrt{2}\phi_c}$, and the “neutral” part, described by the operator ϕ_N^1 . For $N = 4$, ϕ_N^1 is an operator with unit scaling dimension, and its vertex algebra is equivalent to the vertex algebra of the current operator $j \sim \partial\varphi$ in a free boson theory. Thus as far as the vertex algebra of the electron operator is concerned, the electron operator may be written as

$$V_e = j e^{i\sqrt{2}\phi_c}. \quad (75)$$

For an operator \mathcal{O} to be an allowed representation of the vertex algebra, it has to satisfy various consistency conditions, such as the generalized Jacobi identity. In Ref. 10, the allowed representations of the algebra generated by j were systematically studied (without embedding it into a free boson theory). It was found that there is one operator, σ , with scaling dimension $1/16$ and another continuous set of quasiparticles with continuously varying scaling dimensions. These operators are familiar when the algebra is embedded into a free boson theory: σ is the well-known twist operator, whose insertion at a point in space induces a branch cut around which $\varphi \rightarrow -\varphi$. The continuously varying set of operators corresponds to the operators $e^{i\alpha\varphi}$. We see that φ is uncompactified, because all operators of the form $e^{i\alpha\varphi}$ are local with respect to j and therefore correspond to distinct, allowed representations of the electron chiral algebra.

Now notice that $e^{i\alpha\varphi}$ is bosonic when it has integer scaling dimension: $h_\alpha = \alpha^2/2 \in \mathbb{Z}$. Suppose that we add the bosonic operator $\cos(\sqrt{8}\varphi)$ to the vertex algebra. Intuitively, we expect that this will cause only a discrete set of the operators $e^{i\alpha\varphi}$

to now be local with respect to the enlarged vertex algebra, and only a finite set of them will correspond to distinct representations of the algebra. Put another way, inclusion of $\cos(\sqrt{8}\varphi)$ into the vertex algebra has the effect of essentially compactifying the boson φ , which then quantizes the possible values of α . To find all of the allowed, distinct quasiparticles, we want to find all solutions to the consistency conditions¹⁰ for allowed quasiparticle operators for this enlarged algebra. Here, we do not solve these conditions and, instead, save this analysis for future work. Instead, we verify this picture semirigorously using the following arguments.

The operator $\cos(\sqrt{8}\varphi)$ is known to generate the chiral algebra of the $U(1)_8/Z_2$ orbifold CFT. The operator content of the $U(1)_8/Z_2$ CFT is known, so we assume that such an operator content is in one-to-one correspondence with the distinct, allowed operators that satisfy the consistency conditions of the vertex algebra generated by $\cos(\sqrt{8}\varphi)$. This is a reasonable assumption, however, it has not been verified because the operator content of the $U(1)/Z_2$ orbifold CFT was derived using considerations of modular invariance, and not through directly studying the representation theory of the chiral algebra by solving consistency conditions. For example, the $U(1)/Z_2$ CFT contains two twist fields, σ_1 and σ_2 (and their counterparts τ_1 and τ_2 , where $j \times \sigma_i = \tau_i$). The fact that there must be two twist fields can be understood through considerations of modular invariance of the torus partition function, which includes both holomorphic and antiholomorphic sectors of the theory. That there should be two sets of twist fields is less obvious from the point of view of solving consistency conditions of the vertex algebra.

In light of the above assumption, it is now clear what we should do. The allowed quasiparticles will be of the form

$$V_{qp} = \mathcal{O} e^{iQ\sqrt{2}\phi_c}, \quad (76)$$

where \mathcal{O} is a primary operator from the $U(1)_{2N}/Z_2$ orbifold theory. We need to find all possible operators V_{qp} that are local with respect to the operators $V_e = j e^{i\sqrt{2}\phi_c}$ and $\cos(\sqrt{8}\varphi)$. All operators of the form (76) are already local with respect to $\cos(\sqrt{8}\varphi)$ because they form representations of the algebra generated by $\cos(\sqrt{8}\varphi)$, so we only need to worry about their being local with respect to $V_e = j e^{i\sqrt{2}\phi_c}$. Remarkably, carrying this out yields all of the topological properties of the $N = 4$ orbifold states. In particular, we obtain the same results as we did using the prescription used earlier in Secs. IV and V (see Table V and compare it with Table IV). Note that while the CFT labeling of the operators is different, the two prescriptions yield exactly the same topological properties. This agreement is highly nontrivial and only works for $N = 4$, because only for $N = 4$ is the algebra of $j e^{i\sqrt{2}\phi_c}$ the same as the algebra of $\phi_N^1 e^{i\sqrt{2}\phi_c}$.

This adds evidence to the picture presented here, where the orbifold FQH states can be interpreted through the vertex algebra language as though an additional bosonic operator has been added to the chiral algebra. To confirm this picture more rigorously, we would need to systematically solve the consistency conditions on the vertex algebra generated by $j e^{i\sqrt{2}\phi_c}$ and $\cos(\sqrt{8}\varphi)$ and show that the quasiparticle solutions and their properties coincide with those presented here.

TABLE V. Quasiparticles obtained for $N = 4$ orbifold states by embedding the vertex algebra into the $U(1)/Z_2 \times U(1)$ orbifold CFT. This vertex algebra contains the electron operator, which is set to be $V_e = j e^{i\sqrt{2}\varphi_c}$, and the operator $\cos(\sqrt{8}\varphi)$.

$U(1)_{2N}/Z_2$ orbifold FQH states, with $V_e = j e^{i\sqrt{2}\varphi_c}$				
	CFT operator	$\{n_l\}$	$h^{\text{sc}} + h^{\text{ga}}$	Quantum dimension
0	\mathbb{I}	2 0 0 0	0 + 0	1
1	$e^{i1/2\sqrt{2}\varphi_c}$	0 2 0 0	0 + 1/4	1
2	$e^{i\sqrt{2}\varphi_c}$	0 0 2 0	0 + 1	1
3	$e^{i3/2\sqrt{2}\varphi_c}$	0 0 0 2	0 + 9/4	1
4	ϕ_N^1	0 1 0 1	1 + 0	1
5	$\phi_N^1 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	1 + 1/4	1
6	ϕ_N^2	0 1 0 1	1 + 0	1
7	$\phi_N^2 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	1 + 1/4	1
8	ϕ_1	0 1 0 1	1/16 + 0	2
9	$\phi_1 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	1/16 + 1/4	2
10	ϕ_3	0 1 0 1	9/16 + 0	2
11	$\phi_3 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	9/16 + 1/4	2
12	$\sigma_1 e^{i1/4\sqrt{2}\varphi_c}$	1 1 0 0	1/16 + 1/16	2
13	$\sigma_1 e^{i3/4\sqrt{2}\varphi_c}$	0 1 1 0	1/16 + 9/16	2
14	$\tau_1 e^{i1/4\sqrt{2}\varphi_c}$	0 0 1 1	9/16 + 1/16	2
15	$\tau_1 e^{i3/4\sqrt{2}\varphi_c}$	1 0 0 1	9/16 + 9/16	2
16	$\sigma_2 e^{i1/4\sqrt{2}\varphi_c}$	1 1 0 0	1/16 + 1/16	2
17	$\sigma_2 e^{i3/4\sqrt{2}\varphi_c}$	0 1 1 0	1/16 + 9/16	2
18	$\tau_2 e^{i1/4\sqrt{2}\varphi_c}$	0 0 1 1	9/16 + 1/16	2
19	$\tau_2 e^{i3/4\sqrt{2}\varphi_c}$	1 0 0 1	9/16 + 9/16	2
20	ϕ_2	0 1 0 1	1/4 + 0	2
21	$\phi_2 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	1/4 + 1/4	2

Note that since we now have two electron operators, the full pattern-of-zeros characterization should be described by the sequence $\{S_{\vec{a}}\}$, where \vec{a} is now a 2D vector.⁹ Therefore, the results in Table V do not display this full pattern-of-zeros/vertex algebra data and instead only display the pattern of zeros as seen with the electron operator $j e^{i\sqrt{2}\varphi_c}$.

Now that we have two electron operators, we should be able to obtain a double-layer ideal wave function for these $N = 4$ states and an associated ideal Hamiltonian. We save an in-depth study of these issues for future work. Based on the considerations presented here, we expect that the vertex algebra generated by the two electron operators has a unique, finite set of solutions for the quasiparticle structure constants, and therefore that there is a corresponding gapped ideal Hamiltonian.

5. $N = 5$

The $\nu = 2/5$ orbifold FQH state, with $(N, q) = (5, 0)$, is a fermionic state; a bosonic analog can be constructed at $\nu =$

TABLE VI. Properties of the bosonic $(N, q) = (5, -1)$ orbifold states, at $\nu = 2/3$. Compare this to results with the $Z_2|Z_4$ simple-current vertex algebra and pattern-of-zeros solutions studied in Ref. 10. $\phi_k = \cos(k/\sqrt{2N}\varphi)$.

Quasiparticles for $N = 5$ orbifold state ($\nu = 2/3$)				
		$\{n_{\gamma;l}\}$	$h^{\text{sc}} + h^{\text{ga}}$	Quantum dimension
0	\mathbb{I}	2 0 2 0 0 0	0 + 0	1
1	$e^{i2/3\sqrt{v^{-1}}\varphi_c}$	0 2 0 2 0 0	0 + 1/3	1
2	$\phi_N^2 e^{i1/3\sqrt{v^{-1}}\varphi_c}$	0 0 2 0 2 0	5/4 + 1/12	1
3	j	0 0 0 2 0 2	$h = 1 + 0$	1
4	$j e^{i2/3\sqrt{v^{-1}}\varphi_c}$	2 0 0 0 2 0	1 + 1/3	1
5	$\phi_N^1 e^{i1/3\sqrt{v^{-1}}\varphi_c}$	0 2 0 0 0 2	5/4 + 1/12	1
6	$\phi_1 e^{i1/3\sqrt{v^{-1}}\varphi_c}$	2 0 0 2 0 0	1/20 + 1/12	2
7	ϕ_4	0 2 0 0 2 0	4/5	2
8	$\phi_4 e^{i2/3\sqrt{v^{-1}}\varphi_c}$	0 0 2 0 0 2	4/5 + 1/3	2
9	ϕ_2	1 0 1 1 0 1	1/5 + 0	2
10	$\phi_2 e^{i2/3\sqrt{v^{-1}}\varphi_c}$	1 1 0 1 1 0	1/5 + 1/3	2
11	$\phi_3 e^{i1/3\sqrt{v^{-1}}\varphi_c}$	0 1 1 0 1 1	9/20 + 1/12	2
12	$\sigma_1 e^{i1/2\sqrt{v^{-1}}\varphi_c}$	1 1 1 0 1 0	1/16 + 3/16	$\sqrt{5}$
13	$\sigma_2 e^{i1/6\sqrt{v^{-1}}\varphi_c}$	0 1 1 1 0 1	1/16 + 1/48	$\sqrt{5}$
14	$\sigma_2 e^{i5/6\sqrt{v^{-1}}\varphi_c}$	1 0 1 1 1 0	1/16 + 25/48	$\sqrt{5}$
15	$\tau_1 e^{i1/2\sqrt{v^{-1}}\varphi_c}$	0 1 0 1 1 1	9/16 + 3/16	$\sqrt{5}$
16	$\tau_2 e^{i1/6\sqrt{v^{-1}}\varphi_c}$	1 0 1 0 1 1	9/16 + 1/48	$\sqrt{5}$
17	$\tau_2 e^{i5/6\sqrt{v^{-1}}\varphi_c}$	1 1 0 1 0 1	9/16 + 25/48	$\sqrt{5}$

2/3. In Tables VI and VII, we list the properties of these states.

The electron operator is

$$V_e = \phi_N^1 e^{i\sqrt{q+5/2}\varphi_c}, \quad (77)$$

where now ϕ_N^1 has scaling dimension $h_1^{\text{sc}} = 5/4$. The pattern of zeros of this electron operator has also been studied in detail in Ref. 10, in the context of so-called $Z_2|Z_4$ simple-current vertex algebra. We briefly discuss this $N = 5$ case because it contains some novel features that did not arise in the $N = 4$ analysis. In this $N = 5$ case, we see that several of the quasiparticle pattern-of-zeros solutions that are allowed by consistency conditions⁷ do not appear (compare Table VI with Table VII in Ref. 10).

This may be interpreted in the following way. The single-layer ideal wave function with the pattern of zeros of the operator in Eq. (77) is gapless, for the same reason that the $N = 4$ case was gapless: the structure constants of the vertex algebra for quasiparticles have a continuous set of solutions. Driving the ideal wave function away from the critical point, as discussed in the $N = 4$ example, corresponds to adding additional perturbations in the ideal Hamiltonian and, from the perspective of the vertex algebra, amounts to adding additional local operators to the chiral algebra. The quasiparticles must all be local with respect to this enlarged vertex algebra, however,

TABLE VII. Properties of the fermionic $(N, q) = (5, 0)$ orbifold states, at $\nu = 2/5$. $\phi_k = \cos(k/\sqrt{2N}\varphi)$.

Quasiparticles for $N = 5$ orbifold state ($\nu = 2/5$)				
		$\{n_{\gamma;l}\}$	$h^{\text{sc}} + h^{\text{ga}}$	Quantum dimension
0	\mathbb{I}	1 1 0 0 1 1 0 0 0 0	0 + 0	1
1	$e^{i2/5\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 0 1 1 0 0 0	0 + 1/5	1
2	$e^{i4/5\sqrt{\nu^{-1}}\phi_c}$	0 0 1 1 0 0 1 1 0 0	0 + 4/5	1
3	$\phi_N^2 e^{i1/5\sqrt{\nu^{-1}}\phi_c}$	0 0 0 1 1 0 0 1 1 0	5/4 + 1/20	1
4	$\phi_N^2 e^{i3/5\sqrt{\nu^{-1}}\phi_c}$	0 0 0 0 1 1 0 0 1 1	5/4 + 9/20	1
5	j	1 0 0 0 0 1 1 0 0 1	1 + 0	1
6	$j e^{i2/5\sqrt{\nu^{-1}}\phi_c}$	1 1 0 0 0 0 1 1 0 0	1 + 1/5	1
7	$j e^{i4/5\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 0 0 0 1 1 0	1 + 4/5	1
8	$\phi_N^1 e^{i1/5\sqrt{\nu^{-1}}\phi_c}$	0 0 1 1 0 0 0 0 1 1	5/4 + 1/20	1
9	$\phi_N^1 e^{i3/5\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 1 0 0 0 0 1	5/4 + 9/20	1
10	$\phi_1 e^{i1/5\sqrt{\nu^{-1}}\phi_c}$	1 1 0 0 0 1 1 0 0 0	1/20 + 1/20	2
11	$\phi_1 e^{i3/5\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 0 0 1 1 0 0	1/20 + 9/20	2
12	ϕ_4	0 0 1 1 0 0 0 1 1 0	4/5 + 0	2
13	$\phi_4 e^{i2/5\sqrt{\nu^{-1}}\phi_c}$	0 0 0 1 1 0 0 0 1 1	4/5 + 1/5	2
14	$\phi_4 e^{i4/5\sqrt{\nu^{-1}}\phi_c}$	1 0 0 0 1 1 0 0 0 1	4/5 + 4/5	2
15	ϕ_2	0 1 0 0 1 0 1 0 0 1	1/5 + 0	2
16	$\phi_2 e^{i2/5\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 0 1 0 1 0 0	1/5 + 1/5	2
17	$\phi_2 e^{i4/5\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 0 0 1 0 1 0	1/5 + 4/5	2
18	$\phi_3 e^{i1/5\sqrt{\nu^{-1}}\phi_c}$	0 0 1 0 1 0 0 1 0 1	9/20 + 9/20	2
19	$\phi_3 e^{i3/5\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 0 1 0 0 1 0	9/20 + 9/20	2
20	$\sigma_1 e^{i3/10\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 1 0 0 1 0 0	1/16 + 0.113	$\sqrt{5}$
21	$\sigma_1 e^{i7/10\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 0 1 0 0 1 0	1/16 + 0.613	$\sqrt{5}$
22	$\sigma_2 e^{i1/10\sqrt{\nu^{-1}}\phi_c}$	0 0 1 0 1 0 1 0 0 1	1/16 + 0.012	$\sqrt{5}$
23	$\sigma_2 e^{i5/10\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 0 1 0 1 0 0	1/16 + 0.312	$\sqrt{5}$
24	$\sigma_2 e^{i9/10\sqrt{\nu^{-1}}\phi_c}$	0 1 0 0 1 0 1 0 1 0	1/16 + 1.012	$\sqrt{5}$
25	$\tau_1 e^{i3/10\sqrt{\nu^{-1}}\phi_c}$	0 0 1 0 0 1 0 1 0 1	9/16 + 0.113	$\sqrt{5}$
26	$\tau_1 e^{i7/10\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 0 0 1 0 1 0	9/16 + 0.613	$\sqrt{5}$
27	$\tau_2 e^{i1/10\sqrt{\nu^{-1}}\phi_c}$	0 1 0 0 1 0 0 1 0 1	9/16 + 0.012	$\sqrt{5}$
28	$\tau_2 e^{i5/10\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 0 1 0 0 1 0	9/16 + 0.312	$\sqrt{5}$
29	$\tau_2 e^{i9/10\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 0 0 1 0 0 1	9/16 + 1.012	$\sqrt{5}$

certain quasiparticle pattern-of-zeros solutions may become illegal as a result. Put another way, certain quasiparticle pattern-of-zeros solutions that were allowed when the system was thought of as a single-component system may be illegal if the chiral algebra is self-consistently enlarged in a certain way to get a multicomponent system.

6. Conclusion

This concludes our analysis of the orbifold FQH states from the point of view of ideal wave functions and the vertex algebra/pattern-of-zeros approach. The orbifold FQH states provide the first concrete examples in which the operators

of the edge CFT have “sick” pattern-of-zeros solutions. As a result, we find that these states yield profound insights into the vertex algebra framework. Namely, when a certain pattern-of-zeros solution appears to describe a gapless state (due to a continuum of solutions to the quasiparticle structure constants in the vertex algebra), this means that generically there may be a way to self-consistently enlarge the vertex algebra, which physically corresponds to condensing new operators and driving the ideal Hamiltonian away from a critical point. Then the newly enlarged vertex algebra may have a finite number of quasiparticle solutions and there may be a multilayer ideal wave function that captures the topological order of the resulting states. Thus all of the pattern-of-zeros solutions, even when they naively appear to be describing gapless phases, are ultimately relevant in describing incompressible FQH states.

An important direction now is to put the above ideas on a more concrete footing in the vertex algebra framework in order to, for instance, derive the incompressible ideal wave functions that do capture the topological order of the orbifold FQH states.

VIII. RELATION TO EXPERIMENTS AND RELEVANCE TO $\nu = 8/3$ AND $\nu = 12/5$

In both double-layer and wide single-layer quantum wells, several of the (ppq) states, such as the (331) and (330) states, have been routinely realized experimentally.^{44,45} The study presented here suggests that by varying material parameters such as the interlayer tunneling/repulsion, it could be possible to tune through a continuous quantum phase transition into a non-Abelian FQH state.

Since the transition is driven by the condensation of an electrically neutral boson, the charge gap remains nonzero through the transition, which would make detection of the transition difficult through charge transport experiments. Some possible experimental probes are as follows.

The most obvious physical consequence of this transition is that the bulk should become a thermal conductor at the transition, because while the charge gap remains, a neutral mode becomes gapless at the critical point. This would also have a pronounced effect on edge physics; near the transition, the velocity of a neutral mode approaches 0, until at the transition it becomes a gapless excitation in the bulk. Thus this transition should also be detectable through edge tunneling experiments. Furthermore, as discussed in Sec. VI A, the transition should be accompanied by interlayer density fluctuations. Since the density fluctuations carry an electric dipole moment, they can, in principle, be observed through surface acoustic phonons.⁴⁶

One useful physical distinction between the bilayer Abelian states and the orbifold non-Abelian states are that when $N = p - q$ is odd, the minimal electric charge of the quasiparticles becomes halved in the orbifold phase. Thus, for example, the quasiparticle minimal charge can be measured as the interlayer tunneling and interlayer thickness are tuned in a two-component (330) state. An observation of a change in the minimal quasiparticle electric charge from $e/3$ to $e/6$ would indicate a transition to the non-Abelian phase.

Another implication of the results here applies to the single-layer plateaus that have been observed at $\nu = 8/3$ and

$\nu = 12/5$.⁴⁷ Currently, it is believed that the FQH plateaus seen in single layer samples at $\nu = 8/3 = 2 + 2/3$ and $\nu = 12/5 = 2 + 2/5$ might be exotic non-Abelian states.⁴⁸ There are a number of candidate states, including the particle-hole conjugate of the Z_3 parafermion (Read-Rezayi) state⁴⁹ and some hierarchy states formed over the Pfaffian state.^{50,51}

Our study suggests another set of possible states. The orbifold states presented here are neighbors in the phase diagram to more conventional states, such as the (330) and (550) states. These states can exist at $\nu = 8/3$ and $\nu = 12/5$, respectively; in fact, experiments on wide single-layer quantum wells have shown plateaus at $\nu = 8/3$. The fact that the orbifold FQH states are neighbors in the phase diagram to these more conventional bilayer states means that in single-layer samples, the orbifold FQH states should be considered as possible candidates to explain the observed plateaus.

IX. SUMMARY, CONCLUSIONS, AND OUTLOOK

In this paper, we have introduced a set of FQH phases, dubbed the orbifold FQH states, and studied phase transitions between them and conventional Abelian bilayer phases. The orbifold states are labeled by two parameters (N, q) and exist at filling fraction $\nu = 2/(N + 2q)$. The bulk low-energy effective field theory for these phases is the $U(1) \times U(1) \rtimes Z_2$ CS theory. Their edge CFT is a $[U(1) \times U(1)]/Z_2$ orbifold CFT with central charge $c = 2$. These orbifold phases contain an electrically neutral boson whose condensation drives a continuous quantum phase transition to the bilayer (ppq) states. In the $U(1) \times U(1) \rtimes Z_2$ CS theory, this neutral boson carries Z_2 gauge charge and so the effective theory near the transition—where the neutral boson has low energy compared to all other excitations—is a Z_2 gauged Ginzburg-Landau theory, which implies that the transition is in the 3D Ising universality class.

We have introduced a slave-particle gauge theory formulation of these states, which shows how to interpolate between the Abelian bilayer states and the orbifold states. This description provides an interesting example in which Z_2 fractionalization leads to non-Abelian topological phases. Finally, we have seen that the existence of these states sheds considerable light on the pattern-of-zeros/vertex algebra framework for characterizing ideal FQH wave functions. The orbifold states provide the first examples in which the sick pattern-of-zeros solutions are actually relevant for describing incompressible FQH states.

The calculation of the full topological quantum numbers of the quasiparticles relies on a prescription in which we embed

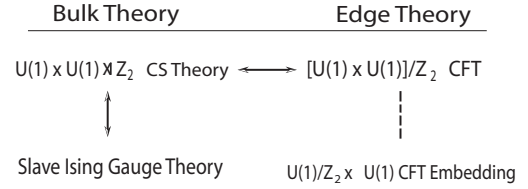


FIG. 2. Here we try to illustrate the different ways of describing the topological order of the orbifold FQH states and how they are related. The $U(1) \times U(1) \rtimes Z_2$ CS theory is a bulk topological field theory, but we do not know how to compute all of the topological properties from this theory. Closely related is the slave Ising theory gauge theory presented in Sec. III, which we believe provides a lattice regularization of the $U(1) \times U(1) \rtimes Z_2$ CS theory. The spectrum of the edge theory for these bulk topological theories is conjectured to be generated by the electron operator $\Psi_e = \cos(\sqrt{N/2}\phi)e^{i\sqrt{(p+q)/2}\phi_c}$. We also conjecture that this edge theory is equivalent to what one would obtain for the edge theory by taking the electron operator to be $\Psi_e = \phi_N^1 e^{i\sqrt{(p+q)/2}\phi_c}$ and embedding it into the holomorphic half of the Z_2 orbifold $\times U(1)_{\text{charge}}$ CFT.

the electron operator in the $U(1)/Z_2 \times U(1)$ CFT. We have not proven rigorously that the results are equivalent to the $[U(1) \times U(1)]/Z_2$ CFT. Let us briefly summarize the successes of the various descriptions of the orbifold FQH states, as reported in Table VIII. The bulk $U(1) \times U(1) \rtimes Z_2$ CS theory can be used to compute the number of quasiparticles and the quantum dimensions of all of the quasiparticles, which can yield the ground-state degeneracy on genus g surfaces. Based on the relation to the neighboring (ppq) states, we can deduce the charges and twists/scaling dimensions of the quasiparticles that have quantum dimensions 1 and 2, but not those of the Z_2 vortices. This relation to the (ppq) states also allows us to deduce certain properties of the fusion rules. Furthermore, by studying the Z_2 vortices in detail, we can also deduce some information about their fusion rules from the $U(1) \times U(1) \rtimes Z_2$ CS theory. The bulk $U(1) \times U(1) \rtimes Z_2$ CS theory is closely related to the slave Ising theory introduced in Sec. III, which allows us to compute the charges of the Z_2 vortices. However, from these bulk theories we do not know how to compute the twists of the Z_2 vortices or all of the fusion rules of the quasiparticles.

The edge theory of the orbifold states is the $[U(1) \times U(1)]/Z_2$ CFT, and the electron operator is $\Psi_e = \cos(\sqrt{N/2}\phi_-)e^{i\sqrt{(p+q)/2}\phi_+}$. However, using this operator we currently can only compute the pattern of zeros of the electron operator, which yields the scaling dimensions of the Abelian

TABLE VIII. Summary of the successes of various descriptions of the orbifold FQH states.

		No. of quasiparticles	Quantum dimensions	Charges rules	Fusion dimensions	Scaling
Bulk	$U(1) \times U(1) \times Z_2$	✓	✓	Some	Some	Some
	Slave Ising theory			✓		
Edge	$U(1)/Z_2 \times U(1)$ CFT prescription	✓	✓	✓	✓	✓
	$[U(1) \times U(1)]/Z_2$ CFT	✓				Some

quasiparticles in the theory.^{7,8} Based on a close relation to the Z_2 orbifold chiral algebra, we conjecture that the topological order can be completely described by setting the electron operator to be $\phi_N^1 e^{i\sqrt{v-1}\phi_c}$ and embedding the electron operator into the $U(1)_{2N}/Z_2 \times U(1)$ CFT. From this prescription, we can compute all topological properties, and they agree with all quantities that can be computed in any other ways.

There are many directions for future research. Conceptually, perhaps the most interesting and important would be to further the understanding of these states through the perspective of ideal wave functions. The ideas presented in this paper regarding the pattern-of-zeros and vertex algebra approaches to ideal wave functions are preliminary and need to be borne out by more concrete calculations. These orbifold states are currently the only FQH states for which we do not have an ideal wave function with the same topological properties.

Another direction is to fill in the logical leaps in the analysis presented here, such as deriving all properties of the edge theory directly from the $[U(1) \times U(1)]/Z_2$ orbifold CFT, without using the prescription in terms of the $U(1)_{2N}/Z_2 \times U(1)$ CFT. Similarly, ways could be developed for computing more topological properties directly from the $U(1) \times U(1) \rtimes Z_2$ CS theory.

An important direction in making contact with experiments is to use the projected trial wave functions developed here using the slave-particle gauge theory to numerically analyze where in the bilayer phase diagram these orbifold FQH states may be favorable. Given their close proximity in the phase diagram to conventional Abelian states that are routinely realized in experimental samples, it is possible that experiments might actually probe this transition. Furthermore, their proximity in the phase diagram to conventional states warrants their inclusion as possible candidates for explaining the single-layer plateaus observed experimentally at $\nu = 8/3$ and $\nu = 12/5$ and whose ultimate nature remains mysterious.

Finally, we note that while the mathematical theory of boson condensation in topological phases has been recently understood, our study provides a physical context and understanding of the physical properties of such phase transitions in the simplest case of the condensed boson having Z_2 fusion rules. More generally, it is known that in tensor category theory, one can start with a tensor category \mathcal{C}_0 and mod out by a subcategory \mathcal{C} , where \mathcal{C} contains the fusion subalgebra that is generated by a boson in the theory. Reference 18 studies many general properties of the topological quantum numbers of such a transition. One direction is to extend our work by studying the physics of such transitions: the physical contexts in which they occur, effective field theories that describe the two phases, and the nature of the transitions.

As a concluding remark we would like to mention that some of the mathematics of the topological order of these orbifold FQH states was recently studied in Ref. 52.

ACKNOWLEDGMENTS

We thank Michael Levin, Zhenhan Wang, and Brian Swingle for discussions. We also thank Tsinghua University for hospitality while part of this work was completed. This research was supported by NSF Grant No. DMR-1005541. M.B. is currently supported by a fellowship from the Simons Foundation.

APPENDIX A: $U(1)/Z_2$ ORBIFOLD CFT

Since the $U(1)/Z_2$ orbifold at $c = 1$ plays an important role in understanding the topological properties of the orbifold FQH states, here we give a brief account of some of its properties. The information here is taken from Ref. 31, where a more complete discussion can be found.

The $U(1)/Z_2$ orbifold CFT, at central charge $c = 1$, is the theory of a scalar boson φ , compactified at a radius R , so that $\varphi \sim \varphi + 2\pi R$, and with an additional Z_2 gauge symmetry: $\varphi \sim -\varphi$. When $\frac{1}{2}R^2$ is rational, that is, $\frac{1}{2}R^2 = p/p'$, with p and p' coprime, then it is useful to consider an algebra generated by the fields $j = i\partial\varphi$, and $e^{\pm i\sqrt{2N}\varphi}$, for $N = pp'$. This algebra is referred to as an extended chiral algebra. The infinite number of Virasoro primary fields in the $U(1)$ CFT can now be organized into a finite number of representations of this extended algebra \mathcal{A}_N . There are $2N$ of these representations, and the primary fields are written as $V_k = e^{ik\varphi/\sqrt{2N}}$, with $k = 0, 1, \dots, 2N - 1$. The Z_2 action takes $V_k \rightarrow V_{2N-k}$.

In the Z_2 orbifold, one now considers representations of the smaller algebra \mathcal{A}_N/Z_2 . This includes the Z_2 invariant combinations of the original primary fields, which are of the form $\phi_k = \cos(k\varphi/\sqrt{2N})$; there are $N + 1$ of these. In addition, there are six new primary fields. The gauging of the Z_2 allows for twist operators that are not local with respect to the fields in the algebra \mathcal{A}_N/Z_2 but, rather, local up to an element of Z_2 . It turns out that there are two of these twisted sectors, and each sector contains one field that lies in the trivial representation of the Z_2 and one field that lies in the nontrivial representation of Z_2 . These twist fields are labeled σ_1 , τ_1 , σ_2 , and τ_2 . In addition to these, an in-depth analysis³¹ shows that the fixed points of the Z_2 action in the original $U(1)$ theory split into a Z_2 invariant and a noninvariant field. We have already counted the invariant ones in our $N + 1$ invariant fields, which leaves two new fields. One fixed point is the identity sector, corresponding to V_0 , which splits into two sectors: 1 and $j = i\partial\varphi$. The other fixed point corresponds to V_N . This splits into two primary fields, which are labeled ϕ_N^i for $i = 1, 2$ and which have scaling dimension $N/4$. In total, there are $N + 7$ primary fields in the Z_2 rational orbifold at “level” $2N$. These fields and their properties are summarized in Table IX.

This spectrum for the Z_2 orbifold is obtained by first computing the partition function of the full Z_2 orbifold CFT defined on a torus, including both holomorphic and antiholomorphic

TABLE IX. Primary fields in the $U(1)_{2N}/Z_2$ orbifold CFT. The label k runs from 1 to $N - 1$.

Label	Scaling dimension	Quantum dimension
\mathbb{I}	0	1
j	1	1
ϕ_N^1	$N/4$	1
ϕ_N^2	$N/4$	1
σ_1	$1/16$	\sqrt{N}
σ_2	$1/16$	\sqrt{N}
τ_1	$9/16$	\sqrt{N}
τ_2	$9/16$	\sqrt{N}
ϕ_k	$k^2/4N$	2

TABLE X. Primary fields in the Z_2 orbifold for $N = 3$, their scaling dimensions, and the Z_4 parafermion fields that they correspond to.

Z_2 orbifold field	Scaling dimension (h)	Z_4 parafermion field
1	0	Φ_0^0
j	1	Φ_4^0
ϕ_N^1	3/4	Φ_2^0
ϕ_N^2	3/4	Φ_6^0
ϕ_1	1/12	Φ_2^2
ϕ_2	1/3	Φ_0^2
σ_1	1/16	Φ_1^1
σ_2	1/16	Φ_{-1}^1
τ_1	9/16	Φ_3^1
τ_2	9/16	Φ_5^1

parts. Then the partition function is decomposed into holomorphic blocks, which are conjectured to be the generalized characters of the \mathcal{A}_N/Z_2 chiral algebra. This leads to the spectrum listed in Table IX. The fusion rules and scaling dimensions for these primary fields are obtained by studying the modular transformation properties of the characters.

The fusion rules are as follows. For N even,

$$j \times j = 1, \quad \phi_N^i \times \phi_N^i = 1, \quad \phi_N^1 \times \phi_N^2 = j. \quad (\text{A1})$$

As mentioned in Ref. 31, the vertex operators ϕ_k have a fusion algebra consistent with their interpretation as $\cos \frac{k}{\sqrt{2N}}\varphi$:

$$\begin{aligned} \phi_k \times \phi_{k'} &= \phi_{k+k'} + \phi_{k-k'} \quad (k' \neq k, N-k), \\ \phi_k \times \phi_k &= 1 + j + \phi_{2k}, \\ \phi_{N-k} \times \phi_k &= \phi_{2k} + \phi_N^1 + \phi_N^2, \quad j \times \phi_k = \phi_k. \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \sigma_i \times \sigma_i &= 1 + \phi_N^i + \sum_{k \text{ even}} \phi_k, \\ \sigma_1 \times \sigma_2 &= \sum_{k \text{ odd}} \phi_k, \quad j \times \sigma_i = \tau_i. \end{aligned} \quad (\text{A3})$$

For N odd, the fusion algebra of 1, j , and ϕ_N^i is Z_4 :

$$j \times j = 1, \quad \phi_N^1 \times \phi_N^2 = 1, \quad \phi_N^i \times \phi_N^i = j. \quad (\text{A4})$$

The fusion rules for the twist fields become

$$\sigma_i \times \sigma_i = \phi_N^i + \sum_{k \text{ odd}} \phi_k, \quad \sigma_1 \times \sigma_2 = 1 + \sum_{k \text{ even}} \phi_k. \quad (\text{A5})$$

The fusion rules for the operators ϕ_k are unchanged.

For $N = 1$, it was observed that the Z_2 orbifold is equivalent to the $U(1)_8$ Gaussian theory. For $N = 2$, it was observed that the Z_2 orbifold is equivalent to two copies of the Ising CFT. For $N = 3$, it was observed that the Z_2 orbifold is equivalent to the Z_4 parafermion CFT of Zamolodchikov and Fateev.³⁷

In Tables X and XI we list the fields from the Z_2 orbifold for $N = 3$ and $N = 2$, their scaling dimensions, and the fields in the Ising² or Z_4 parafermion CFTs that they correspond to.

 TABLE XI. Primary fields in the Z_2 orbifold for $N = 2$, their scaling dimensions, and the fields from Ising² to which they correspond.

Z_2 orbifold field	Scaling dimension (h)	Ising fields ²
1	0	$\mathbb{I} \otimes \mathbb{I}$
j	1	$\psi \otimes \psi$
ϕ_N^1	1/2	$\mathbb{I} \otimes \psi$
ϕ_N^2	1/2	$\psi \otimes \mathbb{I}$
ϕ_1	1/8	$\sigma \otimes \sigma$
σ_1	1/16	$\sigma \times \mathbb{I}$
σ_2	1/16	$\mathbb{I} \otimes \sigma$
τ_1	9/16	$\sigma \otimes \psi$
τ_2	9/16	$\psi \otimes \sigma$

APPENDIX B: Z_N TRANSITIONS BETWEEN ABELIAN STATES

Our analysis of the $N = 1$ orbifold states in Sec. VII B 1 revealed a series of Abelian FQH states that can apparently undergo Z_m phase transitions to other Abelian FQH states. In particular, consider the following two-component states with K matrix and charge vector \mathbf{q} given by

$$K = \begin{pmatrix} 2m^2 & m \\ m & q+1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{B1})$$

For $m > 1$, these states have a neutral boson ϕ with the fusion rule

$$\phi^m = \mathbb{I}. \quad (\text{B2})$$

To see this, observe that ϕ can be described by the integer vector $l_\phi^T = (2m, 1)$. From the formula $Q_\phi = \mathbf{q}^T K^{-1} l_\phi = 0$, we find that ϕ is electrically neutral, while from $\theta_\phi/\pi = l_\phi^T K^{-1} l_\phi = \text{even}$, we find that ϕ is a boson. Finally, from the fact that $ml_\phi^T = (2m^2, m)$, which is the first row in the K matrix, we find that ϕ^m is a local excitation, that is, $\phi^m = \mathbb{I}$.

Based on the analysis in Sec. VII B 1, we expect that the condensation of ϕ will yield the $m = 1$ states and that the transition is in the Z_m universality class. In the case $m = 2$, these are the $N = 1$ orbifold FQH states, which have non-Abelian analogs for more general N . Also, in the case $m = 2$ there is a $U(1) \times U(1) \rtimes Z_2$ CS description that makes the appearance of this discrete Z_2 structure explicit.

We currently do not know whether for $m > 2$ there are also non-Abelian analogs that are separated from a bilayer Abelian phase by a Z_m transition. We also do not know whether there is a way to describe these states in terms of a CS theory with a gauge group that makes the Z_m structure explicit, as there is for $m = 2$.

APPENDIX C: SLAVE ROTOR CONSTRUCTION

While the slave Ising construction presented above is sufficient to describe the bilayer Abelian (ppq) states and the non-Abelian orbifold FQH states, it is a “minimal” slave-particle gauge theory in the sense that it only captures the minimal amount of fluctuations about a given mean-field state in order to see the possibility of the two phases. It is possible to improve the slave-particle description by including

more of these fluctuations about the mean-field states and probing a larger part of the Hilbert space. This can be done by promoting the above slave Ising theory to the following slave rotor description.

We rewrite the electron operators in the following way:

$$\Psi_{i+} = c_{i+}, \quad \Psi_{i-} = e^{i\phi_i} c_{i-}. \quad (\text{C1})$$

In this construction, we have a $U(1)$ gauge symmetry associated with the following local transformations:

$$\phi_i \rightarrow \phi_i + \alpha, \quad c_{i-} \rightarrow e^{-i\alpha} c_{i-}. \quad (\text{C2})$$

This means that the physical states must satisfy

$$e^{i\alpha \hat{L}_i - i\alpha n_{c_{i-}}} = 1 \quad (\text{C3})$$

for any α [there is an arbitrary $U(1)$ phase factor, which we have set to unity here]. The angular momentum $\hat{L}_i \propto i\partial\phi_i$ is conjugate to the field ϕ_i . Equation (C3) implies

$$\hat{L}_i - n_{c_{i-}} = 0. \quad (\text{C4})$$

Note that we will actually want to do a further slave-particle decomposition into partons, as in Eq. (21). For example, for $q = 0$, we decompose c_{\pm} as

$$c_{\pm i} = \prod_{a=1}^N \psi_{ai} \pm \prod_{a=N+1}^{2N} \psi_{ai}. \quad (\text{C5})$$

In this case, the gauge symmetry associated with translating ϕ_i is actually only a Z_2 symmetry:

$$\phi_i \rightarrow \phi_i + \pi, \quad \psi_i \leftrightarrow \psi_{N+i}. \quad (\text{C6})$$

In this case, the constraint on the rotor is actually $\hat{L}_i - n_{c_{i-}} = \text{even}$. Or, alternatively:

$$(-1)^{\hat{L}_i + n_{c_{i-}}} = 1. \quad (\text{C7})$$

Let us set $b_i \equiv e^{i\phi_i}$. Substituting into a model Hamiltonian that includes hopping between sites in the same layer, between sites in different layers, and various interaction terms, we obtain

$$\begin{aligned} H_{\text{kin}} + H_{\text{tun}} &= \sum_{ij} (t_{ij} + t_{ji}^* + T_{ij} + T_{ji}^*) c_{i+}^{\dagger} c_{j+} + \sum_{ij} (t_{ij} + t_{ji}^* - T_{ij} - T_{ji}^*) b_i^* b_j c_{i-}^{\dagger} c_{j-}. \\ H_{\text{int}} &= \frac{1}{2} \sum_{ij} (U_{ij} + V_{ij}) : (n_{i+} n_{j+} + n_{i+} n_{j-} + n_{i-} n_{j+} + n_{i-} n_{j-}) : + \frac{1}{2} \sum_{ij} (U_{ij} - V_{ij}) (b_j b_i c_{+i}^{\dagger} c_{+j}^{\dagger} c_{-j} c_{-i} + b_i^* b_j c_{-i}^{\dagger} c_{+j}^{\dagger} c_{-j} c_{+i} \\ &\quad + b_i^* b_j^* c_{-i}^{\dagger} c_{-j}^{\dagger} c_{+j} c_{+i} + b_i b_j^* c_{+i}^{\dagger} c_{-j}^{\dagger} c_{+j} c_{-i}). \end{aligned} \quad (\text{C8})$$

Note that the above Hamiltonian does not preserve a global $U(1)$ symmetry associated with arbitrary translations of ϕ_i ; there is only a Z_2 symmetry. Thus there are only two distinct phases. The first one is smoothly connected to a situation in which

$$\langle e^{i\phi} \rangle \neq 0, \quad (\text{C9})$$

and the second one is smoothly connected to a situation in which

$$\langle e^{i2\phi} \rangle \neq 0. \quad (\text{C10})$$

The first possibility breaks the Z_2 gauge symmetry, while the second one preserves it. These two possibilities describe precisely the same two phases as the slave Ising theory described above. In the first case, suppose we set $e^{i\phi} = 1$.

Then we are left with the parton construction for the (ppq) states. In the Z_2 unbroken phase, we may set $e^{i2\phi} = 1$, so that $e^{i\phi} = \pm 1 \equiv s_i^z$. Thus, these two phases that we can access in the slave rotor approach are the same phases that we can access from the slave Ising approach. The Z_2 broken phase corresponds to the bilayer (ppq) states, while the Z_2 unbroken phase corresponds to the orbifold FQH states. The slave rotor approach has the advantage of probing more fluctuations around the mean-field states because more of the Hilbert space is being accessed in this decomposition. This may allow for more reliable calculations of the phase diagram.

As in the slave Ising construction, this slave rotor construction also provides trial projected wave functions but provides a larger space of possible trial wave functions that capture the behavior of each of the two phases.

*Corresponding author: maissam@stanford.edu, present address: Department of Physics, Stanford University, Stanford, California 94305, USA.

¹X.-G. Wen, *Quantum Field Theory of Many-Body Systems—From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2004).

²Z. Wang, *Topological Quantum Computation* (American Mathematical Society, Providence, RI, 2010).

³J. Preskill, [<http://www.theory.caltech.edu/~preskill/ph219/topological.ps>] (2004).

⁴G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).

⁵N. Read and E. Rezayi, *Phys. Rev. B* **59**, 8084 (1999).

⁶X.-G. Wen and Z. Wang, *Phys. Rev. B* **77**, 235108 (2008).

- ⁷X.-G. Wen and Z. Wang, *Phys. Rev. B* **78**, 155109 (2008).
- ⁸M. Barkeshli and X.-G. Wen, *Phys. Rev. B* **79**, 195132 (2009).
- ⁹M. Barkeshli and X.-G. Wen, *Phys. Rev. B* **82**, 245301 (2010).
- ¹⁰Y.-M. Lu, X.-G. Wen, Z. Wang, and Z. Wang, *Phys. Rev. B* **81**, 115124 (2010).
- ¹¹A. Y. Kitaev, *Ann. Phys. (NY)* **303**, 2 (2003).
- ¹²M. H. Freedman, A. Kitaev, M. J. Larsen, and Z. Wang, *Bull. Am. Math. Soc.* **40**, 31 (2003).
- ¹³E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, *J. Math. Phys.* **43**, 4452 (2002).
- ¹⁴B. Halperin, *Helv. Phys. Acta* **56**, 75 (1983).
- ¹⁵M. Barkeshli and X.-G. Wen, *Phys. Rev. Lett.* **105**, 216804 (2010).
- ¹⁶X.-G. Wen, *Phys. Rev. Lett.* **84**, 3950 (2000).
- ¹⁷N. Read and D. Green, *Phys. Rev. B* **61**, 10267 (2000).
- ¹⁸F. A. Bais and J. K. Slingerland, *Phys. Rev. B* **79**, 045316 (2009).
- ¹⁹M. Barkeshli and X.-G. Wen, *Phys. Rev. B* **81**, 045323 (2010).
- ²⁰B. Blok and X. G. Wen, *Phys. Rev. B* **42**, 8145 (1990).
- ²¹X.-G. Wen, *Adv. Phys.* **44**, 405 (1995).
- ²²E. Verlinde, *Nucl. Phys. B* **300**, 360 (1988).
- ²³T. Senthil and M. P. A. Fisher, *Phys. Rev. B* **62**, 7850 (2000).
- ²⁴X.-G. Wen, *Phys. Rev. Lett.* **66**, 802 (1991).
- ²⁵X.-G. Wen, *Phys. Rev. B* **60**, 8827 (1999).
- ²⁶X.-G. Wen, *Int. J. Mod. Phys. B* **6**, 1711 (1992).
- ²⁷X.-G. Wen and A. Zee, *Phys. Rev. B* **46**, 2290 (1992).
- ²⁸E. Witten, *Commun. Math. Phys.* **121**, 351 (1989).
- ²⁹G. Moore and N. Seiberg, *Phys. Lett. B* **220**, 422 (1989).
- ³⁰R. Dijkgraaf and E. Witten, *Commun. Math. Phys.* **129**, 393 (1990).
- ³¹R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *Commun. Math. Phys.* **123**, 485 (1989).
- ³²P. D. Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer, New York, 1997).
- ³³B. A. Bernevig and F. D. M. Haldane, *Phys. Rev. Lett.* **100**, 246802 (2008).
- ³⁴E. J. Bergholtz and A. Karlhede, *J. Stat. Mech. Theory Exp.* **2006**, L04001 (2006).
- ³⁵A. Seidel and D.-H. Lee, *Phys. Rev. Lett.* **97**, 056804 (2006).
- ³⁶E. Fradkin and S. H. Shenker, *Phys. Rev. D* **19**, 3682 (1979).
- ³⁷A. Zamolodchikov and V. Fateev, *Sov. Phys. JETP* **62**, 215 (1985).
- ³⁸N. Read, *Phys. Rev. B* **79**, 045308 (2009).
- ³⁹S. H. Simon, E. H. Rezayi, and N. R. Cooper, *Phys. Rev. B* **75**, 075318 (2007).
- ⁴⁰E. Ardonne and K. Schoutens, *Phys. Rev. Lett.* **82**, 5096 (1999).
- ⁴¹E. Ardonne, F. van Lankvelt, A. Ludwig, and K. Schoutens, *Phys. Rev. B* **65**, 041305 (2002).
- ⁴²G. Moore and N. Seiberg, *Nucl. Phys. B* **313**, 16 (1989).
- ⁴³D. Green, Ph.D. thesis, Yale University (2001).
- ⁴⁴T. S. Lay, T. Jungwirth, L. Smrcka, and M. Shayegan, *Phys. Rev. B* **56**, R7092 (1997).
- ⁴⁵Y. W. Suen, H. C. Manoharan, X. Ying, M. B. Santos, and M. Shayegan, *Phys. Rev. Lett.* **72**, 3405 (1994).
- ⁴⁶R. L. Willett, R. R. Ruel, M. A. Paalanen, K. W. West, and L. N. Pfeiffer, *Phys. Rev. B* **47**, 7344 (1993).
- ⁴⁷J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **93**, 176809 (2004).
- ⁴⁸H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **77**, 081301(R) (2008).
- ⁴⁹E. H. Rezayi and N. Read, *Phys. Rev. B* **79**, 075306 (2009).
- ⁵⁰P. Bonderson and J. K. Slingerland, *Phys. Rev. B* **78**, 125323 (2008).
- ⁵¹P. Bonderson, A. E. Feiguin, G. Moller, and J. Slingerland, e-print [arXiv:0901.4965](https://arxiv.org/abs/0901.4965) (2009).
- ⁵²E. C. Rowell and Z. Wang, e-print [arXiv:1009.0241](https://arxiv.org/abs/1009.0241) (2010).
- ⁵³Z.-C. Gu, Z. Wang, and X.-G. Wen, e-print [arXiv:1010.1517](https://arxiv.org/abs/1010.1517) (2010).