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# Transverse momentum dependent quark densities from Lattice QCD

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**Abstract.** We study transverse momentum dependent parton distribution functions (TMDs) with non-local operators in lattice QCD, using MILC/LHPC lattices. We discuss the basic concepts of the method, including renormalization of the gauge link. Results obtained with a simplified operator geometry show visible dipole deformations of spin-dependent quark momentum densities.

**Keywords:** transverse momentum; parton distribution functions; lattice; QCD

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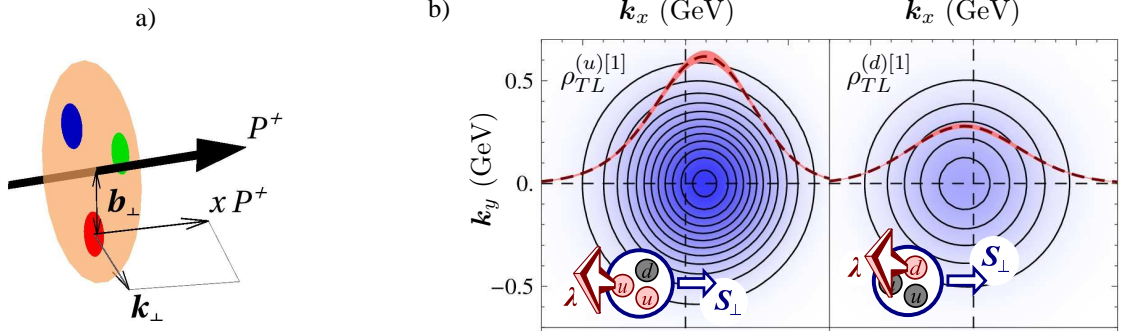
## INTRODUCTION

Generalized parton distribution functions (GPDs) and transverse momentum dependent parton distribution functions (TMDs) provide us with a picture of the internal quark distributions in a nucleon at the instant of an interaction, see illustration Fig. 1 a). GPDs and TMDs have their natural interpretation at large nucleon momentum  $\mathbf{P} = (0, 0, P_z)$ . The quark momentum  $k$  in terms of light cone coordinates  $k^\pm \equiv (k^0 \pm k^3)/\sqrt{2}$ ,  $\mathbf{k}_\perp = (\mathbf{k}_x, \mathbf{k}_y)$  scales like  $k^+ : \mathbf{k}_\perp : k^- \sim P^+ : 1 : (P^+)^{-1}$  with the large momentum component  $P^+$  of the nucleon. TMDs resolve the dependence on  $x \equiv k^+/P^+$  and transverse momentum  $\mathbf{k}_\perp$ , but not on the suppressed component  $k^-$ . In spin-polarized channels at leading twist, TMDs encode dipole- or quadrupole-shaped deformations of the nucleon in the  $\mathbf{k}_\perp$ -plane. We have studied such deformations in first explorative lattice QCD calculations [1, 2, 3], see Fig. 1 and our discussion below. These studies have been motivated by a history of successful lattice computations of  $x$ -moments of GPDs, providing images of the nucleon in the impact parameter,  $\mathbf{b}_\perp$ -, plane, see [4] for a review. A remaining theoretical problem concerns the precise form of the correlator defining TMDs in the continuum, see [5, 6] and references therein. In its basic form, it is given by [7]

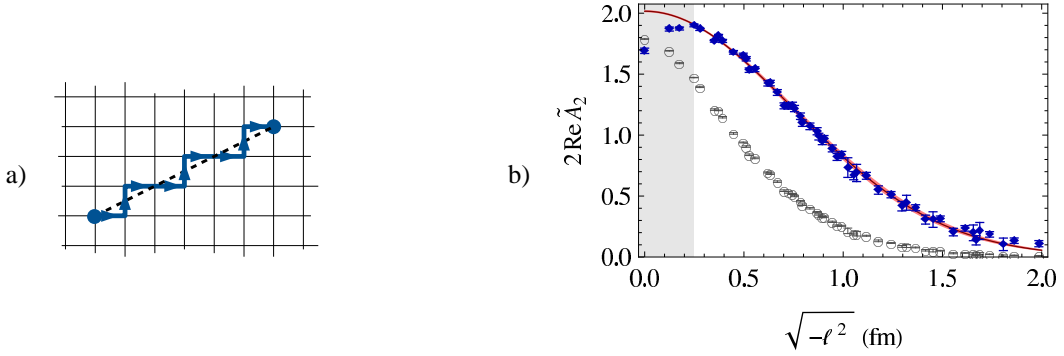
$$\begin{aligned} \Phi_q^{[\Gamma]}(x, \mathbf{k}_\perp; P, S; \mathcal{C}) &\equiv \int dk^- \int \frac{d^4l}{(2\pi)^4} e^{-ik \cdot l} \underbrace{\frac{1}{2} \langle P, S | \bar{q}(l) \Gamma \mathcal{U}[\mathcal{C}_l] q(0) | P, S \rangle}_{\tilde{\Phi}_q^{[\Gamma]}(l, P, S; \mathcal{C})} \Big|_{k^+ = xP^+} \\ &= \underbrace{\frac{1}{P^+} \int \frac{d(l \cdot P)}{2\pi} e^{-i(l \cdot P)x}}_{\not{x}} \underbrace{\int \frac{d^2\mathbf{l}_\perp}{(2\pi)^2} e^{i\mathbf{l}_\perp \cdot \mathbf{k}_\perp} \tilde{\Phi}_q^{[\Gamma]}(l, P, S; \mathcal{C})}_{\not{M}} \Big|_{l^+ = 0} \end{aligned} \quad (1)$$

where  $\Gamma$  is a Dirac matrix. The Wilson line  $\mathcal{U}[\mathcal{C}_l]$  running along a continuous path  $\mathcal{C}_l$  from  $l$  to 0 ensures gauge invariance of the expression. For the SIDIS and Drell-Yan scattering process, the Wilson line extends to infinity along a direction  $v$  that needs to be chosen (almost) lightlike, such that the cross section factorizes into hard, perturbative parts and soft contributions, see, e.g., Ref. [8]. Based on its symmetry transformation properties, the above correlator can be parametrized in terms of TMDs [9, 10, 11], for example

$$2\rho_{TL}^{(q)} \equiv \Phi_q^{[\gamma^+ + \lambda \gamma^+ \gamma^5]} = f_{1,q} + \lambda \frac{\mathbf{k}_\perp \cdot \mathbf{S}_\perp}{m_N} g_{1T,q} + \left[ \frac{\mathbf{S}_j \varepsilon_{ji} \mathbf{k}_i}{m_N} f_{1T,q}^\perp \right]_{\text{odd}}, \quad (2)$$



**FIGURE 1.** a) Illustration of quark degrees of freedom in the nucleon at large momentum. b) Dipole-deformed  $x$ -integrated densities obtained with straight gauge links at a pion mass  $m_\pi \approx 500\text{MeV}$ . The insets display the spin polarization of the quarks (red arrow) and of the nucleon (blue arrow).



**FIGURE 2.** a) Representation of a straight Wilson line (dashed line) as a step-like product of link variables. b) Amplitude  $\tilde{A}_2(l^2, 0)$  for up quarks at a pion mass  $m_\pi \approx 500\text{MeV}$ , using straight gauge links.

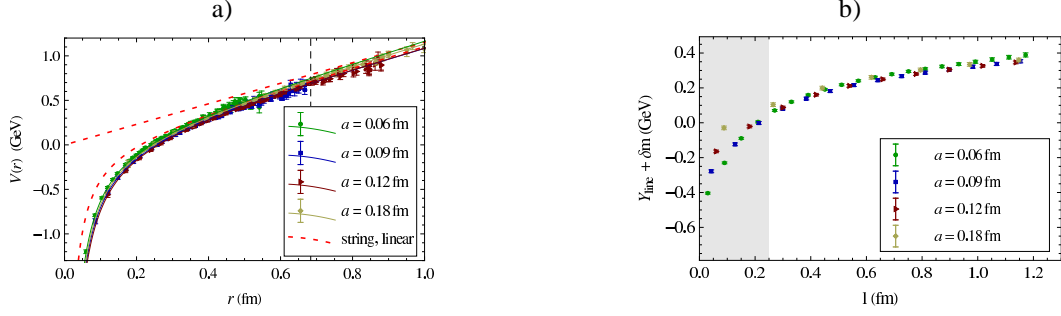
Here  $\lambda$  is the longitudinal quark polarization, and  $\Lambda$  and  $\mathbf{S}_\perp$  are longitudinal and transverse nucleon polarization, respectively. The leading-twist TMDs  $f_{1,q}$ ,  $g_{1T,q}$ ,  $f_{1T,q}^\perp$  are real-valued functions of  $x$  and  $\mathbf{k}_\perp$ . The “naively time-reversal odd” function  $f_{1T,q}^\perp$  switches sign when comparing the SIDIS- with the Drell-Yan process, because the direction  $v$  of the Wilson line changes from future- to past-pointing [12].

## STRAIGHT LINK TMDS FROM THE LATTICE

In light of the uncertainties about the precise form of the continuum correlator, and to develop our methods, our first lattice studies employ a simple operator geometry that does not relate to a specific scattering process: We connect the quark fields with a direct, straight Wilson line. For the resulting “process-independent” TMDs, the T-odd functions such as the Sivers function  $f_{1T,q}^\perp$  vanish exactly.

In our approach, we calculate matrix elements  $\langle P, S | O | P, S \rangle$  from ratios of three- and two-point functions using the same techniques as GPD calculations by the LHP collaboration in Ref. [13]. We also use the same sequential propagators and quark propagators, calculated by LHPC with domain-wall valence fermions on top of asqtad-improved staggered MILC gauge configurations [14, 15, 16] with 2+1 quark flavors at a lattice spacing  $a \approx 0.12\text{fm}$ . The difference with respect to GPD calculations is that we directly insert the non-local operator  $O \equiv \bar{q}(l)\Gamma\mathcal{W}[\mathcal{C}]q(0)$  in our three-point function. The Wilson line  $\mathcal{W}[\mathcal{C}]$  is approximated as a step-like product of HYP-smeared link-variables as illustrated in Fig. 2 a). See also Ref. [2, 3].

The connection between the matrix elements  $\tilde{\Phi}^{[\Gamma]}$  and TMDs is established through a parametrization in terms of Lorentz-invariant amplitudes  $\tilde{A}_i(l^2, l \cdot P)$ . For straight Wilson lines, we obtain in analogy to the parametrization in terms



**FIGURE 3.** a) Static quark potential from MILC lattices at several lattice spacings  $a$ , matched to the string potential at  $r \approx 0.7$  fm. b) Test of the renormalization procedure with straight Wilson lines on a gauge fixed ensemble.

of amplitudes  $A_i(k^2, k \cdot P)$  in Ref. [9] (here our sign conventions follow Ref. [11] with the substitution rule  $k \rightarrow im_N^2 l$ ):

$$\tilde{\Phi}^{[\gamma^\mu]} = 2P^\mu \tilde{A}_2 + 2im_N^2 l^\mu \tilde{A}_3, \quad \tilde{\Phi}^{[\gamma^\mu \gamma^5]} = -2m_N S^\mu \tilde{A}_6 - 2im_N P^\mu (l \cdot S) \tilde{A}_7 + 2m_N^3 l^\mu (l \cdot S) \tilde{A}_8.$$

The TMDs are then obtained by

$$f_1(x, \mathbf{k}_\perp^2) = 2 \not{x} \not{M} \tilde{A}_2(l^2, l \cdot P), \quad g_{1T}(x, \mathbf{k}_\perp^2) = 4m_N^2 \partial_{\mathbf{k}_\perp^2} \not{x} \not{M} \tilde{A}_7(l^2, l \cdot P).$$

In the equations above,  $\not{x}$  only acts on  $l \cdot P$ , while  $\not{M}$  only acts on  $l^2$ . Thus  $x \leftrightarrow l \cdot P$  and  $\mathbf{k}_\perp^2 \leftrightarrow l^2$  are pairs of conjugate variables. Our Euclidean lattice approach is restricted to the determination of amplitudes  $A_i$  for  $l^0 = -il_4 = 0$ , i.e., to the region  $l^2 < 0$ ,  $|l \cdot P| \leq \sqrt{-l^2} |\mathbf{P}|$ , where  $\mathbf{P}$  is the selected three-momentum of the nucleon on the lattice. The limited range in  $|l \cdot P|$  prohibits us from a direct evaluation of  $\not{x}$ . However, first studies of  $x$ - and  $\mathbf{k}_\perp$ - correlations are possible [17, 3]. Moreover,  $x$ -integrated TMDs and densities are directly accessible: Integrating Eq. (1) with respect to  $x$  removes  $\not{x}$  and sets  $l \cdot P$  to zero. Correspondingly, the  $x$ -integral of, e.g.,  $f_1$  becomes  $\int_{-1}^1 dx f_1(x, \mathbf{k}_\perp^2) \equiv f_1^{[1]}(\mathbf{k}_\perp^2) = 2 \not{M} \tilde{A}_2(l^2, 0)$ . In Fig. 2 b), open symbols correspond to unrenormalized lattice data for  $\tilde{A}_2(l^2, 0)$ .

To obtain results independent of our lattice spacing  $a$  and our lattice action, we must renormalize our data. The Wilson line  $\mathcal{W}[\mathcal{C}]$  introduces a length dependent renormalization factor  $\exp(-\delta m \sqrt{-l^2})$  [18, 19, 20]. To fix  $\delta m$ , we follow the strategy of Refs. [21, 22], and match the renormalized static quark potential  $V^{\text{ren}}(r) = V(r) + 2\delta m$  to the string potential  $V_{\text{string}} = \sigma r - \pi/(12r)$  [23] at a matching point  $r = 1.5r_0 \approx 0.7$  fm. In Fig. 3 a), we test the method for several lattice spacings  $a$  on four MILC lattices with similar pion masses  $m_\pi \approx 500$  MeV. The renormalized lattice data agree very well with each other and are approximated well by the string potential (red dashed curve) near the matching point, indicated by a vertical dashed line. The procedure implements a gauge-invariant renormalization condition that we can formulate as the demand that the static quark potential asymptotically approach a straight line  $\sigma r$  through the origin (shown as a red dashed line). In connection with TMDs, we lack at present an interpretation of this renormalization condition as a physical renormalization or factorization scale. In Figure 3 b), we check the applicability of the approach to Wilson lines by plotting  $Y_{\text{line}}^{\text{ren}}(l) = \ln(U_{l-a/2}/U_{l+a/2})/a + \delta m$ , where  $U_l$  is the expectation value of the color trace of a straight Wilson line of length  $l$  evaluated on a Landau gauge fixed ensemble, and where the length dependent renormalization has been carried out with the values  $\delta m$  obtained from the static quark potential. Only at short lengths,  $l \lesssim 0.25$  fm, we find significant differences between lattice data from different lattice spacings, a sign of lattice cutoff effects. For our TMD calculations discussed below we exclude data obtained in this region from our fits. For  $l \gtrsim 0.25$  fm, we assume that renormalization of the lattice operator can be carried out as in the continuum,  $O^{\text{ren}} = Z_{\Psi, z}^{-1} \exp(-\delta m \sqrt{-l^2}) O$ , where the renormalization constants  $Z_{\Psi, z}^{-1}$  and  $\delta m$  are independent of the Dirac structure  $\Gamma$  [19].

Figure 2 b) shows the renormalized lattice data for  $\tilde{A}_2(l^2, 0)$  as solid data points. The curve and statistical error band correspond to a Gaussian fit to this data in the range  $\sqrt{-l^2} \geq 0.25$  fm. Note that the renormalization constant  $Z_{\Psi, z}^{-1}$  has been fixed (in the isovector,  $u-d$ -channel) such that the  $x$ - $\mathbf{k}_\perp$ -integrated Gaussian density of unpolarized quarks yields the correct total number of valence quarks,  $\int d^2 \mathbf{k}_\perp f_{1, u-d}^{[1]} = 1$ . Similar fits for  $\tilde{A}_7$  enable us to calculate the “worm-gear” function  $g_{1T}^{[1]}$ , and correspondingly, the dipole deformed  $x$ -integrated density  $\rho_{TL}^{(q)[1]}$  defined in Eq. (2) and shown in Fig. 1 b). While the widths of our distributions depend strongly on our renormalization condition for  $\delta m$ , the average

transverse quark momentum shift can be expressed in terms of ratios of the Gaussian amplitudes at  $l^2=0$ :

$$\langle \mathbf{k}_x \rangle_{TL} \equiv \frac{\int d^2 \mathbf{k}_\perp \mathbf{k}_x \rho_{TL}^{[1]} \Big|_{\lambda=1, \mathbf{s}_\perp=(1,0)}}{\int d^2 \mathbf{k}_\perp \rho_{TL}^{[1]}} = m_N \frac{\int d^2 \mathbf{k}_\perp \mathbf{k}_\perp^2 / (2m_N^2) g_{1T}^{[1]}(\mathbf{k}_\perp)}{\int d^2 \mathbf{k}_\perp f_1^{[1]}(\mathbf{k}_\perp)} = -m_N \frac{\tilde{A}_7(0,0)}{\tilde{A}_2(0,0)} = \begin{cases} 67(5) \text{ MeV} & (\text{up}) \\ -30(5) \text{ MeV} & (\text{down}) \end{cases}$$

(errors statistical only). In these ratios, renormalization factors largely cancel. Reference [24] reveals a remarkable similarity of our results with a light-cone constituent quark model [25], despite the unphysically large quark masses employed in our lattice calculation: They find  $\langle \mathbf{k}_x \rangle_{TL} = 55.8 \text{ MeV}$  for up-, and  $\langle \mathbf{k}_x \rangle_{TL} = -27.9 \text{ MeV}$  for down-quarks.

## CONCLUSIONS AND OUTLOOK

We have performed first lattice studies of TMDs using non-local operators with a simplified, straight gauge link. Resulting average momentum shifts  $\langle \mathbf{k}_x \rangle_{TL}$  corroborate model results. An ongoing project with staple-shaped gauge links can potentially address TMDs specific to SIDIS or the Drell-Yan process, including T-odd functions responsible for single-spin asymmetries.

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