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$\mathcal{H}_2$ -Optimal Decentralized Control over Posets: A State Space Solution for State-Feedback

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Abstract—We develop a complete state-space solution to $\mathcal{H}_2$ -optimal decentralized control of poset-causal systems with state-feedback. Our solution is based on the exploitation of a key separability property of the problem, that enables an efficient computation of the optimal controller by solving a small number of uncoupled standard Riccati equations. Our approach gives important insight into the structure of optimal controllers, such as degree bounds that depend on the structure of the poset. A novel element in our state-space characterization of the controller is a remarkable pair of transfer functions, that belong to the incidence algebra of the poset, are inverses of each other, and are intimately related to estimation of the state along the different paths on the poset.

I. I

Finding computationally efficient algorithms to design decentralized controllers is a challenging area of research (see [5], [1] and the references therein). Current research suggests that while the problem is hard in general, certain classes with special information structures are tractable via convex optimization techniques. In past work, the authors have argued that communication structures modeled by partially ordered sets (or *posets*) provide a rich class of decentralized control systems (which we call *poset causal* systems) that are amenable to such an approach [5]. Posets have appeared in the control theory literature earlier in the context of team theory [3], and specific posets (chains) have been studied in the context of decentralized control [7].

While it is possible to design optimal decentralized controllers for a fairly large class of systems known as quadratically invariant systems in the Youla domain [4], there are some important drawbacks with such an approach. Typically Youla domain techniques are not computationally efficient, and the degree of optimal controllers synthesized with such techniques is not always well-behaved. Moreover, such approaches typically do not provide insight into the structure of the optimal controller. This drawback emphasizes the need for *state-space techniques* to synthesize optimal decentralized controllers. State space techniques are usually computationally efficient, and provide degree bounds for optimal controllers.

In this paper we consider the problem of designing $\mathcal{H}_2$ optimal decentralized controllers for poset causal systems. The main contributions in the paper are as follows:

- We show a certain crucial separability property of the problem under consideration. This result is outlined in Theorem 1. This makes it possible to decompose the decentralized control problem over posets into a set of standard centralized control problems.

- We give an explicit state-space solution procedure in Theorem 2. To construct the solution, one needs to solve Riccati equations (corresponding to the different sub-problems). Using the solutions of these Riccati equations, we construct certain block matrices and provide a state-space realization of the controller.
- We provide bounds on the degree of the optimal controller in terms of a parameter $\sigma_{\mathcal{P}}$ that depends only on the order-theoretic structure of the poset.
- In Theorem 3 we describe the structural form of the optimal controller. We introduce a novel pair of transfer functions (Φ, Γ) which are inverses of each other, and which capture the estimation structure in the optimal controller. The object Φ corresponds to direct estimates of unknown states, whereas Γ corresponds to estimation errors. These objects enable us to decompose the optimal controller into local controllers, shedding light on its structure.

In an interesting paper by Swigart and Lall [6], the authors consider the $\mathcal{H}_2$ optimal controller synthesis problem over a particular poset with two nodes. Their approach is restricted to the finite time horizon setting, and uses a particular decomposition of certain optimality conditions. In this setting, they synthesize optimal controllers and provide insight into the structure of the optimal controller. By using our new separability condition (which is related to their decomposition property, but which we believe to be more fundamental) we significantly generalize those results in this paper. We provide a solution for *all posets* and for the *infinite time horizon*. In recent work [4], Rotkowitz and Lall proposed a state-space technique to solve $\mathcal{H}_2$ optimal control problems for quadratically invariant systems (which could be used for poset-causal systems). An important drawback of their reformulation is that one would need to solve larger Riccati equations. Our approach for poset causal systems is more efficient computationally. Moreover, our approach also provides insight into the structure of the optimal controllers.

The rest of this paper is organized as follows. In Section II we introduce the necessary preliminaries regarding posets, the control theoretic framework and notation. In Section III we describe our solution strategy. In Section IV we present the main results. We devote Section V to a discussion of the main results. Due to space constraints, we omit proofs of the main results in this paper.

II. P

A. Posets

Definition 1: A partially ordered set (or *poset*) $\mathcal{P} = (P, \leq)$ consists of a set P along with a binary relation $\leq$ which has the following properties:

- 1) $a \leq a$ (reflexivity),
- 2) $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry),
- 3) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

In this paper we will deal with finite posets (i.e. $|P|$ is finite).

Example 1: An example of a poset with three elements (i.e., $P = \{1, 2, 3\}$) with order relations $1 \leq 2$ and $1 \leq 3$ is shown in Figure 1(b).

Fig. 1. Hasse diagrams of some posets.

Let $\mathcal{P} = (P, \leq)$ be a poset and let $p \in P$. We define $\uparrow p = \{q \in P | p \leq q\}$ (in order theory, this set is known as a *principal filter*). Let $\uparrow\uparrow p = \{q \in P | p \leq q, q \neq p\}$ (called the *strict principal filter*). Similarly, let $\downarrow p = \{q \in P | q \leq p\}$ (called a *principal ideal*). Define an interval $[i, j] = \{p \in P | i \leq p \leq j\}$. A *chain* is a subset $C \subseteq P$ which is *totally ordered* (i.e. any two elements of C are comparable). In the preceding example, $\uparrow 1 = \{1, 2, 3\}$, whereas $\uparrow 2 = \{2\}$. The set $\{1, 2\}$ is a chain.

Definition 2: Let $(P, \leq)$ be a poset. Let $\mathbb{Q}$ be a ring. The set of all functions $f : P \times P \rightarrow \mathbb{Q}$ with the property that $f(x, y) = 0$ if $y \not\leq x$ is called the *incidence algebra* of $\mathcal{P}$ over $\mathbb{Q}$. It is denoted by $I(\mathcal{P})$.

When the poset $\mathcal{P}$ is finite, the set of functions in the incidence algebra may be thought of as matrices with a specific sparsity pattern given by the order relations of the poset. An example of a member in $I(\mathcal{P})$ for the poset from Example 1 (Fig. 1(b)) is:

$$\zeta_{\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Given two functions $f, g \in I(\mathcal{P})$, their sum $f + g$ and scalar multiplication cf are defined as usual. The product $h = f \cdot g$ is defined by $h(x, y) = \sum_{z \in P} f(x, z)g(z, y)$. Note that above definition of function multiplication is made so that it is consistent with standard matrix multiplication.

Lemma 1: Let $\mathcal{P}$ be a poset. Under the usual definition of addition and multiplication as defined in (1) the incidence algebra is an associative algebra (i.e. it is closed under addition, scalar multiplication and function multiplication).

Proof: The proof is standard, see for example [5]. ■

B. Control Theoretic Preliminaries

We consider the following state-space system in discrete time:

$$\begin{aligned} x[t+1] &= Ax[t] + w[t] + Bu[t] \\ z[t] &= Cx[t] + Du[t]. \end{aligned} \quad (1)$$

While we present the discrete time case here, we wish to emphasize that analogous results hold in continuous time in a straightforward manner. In this paper we consider what we will call *poset causal systems*. We think of the system matrices (A, B, C, D) to be partitioned into blocks in the following natural way. Let $\mathcal{P} = (P, \leq)$ be a poset with $P = \{1, \dots, s\}$. We think of this system as being divided into s sub-systems, with sub-system i having some states $x_i[t] \in \mathbb{R}^{n_i}$, and control inputs $u_i[t] \in \mathbb{R}^{m_i}$ for $i \in \{1, \dots, s\}$. The external output is $z[t] \in \mathbb{R}^p$. The signal $w[t]$ is a disturbance signal. (To use certain standard state-space factorization results, we assume that $C^T D = 0$ and $D^T D > 0$). The states and inputs are partitioned in the natural way such that the sub-systems correspond to elements of the poset $\mathcal{P}$ with $x[t] = [x_1[t] | x_2[t] | \dots | x_s[t]]^T$, and $u[t] = [u_1[t] | u_2[t] | \dots | u_s[t]]^T$. This naturally partitions the matrices A, B, C, D into appropriate blocks so that $A = [A_{ij}]_{i,j \in P}$, $B = [B_{ij}]_{i,j \in P}$, $C = [C_j]_{j \in P}$ (partitioned into columns), $D = [D_j]_{j \in P}$. (We will throughout deal with matrices at this block-matrix level, so that A_{ij} will unambiguously mean the (i, j) block of the matrix A .) Using these block partitions, one can define the incidence algebra at the block matrix level in the natural way. We denote by $\mathcal{I}_A(\mathcal{P}), \mathcal{I}_B(\mathcal{P})$ the block incidence algebras corresponding to the block partitions of A and B .

Definition 3: We say that a state-space system is $\mathcal{P}$ -*poset causal* (or simply *poset causal*) if $A \in \mathcal{I}_A(\mathcal{P})$ and $B \in \mathcal{I}_B(\mathcal{P})$.

We refer to such systems as being *poset causal* due to the following interesting property among the sub-systems. If an input is applied to sub-system i via u_i , the effect of the input is seen by the states x_j for all sub-systems $j \in \uparrow i$. Thus $\uparrow i$ may be seen as the cone of influence of input i . We refer to this causality-like property as *poset causality*. For most of this paper we will deal with poset causal systems (with respect to some arbitrary but fixed finite poset $\mathcal{P}$).

It is possible to show that a poset-causal system (1) is stabilizable by a poset-causal controller $K \in \mathcal{I}(\mathcal{P})$ if and only if the (A_{ii}, B_{ii}) are stabilizable for all $i \in P$. In this paper, we make the following important assumption about the stabilizability of the sub-systems:

Assumption 1

Given the poset causal system of the form (1), we assume that the sub-systems (A_{ii}, B_{ii}) are stabilizable for all $i \in P$.

A direct consequence of this assumption is that it allows us to reduce the problem under consideration to a problem with a stable plant, without loss of generality.

Lemma 2 (Stabilizability): Given a poset causal system of the form (1). Suppose that (A_{ii}, B_{ii}) are stabilizable for all $i \in P$. Then without loss of generality, one can assume that the overall plant is stable.

Given this assumption, we can assume henceforth (without

loss of generality) that A is in fact stable. In what follows, we solve the problem by solving a set of Riccati equations (corresponding to certain sub-problems), and producing an optimal Youla parameter Q^* (from which the controller can be recovered). Assumption 1 allows us to ensure that all of these Riccati equations have well-defined solutions, and that the parameter Q^* renders the closed loop internally stable.

The system (2) may be viewed as a map from the inputs w, u to outputs z, x via

$$\begin{aligned} z &= P_{11}w + P_{12}u \\ x &= P_{21}w + P_{22}u \end{aligned}$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} C(zI - A)^{-1} & C(zI - A)^{-1}B + D \\ (zI - A)^{-1} & (zI - A)^{-1}B \end{bmatrix} \quad (2)$$

$$= \left[\begin{array}{c|cc} A & I & B \\ \hline C & 0 & D \\ I & 0 & 0 \end{array} \right].$$

A controller $u = Kx$ induces a map from w to z via:

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w.$$

Let P_{cl} denote the closed loop map. The objective function of interest is the $\mathcal{H}_2$ norm [9] of the system which we denote by $\|\cdot\|$.

Information Constraints on the Controller

What makes decentralized control problems hard is the presence of constraints of the form $K \in S$ (typically, S is a subspace of sparsity constraints on the controller). This paper focuses on the problem of synthesizing a poset-causal controller. This translates into a sparsity (subspace) constraint on the set S . The decentralization constraint of interest in this paper is one where the controller mirrors the structure of the plant, and is therefore also in the block incidence algebra $\mathcal{I}_K(\mathcal{P})$. This translates into the requirement that $S = \mathcal{I}_K(\mathcal{P})$, i.e. that input u_i only has access to x_j for $j \in \downarrow i$.

Problem Statement

Given the poset causal system (2) with poset $(P, \leq)$, $|P| = s$, solve the optimization problem:

$$\begin{aligned} &\text{minimize} && \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|^2 \\ &\text{subject to} && K \in S \\ &&& K \text{ stabilizing.} \end{aligned} \quad (3)$$

Note that for poset causal systems, the matrices A and B are both in the block incidence algebra. As a consequence of (2), P_{21} and P_{22} are also in the incidence algebra. This structure, which follows from the closure properties of an incidence algebra, will be extensively used.

Problem (3) as stated has a nonconvex objective function. Typically [5], [4], this is convexified by a bijective change of parameters given by $R := K(I - P_{22}K)^{-1}$. For poset causal systems with state feedback we can use a slightly different parametrization. Since $P_{21}, P_{22} \in \mathcal{I}(\mathcal{P})$ the optimization problem (3) maybe be reparametrized as follows. Set $Q :=$

$K(I - P_{22}K)^{-1}P_{21}$. Note that the map $K \mapsto K(I - P_{22}K)^{-1}P_{21}$ is bijective (provided the inverse exists). Given Q , K can be recovered using

$$K = QP_{21}^{-1}(I + P_{22}QP_{21}^{-1})^{-1}. \quad (4)$$

Moreover, since I, P_{21}, P_{22} all lie in the incidence algebra, $K \in S$ if and only if $Q \in S$. Using this reparametrization the optimization problem can be recast as:

$$\begin{aligned} &\text{minimize} && \|P_{11} + P_{12}Q\|^2 \\ &\text{subject to} && Q \in S \\ &&& Q \text{ stabilizing.} \end{aligned} \quad (5)$$

(We note that $P_{21} = (zI - A)^{-1}$, and hence (4), which involves P_{21}^{-1} may potentially be improper. However, we can *prove* that for the optimal Q in (5), this expression is proper and corresponds to a rational controller K .)

Given a system (A, B, C, D) with P_{11}, P_{12} given by (2) the solution to the centralized problem

$$\begin{aligned} &\text{minimize} && \|P_{11} + P_{12}Q\|^2 \\ &&& Q \text{ stabilizing} \end{aligned} \quad (6)$$

is given by a Riccati equation [9]. Let P, K be obtained from the unique positive definite solution to the Riccati equation:

$$P = C^T C + A^T P A - A^T P B (D^T D + B^T P B)^{-1} B^T P A \quad (7)$$

$$K = (D^T D + B^T P B)^{-1} B^T P A \quad (8)$$

$$\begin{aligned} Q &= -K(zI - (A - BK))^{-1} \\ &= \left[\begin{array}{c|c} A - BK & I \\ \hline -K & 0 \end{array} \right]. \end{aligned} \quad (9)$$

(We will often refer to the trio of equations (7), (8), (9) by $(K, Q, P) = \text{Ric}(A, B, C, D)$.)

C. Notation

Given a matrix Q , let $Q(j)$ denote the j^{th} column of Q . Given a sparsity constraint of the form $Q \in S$ let S^j be the set of column vectors with sparsity constraints inherited from the columns of S in the natural way:

$$S^j = \{Q(j) | Q(j)_i = 0 \text{ if } S_{ij} = 0\}.$$

(Note that $Q(j)_i = Q_{ij}$, from which this follows). Given the data (A, B, C, D) , we will often need to consider sub-matrices or embed a sub-matrix into a full dimensional matrix by zero padding. Some notation for that purpose we will use is the following:

- 1) Define $Q^{\uparrow j} = [Q_{ij}]_{i \in \uparrow j}$ (so that it is the j^{th} column shortened to include only the nonzero entries).
- 2) Also define $A(\uparrow j) = [A(i)]_{i \in \uparrow j}$ so that it is the sub-matrix of A containing exactly those columns corresponding to the set $\uparrow j$.
- 3) Define $A(\uparrow j, \uparrow j) = [A_{kl}]_{k, l \in \uparrow j}$ so that it is the $(\uparrow j, \uparrow j)$ sub-matrix of A (containing exactly those rows and columns corresponding to the set $\uparrow j$).
- 4) Sometimes, given a block $|\uparrow j| \times |\uparrow j|$ matrix we will need to embed it into a block matrix indexed by the original poset (i.e. a $s \times s$ matrix) by padding it with

zeroes. Given K (a block $|\uparrow j| \times |\uparrow j|$ matrix) we define:

$$[\hat{K}]_{lm} = \begin{cases} K_{lm} & \text{if } l, m \in \uparrow j \\ 0 & \text{otherwise.} \end{cases}$$

- 5) $E_i = [0 \ \dots \ I \ \dots \ 0]^T$ be the tall matrix (indexed with the elements of the poset) with an identity in the i^{th} row.
- 6) Let $S \subseteq P$. Define $E_S = [E_i]_{i \in S}$. Note that given a block $s \times s$ matrix M , $ME_{\uparrow j} = M(\uparrow j)$ is a matrix containing the columns indexed by $\uparrow j$.
- 7) Given matrices $A_i, i \in P$, we define the block diagonal matrix:

$$\text{diag}(A_i) = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_s \end{bmatrix}.$$

Note that the poset $\mathcal{P}$ has a linear extension (i.e. a total order on P which is consistent with the partial order $\leq$). For convenience, we fix such a linear extension, and all indexing of our matrices throughout the paper will be consistent with this linear extension (so that members of the incidence algebra are lower triangular).

Example 2: Let $\mathcal{P}$ be the poset shown in Fig. 1(d). Let $A \in \mathcal{I}(\mathcal{P})$. Note that $\uparrow 2 = \{2, 4\}$. Then

$$A(\uparrow 2) = \begin{bmatrix} A_{12} & A_{14} \\ A_{22} & A_{24} \\ A_{32} & A_{34} \\ A_{42} & A_{44} \end{bmatrix}, \text{ and } A(\uparrow 2, \uparrow 2) = \begin{bmatrix} A_{22} & A_{24} \\ A_{42} & A_{44} \end{bmatrix}.$$

III. S S

In this section we illustrate the main solution strategy via a simple example. Consider the decentralized control problem for the poset in Example 1 (Fig. 1(b)). Using the canonical reformulation (6) the problem may be recast as:

$$\begin{aligned} \min \quad & \left\| P_{11} + P_{12} \begin{bmatrix} Q_{11} & 0 & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & 0 & Q_{33} \end{bmatrix} \right\|^2 \\ \text{s.t.} \quad & Q \text{ stabilizing.} \end{aligned}$$

Note that $P_{12}(\uparrow 1) = P_{12}$, $P_{12}(\uparrow 2) = P_{12}(2)$ (second column of P_{12}), and $P_{12}(\uparrow 3) = P_{12}(3)$. Similarly $Q^{\uparrow 1} = [Q_{11} \ Q_{21} \ Q_{31}]^T$, $Q^{\uparrow 2} = Q_{22}$, and $Q^{\uparrow 3} = Q_{33}$. Due to the column-wise separability of the $\mathcal{H}_2$ norm, the problem can be recast as:

$$\begin{aligned} \min. \quad & \left\| P_{11}(1) + P_{12} \begin{bmatrix} Q_{11} \\ Q_{21} \\ Q_{31} \end{bmatrix} \right\|^2 + \|P_{11}(2) + P_{12}(2)Q_{22}\|^2 \\ & + \|P_{11}(3) + P_{12}(3)Q_{33}\|^2 \\ \text{s.t.} \quad & Q \text{ stabilizing.} \end{aligned}$$

Since the sets of variables appearing in each of the three quadratic terms is different, the problem now may be decoupled into three separate sub-problems, each of which is a standard centralized control problem. For instance, the solution to the second sub-problem can be obtained by noting the realizations of $P_{11}(2)$ and $P_{12}(2)$ and then using (9). In

this instance,

$$(K, Q_{22}^*, P) = \text{Ric}(A_{22}, B_{22}, C_2, D_2).$$

In a similar way, the entire matrix Q^* can be obtained, and by design $Q^* \in S$ (and is stabilizing). To obtain the optimal K^* , one can use (4). In fact, using (4) it is possible to give an explicit state-space formula for K^* , this is the main content of Theorem 2 in the next section.

IV. M R

In this section, we present the main results of the paper.

A. Problem Decomposition and Computational Procedure

Theorem 1 (Decomposition Theorem): A problem of the form (5) is equivalent to solving:

$$\begin{aligned} \text{minimize} \quad & \|P_{11}(j) + P_{12}(\uparrow j)Q^{\uparrow j}\|^2 \\ \text{subject to} \quad & Q^{\uparrow j} \text{ stabilizing} \\ & \text{for all } j \in P. \end{aligned} \quad (10)$$

Theorem 1 is essentially the first step towards a state-space solution. The advantage of this equivalent reformulation of the problem is that we now have $s = |P|$ sub-problems, each over a different set of variables (thus the problem is *decomposed*). Moreover, each sub-problem corresponds to a particular *standard centralized* control problem, and thus the optimal Q in (3) can be computed by simply solving each of these sub-problems.

Our next theorem provides an efficient computational technique to obtain the required state-space solution. To obtain the solution, one needs to solve Riccati equations corresponding to the sub-problems we saw in Theorem 1. We combine these solutions to form certain simple block matrices, and after simple LFT transformations, one obtains the optimal controller K^* .

Before we state the theorem, we introduce some relevant notation. Let

$$(K(\uparrow j, \uparrow j), Q(j), P(j)) = \text{Ric}(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)).$$

Let $\mathcal{A}(j) = A(\uparrow j, \uparrow j) - B(\uparrow j, \uparrow j)K(\uparrow j, \uparrow j)$. We define the following quantities:

$$\begin{aligned} A_\Phi(j) &= E_{\{2, \dots, |\uparrow j|\}}^T \mathcal{A}(j) E_{\{2, \dots, |\uparrow j|\}} \\ A_\Phi &= \text{diag}(A_\Phi(j)) \\ B_\Phi(j) &= E_{\{2, \dots, |\uparrow j|\}}^T \mathcal{A}(j) E_1 \end{aligned}$$

$$\begin{aligned} B_\Phi &= \text{diag}(B_\Phi(j)) \\ C_\Phi &= \begin{bmatrix} E_{\uparrow \uparrow 1} & E_{\uparrow \uparrow 2} & \dots & E_{\uparrow \uparrow s} \end{bmatrix}. \end{aligned}$$

(Using the notation developed earlier, $A_\Phi(j)$ is the sub-matrix of $\mathcal{A}(j)$ with the first set of rows and columns corresponding to sub-system j being eliminated.)

Theorem 2 (Computation of Optimal Controller): Given the poset causal system (1). Let

$$(K(\uparrow j, \uparrow j), Q(j), P(j)) = \text{Ric}(A(\uparrow j, \uparrow j), B(\uparrow j, \uparrow j), C(\uparrow j), D(\uparrow j)).$$

Construct the following matrices:

$$\mathbf{B} = \begin{bmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & E_1 \end{bmatrix},$$

$$T = \text{diag} \left(\begin{bmatrix} E_2 & \dots & E_{|\uparrow j|} \end{bmatrix} \right),$$

$$C_Q = \begin{bmatrix} -\hat{K}(\uparrow 1, \uparrow 1)E_{\uparrow 1} & \dots & -\hat{K}(\uparrow s, \uparrow s)E_{\uparrow s} \end{bmatrix}.$$

(In $\mathbf{B}$, the j^{th} diagonal block E_1 has $|\uparrow j|$ number of block rows. To be precise, it is a $(\sum_{k \in \uparrow j} n_k) \times n_j$ matrix with the first $n_j \times n_j$ block as the identity and the rest zeroes. T is simply the standard basis for the kernel of $\mathbf{B}^T$.) Then the optimal controller is given by

$$K^* = \left[\frac{A_\Phi - B_\Phi C_\Phi}{C_Q(T - \mathbf{B}C_\Phi)} \mid \frac{B_\Phi}{C_Q \mathbf{B}} \right]. \quad (11)$$

Lastly, the controller $K^* \in \mathcal{S}$ and is stabilizing.

Recall that n_i denotes the degree of the i^{th} sub-system in (1). Let $n_{\max} = \max_i n_i$ be the largest degree of the sub-systems. Let $n(\uparrow i) = \sum_{j \in \uparrow i} n_j$. Let $\sigma_P = \sum_{j \in P} |\uparrow j|$ (note that this is a purely combinatorial quantity, dependent only on the poset). As we mentioned in the introduction, one of the advantages of state-space techniques is that they provide graceful degree bounds for the optimal controller. As a consequence of Theorem 2 we have the following:

Corollary 1 (Degree Bounds): The degree of the optimal controller d^* is bounded above by

$$d^* \leq \sum_{j \in P} (n(\uparrow j) - n_j).$$

In particular, $d^* \leq (\sigma_P - s)n_{\max}$.

B. Structural Form of the Optimal Controller

Having established the computational aspects, we now turn to some structural aspects of the optimal controller. We first introduce a pair of very important objects (Φ, Γ) . Define the block $s \times s$ transfer function matrices (Φ, Γ) via:

$$\Phi = \left[\frac{A_\Phi}{C_\Phi} \mid \frac{B_\Phi}{I} \right], \quad \Gamma = \left[\frac{A_\Phi - B_\Phi C_\Phi}{C_\Phi} \mid \frac{-B_\Phi}{I} \right].$$

Note that both Φ and Γ are invertible (since their “ D ” matrices are equal to I), and in fact, they are inverses of each other, i.e., $\Gamma\Phi = \Phi\Gamma = I$. We will denote the entries $\Phi_{ij} = \Phi_{i \leftarrow j}$ and similarly $\Gamma_{ij} = \Gamma_{i \leftarrow j}$. We note that $\Phi_{i \leftarrow i} = \Gamma_{i \leftarrow i} = I$ (this can be seen from the fact that the corresponding entries in the “ C ” matrices of the transfer functions is zero). One can show (though it is not immediately obvious from the state-space realizations), that $\Phi, \Gamma \in \mathcal{I}(\mathcal{P})$.

It is possible to show that this state-space realization of $\Phi_{i \leftarrow k}$, in fact, corresponds to a *specific observer* that, given a state or signal at x_k at subsystem k , predicts the state or signal x_l at subsystem l . These predictions, however, will not be perfect. The object Γ has the interesting dual interpretation of capturing these prediction errors. The vector $e = \Gamma x$ is a vector of certain generalized errors between the true state and the predicted state. Before stating the next theorem, we

introduce the matrix K_Φ , which is defined column-wise via:

$$K_\Phi(j) = \hat{K}(\uparrow j, \uparrow j)\Phi(j).$$

Theorem 3 (Structure of Optimal Controller): The optimal controller is of the form:

$$u[t] = -K_\Phi \Gamma x[t] = -K_\Phi e[t].$$

Remark The above structural form of the controller suggests that the controller at i forms different error estimates of all the states e_l , and fuses them together with appropriate gains to obtain the control input u_i . There are two types of operations, those corresponding to Φ which correspond to prediction and those corresponding to Γ which correspond to error terms. Note that while $\hat{x}_j = \Phi_{j \leftarrow i} x_i$ forms a *direct estimate* of x_j from x_i , $e_j = \Gamma_{j \leftarrow i} x_i$ is a filter that aggregates this estimate along the different paths to compute a prediction error.

V. D E

A. The Nested Case

Consider the poset on two elements $\mathcal{P} = (\{1, 2\}, \leq)$ with the only order relation being $1 \leq 2$ (Fig. 1(a)). This is the poset corresponding to the communication structure in the “Two-Player Problem” considered in [6]. We show that their results are a specialization of our general results in Section IV restricted to this particular poset.

We begin by noting that from the problem of designing a nested controller can be recast as:

$$\begin{aligned} \min \quad & \left\| P_{11} + P_{12} \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} \right\|^2 \\ \text{s.t.} \quad & Q \text{ stabilizing.} \end{aligned}$$

By Theorem 1 this problem can be recast as:

$$\begin{aligned} \min. \quad & \left\| P_{11}^1 + P_{12} \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \right\|^2 + \left\| P_{11}^2 + P_{12}^2 Q_{22} \right\|^2 \\ \text{s.t.} \quad & Q \text{ stabilizing.} \end{aligned}$$

We wish to compare this to the results obtained in Swigart and Lall [6]. It is possible to obtain precisely this same decomposition in the finite time horizon where the $\mathcal{H}_2$ norm can be replaced by the Frobenius norm and separability can be used to decompose the problem. For each of the sub-problems, the corresponding optimality conditions may be written (since they correspond to simple constrained-least squares problems). These optimality conditions correspond exactly to the decomposition of optimality conditions they obtain (the crucial Lemma 3 in their paper). We point out that the decomposition is a simple consequence of the separability of the Frobenius norm.

Let us now examine the structure of the optimal controller via Theorem 3. Note that $\uparrow 1 = \{1, 2\}$ and $\uparrow 2 = \{2\}$. Based on Theorem 2, we are required to solve $(P(1), K) = \text{Ric}(A, B, C, D)$, and $(P(2), J) = \text{Ric}(A_{22}, B_{22}, C_2, D_2)$. Noting that in this example $\Gamma_{2 \leftarrow 1} = -\Phi_{2 \leftarrow 1}$, a straightforward

application of Theorem 3 yields the following:

$$\begin{aligned} u_1[t] &= -(K_{11} + K_{12}\Phi_{2\leftarrow 1})x_1[t] \\ u_2[t] &= -(K_{21} + K_{22}\Phi_{2\leftarrow 1})x_1[t] - J(x_2[t] - \Phi_{21}x_1[t]), \end{aligned}$$

which is precisely the structure of the optimal controller given in [6] (though they present the results in a finite-time horizon framework). It is possible to show (as Swigart et. al indeed do in [6]) that $\Phi_{2\leftarrow 1}$ is an predictor of x_2 based on x_1 . Thus the controller for u_1 predicts the state of x_2 from x_1 , uses it as a surrogate for the actual state, and uses the gain K_{21} in the feedback loop. The controller for u_2 (perhaps somewhat surprisingly) also estimates the state x_2 based on x_1 using $\hat{x}_2 = \Phi_{21}x_1$ (this can be viewed as a “simulation” of the controller for u_1). The prediction error for state 2 is then given by $e_2 = x_2 - \hat{x}_2 = x_2 - \Phi_{21}x_1$. The control law for u_2 may be rewritten as

$$u_2 = -(K_{21}x_1 + K_{22}\hat{x}_2 + Je_2).$$

Thus this controller uses predictions of x_2 based on x_1 along with prediction errors in the feedback loop. This prediction of states higher up in the poset is prevalent in such poset-causal systems, which results in somewhat larger order controllers.

Analogous to the results in [6], it is possible to derive the results in this paper for the finite time horizon case (this is a special case corresponding to FIR plants in our discrete-time setup). We do not devote attention to the finite time horizon case in this paper due to space constraints, but just mention that similar results follow in a straightforward manner.

B. Discussion Regarding Computational Complexity

Note that the main computational step in the procedure presented in Theorem 2 is the solution of the s sub-problems. The j^{th} sub-problem requires the solution of a Riccati equation of size at most $|\uparrow j|n_{\max} = O(s)$ (when the degree $n_{\max}$ is fixed). Assuming the complexity of solving a Riccati equation using linear algebraic techniques is $O(s^4)$ [2] the complexity of solving s of them is at most $O(s^5)$. We wish to compare this with the only other known state-space technique that works on all poset-causal systems, namely the results of Rotkowitz and Lall [4]. In this paper, they transform the problem to a standard centralized problem using Kronecker products. In the final computational step, one would be required to solve a single large Riccati equation of size $O(s^2)$, resulting in a computational complexity of $O(s^8)$.

C. Discussion Regarding Degree Bounds

It is insightful to study the asymptotics of the degree bounds in the setting where the sub-systems have fixed degree and the number of sub-systems s grows. As an immediate consequence of the corollary, the degree of the optimal controller (assuming that the degree of the sub-systems $n_{\max}$ is fixed) is at most $O(s^2)$ (since $n(\uparrow j) \leq s$). In fact, the asymptotic behavior of the degree can be sub-quadratic. Consider a poset $(\{1, \dots, s\}, \leq)$ with the only order relations being $1 \leq i$ for all i . Here $|\uparrow 1| = s$, and $|\uparrow i| = 1$

for all $i \neq 1$. Hence, $\sum_j |\uparrow j| - s \leq s$, and thus $d^* \leq sn_{\max}$. In this sense, the degree of the optimal controller is governed by the poset parameter $s\sigma_p$.

D. Theorem 3 and Decomposition Structure of Controller

Theorem 3 implies an interesting decomposition structure of the optimal controller. Recall that we have the following form of the optimal controller: $u[t] = -K_\Phi \Gamma x[t]$. Recalling that $e = \Gamma x$ and $\Phi(j)$ is the j^{th} column of Φ the preceding maybe expression may be decomposed in the following form: $u = -\sum_{j \in P} \hat{K}(\uparrow j, \uparrow j)\Phi(j)e_j$. The signal $\hat{K}(\uparrow j, \uparrow j)\Phi(j)e_j$ can be interpreted as a local control signal or a local control law used by subsystem j . The local control law takes the error signal e_j , forms prediction errors via $\Phi_{ij}e_j$, and then uses the static feedback obtained via the j^{th} Riccati equation $K(\uparrow j, \uparrow j)$ to form the control signal. The final control signal is the sum of local control signals corresponding to the different subsystems. Since the individual error signals e_i at each subsystem (in the case of minimal elements, these are states) are multiplied by $\Phi(j)$ (which plays the role of propagating errors), and these are then multiplied by the feedback gains $\hat{K}(\uparrow j, \uparrow j)$ to form the local control signal, the final control law is an *aggregation of local control laws* at the different subsystems. As an example, consider the poset from Fig. 1(d). (For simplicity, we let $K_j = \hat{K}(\uparrow j, \uparrow j)$). The control law may be decomposed into local controllers as:

$$u = K_1 \begin{bmatrix} I \\ \Phi_{21} \\ \Phi_{31} \\ \Phi_{41} \end{bmatrix} x_1 + K_2 \begin{bmatrix} 0 \\ I \\ 0 \\ \Phi_{42} \end{bmatrix} e_2 + K_3 \begin{bmatrix} 0 \\ 0 \\ I \\ \Phi_{43} \end{bmatrix} e_3 + K_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} e_4.$$

Each term in the above expression has the natural interpretation of a local control signal corresponding to a subsystem, and the final controller can be viewed as an aggregation of these.

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- [1] V. D. Blondel and J. N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36(9):1249–1274, 2000.
- [2] R. B. Carroll and D. A. Brask. A practical implementation for solutions to the algebraic matrix Riccati equation in a LQCM setting. *Journal of Economic Dynamics and Control*, 23(1):1 – 7, 1998.
- [3] Y.-C. Ho and K.-C. Chu. Team decision theory and information structures in optimal control problems-part I. *IEEE Transactions on Automatic Control*, 17(1):15–22, 1972.
- [4] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2006.
- [5] P. Shah and P. A. Parrilo. A partial order approach to decentralized control. In *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008.
- [6] J. Swigart and S. Lall. An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator. *Submitted to American Control Conference*, (available at <http://junction.stanford.edu/~lall/publications/tpss>), 2010.
- [7] P. Voulgaris. Control of nested systems. In *Proceedings of the American Control Conference*, 2000.
- [8] K. Zhou and J. C. Doyle. *Essentials of Robust Control*. Prentice Hall, 1998.