Pattern Avoidance for Alternating Permutations and Reading Words of Tableaux

by

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Abstract

We consider a variety of questions related to pattern avoidance in alternating permutations and generalizations thereof. We give bijective enumerations of alternating permutations avoiding patterns of length 3 and 4, of permutations that are the reading words of a "thickened staircase" shape (or equivalently of permutations with descent set $\{k, 2k, 3k, \ldots\}$) avoiding a monotone pattern, and of the reading words of Young tableaux of any skew shape avoiding any of the patterns 132, 213, 312, or 231. Our bijections include a simple bijection involving binary trees, variations on the Robinson-Schensted-Knuth correspondence, and recursive bijections established via isomorphisms of generating trees.

Thesis Supervisor: Alexander Postnikov
Title: Associate Professor

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Chapter 1

Introduction

The problem of enumerating permutations avoiding a pattern or collection of patterns dates to the work of MacMahon [24] in the 1910s and is present in the works of Knuth [17] and others. The first systematic study of pattern avoidance was the paper [30] of Simion and Schmidt, which also appears to have established the "pattern" terminology. Since then, the field has exploded, with numerous extensions and variations. In [25], Mansour initiated the study of pattern avoidance in *alternating* permutations, giving a number of generating-functional proofs of enumerative pattern avoidance results for alternating permutations avoiding the pattern 132 and possibly other patterns. In this thesis, we extend this line of research to consider a broader class of enumerative questions related to pattern avoidance in alternating permutations. Our main results demonstrate interesting connections between pattern avoidance in alternating permutations and in the set of all permutations, show that pattern avoidance in alternating permutations exhibits attractive enumerative properties, and provides a fascinating general setting for the study of pattern avoidance that includes pattern avoidance in alternating permutations and pattern avoidance in all permutations as special cases.

In the remainder of this introduction, we lay out the definitions that we use throughout this thesis, provide a few historical comments and a discussion of related work in the literature, and give an overview of the main results of this thesis.

1.1 Permutations, alternating permutations, patterns, and tableaux

A **permutation** w of length n is a sequence containing each element of $[n] = \{1, 2, \ldots, n\}$ exactly once. The set of permutations of length n is denoted S_n. Given a word $w = w_1 w_2 \cdots w_n$ and a permutation $p = p_1 \cdots p_k \in S_k$, we say that w **contains** p **as a pattern** if there exists a set of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that the subsequence $w_{i_1} w_{i_2} \cdots w_{i_k}$ of w is **order-isomorphic** to p, that is, if for all $\ell, m \in [k]$ we have $w_{i_\ell} < w_{i_m}$ if and only if $p_\ell < p_m$. Otherwise, w is said to **avoid** p. Given a pattern p and a set S of permutations, we denote by $S(p)$ the set of elements of S that avoid p. For example, $S_n(123)$ is the set of permutations of length n avoiding the pattern 123, that is, the set of permutations of length n with no three-term increasing subsequence. In the special case that $p = 12\cdots k$ is an identity permutation, we will call p **monotone** and may refer to **monotone pattern avoidance** when we consider questions relating to avoidance of a monotone pattern.

A permutation $w = w_1 w_2 \cdots w_n$ is **alternating** if $w_1 < w_2 > w_3 < w_4 > \ldots$. Note that in the terminology of [37], these "up-down" permutations are *reverse alternating* while alternating permutations are "down-up" permutations. Luckily, this difference in convention doesn't matter: any pattern result on either set can be translated into a result on the other via **complementation**, i.e., by considering w^c such that $w^c_i = n+1-w_i$. Then results for the pattern 132 and up-down permutations would be replaced by results 312 and down-up permutations, and so on. We denote by A_n the set of alternating permutations of length n and by A'_n the set of reverse-alternating (that is, down-up) permutations of length n.

A **partition** is a weakly decreasing, finite sequence of nonnegative integers. We consider two partitions that differ only in the number of trailing zeroes to be the same. We write partitions in sequence notation, as $\langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$, or to save space, with exponential notation instead of repetition of equal elements. Thus, the partition $\langle 5, 5, 3, 3, 2, 1 \rangle$ may be abbreviated $\langle 5^2, 3^2, 2, 1 \rangle$. If the sum of the entries of λ is equal to m then we write $\lambda \vdash m$.

Given a partition $\lambda = \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$, the **Young diagram** of shape λ is a left-justified array of $\lambda_1 + \ldots + \lambda_n$ boxes with λ_1 in the first row, λ_2 in the second row, and so on. We identify each partition with its Young diagram and speak of them interchangeably. Given two Young diagrams λ and μ such that the diagram of μ fits inside the diagram of λ when the diagrams are arranged so that their first rows and first columns coincide, the **skew Young diagram** λ/μ is the diagram that results when we remove the boxes of μ from those of λ. If λ/μ is a skew Young diagram with m boxes, a **standard Young tableau** of shape λ/μ is a filling of the boxes of λ/μ with $[m]$ so that each element appears in exactly one box, and entries increase along rows and columns. We denote by $\mathrm{sh}(T)$ the shape of the standard Young tableau T, by $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of shape λ, and by $f^{\lambda/\mu}$ the number of standard Young tableaux of shape λ/μ, so $f^\lambda = |\,\mathrm{SYT}(\lambda)|$. We draw all diagrams and tableaux in "English notation", with the first row on top and the first column at the left, and we identify boxes using matrix coordinates, so the box in the first row and second column is numbered $(1, 2)$.

Given a standard Young tableau, we can form the **reading word** of the tableau by reading the last row from left to right, then the next-to-last row, and so on. In English notation, this means we read rows left to right, starting with the bottom row and working up. An example of a tableau and its reading word is given in Figure 1-1. The other "usual" reading order, from right to left then top to bottom in English notation, is simply the reverse of our reading order. Consequently, any pattern avoidance result in our case carries over to the other reading order by taking the **reverse** of all permutations and patterns involved, i.e., by replacing $w = w_1 \cdots w_n$ with $w^r = w_n \cdots w_1$.

We make note of two more operations on Young diagrams and tableaux. Given a partition λ, the **conjugate partition** λ' is defined so that the ith row of λ' has

				2	3	6
			1	5	9	
		4	11	12		
	8	13	15			
7	10	14				

Figure 1-1: A standard skew Young tableau whose reading word is the permutation 7 10 14 8 13 15 4 11 12 1 5 9 2 3 6.

the same length as the ith column of λ for all i. The conjugate of a skew Young diagram λ/μ is defined by $(\lambda/\mu)' = \lambda'/\mu'$. Given a standard skew Young tableau T of shape λ/μ, the conjugate tableau T' of shape $(\lambda/\mu)'$ is defined to have the entry a in box (i, j) if and only if T has the entry a in box (j, i). Geometrically, all these operations can be described as "reflection through the main diagonal." Given a skew Young diagram λ/μ, rotation by 180° gives a new diagram $(\lambda/\mu)^*$. Given a tableaux T with n boxes, we can form T^*, the **rotated-complement** of T, by rotating T by 180° and replacing the entry i with $n + 1 - i$ for each i. Observe that the reading word of T^* is exactly the reverse-complement of the reading word of T.

The **Schensted insertion algorithm**, or equivalently the **RSK correspondence**, is an extremely powerful tool relating permutations to pairs of standard Young tableaux. For a description of the bijection and a proof of its correctness and some of its properties, we refer the reader to [34, Chapter 7]. Our use of notation follows that source, so in particular we denote by $T \leftarrow i$ the tableau that results when we (row-) insert i into the tableau T. Particular properties of RSK will be quoted as needed.

Perhaps the most notable sequence of integers in enumerative combinatorics is the sequence of **Catalan numbers**. The nth Catalan number C_n is defined by $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$. These numbers appear in an extraordinary number of situations of combinatorial interest; see [34, Problem 6.19] and [38] for around 200 examples.

1.2 History

The earliest result on pattern avoidance is MacMahon's 1915 enumeration of 123-avoiding permutations in his book *Combinatory Analysis* [24]. In particular, MacMahon proved that $|S_n(123)| = C_n$ for all n. However, this work seems not to have led to any subsequent research. More than a half-century later, Knuth [17] enumerated 231-avoiding permutations in the context of sortability of lists in computer science, proving $|S_n(213)| = C_n$. Subsequent work (e.g. [27, 28]) focused on classes of pattern-avoiding permutations defined by similar sortability criteria, rather than by pattern avoidance *per se*. The jump to the enumerative study of pattern avoidance in and of itself came in the 1985 paper of Simion and Schmidt, which introduced the word "pattern", gave the first direct bijective proof of the equality $|S_n(123)| = |S_n(132)|$, enumerated permutations avoiding any collection of patterns of length 3, and initiated the study of pattern avoidance in interesting subsets of S_n (in particular, in even and odd permutations, and in involutions).

Since the paper of Simion and Schmidt, the field of pattern avoidance has exploded. Interest has been driven in part by connections to algebraic combinatorics (e.g., Schubert varieties [19] and Schubert polynomials [20] have certain nice properties if and only if w avoids certain patterns; see also Tenner's database of pattern avoidance [39]) and in part by the fact that pattern avoidance is very fertile ground for enumeration. We mention a few highlights most relevant to the work in this thesis, omitting much other work (in particular, outside of this parenthetical we do not mention variations in the notion of *pattern*, such as the dashed patterns of Babson and Steingrímsson [3] and the bivincular patterns of Bousquet-Mélou, Claesson, Dukes and Kitaev [9], nor the large literature on permutations avoiding multiple patterns simultaneously); a thorough treatment is available in the recent book of Kitaev [16]. In [13], Gessel gave a formula for the number of 1234-avoiding permutations of length n and an expression for the generating function for the number of $12\cdots k$-avoiding permutations of length n for any k and n. Subsequent work by Stankova and West [31, 40, 41, 32, 33] established that actually $S_n(1234)$ is in bijection with $S_n(p)$ for eleven other permutations $p \in S_4$; moreover, every permutation p in S_4 satisfies for every n one of the three equalities $|S_n(p)| = |S_n(1234)|$, $|S_n(p)| = |S_n(1324)|$ or $|S_n(p)| = |S_n(1342)|$. Thus, S_4 splits into three *Wilf-equivalence classes*. The third of these classes was enumerated by Bóna [7]; the enumeration of the second is still open. Further work on Wilf-equivalence by Stankova and West [33], Babson and West [4], and Backelin, West, and Xin [5] has established a large class of Wilf-equivalences among permutations of arbitrary length, but the problems of determining the complete Wilf-equivalence classification for patterns of arbitrary length and of giving enumerations of these classes remain wide open.

The study of alternating permutations dates to the work of André [1, 2] in the late 19th century. The number $|A_n|$ of alternating permutations of length n is known variously as an Euler number, zigzag number or André number. These numbers are also known as tangent and secant numbers due to their appearance in the formula

$$\tan x + \sec x = \sum_{n \geq 0} |A_n| \cdot \frac{x^n}{n!}.$$

Alternating permutations have a variety of interesting combinatorial properties; see [36] for a recent survey.

The study of pattern avoidance in alternating permutations began with Mansour [25], who used generating-functional techniques to enumerate alternating permutations avoiding 132 and possibly other patterns. Among other results, Mansour showed that $|A_n(132)|$ is a Catalan number for all n. Around the same time, Deutsch and Reifegerste showed (as reported by Stanley, [38, Problem h^7]) that also $|A_{2n}(123)| = C_n$. One can also show with similar arguments that actually $|A_n(p)|$ is a Catalan number for every nonnegative integer n and every pattern $p \in S_3$. Further related work includes the enumeration by Ouchterlony [26] of *doubly-alternating* permutations (i.e., alternating permutations with alternating inverses) avoiding the pattern 1234 and Stanley's umbral analogue [36] of Gessel's determinantal formula for permutations avoiding a monotone pattern. (Unfortunately, these umbral formulas are extremely difficult to work with; for example, it is not clear how to extract from them the equality $|A_{2n}(123)| = C_n$.)

1.3 Summary of results

The result $|S_n(132)| = C_n$ has a simple recursive proof, as does the analogous result $|A_{2n+1}(132)| = C_n$. In Chapter 2, we give a bijection φ between $S_n(132)$ and $A_{2n+1}(132)$, using binary trees as an intermediate object. This bijection interacts nicely with pattern containment, and this leads to the following result.

Theorem 2.2.1. *For any collection of* 132*-avoiding patterns* $p_1, \ldots, p_k$*, choose patterns* $q_1, \ldots, q_k$ *such that* q_i *is contained in* $\varphi(p_i)$ *and there is an instance of* p_i *contained in* q_i *containing no left-to-right minima of* q_i*. Then* φ *is a bijection between* $S_n(132, p_1, \ldots, p_k)$ *and* $A_{2n+1}(132, q_1, \ldots, q_k)$*, and in particular these sets are equinumerous.*

Thus, pattern avoidance for alternating permutations is closely related to, and in some cases actually reduces to, pattern avoidance for all permutations.

In Chapter 3, we define a set $\mathcal{L}_{n,k}$ of permutations that generalizes both the set of all permutations and the set of alternating permutations of even length. There are several nice proofs of the equality $|S_n(123)| = C_n$ including a clever application of the RSK algorithm ([34, Problem 6.19(ee)]); we give a bijective proof of the following generalization of this result.

Theorem 3.3.4. *There is a bijection between* $\mathcal{L}_{n,k}(12\cdots(k+1)(k+2))$ *and the set of standard Young tableaux of shape* $\langle (k+1)^n \rangle$ *and so*

$$|\mathcal{L}_{n,k}(12\cdots(k+1)(k+2))| = f^{\langle (k+1)^n \rangle}.$$

For $k = 1$ this is a rederivation of the equality $|S_n(123)| = C_n$ while for $k = 2$ it implies the following.

Corollary. *We have* $|A_{2n}(1234)| = f^{\langle 3^n \rangle} = \dfrac{2(3n)!}{n!(n+1)!(n+2)!}$ *for all* $n \geq 0$*.*

This is the first computation of any expression of the form $|A_{2n}(p)|$ or $|A_{2n+1}(p)|$ for $p \in S_4$. In Theorem 3.4.2, we prove the complementary result for alternating permutations of odd length, while in Theorem 3.5.1 we generalize the work of Ouchterlony [26] to this setting.

Much of the work of West, Stankova and coauthors in proving Wilf-equivalence relations has made use of the recursive/bijective technique of *generating trees*. In Chapter 4, we adopt this technique to prove several results about pattern avoidance in alternating permutations. Our main results in this chapter are a second bijective proof of Theorem 3.3.4 and the first Wilf-type equivalence for alternating permutations.

Theorem 4.4.5. *For all $n \geq 1$ we have*

$$|A_{2n}(2143)| = |\operatorname{SYT}(n,n,n)| = |A_{2n}(1234)| = \frac{2 \cdot (3n)!}{n!(n+1)!(n+2)!}.$$

We also prove the complementary result for $A_{2n+1}(2143)$, and we describe recent work of Jagadeesan settling a conjecture concerning the enumeration of $A_{2n}(3412)$. Chapter 4 ends with a collection of open problems.

Finally, in Chapter 5, we consider a new setting for pattern avoidance that is substantially more general than just the setting of $\mathcal{L}_{n,k}$. For any skew Young diagram λ/μ, we consider the set of reading words of standard tableaux of shape λ/μ. We give a bijective proof of the following remarkable enumeration of the 213-avoiding members of this set.

Theorem 5.1.1. *The number of tableaux of skew shape λ/μ whose reading words avoid the pattern 213 is equal to the number of partitions whose Young diagram is contained in that of μ (subject to a minor technical restriction).*

It is far from obvious that any result of this sort should be true; note also the fascinating fact that this enumeration depends only on μ, so in particular one may add additional boxes to λ without creating any new 213-avoiding reading words. We also give similar results for the patterns 132, 312 and 231.

The substantial majority of this work has been published as [21, 22, 23] and in slightly rougher form on the arXiv.

Chapter 2

Alternating permutations avoiding the pattern 132

The set $S_{n+1}(132)$ of permutations of length $n+1$ avoiding the pattern 132 can be easily enumerated, as follows: for any $w \in S_{n+1}(132)$ with $w_{k+1} = n+1$ and for any i, j such that $1 \le i < k+1 < j \le n+1$, we must have $w_i > w_j$ or else $w_i w_{k+1} w_j$ is a 132-pattern contained in w. Thus, $\{w_1, \ldots, w_k\} = \{n-k+1, \ldots, n\}$, $\{w_{k+2}, \ldots, w_{n+1}\} = \{1, \ldots, n-k\}$, and the words $w_1 \cdots w_k$ and $w_{k+2} \cdots w_n$ are themselves 132-avoiding. Conversely, it's easy to check that if for some k the permutation $v = v_1 \cdots v_{n+1}$ is such that $v_{k+1} = n+1$, $\{v_{k+2}, \ldots, v_{n+1}\} = \{1, \ldots, n-k\}$, $\{v_1, \ldots, v_k\} = \{n-k+1, \ldots, n\}$ and the words $v_1 \cdots v_k$ and $v_{k+2} \cdots v_{n+1}$ are 132-avoiding then $v \in S_{n+1}(132)$. It follows immediately that

$$|S_{n+1}(132)| = \sum_{k=0}^{n} |S_k(132)| \cdot |S_{n-k}(132)|$$

with initial value $|S_0(132)| = 1$ and so by induction $|S_n(132)|$ is equal to the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

A nearly identical argument can be applied to enumerate $A_{2n+1}(132)$: one need only note that the position of the maximal element $2n+3$ in a permutation $w \in A_{2n+3}(132)$ must be even, so

$$|A_{2n+3}(132)| = \sum_{k=0}^{n} |A_{2k+1}(132)| \cdot |A_{2n-2k+1}(132)|$$

with $|A_1(132)| = 1$. Thus $|A_{2n+1}(132)| = C_n$ and so also $|A_{2n+1}(132)| = |S_n(132)|$. In fact, these computations imply the existence of a recursive bijection between $S_n(132)$ and $A_{2n+1}(132)$. In this chapter, we construct this bijection directly (that is, non-recursively), using binary trees as an intermediate structure.

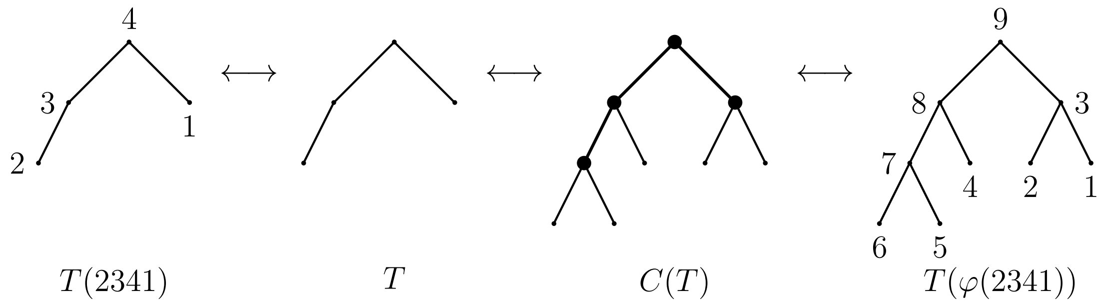


Figure 2-1: Applying φ to 2341 gives 675849231. The subsequence 7893 of even-indexed entries is order-isomorphic to 2341.

2.1 Bijection via binary trees

Given any permutation $w \in S_n$, we can bijectively associate a labeled, decreasing binary tree $T(w)$ as in [37, p. 51]: if w_k is the maximal entry of w, we label the root with w_k, let the left subtree below the root be $T(w_1 \cdots w_{k-1})$ and let the right subtree be $T(w_{k+1} \cdots w_n)$. Note that for any $i \in [n]$, the vertex labeled w_i has a left child in $T(w)$ if and only if $i > 1$ and $w_i > w_{i-1}$ and has a right child if and only if $i < n$ and $w_i > w_{i+1}$. In particular, $T(w)$ is **complete** (i.e., every vertex has either zero or two children) if and only if w is an alternating permutation of odd length. (Note that in some sources, a complete binary tree must have all leaves at the same height; this is not the case here.)

If w is 132-avoiding then for any vertex v in $T(w)$, each label on the left subtree at v is larger than every label on the right subtree at v. (This is essentially the same observation that we used to derive our recursion for $|S_n(132)|$ above.) Erasing all labels gives a bijection between decreasing, labeled binary trees with this property and unlabeled binary trees: to invert when given an unlabeled binary tree on n nodes, note that the label of the root must be n and the labels on its left subtree larger than the labels on its right subtree, and continue recursively. We will consider this bijection to be an identification, i.e., we will treat unlabeled trees and trees labeled in this way as interchangeable.

For any unlabeled binary tree B we may construct the **completion** $C(B)$ by adding vertices to B so that every vertex of B has two children in $C(B)$ and every vertex of $C(B)$ that is not also a vertex of B has zero children in $C(B)$. This operation is a bijection between unlabeled binary trees on n nodes and unlabeled complete binary trees on $2n+1$ nodes; to reverse it, erase all leaves.

We can use these three operations to construct a bijection between $S_n(132)$ and $A_{2n+1}(132)$: beginning with any permutation $w \in S_n(132)$, construct the labeled tree $T(w)$. Erase the labels from this tree and take its completion. Take the associated labeled tree and apply the inverse of the bijection T. The resulting permutation is an alternating, 132-avoiding permutation of length $2n+1$, i.e., an element of $A_{2n+1}(132)$. Each step is bijective, so the composition of steps is bijective. We denote this bijection by $\varphi : S_n(132) \to A_{2n+1}(132)$. Figure 2-1 illustrates the application of φ to the permutation 2341.

Note that φ^{-1} has a particularly nice form: erasing the leaves of $T(\varphi(w))$ is the same as erasing the odd-indexed entries in $\varphi(w)$, so w is just the permutation order-isomorphic to the sequence $(\varphi(w)_2, \varphi(w)_4, \ldots)$ and $\varphi(w)$ is the unique member of $A_{2n+1}(132)$ whose sequence of even-indexed entries is order-isomorphic to w. It follows immediately that if w contains a pattern p then $\varphi(w)$ does as well. In fact, we can say substantially more:

Proposition 2.1.1. *If a permutation $w \in S_n(132)$ contains the pattern $p \in S_k(132)$ then $\varphi(w) \in A_{2n+1}(132)$ contains $\varphi(p) \in A_{2k+1}(132)$.*

Note that the restriction $p \in S_k(132)$ is actually no restriction at all: since pattern containment is transitive, if w avoids 132 then w automatically avoids all patterns that contain 132.

Proof. Suppose w contains p. Choose a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $w_{i_1} w_{i_2} \cdots w_{i_k}$ is an instance of p contained in w. Note that each vertex of $T(w)$ corresponds naturally via our construction to a vertex of $T(\varphi(w))$, and similarly for $T(p)$ and $T(\varphi(p))$. For each j, denote by v_j the vertex of $T(\varphi(w))$ corresponding to the vertex labeled w_{i_j} in $T(w)$ and denote by v'_j the vertex of $T(\varphi(p))$ corresponding to the vertex labeled p_j in $T(p)$.

We build a set of $2k+1$ vertices of $T(\varphi(w))$ as follows: first, we include the vertices v_j for $1 \leq j \leq k$. Then, for each j, we add the left child of v_j to our set if and only if v'_j has a left child in $T(\varphi(p))$ but the vertex labeled p_j in $T(p)$ does not – that is, take the left child of v_j if and only if a new left child was added to the vertex labeled p_j when passing from $T(p)$ to $T(\varphi(p))$. Do the same for right children. Figure 2-2 illustrates this process for $p = 4312$ and $w = 5647231$.

The relative order on the labels of vertices in this collection is determined entirely by the relative position of these vertices in the tree,[1] and the relative positions of these vertices in $T(\varphi(w))$ are the same as their relative positions in $T(\varphi(p))$. Thus the labels on these vertices form a $\varphi(p)$-pattern contained in $\varphi(w)$. $\square$

The converse of this result is also true, and we prove a modest strengthening of it.

Proposition 2.1.2. *Suppose $\varphi(w) \in A_{2n+1}(132)$ contains a pattern q and q contains p, with the additional restriction that there is some subsequence of q order-isomorphic to p that contains no left-to-right minima of q. Then $w \in S_n(132)$ contains p.*

Proof. If $\varphi(w)_{2k-1} < \varphi(w)_{2k+1}$ then $\varphi(w)_{2k-1} < \varphi(w)_{2k+1} < \varphi(w)_{2k}$ is a 132-pattern contained in $\varphi(w)$, a contradiction, so $\varphi(w)_1 > \varphi(w)_3 > \varphi(w)_5 > \ldots$ and the left-to-right minima of $\varphi(w)$ are exactly the odd-indexed entries. Select an instance of q in $\varphi(w)$ and choose the special subsequence corresponding to p. No entry of

[1]By the "relative position" of vertices a and b, we mean the following: either one of a and b is a descendant of the other, in which case the ancestor has the larger label, or they have some nearest common ancestor c. In the latter case, we say a is "relatively to the left" of b if a lies in the left subtree of c and b lies in the right subtree of c. By the 132-avoidance of the permutations associated to these trees, "being relatively left" implies "having a larger label."

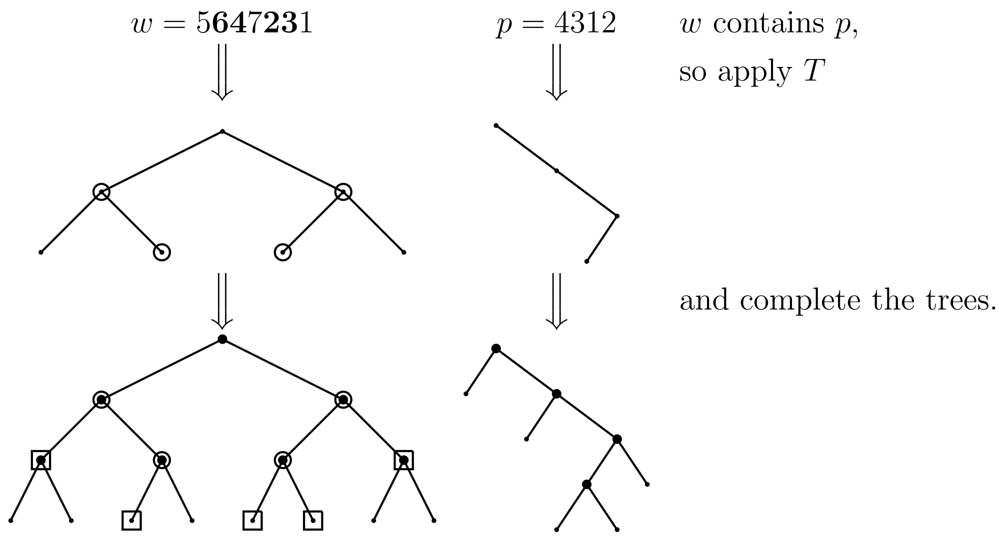


Figure 2-2: The subsequence 6423 of $w = 5647231$ is an instance of $p = 4312$. We apply T and complete both trees. For each (circled) node of $C(T(w))$ corresponding to a node of $T(p)$ we select (box) a child if and only if the corresponding child was added to $T(p)$ in the completion $C(T(p))$. The resulting subsequence 13 14 9 10 5 6 4 7 3 is an instance of $\varphi(p) = 896734251$ in $\varphi(w) = 12\,13\,11\,14\,9\,10\,8\,15\,5\,6\,4\,7\,2\,3\,1$.

this subsequence is a left-to-right minimum in our instance of q, so no entry of this subsequence is a left-to-right minimum in $\varphi(w)$. Then every entry of this subsequence (an instance of the pattern p) occurs among the even-indexed entries of $\varphi(w)$. Since the sequence of even-indexed entries is order-isomorphic to w, p is contained in w. □

The converse of Proposition 2.1.1 follows because the even-indexed entries of $\varphi(p)$ are an instance of p including no left-to-right minima in $\varphi(p)$, so if $\varphi(w)$ contains $\varphi(p)$ then w contains p.

2.2 Consequences

We can rephrase the results of the previous section as follows:

Theorem 2.2.1. *For any collection of 132-avoiding patterns $p_1, \ldots, p_k$, choose patterns $q_1, \ldots, q_k$ such that q_i is contained in $\varphi(p_i)$ and there is an instance of p_i contained in q_i containing no left-to-right minima of q_i. Then φ is a bijection between $S_n(132, p_1, \ldots, p_k)$ and $A_{2n+1}(132, q_1, \ldots, q_k)$, and in particular these sets are equinumerous.*

Note that $\varphi(p_i)$ is always a valid choice for q_i, but that there may also be others. Also, because φ is a bijection on classes of pattern-avoiding permutations and on their complements, this restatement does not actually capture the full strength of our previous results.

Proof. We have by Proposition 2.1.1 that the image of $S_n(132, p_1, \ldots, p_k)$ under φ contains $A_{2n+1}(132, \varphi(p_1), \ldots, \varphi(p_k))$ while we have by Proposition 2.1.2 that φ is an injection from $S_n(132, p_1, \ldots, p_k)$ to $A_{2n+1}(132, q_1, \ldots, q_k)$. By transitivity of pattern containment, $A_{2n+1}(132, \varphi(p_1), \ldots, \varphi(p_k)) \supseteq A_{2n+1}(132, q_1, \ldots, q_k)$. Thus in fact φ is a bijection between $S_n(132, p_1, \ldots, p_k)$ and $A_{2n+1}(132, q_1, \ldots, q_k)$, as claimed. $\square$

As a result of this theorem, a large class of enumeration problems for pattern-avoiding alternating permutations can be expressed as enumeration problems for pattern-avoiding (standard) permutations. We give three examples involving a single pattern.

Corollary 2.2.2. *For all integers $n \geq 0$ and $k \geq 1$, we have*

$$|S_n(132, 12\cdots k)| = |A_{2n+1}(132, 12\cdots(k+1))|.$$

As a result, $w \in S_n(132)$ has longest increasing subsequence of length k if and only if $\varphi(w) \in A_{2n+1}(132)$ has longest increasing subsequence of length $k+1$.

Proof. We have that $\varphi(12\cdots k) = (k+1)(k+2)k(k+3)\cdots 2(2k+1)1$ contains the subsequence $(k+1)(k+2)\cdots(2k)(2k+1)$. This is an instance of the pattern $q = 12\cdots k(k+1)$ and q contains an instance of $12\cdots k$ not including any left-to-right minima of q, so by Theorem 2.2.1, φ is a bijection between $S_n(132, 12\cdots k)$ and $A_{2n+1}(132, 12\cdots k(k+1))$.

For the second half of the claim, note that the set of permutations in $S_n(132)$ with longest increasing subsequence of length k is $S := S_n(132, 12\cdots k(k+1)) \setminus S_n(132, 12\cdots k)$. Since

$$A_{2n+1}(132, 12\cdots(k+1)) \subset A_{2n+1}(132, 12\cdots(k+1)(k+2))$$

and

$$S_n(132, 12\cdots k) \subset S_n(132, 12\cdots k(k+1)),$$

φ is a bijection between S and $A_{2n+1}(132, 12\cdots(k+1)(k+2)) \setminus A_{2n+1}(132, 12\cdots(k+1))$, the set of 132-avoiding alternating permutations with longest increasing subsequence of length $k+1$. $\square$

For $k = 1$ the resulting sequence is $|A_{2n+1}(132, 123)| = |S_n(132, 12)| = 1$. For $k = 2$ it's $|A_{2n+1}(132, 1234)| = |S_n(132, 123)| = \lceil 2^{n-1} \rceil$ (see [30]). For $k = 3$ it's the even Fibonacci sequence $|A_{2n+1}(132, 12345)| = |S_n(132, 1234)| = F_{2n-2}$ defined with initial terms $F_0 = F_1 = 1$ (see [41]), and so on. These values were calculated directly for the alternating permutations in [25, Section 2].

Corollary 2.2.3. *For all integers $n \geq 0$ and $k \geq 2$ we have*

$$|A_{2n+1}(132, 341256\cdots(k+2))| = |S_n(132, 2134\cdots k)|.$$

Proof. We have

$$\varphi(2134\cdots k) = (k+2)(k+3)k(k+1)(k-1)(k+4)(k-2)(k+5)\cdots 2(2k+1)1,$$

which contains the subsequence $(k+2)(k+3)k(k+1)(k+4)(k+5)\cdots(2k)(2k+1)$, an instance of the pattern $q = 341256\cdots(k+2)$. This q contains the subsequence $4256\cdots(k+2)$, an instance of $2134\cdots k$ containing no left-to-right minima of q. The result follows from Theorem 2.2.1. □

The resulting sequences are the same as in the previous case (see [30, 41]).

Note that in both Corollaries 2.2.2 and 2.2.3 we chose the shortest possible pattern q associated with $p = 123\cdots k$ or $p = 213\cdots k$, respectively. In both cases there are many other possible choices for q of various lengths: we could take any q' that is order-isomorphic to a subsequence of $\varphi(p)$ that contains our selected instance of q and we would arrive at the same set of pattern-avoiding alternating permutations. In our next example, the choice of q is much more restricted.

Corollary 2.2.4. *For all integers $n \geq 0$ and $k \geq 2$, we have*

$$|A_{2n+1}(132, (2k-1)(2k)(2k-3)(2k-2)\cdots 3412)| = |S_n(132, k(k-1)\cdots 21)|.$$

Proof. We have $\varphi(k(k-1)\cdots 21) = (2k)(2k+1)(2k-2)(2k-1)\cdots 45231$, and taking q to be the permutation order-isomorphic to $\varphi(p)$ with the final 1 removed gives the result. □

For $k = 2$ the resulting sequence is $|A_{2n+1}(132, 3412)| = |S_n(132, 21)| = 1$ and for $k = 3$ it's $|A_{2n+1}(132, 563412)| = |S_n(132, 321)| = \frac{n^2-n+2}{2}$ (see [30]). The q selected in this instance is the only possible choice other than $q' = \varphi(p)$.

Finally, our results can be used to show equalities among sets of pattern-avoiding alternating permutations that might not otherwise be obvious. For example, in [25] it was shown by generating-functional methods that for all n and k we have $|A_n(132, 12\cdots k)| = |A_n(132, 2134\cdots k)|$ and a bijective proof was requested. Using the fact that the sets counted by these numbers are equinumerous, we can easily show that actually they are equal for n odd.

Corollary 2.2.5. *For all integers n, k such that $n \geq 0$ and $k \geq 1$, the three sets $A_{2n+1}(132, 12\cdots(k+1))$, $A_{2n+1}(132, \varphi(12\cdots k))$ and $A_{2n+1}(132, 2134\cdots k(k+1))$ are equal.*

Proof. The equality of the first two sets follows from the proof of Corollary 2.2.2: we have that φ is a bijection between $S_n(132, 12\cdots k)$ and $A_{2n+1}(132, \varphi(12\cdots k))$ and also between $S_n(132, 12\cdots k)$ and $A_{2n+1}(132, 12\cdots k(k+1))$, so the two sets of alternating permutations must certainly be equal. Also, one containment between the last two sets is immediate: the pattern

$$\varphi(12\cdots k) = (k+1)(k+2)k(k+3)(k-1)(k+4)\cdots 2(2k+1)1$$

contains the subsequence $(k+2)k(k+3)(k+4)\cdots(2k+1)$ that is order-isomorphic to $2134\cdots k(k+1)$, so

$$A_{2n+1}(132, \varphi(12\cdots k)) \supseteq A_{2n+1}(132, 2134\cdots k(k+1)).$$

Thus $A_{2n+1}(132, 12\cdots k(k+1)) \supseteq A_{2n+1}(132, 2134\cdots k(k+1))$. Since these two sets have the same cardinality, they must actually be equal. □

2.3 Even-length alternating permutations avoiding 132

Our preceding results all concern alternating permutations of odd length. However, there is a simple relationship between 132-avoiding alternating permutations of length $2n+1$ and 132-avoiding alternating permutations of length $2n$. By the observation at the beginning of the proof of Proposition 2.1.2, if $w \in A_{2n+1}(132)$ then $w_{2n+1} = 1$. Thus, from any 132-avoiding alternating permutation w of length $2n+1$ we may build a 132-avoiding alternating permutation w' of length $2n$ by removing the last entry of w and subtracting 1 from each other entry, i.e., $w' = (w_1 - 1)(w_2 - 1)\cdots(w_{2n} - 1)$. Conversely, given any 132-avoiding alternating permutation w' of length $2n$ we can build a 132-avoiding alternating permutation w of length $2n+1$ by adding 1 to each entry and appending a 1, i.e., $w = (w'_1 + 1)(w'_2 + 1)\cdots(w'_{2n} + 1)1$. It follows immediately that $|A_{2n}(132)| = |A_{2n+1}(132)|$. Moreover, $w \in A_{2n+1}(132)$ avoids a pattern $p = p_1 \cdots p_{k-1}p_k$ with $p_k = 1$ if and only if w' avoids $p' = (p_1 - 1)\cdots(p_{k-1} - 1)$, while if $p_k \neq 1$ then w avoids p if and only if w' avoids p. These comments prove the following result.

Theorem 2.3.1. *Given patterns $p_1, \ldots, p_i$ ending in 1 and patterns $q_1, \ldots, q_j$ not ending in 1, we have*

$$|A_{2n+1}(132, p_1, \ldots, p_i, q_1, \ldots, q_j)| = |A_{2n}(132, p'_1, \ldots, p'_i, q_1, \ldots, q_j)|.$$

In particular, every enumeration problem concerning 132-avoiding alternating permutations of even length reduces to a problem concerning permutations of odd length.

Chapter 3

Generalized alternating permutations avoiding a monotone pattern

In this chapter, we consider monotone pattern avoidance for a natural generalization of alternating permutations that we denote $\mathcal{L}_{n,k}$, and in particular we enumerate $A_{2n}(123)$ (originally counted by Deutsch and Reifegerste) and $A_{2n}(1234)$. The latter enumeration is the first enumeration of alternating permutations avoiding any pattern of length four. The main tool used in this enumeration is a variation on the RSK correspondence closely related to work of Ouchterlony [26] on doubly-alternating permutations (i.e., those whose inverses are also alternating). A second proof of this result, using generating trees, is given in Chapter 4.

The permutations in $\mathcal{L}_{n,k}$, which may be viewed as reading words of certain staircase-shaped tableaux, seem interesting in their own right; for example, they have been enumerated (without any pattern restriction) by Baryshnikov and Romik [6], and the resulting formulas are attractive (at least for small values of k). A discussion of pattern avoidance in the more general context of readings words of tableaux of any skew shape may be found in Chapter 5.

3.1 Generalized alternating permutations

For positive integers n and k, define $\mathcal{L}_{n,k}$ to be the subset of S_{nk} consisting of permutations $w = w_{1,1}w_{1,2}\cdots w_{1,k}w_{2,1}\cdots w_{n,k}$ that satisfy the conditions

L1. $w_{i,j} < w_{i,j+1}$ for all $1 \le i \le n$, $1 \le j \le k-1$, and

L2. $w_{i,j+1} > w_{i+1,j}$ for all $1 \le i \le n-1$, $1 \le j \le k-1$.

Note in particular that $\mathcal{L}_{n,1} = S_n$ (we have no restrictions in this case) and $\mathcal{L}_{n,2} = A_{2n}$. Thus, $\mathcal{L}_{n,k}$ is one possible generalization of alternating permutations.

Observe that $\mathcal{L}_{n,k}$ may be described easily in terms of reading words of tableaux: the reading words of the tableaux of shape $\langle n, n-1, \ldots, 2, 1\rangle / \langle n-1, n-2, \ldots, 1\rangle$

are all of S_n, and similarly $\mathcal{L}_{n,k}$ is equal to the set of reading words of standard skew Young tableaux of shape $\langle n+k-1, n+k-2, \ldots, k\rangle/\langle n-1, n-2, \ldots, 1\rangle$.

For any k and n, $|\mathcal{L}_{n,k}(12\cdots k)| = 0$. Thus, for monotone pattern-avoidance in $\mathcal{L}_{n,k}$ one should consider patterns of length $k+1$ or longer.

Note 1. If $w = w_{1,1}\cdots w_{n,k} \in S_{nk}$ satisfies L1 and also avoids $12\cdots(k+2)$ then it automatically satisfies L2, since a violation $w_{i,j+1} < w_{i+1,j}$ of L2 leads immediately to a $(k+2)$-term increasing subsequence $w_{i,1} < \ldots < w_{i,j+1} < w_{i+1,j} < \ldots < w_{i+1,k}$. Consequently, we can also describe $\mathcal{L}_{n,k}(1\cdots(k+2))$ (respectively, $\mathcal{L}_{n,k}(1\cdots(k+1))$) as the set of permutations in $S_{nk}(1\cdots(k+2))$ (respectively, $S_{nk}(1\cdots(k+1))$) whose descent set is (or in fact, is contained in) $\{k, 2k, \ldots, (n-1)k\}$. Thus, in Sections 3.2 and 3.3 we could replace $\mathcal{L}_{n,k}$ by permutations with descent set $\{k, 2k, \ldots\}$ without changing the content of any theorems. ■

3.2 The pattern $12\cdots(k+1)$

In this section we give the simplest of the bijections in this chapter.

Proposition 3.2.1. *There is a bijection between $\mathcal{L}_{n,k}(12\cdots(k+1))$ and the set of standard Young tableaux of shape $\langle k^n\rangle$.*

We have $f^{\langle n\rangle} = f^{\langle 1^n\rangle} = 1$ and $f^{\langle n,n\rangle} = f^{\langle 2^n\rangle} = \frac{1}{n+1}\binom{2n}{n} = C_n$, the nth Catalan number. By the hook-length formula [34, 29] we have

$$f^{\langle k^n\rangle} = \frac{(kn)! \cdot 1! \cdot 2! \cdots (k-1)!}{n! \cdot (n+1)! \cdots (n+k-1)!}.$$

Then Proposition 3.2.1 says $|\mathcal{L}_{n,k}(12\cdots(k+1))| = f^{\langle k^n\rangle}$. For $k=1$, this is the uninspiring result $|S_n(12)| = 1$. For $k=2$, it tells us $|A_{2n}(123)| = C_n$, a result that Stanley [38, Problem h^7] attributes to Deutsch and Reifegerste.

Proof. Given a permutation $w \in \mathcal{L}_{n,k}(12\cdots(k+1))$, define a tableau T of shape $\langle k^n\rangle$ by $T_{i,j} = w_{n+1-i,j}$. We claim that this is the desired bijection. By L1 we have $T_{i,j} = w_{n-i+1,j} < w_{n-i+1,j+1} = T_{i,j+1}$ for all $1 \le i \le n$, $1 \le j \le k-1$. Thus, the tableau T is increasing along rows. Suppose for sake of contradiction that there is some place at which T fails to increase along a column. Then there are some i, j such that $w_{n-i+1,j} = T_{i,j} > T_{i+1,j} = w_{n-i,j}$. Then $w_{n-i,1} < w_{n-i,2} < \ldots < w_{n-i,j} < w_{n-i+1,j} < w_{n-i+1,j+1} < \ldots < w_{n-i+1,k}$ is an instance of $12\cdots(k+1)$ contained in w, a contradiction, so T must also be increasing along columns. Since w contains each of the values between 1 and nk exactly once, T does as well, so T is a standard Young tableau.

Conversely, suppose we have a standard Young tableau T of shape $\langle k^n\rangle$. Define a permutation $w = w_{1,1}\cdots w_{n,k} \in S_{nk}$ by $w_{i,j} = T_{n-i+1,j}$. If we show that the image of this map lies in $\mathcal{L}_{n,k}(1\cdots(k+1))$ we will be done, as in this case the two maps are clearly inverses. Note that

$$w_{i+1,j} = T_{n-i,j} < T_{n-i+1,j} = w_{i,j} \tag{3.1}$$

and

$$w_{i,j} = T_{n-i+1,j} < T_{n-i+1,j+1} = w_{i,j+1} \tag{3.2}$$

because T is increasing along columns and along rows, respectively. Condition 3.2 is exactly L1, while the two conditions together give us that $w_{i+1,j} < w_{i,j} < w_{i,j+1}$, and this is L2. Now choose any $k+1$ entries of w. By the pigeonhole principle, this set must include two entries with the same second index, and by Equation (3.1) these two will form an inversion. Thus no subsequence of w of length $k+1$ is monotonically increasing and so w avoids the pattern $12\cdots(k+1)$. Thus $w \in \mathcal{L}_{n,k}(1\cdots(k+1))$ and we have a bijection between the set $\mathcal{L}_{n,k}(12\cdots(k+1))$ and the set of Young tableaux of shape $\langle k^n \rangle$, as desired. □

For example, we have under this bijection the correspondences

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6&7\\\hline 4&8&9\\\hline\end{array} \quad\longleftrightarrow\quad 489267135 \in \mathcal{L}_{3,3}(1234)$$

and

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&7\\\hline 6&8\\\hline\end{array} \quad\longleftrightarrow\quad 68472513 \in \mathcal{L}_{4,2}(123).$$

Both directions of this bijection are more commonly seen with other names. The map that sends $w \mapsto T$ is actually the Schensted insertion algorithm used in the RSK correspondence. (For any $w \in \mathcal{L}_{n,k}(1\cdots(k+1))$, the recording tableau is the tableau whose first row contains $\{1,\ldots,k\}$, second row contains $\{k+1,\ldots,2k\}$, and so on.) The map that sends $T \mapsto w$ is the reading-word map defined in Section 1.1.

3.3 The pattern $12\cdots(k+1)(k+2)$

In this section, we enumerate $\mathcal{L}_{n,k}(12\cdots(k+1)(k+2))$ using a modification of the RSK insertion algorithm. Our modification is an extension of the bijection devised by Ouchterlony in [26]. Recall that the RSK insertion algorithm is a map between S_n and the set of pairs (P,Q) of standard Young tableaux such that $\mathrm{sh}(P) = \mathrm{sh}(Q) \vdash n$ with the following properties:

Theorem 3.3.1 ([34, 7.11.2(b)]). *If P is a standard Young tableau and $j < k$ then the insertion path of j in $P \leftarrow j$ lies strictly to the left of the insertion path of k in $(P \leftarrow j) \leftarrow k$, and the latter insertion path does not extend below the former.*

Theorem 3.3.2 ([34, 7.23.11]). *If $w \in S_n$ and $w \xrightarrow{RSK} (P,Q)$ with $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda$, then λ_1 is the length of the longest increasing subsequence in w.*

Now we describe a bijection from $\mathcal{L}_{n,k}(12\cdots(k+2))$ to pairs (P,R) of standard Young tableaux such that P has nk boxes, R has n boxes, and the shape of R can

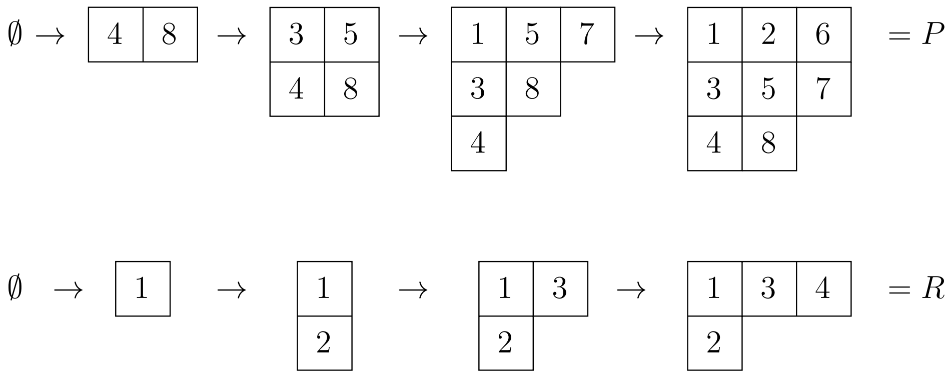


Figure 3-1: An application of our modified version of RSK to the permutation $48351726 \in \mathcal{L}_{4,2}(1234)$. Note that only every other insertion step is shown in the construction of P.

be rotated 180° and joined to the shape of P to form a rectangle of shape $\langle (k+1)^n \rangle$. (In other words, $\mathrm{sh}(P)'_i + \mathrm{sh}(R)'_{k+2-i} = n$ for $1 \leq i \leq k+1$.) Observe that the set of such pairs of tableaux is in natural bijection with the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$: given a tableau of shape $\langle (k+1)^n \rangle$, break off the portion of the tableau filled with $nk+1, \ldots, n(k+1)$, rotate it 180° and replace each value i that appears in it with $nk+n+1-i$.

Given a permutation $w = w_{1,1}w_{1,2}\cdots w_{1,k}w_{2,1}\cdots w_{n,k}$, let $P_0 = \emptyset$ and let $P_i = (\cdots((P_{i-1} \leftarrow w_{i,1}) \leftarrow w_{i,2})\cdots) \leftarrow w_{i,k}$ for $1 \leq i \leq n$. Define $P := P_n$, so P is the usual RSK insertion tableau for w. Define R as follows: set $R_0 = \emptyset$ and $\lambda_i = \mathrm{sh}(P_i)$. Observe that by Theorem 3.3.1, λ_i/λ_{i-1} is a horizontal strip of size k and that by Theorem 3.3.2, λ_i/λ_{i-1} stretches no further right than the $(k+1)$th column. Thus there is a unique j such that λ_i/λ_{i-1} has a box in the ℓth column for all $\ell \in [k+1] \setminus \{j\}$. Let R_i be the shape that arises from R_{i-1} by adding a box filled with i in the $(k+2-j)$th column, and define $R = R_n$. This map is illustrated in Figure 3-1.

Proposition 3.3.3. *The map just described is a bijection between $\mathcal{L}_{n,k}(12\cdots(k+2))$ and the set of pairs (P,R) of standard Young tableaux such that P has nk boxes, R has n boxes, and $\mathrm{sh}(R)$ can be rotated and joined to $\mathrm{sh}(P)$ to form a rectangle of shape $\langle (k+1)^n \rangle$.*

Proof. By construction, it is clear that P is a standard Young tableau with nk boxes and that R is a shape with n boxes filled with $[n]$ such that rotating R by 180° we may join it to P in order to get a rectangle of shape $\langle (k+1)^n \rangle$. So it is left to show that R is actually a standard Young tableau and that this process is a bijection.

The tableau R is constructed in such a way that it is automatically increasing down columns. Also, R increases across rows unless at some intermediate stage it is not of partition shape, i.e., for some i, j we have that R_i contains more boxes in column $j+1$ than in column j. But in this case it follows from the construction of R that P_i has more boxes in column $k+2-j$ than in column $k+1-j$. Since we have by properties of RSK that every P_i is of partition shape, this is absurd, and so R really is a standard Young tableau.

Finally, we need that this algorithm is invertible and that its inverse takes pairs of tableaux of the given sort to permutations with the appropriate restrictions. Invertibility is immediate, since from a pair (P, R) of standard Young tableaux of appropriate shapes we can construct a pair of standard Young tableaux (P, Q) of the same shape with $w \mapsto (P, R)$ under our algorithm exactly when $w \xrightarrow{\text{RSK}} (P, Q)$: if R has entry i in column j, place the entries $ki - k + 1, ki - k + 2, \ldots, ki$ into columns $1, \ldots, k + 1 - j, k + 3 - j, \ldots, k + 1$ of Q, respectively. Moreover, by Theorem 3.3.1 we have that the preimage under RSK of this pair (P, Q) must consist of n runs of k elements each in increasing order, i.e., it must satisfy L1, and by Theorem 3.3.2 it must have no increasing subsequence of length $k + 2$. Then by the remarks in Section 3.1 following the definition of $\mathcal{L}_{n,k}$ we have that the preimage satisfies L2 as well. This completes the proof. $\square$

This immediately implies the main result of this chapter.

Theorem 3.3.4. *There is a bijection between $\mathcal{L}_{n,k}(12 \cdots (k+1)(k+2))$ and the set of standard Young tableaux of shape $\langle (k+1)^n \rangle$ and so*

$$|\mathcal{L}_{n,k}(12 \cdots (k+1)(k+2))| = f^{\langle (k+1)^n \rangle}.$$

Corollary. *We have* $|A_{2n}(1234)| = f^{\langle 3^n \rangle} = \dfrac{2(3n)!}{n!(n+1)!(n+2)!}$ *for all* $n \geq 0$.

3.4 An analogue of alternating permutations of odd length

The results of Sections 3.2 and 3.3 concern $\mathcal{L}_{n,k}$, a generalization of the set A_{2n} of alternating permutations of even length. We now define a set $\mathcal{L}_{n,k;r}$ that is one possible associated analogue of alternating permutations of odd length. We briefly describe the changes that need to be made to the bijections of Sections 3.2 and 3.3 in order to have them apply in this context and the analogous theorems that result. The proofs are very similar to our preceding work, and we trust that the interested reader can work out the details for herself.

For $k \geq 1$ and $0 \leq r \leq k - 1$, define $\mathcal{L}_{n,k;r}$ to be the set of permutations $w = w_{0,2} w_{0,3} \cdots w_{0,r+1} w_{1,1} \cdots w_{n,k}$ in S_{nk+r} that satisfy the conditions

L1′. $w_{i,j} < w_{i,j+1}$ for all i, j such that $1 \leq i \leq n$, $1 \leq j \leq k - 1$ or $i = 0$, $2 \leq j \leq r$, and

L2′. $w_{i,j+1} > w_{i+1,j}$ for all i, j such that $1 \leq i \leq n - 1$, $1 \leq j \leq k - 1$ or $i = 0$, $1 \leq j \leq r$.

Note in particular that $\mathcal{L}_{n,k;0} = \mathcal{L}_{n,k}$ and that $\mathcal{L}_{n,2;1}$ is the set A'_{2n+1} of down-up alternating permutations of length $2n + 1$, i.e., those for which $w_1 > w_2 < w_3 >$

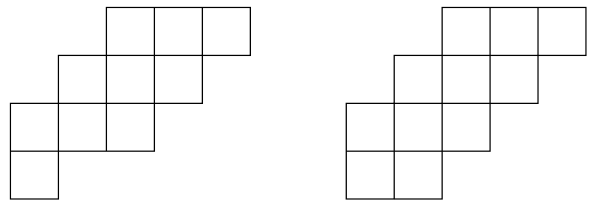


Figure 3-2: The Young diagrams whose standard Young tableaux have reading words $\mathcal{L}_{3,3;1}$ and $\mathcal{L}_{3,3;2}$.

$\ldots < w_{2n+1}$. As with $\mathcal{L}_{n,k}$, these permutations are the reading words of tableaux of a certain shape, which is illustrated in Figure 3-2.

The following results extend our work in the preceding sections to this context.

Proposition 3.4.1. *There exists a bijection between* $\mathcal{L}_{n,k;r}(12\cdots(k+1))$ *and the set of standard Young tableaux of shape* $\langle k^n, r\rangle$.

The bijection of Section 3.2 carries over in the obvious way.

Theorem 3.4.2. *There exists a bijection between* $\mathcal{L}_{n,k;r}(12\cdots(k+2))$ *and the set of standard Young tableaux of shape* $\langle (k+1)^{n-1}, k, r\rangle$.

Observe that this coincides with the result of Theorem 3.3.4 in the case $r = 0$, since every standard Young tableau of shape $\langle (k+1)^{n-1}, k\rangle$ can be extended uniquely to a standard Young tableau of shape $\langle (k+1)^n\rangle$ by adding a single box filled with $nk+n$.

Proof idea. We describe how to modify the bijection of Section 3.3 so that it works in the context of $\mathcal{L}_{n,k;r}(12\cdots(k+2))$. First, we note that standard Young tableaux of shape $\langle (k+1)^{n-1}, k, r\rangle$ are in bijection with pairs (P, R) of tableaux such that $\mathrm{sh}(P)$ is a partition of $nk+r$, $\mathrm{sh}(R)$ is a skew shape $\mu/\langle k+1-r, 1\rangle$ with $n-1$ boxes, and $\mathrm{sh}(R)$ can be rotated and joined to $\mathrm{sh}(P)$ to form the partition $\langle (k+1)^{n-1}, k, r\rangle$. Our bijection is between $\mathcal{L}_{n,k;r}(12\cdots(k+2))$ and pairs of tableaux of this form.

Suppose that we are given a permutation $w \in \mathcal{L}_{n,k;r}(12\cdots(k+2))$. To build the associated pair of tableaux, we make the following changes to the algorithm given just before the proof of Proposition 3.3.3: we again make use of RSK, this time with intermediate tableau $P_{-1}, \ldots, P_n = P$ and $R_1, \ldots, R_n = R$. We set $P_{-1} = \emptyset$ and let P_0 be the result of inserting the first r values of w into P_{-1}. For $0 \le i \le n-1$, we build P_{i+1} from P_i by inserting the next k values of w. The R_i are no longer standard Young tableaux but are instead standard skew Young tableaux, with $R_1 = \langle k+1-r, 1\rangle/\langle k+1-r, 1\rangle$. For $1 \le i \le n-1$, we build R_{i+1} from R_i by the same column-pairing process as before, always preserving the removed shape $\langle k+1-r, 1\rangle$. Thus for each i we have $R_i = \mu/\langle k+1-r, 1\rangle$ for some $\mu \vdash k+1-r+i$. (In particular, the first box is added to R_1 to form R_2 after $2k+r$ values have been inserted into P.)

To invert this process, we proceed as in Proposition 3.3.3 for $n-1$ steps, until R has been exhausted. At that point, what remains of P will be of shape $\langle k, r\rangle$. Then we set $w_{0,2}w_{0,3}\cdots w_{0,r+1}$ equal to the second row and $w_{1,1}w_{1,2}\cdots w_{1,k}$ equal to the first row of (what remains of) P. □

As an immediate corollary we have that the number of down-up alternating permutations of length $2n+1$ avoiding 1234 is the same as the number of standard Young tableaux of shape $\langle 3^{n-1}, 2, 1\rangle$. Taking reverse-complements of permutations gives a bijection between the set of down-up alternating permutations of length $2n+1$ avoiding 1234 and the set $A_{2n+1}(1234)$ of (up-down) alternating permutations of length $2n+1$ avoiding 1234. Applying the hook-length formula yields the following result.

Corollary 3.4.3. *We have* $|A_{2n+1}(1234)| = \dfrac{16(3n)!}{(n-1)!(n+1)!(n+3)!}$ *for all* $n \geq 1$.

3.5 Generalized doubly-alternating permutations

In [26], a variety of enumerative results were obtained for pattern avoidance in **doubly-alternating permutations**. These are permutations w such that both w and w^{-1} are alternating.[1] One of these results in particular is similar in flavor to the results we have discussed so far, namely:

Theorem ([26, Theorem 5.2])**.** *There exists a bijection between the set of doubly-alternating permutations of length* $2n$ *that avoid the pattern* 1234 *and* $S_n(1234)$.

We extend this result as follows:

Theorem 3.5.1. *For all* $n, k \geq 1$*, the set of permutations* $w \in \mathcal{L}_{n,k}(1\cdots(k+2))$ *such that* $w^{-1} \in \mathcal{L}_{n,k}(1\cdots(k+2))$ *is in bijection with* $S_n(1\cdots(k+2))$.

Proof. Our proof is nearly identical to that of [26]. Recall the argument of Proposition 3.3.3: we showed that if $w \in \mathcal{L}_{n,k}(1\cdots(k+2))$ and $w \xrightarrow{\text{RSK}} (P, Q)$ then we can encode Q, a tableau with nk boxes, as R, a tableau with only n boxes, by replacing a set of boxes in columns $1, \ldots, j-1, j+1, \ldots, k+1$ by a single box in column $k+2-j$. We refer to this as the **pairing** of the column $k+2-j$ in R with the columns $[k+1] \setminus \{j\}$ in Q.

Suppose that $w^{-1} \in \mathcal{L}_{n,k}(1\cdots(k+2))$. We have by [34, Theorem 7.13.1] that $w^{-1} \xrightarrow{\text{RSK}} (Q, P)$. Then it follows that we can perform the same column-pairing procedure to encode P as a tableau S with n boxes. We have $\text{sh}(P) = \text{sh}(Q)$, so by the nature of the pairing we have that $\text{sh}(S) = \text{sh}(R)$. Thus R and S are standard Young tableaux of the same shape, with n boxes and at most $k+1$ columns. Moreover, this map is a bijection between $\{w \mid w, w^{-1} \in \mathcal{L}_{n,k}(1\cdots(k+2))\}$ and the set of all such pairs of tableaux. Finally, RSK is a bijection between the set of these pairs of tableaux and $S_n(1\cdots(k+2))$, so we have the desired bijection. □

[1] These permutations have some independent interest; for example, Ira Gessel has conjectured (unpublished) that for fixed n, the doubly-alternating permutations are the maximally-sized sets of the form $\beta_n(S,T) = \{w \in S_n \mid \text{Des}(w) = S, \text{Des}(w^{-1}) = T\}$, where Des gives the descent set of a permutation. Foulkes [11] and Stanley [35] enumerated doubly-alternating permutations using symmetric functions.

It is possible to exploit various properties of RSK to give other results of a similar flavor, of which we give one example. If we apply RSK to an involution $w \in \mathcal{L}_{n,k}(1 \cdots (k+2))$, the result is a pair (P, P) of equal tableaux. In this case the associated tableaux R and S described in the proof of the preceding theorem are also equal. Thus the image of w under the bijection of Theorem 3.5.1 is also an involution, and, moreover, every involution in $S_n(1 \cdots (k+2))$ is the image of some involution in $\mathcal{L}_{n,k}(1 \cdots (k+2))$ under this map. This argument implies the following result.

Corollary 3.5.2. *The number of involutions in* $\mathcal{L}_{n,k}(1 \cdots (k+2))$ *is equal to the number of involutions in* $S_n(1 \cdots (k+2))$.

Chapter 4

Generating trees for pattern avoidance in alternating permutations

In Chapter 3, we gave a bijection between $A_{2n}(1234)$ and the set of standard Young tableaux of shape $\langle 3^n \rangle$ using RSK. In this chapter, we reprove this identity via an isomorphism of generating trees, i.e., we give a recursive bijection between the two sets. We then show that this approach can be extended to enumerate other classes of pattern-avoiding alternating permutations. In particular, we show that $A_{2n}(2143)$ is in bijection with standard Young tableaux of shape $\langle n, n, n \rangle$ (and so also with $A_{2n}(1234)$) and that $A_{2n+1}(2143)$ is in bijection with *shifted* standard Young tableaux of shape $\langle n+2, n+1, n \rangle$.

In Section 4.1, we provide the notation and terminology that we use in our discussion of generating trees. In Section 4.2, we describe a generating tree for standard Young tableaux of certain shapes and show that it obeys a simple two-parameter labeling scheme. In Section 4.3, we show that alternating permutations of even length with no four-term increasing subsequence share the same generating tree as the tableaux of Section 4.2. In Section 4.4, we show that alternating permutations of even length avoiding the pattern 2143 also share this generating tree. In Section 4.5, we show that similar methods can be used to enumerate alternating permutations of odd length avoiding the pattern 2143, and that the resulting bijection is with shifted Young tableaux of certain shapes. In Section 4.7 we give several conjectures and open problems relating to pattern avoidance in alternating permutations.

4.1 Generating trees

In this section, we provide important definitions and notation that will be used throughout the rest of this chapter.

Given a sequence $\{\Sigma_n\}_{n\geq 1}$ of finite, nonempty sets with $|\Sigma_1| = 1$, a **generating tree** for this sequence is a rooted, labeled tree such that the vertices at level n are the elements of Σ_n and the label of each vertex determines the multiset of labels of its children. In other words, a generating tree is one particular type of recursive structure in which heredity is determined by some local data. We are particularly interested in generating trees for which the labels are (much) simpler than the objects they are labeling. In this case, we may easily describe a generating tree by giving the label L_1 of the **root vertex** (the element of Σ_1) and the **succession rule** $L \mapsto S$ that gives the set S of labels of the children in terms of the label L of the parent.

Beginning with work of Chung, Graham, Hoggatt, and Kleiman [10], generating trees have been put to good use in the study of pattern-avoiding permutations, notably in the work of West (see, e.g., [40, 41]). The usual approach has been to consider subtrees of the tree of all permutations given by the rule that $v \in S_{n+1}$ is a child of $u \in S_n$ exactly when erasing the entry $n+1$ from v leaves the word u. Because we are interested here in alternating permutations and the permutation that results from erasing the largest entry of an alternating permutation typically is not alternating, this tree is unsatisfactory. However, the inverse tree, in which we arrive at the children of u by inserting arbitrary values in the last position (rather than inserting the largest value in an arbitrary position), is well-suited to the case of alternating permutations. This motivates the following definitions.

Given a permutation $u \in S_n$ and an element $i \in [n+1]$, there is a unique permutation $v = v_1 v_2 \cdots v_n v_{n+1} \in S_{n+1}$ such that $v_{n+1} = i$ and the word $v_1 v_2 \cdots v_n$ is order-isomorphic to u. We denote this permutation by $u \hookleftarrow i$ and refer to it as the **extension** of u by i. In other words, the operation of extending u by i replaces each entry $c \geq i$ in u by $c+1$ and then attaches i to the end of the result. For example, $3142 \hookleftarrow 3 = 41523$.

Given a pattern p and a permutation $w \in S_n(p)$, we say that $c \in [n+1]$ is **active** or **an active value** for w (with respect to p) if $w \hookleftarrow c$ avoids p. (This terminology is borrowed from the "usual" case, in which a position is said to be active if one can insert $n+1$ into that position while preserving pattern avoidance.)

There is a natural generating tree structure on alternating permutations of even length analogous to the tree on all permutations mentioned above: given an alternating permutation u of length $2n$, its children are precisely the alternating permutations v of length $2n+2$ such that the prefix of v of length $2n$ is order-isomorphic to u. (Of course, there is also a similar tree for alternating permutations of odd length.) Since pattern containment is transitive, the set $\bigcup_{n \geq 1} A_{2n}(p)$ of alternating permutations of even length avoiding the pattern p is the set of vertices of a subtree. It is these trees (for the patterns 1234 and 2143) that we will consider in Sections 4.3 and 4.4.

4.2 A generating tree for tableaux

For $n \geq 1$, the collection of standard Young tableaux of shape $\langle n^{k+1} \rangle$ has a simple associated generating tree: given a tableau $S \in \mathrm{SYT}(n^{k+1})$, its children are precisely the tableaux $T \in \mathrm{SYT}((n+1)^{k+1})$ such that removing the last column of T leaves a tableau that is order-isomorphic to S. Notice that the shape of the tree below S is determined entirely by the entries of the last column of S, and in particular by the entries $S(1,n)$ through $S(k,n)$ (since $S(k+1,n) = nk+n$ for all $S \in \mathrm{SYT}(n^{k+1})$).

Thus, we wish to choose a labeling for our tree that captures exactly this information. Our choice (one of several reasonable options) is to assign to each $S \in \mathrm{SYT}(n^{k+1})$ the label $(a_1, \ldots, a_k) = (nk + n + 1 - S(k, n), \ldots, nk + n + 1 - S(1, n))$. This provides a k-label generating tree for

$$\bigcup_{n \geq 1} \mathrm{SYT}(n^{k+1}),$$

whose root (the unique standard Young tableau of shape $\langle 1, \ldots, 1 \rangle$) has the label $(k + 2 - k, \ldots, k + 2 - 1) = (2, 3, \ldots, k + 1)$.

Proposition 4.2.1. *The generating tree that we have just described for standard Young tableaux with all columns of length $k + 1$ obeys the rule*

$$(a_1, \ldots, a_k) \mapsto \{(x_1, \ldots, x_k) \mid 2 \leq x_1 < x_2 < \ldots < x_k \text{ and } x_i \leq a_i + i \text{ for all } i\}.$$

For example, when $k = 2$ the root tableau $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ has label $(2, 3)$ and has five children,

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array} \text{ and } \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}\,,$$

which have labels $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$ and $(3, 5)$, respectively.

Proof. Choose a standard Young tableau $S \in \mathrm{SYT}(n^{k+1})$ with label $(a_1, \ldots, a_k)$ and a child $T \in \mathrm{SYT}((n + 1)^{k+1})$ of S with label $(x_1, \ldots, x_k)$. We wish to show that $2 \leq x_1 < \ldots < x_k$ and $x_i \leq a_i + i$. Expressing these conditions in terms of the entries of S and T, we must show that

$$T(1, n + 1) < T(2, n + 1) < \ldots < T(k, n + 1) \leq nk + n + k \tag{4.1}$$

and that for all i,

$$i + S(i, n) \leq T(i, n + 1). \tag{4.2}$$

The proof of Equation (4.1) is straightforward: the first $k - 1$ inequalities hold because T is a standard Young tableau, while the last holds because $T(k, n + 1) < T(k + 1, n + 1) = nk + n + k + 1$.

To prove Equation (4.2), we compute that

$$\begin{aligned} T(i, n + 1) &= \Big|\{(\ell, m) \mid T(\ell, m) \leq T(i, n + 1)\}\Big| \\ &\geq \Big|\{(\ell, m) \mid T(\ell, m) \leq T(i, n) \text{ and } m \leq n\} \cup \{(1, n + 1), \ldots, (i, n + 1)\}\Big| \\ &= \Big|\{(\ell, m) \mid S(\ell, m) \leq S(i, n)\} \cup \{(1, n + 1), \ldots, (i, n + 1)\}\Big| \\ &= S(i, n) + i, \end{aligned}$$

as desired. $\square$

In the particular case $k = 2$ (tableaux of shape $\langle n, n, n\rangle$, which correspond to $A_{2n}(1234)$), this gives the following result.

Proposition 4.2.2. *The generating tree that we have just described for standard Young tableaux with all columns of length three obeys the rule*

$$(a, b) \mapsto \{(x, y) \mid 2 \leq x \leq a + 1 \text{ and } x + 1 \leq y \leq b + 2\}.$$

4.3 Generating tree for $\mathcal{L}_{n,k}(1 \cdots (k + 2))$

Recall from Section 4.1 that for any pattern p, there is a natural generating tree for the set $\bigcup_n A_{2n}(p)$ of p-avoiding alternating permutations of even length: given a p-avoiding alternating permutation u of length $2n$, its children are precisely the p-avoiding alternating permutations v of length $2n + 2$ such that the prefix of v of length $2n$ is order-isomorphic to u. In this section, we study this tree for the pattern $p = 1234$, and more generally we study the generating tree for $\bigcup_n \mathcal{L}_{n,k}(1 \cdots (k + 2))$. These generating trees are closely related to the generating tree for $S_n(1234)$ discussed in [40] and [8].

Compared with the proofs of Chapter 3, the proofs in this section are somewhat technical and not entirely enlightening.

Given a permutation $w \in \mathcal{L}_{n,k}(1 \cdots (k+2))$, associate to w the label $(a_2, \ldots, a_{k+1})$ as follows: for $2 \leq j \leq k + 1$, let a_j be the smallest entry of w that is the largest (equivalently, last) entry in a j-term increasing subsequence, or $nk + 1$ if there is no such entry.[1] Equivalently, a_j is the last-occurring entry of w that is the largest term in a j-term increasing subsequence of w but is not the largest term in a $(j + 1)$-term increasing subsequence. We show in a series of steps that with this labeling, the generating tree for $\bigcup_n \mathcal{L}_{n,k}(1 \cdots (k + 2))$ obeys the rule

$$(a_2, \ldots, a_{k+1}) \mapsto \{(x_2, \ldots, x_{k+1}) \mid 1 < x_2 < \ldots < x_{k+1} \text{ and } x_i \leq a_i + i - 1\}. \quad (4.3)$$

(This tree is easily seen to be isomorphic to the tree in Section 4.2.) As a first step, we show that entries of the label of an individual permutation are in strictly increasing order.

Proposition 4.3.1. *If $u \in \mathcal{L}_{n,k}(1 \cdots (k + 2))$ has label $(a_2, \ldots, a_{k+1})$ then we have $1 < a_2 < \ldots < a_{k+1} \leq nk + 1$.*

Proof. Fix $u \in \mathcal{L}_{n,k}(1 \cdots (k + 2))$ with label $(a_2, \ldots, a_{k+1})$.

By definition, a_2 is not the smallest entry of the permutation u, so $1 < a_2$ is automatic. Likewise, $a_{k+1} \leq nk + 1$ follows immediately from the definitions. We now tackle the other inequalities.

Every permutation in $\mathcal{L}_{n,k}$ contains increasing subsequences of every length up to (and including) k. For any j such that $2 \leq j \leq k - 1$ and any increasing subsequence

[1] As we see in the next result, the piecewise nature of this definition is relevant only for a_{k+1}. As it happens, we can also give a non-piecewise definition for a_{k+1}: it is exactly the number of safe values for w. However, this definition does not seem to be more convenient for our proofs.

of u of length $j+1$, the jth term of this subsequence is strictly less than the $(j+1)$th. In particular, this is true in the subsequence whose $(j+1)$th term is a_{j+1}. It follows immediately that $a_j < a_{j+1}$ for $2 \leq j \leq k-1$.

If u contains an increasing subsequence of length $k+1$, it also follows from the same argument that $a_k < a_{k+1}$. Alternatively, if u contains no $(k+1)$-term increasing subsequence then $a_{k+1} = nk+1$ while $a_k \in [nk]$, so $a_k < a_{k+1}$ in this case, as well. □

In subsequent proofs, we consider several cases of a permutation repeatedly extended by a sequence of values. The following technical lemma is useful in such situations; it tells us how many times an entry in the permutation will be "bumped up" by these extensions.

Lemma 4.3.2. *Suppose u is a permutation and $c_1 < c_2 < \ldots < c_\ell$ are any values such that we can define $v = (\cdots((u \hookleftarrow c_1) \hookleftarrow c_2)\cdots) \hookleftarrow c_\ell$. For $t \geq 1$, we have $v_i - u_i \geq t$ if and only if $c_t \leq u_i + t - 1$.*

Proof. The proof is a simple induction. □

We now move on to directly study the generating tree of interest. First, we show that all children of a permutation in $\bigcup_n \mathcal{L}_{n,k}(1\cdots(k+2))$ have labels among the set of values claimed in Equation (4.3).

Proposition 4.3.3. *Suppose that $u \in \mathcal{L}_{n,k}(1\cdots(k+2))$ has label $(a_2, \ldots, a_{k+1})$. If $v \in \mathcal{L}_{n+1,k}(1\cdots(k+2))$ is a child of u with label $(x_2, \ldots, x_{k+1})$, then $x_j \leq a_j + j - 1$ for all j.*

Proof. Fix $u \in \mathcal{L}_{n,k}(1\cdots(k+2))$ with label $(a_2, \ldots, a_{k+1})$ and fix a child $v \in \mathcal{L}_{n+1,k}(1\cdots(k+2))$ of u with label $(x_2, \ldots, x_{k+1})$. Choose j such that $2 \leq j \leq k+1$. We wish to show that $x_j \leq a_j + j - 1$.

If $j = k+1$ and u contains no $(k+1)$-term increasing subsequence then $a_j = a_{k+1} = nk+1$ and by definition $x_{k+1} \leq (n+1)k+1 = a_{k+1}+k$, which is the desired inequality. In all other cases (i.e., if $j \leq k$, or if $j = k+1$ but u does contain a $(k+1)$-term increasing subsequence), we have that $a_j \in [nk]$. We handle these cases now.

Since the last k entries of v occur in increasing order, we have that

$$v = (\cdots((u \hookleftarrow v_{nk+1}) \hookleftarrow v_{nk+2})\cdots) \hookleftarrow v_{nk+k}.$$

Thus, for $i \in [nk]$, we have by Lemma 4.3.2 that $v_i - u_i \geq t$ if and only if $v_{nk+t} \leq u_i + t - 1$.

Now choose i such that $u_i = a_j$, and consider the possible values of v_i. If $v_{nk+j} > u_i + (j-1) = a_j + j - 1$ then, by the preceding paragraph, $v_i \leq a_j + j - 1$; that is, u_i gets bumped fewer than j times in this case. Otherwise, $v_{nk+j} \leq u_i + (j-1) = a_j + j - 1$. In either case, we have an entry of v (in the first case, v_i; in the second, v_{nk+j}) that is the largest term in a j-term increasing subsequence of v and is no larger than $a_j + j - 1$. By the definition of x_j, it follows immediately that $x_j \leq a_j + j - 1$, as desired. □

In the next result, we establish all the other facts necessary to completely describe the generating tree for $\bigcup_n \mathcal{L}_{n,k}(1\cdots(k+2))$.

Proposition 4.3.4. *If $u \in \mathcal{L}_{n,k}(1\cdots(k+2))$ has label $(a_2,\ldots,a_{k+1})$ and the tuple $(x_2,\ldots,x_{k+1})$ is such that $1 < x_2 < \ldots < x_{k+1}$ and $x_j \le a_j + j - 1$ for all j, then there exists a unique child $v \in \mathcal{L}_{n,k}(1\cdots(k+2))$ of u with label $(x_2,\ldots,x_{k+1})$.*

Proof. Fix a permutation $u \in \mathcal{L}_{n,k}(1\cdots(k+2))$ with label $(a_2,\ldots,a_{k+1})$ and fix a k-tuple $(x_2,\ldots,x_{k+1})$ such that $1 < x_2 < \ldots < x_{k+1}$ and $x_j \le a_j + j - 1$ for all j. We wish to construct a child $v \in \mathcal{L}_{n,k}(1\cdots(k+2))$ of u with label $(x_2,\ldots,x_{k+1})$. For convenience, define $a_1 = x_1 = 1$. We show that we recover the desired v by extending u successively by k of the $k+1$ values $x_1, \ldots, x_{k+1}$.

The set of indices i at which we have $x_i = a_i + i - 1$ is nonempty (it contains 1); let m be its largest member. Define

$$v = (\cdots(((\cdots(u \hookleftarrow x_1)\cdots) \hookleftarrow x_{m-1}) \hookleftarrow x_{m+1})\cdots) \hookleftarrow x_{k+1}. \tag{4.4}$$

We wish to show that v is in $\mathcal{L}_{n+1,k}(1\cdots(k+2))$, that v has label $(x_2,\ldots,x_{k+1})$, and that there is no other child of u with this label. In the following lemma, we break the first two of these claims down into a sequence of

Lemma 4.3.5. *Let u, a, x be as above and let v be defined as in Equation (4.4). For any j such that $a_j \in [nk]$, let i_j be the index such that $u_{i_j} = a_j$. (In other words, $i_j = u^{-1}_{a_j}$.) The following are true of the permutation v.*
1. If $j < m$ then $v_{i_j} > x_j$.
2. For $j < m$, $v_{nk+j} = x_j$ is not the largest entry in a $(j+1)$-term increasing subsequence of v. However, it is the largest entry in a j-term increasing subsequence.
3. If $j > m$ then $v_{i_j} > x_j$.
4. For $j > m$, $v_{nk+j-1} = x_j$ is not the largest entry in a $(j+1)$-term increasing subsequence of v.
5. v avoids $1\cdots(k+2)$.
6. $v \in \mathcal{L}_{n+1,k}(1\cdots(k+2))$.
7. $v_{i_m} = x_m$.
8. v has label $(x_2,\ldots,x_m)$.

Proof of lemma. We prove the results in the order in which they are stated.
1. Since $j < m$ we have $x_j \le a_j + j - 1 = u_{i_j} + j - 1$. Thus, by Lemma 4.3.2, $v_{i_j} \ge u_{i_j} + j > x_j$, as desired.
2. Any $(j+1)$-term increasing subsequence of v ending at v_{nk+j} would contain, for some ℓ, exactly ℓ terms from the first nk entries of v and $j - \ell + 1$ terms from the last k entries of v. Thus, the ℓth term of this subsequence is at least v_ℓ while the $(\ell+1)$th term is at most $v_{nk+\ell} = x_\ell$. By **1**, this is a contradiction and no such subsequence exists. However, v does contain the j-term increasing subsequence $v_{nk+1}, v_{nk+2}, \ldots, v_{nk+j}$.
3. Since $j > m$ we have by the definition of m that $x_j \le a_j + j - 2 = u_{i_j} + j - 2$. Because $j > m$, x_j is the $(j-1)$th value by which u is extended; thus, by Lemma 4.3.2, $v_{i_j} \ge u_{i_j} + j - 1 > x_j$, as desired.

4. If i_j exists, the proof is essentially the same as that of **2**. Otherwise, we have $j = k+1$, $a_j = nk+1$. In this case, u contains no $(k+1)$-term increasing subsequence. In fact, $(\cdots(u \hookleftarrow x_1)\cdots) \hookleftarrow x_k$ contains no $(k+1)$-term increasing subsequence by the first part of this proof in the case $j = k$. Thus certainly $x_{k+1} = v_{nk+k}$ can't be the last term in a $(k+2)$-term increasing subsequence of v.
5. The first $[nk]$ entries of v do not contain a $(k+2)$-term increasing subsequence because u avoids $1\cdots(k+2)$. Thus, any such subsequence must contain some of the last k entries of v. By moving the last entry, we see that v contains $1\cdots(k+2)$ if and only if there is a $(k+2)$-term increasing subsequence of v ending in $v_{nk+k} = x_{k+1}$; but by **4** there is no such subsequence.
6. This follows immediately from **5** and Note 1.
7. We have $x_{m-1} \leq x_m - 1 = a_m + m - 2 = u_{i_m} + m - 2$, so by Lemma 4.3.2 we have $v_{i_m} \geq u_{i_m} + m - 1 = x_m$. Moreover, $x_{m+1} > x_m = a_m + m - 1$ so again by Lemma 4.3.2 we have $v_{i_m} < u_{i_m} + m$. The result follows.
8. From **6**, the statement makes sense. For $j < m$, its truth is established by **2** and the alternate characterization of the labeling scheme given in the third paragraph of this section. For $j \geq m$, its truth follows from **4**, **7** and the same characterization of the labeling scheme. □

Finally, any child of u has the form $v = (\cdots(u \hookleftarrow c_1)\cdots) \hookleftarrow c_k$ for some sequence $c_1 < \ldots < c_k$. If the c_i do not obey the necessary inequalities, it follows not only that the proofs above fail but actually that we can write down a $(k+2)$-term increasing subsequence of v: if $c_j > a_{j+1} + j$ then $v_{i_{j+1}} < c_j$ and so there is a $(k+2)$-term increasing subsequence of v of the form $\ldots, v_{i_{j+1}}, c_j, \ldots, c_k$. Thus, the children of u of the form above are all such children, and so they have distinct labels, as claimed. □

Note 2. In the special case that $k = 2$ (i.e., alternating permutations), we have that if $w \in A_{2n}(1234)$ has label (a, b) then the collection of labels of children of w is exactly the set of (x, y) such that $2 \leq x \leq a + 1$ and $x + 1 \leq y \leq b + 2$, each pair occurring with multiplicity one. (Here a is the smallest entry of w that is preceded by an even smaller entry, and b is the number of active values for w with respect to 1234.) ■

Finally, combining Propositions 4.2.1, 4.3.3 and 4.3.4, we immediately recover our main theorem.

Theorem 3.3.4. *There is a bijection between* $\mathcal{L}_{n,k}(12\cdots(k+1)(k+2))$ *and the set of standard Young tableaux of shape* $\langle n^{k+1}\rangle$ *and so*

$$|\mathcal{L}_{n,k}(12\cdots(k+1)(k+2))| = f^{\langle n^{k+1}\rangle}.$$

Note 3. Observe that (after taking transposes of tableaux) this bijection is not the same as the bijection of Section 3.3. For example, consider the permutation $35184726 \in A_8(1234)$. Under RSK, we have

$$35184726 \overset{\text{RSK}}{\longleftrightarrow} \left(\begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 3 & 4 & 7 \\ \hline 5 & 8 \\ \cline{1-2} \end{array}\, , \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} \end{array} \right)$$

$$12 \quad \longrightarrow \quad 2314 \quad \longrightarrow \quad 241635 \quad \longrightarrow \quad 35184726$$

$$\text{label } (2,3) \qquad \text{label } (3,4) \qquad \text{label } (3,5) \qquad \text{label } (2,6)$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 8 & 11 \\ \hline 6 & 9 & 10 & 12 \\ \hline \end{array}$$

Figure 4-1: Corresponding branches in the generating trees for $\mathcal{L}_{n,2}(1234)$ and $\mathrm{SYT}(n^3)$.

and so (taking the transpose) we have the correspondence

$$35184726 \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 10 \\ \hline 2 & 4 & 8 & 11 \\ \hline 6 & 7 & 9 & 12 \\ \hline \end{array}$$

under the bijection of Section 3.3. On the other hand, the portion of the generating trees shown in Figure 4-1 shows that

$$35184726 \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 8 & 11 \\ \hline 6 & 9 & 10 & 12 \\ \hline \end{array}$$

under the bijection implied by the generating tree equivalence in this section. Moreover, the two bijections are not related by any of the obvious symmetries of either permutations or tableaux (e.g., reverse-complementation or rotated-complementation). However, it is true that given a permutation $w \in \mathcal{L}_{n,k}(1 \cdots (k+2))$, the first row of the tableau associated to it via the bijection of Section 3.3 (the first column after taking transpose) is precisely $(1, a_2, a_3, \ldots, a_{k+1})$. ■

Note 4. We can also reprove the more general Theorem 3.4.2 using generating trees. We again change slightly the set of tableaux under consideration and work with those of skew shape $\lambda/\mu = \langle (n+1)^{k+1} \rangle / \langle 2, 1^{k-r} \rangle$. For both $\mathcal{L}_{n,k;r}(1 \cdots (k+2))$ and $\mathrm{SYT}(\lambda/\mu)$, we use the same labeling scheme when $n \geq 2$ as in the rest of this section. In both cases, it's easy to see that the resulting generating tree rules will be identical. Thus, all that's left to check is that the trees are isomorphic for the first two levels and that the nodes on the second level have corresponding labels in the two trees. These facts are not difficult to prove, so we omit the proof here. ■

We now turn our attention to the case of alternating permutations and to the pattern 2143.

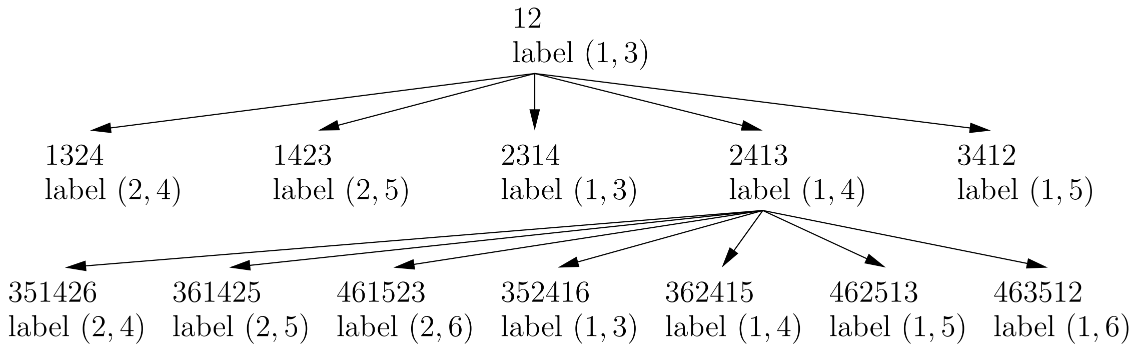


Figure 4-2: An initial portion of the generating tree for 2143-avoiding alternating permutations of even length.

4.4 Generating tree for $A_{2n}(2143)$

In this section, we show that alternating permutations of even length avoiding 2143 have a generating tree isomorphic to the trees for $\bigcup_{n\geq 1} A_{2n}(1234)$ and $\bigcup_n \mathrm{SYT}(n,n,n)$ described in the preceding sections. A portion of this tree is shown in Figure 4-2.

Given a permutation $w \in A_{2n}(2143)$, assign to it a label (a, b) where $a = w_{2n-1}$ is the next-to-last entry of w and b is the number of active values for w in $[2n+1]$.

For example, the permutation $w = 68142537 \in A_8(2143)$ has active values 1, 2, 3, 4, 8 and 9 because the permutations 792536481, 791536482, 791526483, 791526384, 691425378 and 681425379 avoid 2143 while the permutations 691425387, 791425386 and 791426385 contain it. Thus for this permutation we have $a = 3$ and $b = 6$.

As a second example, the permutation $w = 35462718 \in A_8(2143)$ has active values 1, 2 and 9. Thus $a = 1$ and $b = 3$.

We have that the root $12 \in A_2(2143)$ has label $(1, 3)$. We will show that under this labeling the generating tree for $\bigcup_n A_{2n}(2143)$ obeys the rule

$$(a, b) \mapsto \{(x, y) \mid 1 \leq x \leq a+1 \text{ and } x+2 \leq y \leq b+2\},$$

and we will use this result to establish an isomorphism between this generating tree and those discussed in the preceding sections. We break the proof of this result into several smaller pieces: Propositions 4.4.2 and 4.4.4 form the meat of the argument establishing the succession rule, while Propositions 4.4.1 and 4.4.3 are helpful technical lemmas. We begin with a simple observation that will be of use in the subsequent proofs.

Note 5. Given a permutation $w \in S_n(2143)$, we have that $c \in [n+1]$ is not active for w if and only if there exist $i < j < k$ such that $w_j < w_i < c \leq w_k$. ■

Proposition 4.4.1. *If $u \in A_{2n}(2143)$ and $u_{2n-1} = a$ then $\{1, 2, \ldots, a+1\}$ are active values for u.*

Proof. Fix $u \in A_{2n}(2143)$ with $u_{2n-1} = a$, choose $c \leq a+1$ and let $v = u \hookleftarrow c \in S_{2n+1}$. We wish to show that v avoids 2143, so suppose otherwise. Then there exist $i < j <$

$k < 2n+1$ such that $v_i v_j v_k c$ is an instance of 2143 in v. We use this (suppositional) instance to construct an instance of 2143 in u; this contradiction establishes that c is active for u. In particular, we show that $u_i u_j u_{2n-2} u_{2n-1}$ is an instance of 2143 in u by showing that $v_i v_j v_{2n-2} v_{2n-1}$ is an instance of 2143 in v. In order to do this, it suffices to show that $j < 2n-2$ (so that $v_i v_j v_{2n-2} v_{2n-1}$ is a subsequence of v) and that $v_i < v_{2n-1}$ (so that this subsequence is order-isomorphic to 2143).

Since $v_i v_j v_k c$ is an instance of 2143 and $c \leq a+1$, we have that $v_j < v_i < c \leq a+1$ and thus $v_i \leq a = u_{2n-1} \leq v_{2n-1}$. There are at least three entries to the right of v_i in v but only two to the right of v_{2n-1}, so $v_i \neq v_{2n-1}$ and thus $v_i < v_{2n-1}$, one of the two conditions we need. It follows that $v_j < v_i < v_{2n-1} < v_{2n-2}$ and similarly $v_j < v_{2n}$, so v_j is smaller than all of $v_{2n-2}, v_{2n-1}, v_{2n}$ and v_{2n+1}. These entries form a suffix of v, so v_j must occur at an earlier position in v. That is, we have $j < 2n-2$, the second necessary condition. Thus $v_i v_j v_{2n-2} v_{2n-1}$ is an instance of 2143 in v and so $u_i u_j u_{2n-2} u_{2n-1}$ is an instance of 2143 in u. Since u avoids 2143, this is a contradiction, so actually v avoids 2143 and c is active for u, as desired. □

Proposition 4.4.2. *If $u \in A_{2n}(2143)$ has label (a,b) and w is a child of u with label (x,y) then $1 \leq x \leq a+1$ and $x+2 \leq y \leq b+2$.*

Proof. Suppose that permutations $u \in A_{2n}$ and $w \in A_{2n+2}$ have the property that the first $2n$ entries of w are order-isomorphic to u, and set $a = u_{2n-1}$. We now demonstrate that if $w_{2n+1} > a+1$ then w contains 2143; this will allow us to conclude the first inequality. Suppose that $w_{2n+1} > a+1$; then also $w_{2n} > a+1$ and $w_{2n+2} > a+1$, while $w_{2n-1} = a$. Thus $w^{-1}(a+1) \notin \{2n-1, 2n, 2n+1, 2n+2\}$ and, defining $i = w^{-1}(a+1)$, we have $i < 2n-1$. Then $w_i w_{2n-1} w_{2n} w_{2n+1}$ is an instance of 2143 in w. Taking the contrapositive, if w avoids 2143 then $w_{2n+1} \leq a+1$. Now fix $u \in A_{2n}(2143)$ with label (a,b) and a child $w \in A_{2n+2}(2143)$ of u with label (x,y); the preceding argument shows that $x \leq a+1$. Clearly $x \geq 1$, so we have proved the first half of our assertion. We now proceed to bound y, the number of active values of w.

Define a one-to-one function $f\colon [2n] \to [2n+2]$ by

$$f(z) = \begin{cases} z, & z < w_{2n+1} \\ z+1, & w_{2n+1} \leq z < w_{2n+2} \\ z+2, & w_{2n+2} \leq z, \end{cases}$$

so $f(u_\ell) = w_\ell$ for all $\ell \in [2n]$. We show that f is a map from nonactive values of u to nonactive values of w; from this fact we may establish the upper bound on y. To this end, choose any $c \in [2n]$ that is not active for u, and choose $i < j < k$ such that $u_j < u_i < c \leq u_k$. We have $(w_i, w_j, w_k, f(c)) = (f(u_i), f(u_j), f(u_k), f(c))$. One can easily see that f preserves order, so $w_j < w_i < f(c) \leq w_k$ and thus $f(c)$ is not active for w. Thus for each of the $2n-b$ choices of a nonactive value c for u we have a corresponding nonactive value $f(c)$ for w and so w has at most $(2n+2)-(2n-b) = b+2$ active values, i.e., $y \leq b+2$.

Finally, we have by Proposition 4.4.1 that $\{1, 2, \ldots, x+1\}$ are active values for w. We also have that $2n+3$ is active for w and that $2n+3 \notin \{1, 2, \ldots, x+1\}$, so there

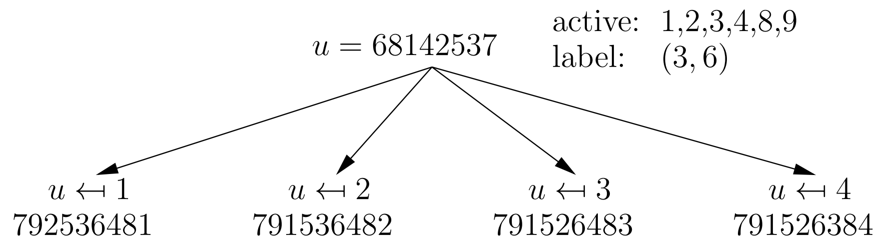


Figure 4-3: The permutation $68142537 \in A_8(2143)$ may be extended by 1, 2, 3 or 4 to give an alternating, 2143-avoiding permutation of length 9. Each of the four extensions have active values 1, 2, 3, 4, 5, 9 and 10.

are at least $x+2$ active values for w. Thus $x+2 \leq y$, which completes the proof of our claim. $\square$

So far, we have shown that the only possible labels for a child of a 2143-avoiding permutation are those claimed. The next result shows how the active values of an extension of an alternating 2143-avoiding permutation relate to those of the permutation itself.

Proposition 4.4.3. *Suppose $u \in A_{2n}(2143)$ has label (a,b) and active values $s_1 < s_2 < \ldots < s_b$. If $x \leq a+1$ and $v = u \hookleftarrow x$ then $v \in A_{2n+1}(2143)$ and v has active values $1, s_1+1, s_2+1, \ldots, s_b+1$.*

For example, $u = 68142537 \in A_8(2143)$ has label $(3,6)$ and active values 1, 2, 3, 4, 8 and 9. For $x = 4 \leq 3+1$ we have $v = 791526384$ with active values 1, 2, 3, 4, 5, 9 and 10. Figure 4-3 shows the other possible extensions of u.

Proof. Choose $u \in A_{2n}(1234)$ with label (a,b) and choose $x \leq a+1$. Let $s_1 < \ldots < s_b$ be the active values for u and let $v = u \hookleftarrow x$. Since $v_{2n+1} = x \leq a+1 \leq v_{2n-1}+1$ and $v_{2n-1} < v_{2n}$, we have $v_{2n+1} \leq v_{2n}$ and so actually $v_{2n+1} < v_{2n}$. Thus v is alternating. Also, by Proposition 4.4.1, x is an active value for u and so v avoids 2143. This proves the first part of our claim; to finish, we must show that the set of active values for v is $\{1, s_1+1, \ldots, s_b+1\} = \{1, 2, \ldots, x, s_x+1, s_{x+1}+1, \ldots, s_b+1\}$. We first show that each of these values is in fact an active value for v by considering two cases depending on how the value in question compares to x.

Case 1. Fix $m \geq x$ so that $s_m \geq x$ is an active value for u. We wish to show that s_m+1 is an active value for v. Let $w = w_1 \cdots w_{2n+2}$ be the result of extending v by s_m+1, and suppose for sake of contradiction that w contains an instance $w_i w_j w_k w_\ell$ of 2143. We use this suppositional subsequence to find an instance of 2143 in v. Since v avoids 2143, we must have $\ell = 2n+2$, so $w_\ell = w_{2n+2} = s_m+1$. As $w_{2n+2} = s_m+1 > x = v_{2n+1}$, extending v by s_m+1 does not change the value of the $(2n+1)$th entry and thus $w_{2n+1} = v_{2n+1} = x$, whence $w_{2n+2} > w_{2n+1}$. Since $w_k > w_{2n+2}$, we have $k \neq 2n+1$ and so w_{2n+1} is not part of our instance of 2143. Let v' be the permutation order-isomorphic to $w_1 w_2 \cdots w_{2n} w_{2n+2}$; we've shown that $v'_i v'_j v'_k v'_{2n+1}$ is an instance of 2143 in v'. However, we also have that $v' = u \hookleftarrow s_m$ and that s_m is active for u. This is a contradiction, so w cannot contain an instance of 2143, and we conclude that $s_m + 1$ is an active value for v by definition.

Case 2. Fix $c \leq x$ so that $c = s_c$ is an active value for u. We wish to show that c is an active value for v. Let $w = w_1 \cdots w_{2n+1} w_{2n+2}$ be the result of extending v by c, and suppose for sake of contradiction that w contains an instance $w_i w_j w_k w_\ell$ of 2143. Again we use this instance to produce an instance of 2143 in v. If $\{k, \ell\} \neq \{2n+1, 2n+2\}$ then we can conclude by an argument nearly identical to the previous case, so assume that $k = 2n + 1$ and $\ell = 2n + 2$. Since $w_i < w_k = w_{2n+1}$ and $w_j < w_k = w_{2n+1}$ and $w_{2n} > w_{2n+1}$, we have $2n \notin \{i, j\}$ and so $w_i w_j w_{2n} w_{2n+2}$ is another instance of 2143 in w. But now we have an instance not including w_{2n+1} and so we may proceed as in the previous case. We conclude that c is an active value for v.

Finally, we show that these values are the only active values for v. In particular, we must show that for every $c > x$ that is not active for u, $c + 1$ is not active for v. Fix such a c. Since c is not active for u, there exist $i < j < k$ such that $u_j < u_i < c \leq u_k$. We have $v_\ell \leq u_\ell + 1$ for all $\ell \in [2n]$, so $v_j < v_i < c + 1$. Since $u_k > c - 1 \geq x$, we have $v_k = u_k + 1$. Thus $v_j < v_j < c + 1 \leq u_k + 1 = v_k$ and so $c + 1$ is not an active value for v. We conclude that the active values for v are exactly $1, 2, \ldots, x, s_x + 1, s_{x+1} + 1, \ldots, s_b + 1$, as claimed. □

Finally, we combine the preceding results to establish that 2143-avoiding alternating permutations of even length have the claimed generating tree.

Proposition 4.4.4. *If $u \in A_{2n}(2143)$ has label (a, b) and x, y are such that $1 \leq x \leq a + 1$, $x + 2 \leq y \leq b + 2$, then there is a unique child $w \in A_{2n+2}(2143)$ of u with label (x, y).*

Proof. Choose a permutation $u \in A_{2n}(2143)$ with label (a, b) and choose (x, y) such that $1 \leq x \leq a + 1$ and $x + 2 \leq y \leq b + 2$. We will construct a child of u with label (x, y). This process is partially illustrated in Figure 4-4.

Let $s_1 < \ldots < s_b$ be the active values for u. Define $v = v_1 v_2 \cdots v_{2n} v_{2n+1}$ and $w = w_1 w_2 \cdots w_{2n} w_{2n+1} w_{2n+2}$ by $v = u \hookleftarrow x$ and $w = v \hookleftarrow (s_{b+2+x-y} + 1)$. We claim that w is the desired permutation. We must show that w belongs to $A_{2n+2}(2143)$ and that its label really is (x, y).

It follows from Proposition 4.4.3 that $v \in A_{2n+1}(2143)$. Because $y \leq b + 2$, we have $b + 2 + x - y \geq x$ and so $s_{b+2+x-y} + 1 > s_x = x = v_{2n+1}$. Thus w is alternating. We also have from Proposition 4.4.3 that $s_{b+2+x-y} + 1$ is an active value for v, so $w \in A_{2n+2}(2143)$. We have left to show that w has label (x, y).

Since $w_{2n+2} > v_{2n+1}$, extending v by w_{2n+2} leaves the value of the entry in the $(2n + 1)$th position unchanged and so $w_{2n+1} = v_{2n+1} = x$. It remains to show that w has exactly y active values. We claim that the active values for w are precisely $1 < 2 < \ldots < x+1 < s_{b+2+x-y}+2 < s_{b+3+x-y}+2 < \ldots < s_b+2$. By Proposition 4.4.1, we already know that $1, 2, \ldots, x + 1$ are active for w.

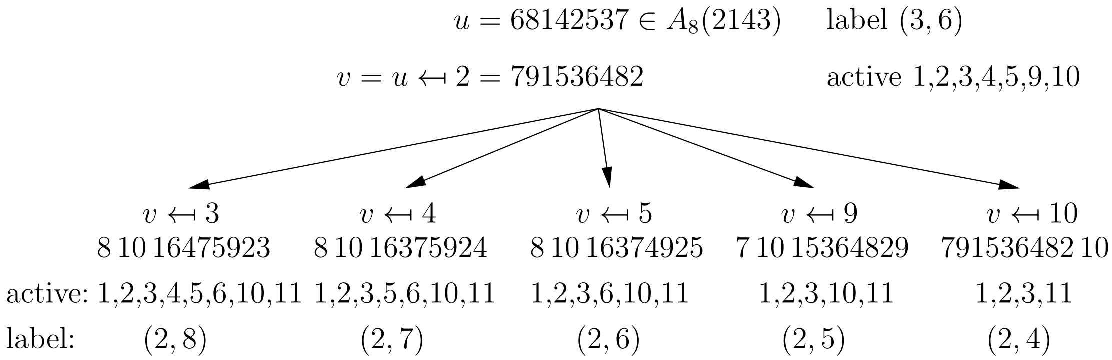


Figure 4-4: The five children of 68142537 in $A_{10}(2143)$ with next-to-last entry 2.

For any fixed i such that $b+2+x-y \le i \le b$, let $z = w \hookleftarrow (s_i+2)$. We wish to show that z avoids 2143. We proceed by the same argument as in Case 1 of the proof of Proposition 4.4.3: since $s_i + 2 > w_{2n+2}$, we have $z_{2n+2} = w_{2n+2} < z_{2n+3}$. It follows that z_{2n+2} and z_{2n+3} cannot be part of the same 2143 pattern in z. Thus z contains an instance of 2143 if and only if $z' = z_1 \cdots z_{2n+1} z_{2n+3}$ does. However, z' is order-isomorphic to $v \hookleftarrow (s_i+1)$. Since s_i+1 is an active value for v, z' avoids 2143 and so z avoids 2143. Thus s_i+2 is an active value for w. There are $b-(b+2+x-y)+1 = y-x-1$ such active values. We must show that there are no other active values for w larger than $x+1$.

For any fixed $c > s_{b+2+x-y}+2$ that is not of the form s_i+2, we wish to show that $w \hookleftarrow c$ contains 2143. We proceed by the same arguments that follow Case 2 of the proof of Proposition 4.4.3: if $v_i v_j v_k$ are entries of v that can be used to form a 2143 pattern when v is extended by $c-1$ then $w_i w_j w_k$ can be used to form a 2143 pattern when w is extended by c.

Finally, for any fixed c such that $x+1 < c < s_{b+2+x-y}+2$, let $z = w \hookleftarrow c$. We wish to show that z contains 2143. We have that $z_{2n+1} = x$, $z_{2n+2} = s_{b+2+x-y}+2 > c$, and $z_{2n+3} = c > x+1$, and we know that there exists $i < 2n+1$ such that $z_i = x+1$. Then $z_i z_{2n+1} z_{2n+2} z_{2n+3} = (x+1)x(s_{b+2+x-y}+2)c$ is an instance of 2143 in z, so c is not active for w.

The preceding four paragraphs account for all elements $[2n+3]$. We've shown that exactly $(x+1)+(y-x-1) = y$ of these values are active for w. Putting everything together, we have that $w \in A_{2n+2}(2143)$ is a child of u with y active values and $w_{2n+1} = x$, i.e., w has label (x,y). In fact, it follows from our proof that every child of u has a distinct label: we've exhausted the possible pairs of values for w_{2n+1}, w_{2n+2} such that $w \in A_{2n+2}(2143)$ is a child of u. $\square$

Theorem 4.4.5. *For all $n \ge 1$ we have*

$$|A_{2n}(2143)| = |\operatorname{SYT}(n,n,n)| = |A_{2n}(1234)| = \frac{2 \cdot (3n)!}{n!(n+1)!(n+2)!}.$$

Proof. Proposition 4.2.1 shows that the generating tree for $\bigcup_{n\ge1} \operatorname{SYT}(n,n,n)$ has root $(2,3)$ and rule

$$(a,b) \mapsto \{(x,y) \mid 2 \le x \le a+1 \text{ and } x+1 \le y \le b+2\}$$

while Propositions 4.4.2 and 4.4.4 together show that the generating tree for the set $\bigcup_{n\ge 1} A_{2n}(2143)$ has root $(1,3)$ and rule

$$(a,b) \mapsto \{(x,y) \mid 1 \le x \le a+1 \text{ and } x+2 \le y \le b+2\}.$$

These two trees are isomorphic: replacing each label (a,b) in the first tree with $(a-1,b)$ results in the second tree. Thus, there is a recursive bijection between $A_{2n}(2143)$ and $\mathrm{SYT}(n,n,n)$, and we have the first claimed equality. Theorem 3.3.4 completes the proof. □

4.5 Generating tree for $A_{2n+1}(2143)$

Turning our attention to alternating permutations of odd length, we find that the generating trees of $A_{2n+1}(1234)$ and $A_{2n+1}(2143)$ are not isomorphic. Indeed, the two sequences enumerate differently: all sixteen alternating permutations of length five avoid 1234, but only twelve of them avoid 2143. Although this initially seems disappointing, it turns out that we can still use the methods of the preceding section to enumerate 2143-avoiding alternating permutations of odd length.

As in [22], it is convenient to consider the set A'_{2n+1} of *down-up* alternating permutations of odd length rather than up-down alternating permutations; note that results in either case may be translated into results in the other via reverse-complementation. Arguments very similar to those of Section 4.4 show that if we associate to the permutation $w \in A'_{2n+1}(2143)$ the label (a,b), where $a = w_{2n}$ and b is the number of active values for w then the generating tree for $\bigcup_{n\ge 0} A'_{2n+1}(2143)$ has root $1 \in A'_1(2143)$ with label $(0,2)$ and satisfies the rule

$$(a,b) \mapsto \{(x,y) \mid 1 \le x \le a+1 \text{ and } x+2 \le y \le b+2\}.$$

The two children $213, 312 \in A'_3(2143)$ of the root have labels $(1,3)$ and $(1,4)$, respectively, and so on.

We seek to enumerate $A'_{2n+1}(2143)$ by aping our approach for even-length permutations, i.e., by finding a family of objects with isomorphic generating tree that we already know how to enumerate. In the case at hand, these objects turn out to be **shifted standard Young tableaux** (henceforward SHSYT) of shape $\langle n+2, n+1, n\rangle$. (See [18] or [14, Chapter 10] for definitions, etc.)

Given a SHSYT T of shape $\langle n+2, n+1, n\rangle$, assign to it the label (a,b), where $a = 3n+4-T(2,n+2)$ and $b = 3n+4-T(1,n+2)$. (Note that for $n \ge 1$, we have $T(3,n+2) = 3n+3$ for every SHSYT of this shape, so this label captures all the information we need to reconstruct the last column of T.) Then the root of the tree is the unique SHSYT $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$ of shape $\langle 2,1\rangle$, which has label $(1,2)$. Its children are the two SHSYT

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 \\ \cline{3-3} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \multicolumn{2}{c|}{} & 6 \\ \cline{3-3} \end{array}$$

of shape $\langle 3, 2, 1\rangle$, which have labels $(2, 3)$ and $(2, 4)$, respectively. In subsequent layers of the tree, the succession rule is identical to the rule for rectangular tableaux; indeed, the subtree below an SHSYT of shape $\langle n+2, n+1, n\rangle$ depends only on its last column (or equivalently, on its associated label), and so might as well be the subtree of an SYT of shape $\langle (n + 1)^3\rangle$ with the same label.

It follows immediately that $|A'_{2n+1}(2143)|$ (and so also $|A_{2n+1}(2143)|$) is the number of SHSYT of shape $\langle n+2, n+1, n\rangle$. As in the case of standard Young tableaux, there is a simple hook-length formula for SHSYT (see, e.g., [18] or [14, pp. 187-190]). In our case, it gives the following result.

Proposition 4.5.1. *For $n \geq 0$ we have*

$$|A_{2n+1}(2143)| = \frac{2(3n + 3)!}{n!(n + 1)!(n + 2)!(2n + 1)(2n + 2)(2n + 3)}.$$

4.6 Generating tree for $A_{2n}(3412)$

In this section we describe enumeration for alternating permutations of even length avoiding the pattern 3412, which was conjectured by the present author and proved by Ravi Jagadeesan in not-yet-published work. (Alternating permutations of odd length avoiding 3412 are the reversals of alternating permutations of odd length avoiding 2143, so are counted in Section 4.5.) Given a permutation $w \in A_{2n}(3412)$, assign to it the label (a, b) where

$$a = 2n - \max\{w_i \mid \exists j > i \text{ s.t. } w_i < w_j\}$$

and b is the number of active values (with respect to 3412) for w in $[2n + 1]$. This labeling is a the labeling scheme of a generating tree for $\bigcup_{n\geq 1} A_{2n}(3412)$.

Proposition 4.6.1 (Jagadeesan, private communication)**.** *If $w \in A_{2n}(3412)$ is a permutation with label (a, b), then the collection of labels of children of w is exactly the set*

$$\{(x, y) \mid 1 \leq x \leq a + 1 \text{ and } a + 3 \leq y \leq b + 2\},$$

each pair occurring with multiplicity one.

The root of this tree is the permutation $12 \in A_2(3412)$ with label $(1, 3)$. Its children are the permutations 1324, 1423, 2314 and 2413 in $A_4(3412)$ with labels $(1, 5)$, $(2, 5)$, $(1, 4)$ and $(2, 4)$, respectively.

This result leads to a bijection with a family of truncated shifted tableaux and thus to the enumeration of $A_{2n}(3412)$. We outline the other portions of this argument now.

For $n \geq 1$, consider the set of tableaux with three rows, $n+1$ boxes in each row, each row shifted one box to the right of the row above it, such that every entry is smaller than the entries below it and to the right. For example,

$$\begin{array}{|c|c|c|c|cc} \hline 1 & 2 & 4 & 6 & & \\ \hline & 3 & 5 & 8 & 9 & \\ \hline & & 7 & 10 & 11 & 12 \\ \hline \end{array} \tag{4.5}$$

is such a tableau. The set of these tableaux have a natural generating tree structure in which the tableau T is a child of the tableau S that we get by removing the $(1,1)$, $(2,2)$ and $(3,3)$ entries and applying an order-isomorphism. For example, the tableau of Equation (4.5) is a child of

$$\begin{array}{ccccc} \hline 1 & 2 & 4 & & \\ \hline & 3 & 5 & 6 & \\ \hline & & 7 & 8 & 9 \\ \hline \end{array}.$$

The label of a tableau T is the pair $(T(2,2), T(3,3))$. Thus, the root tableau $\begin{array}{cccc} 1 & 2 & & \\ & 3 & 4 & \\ & & 5 & 6 \end{array}$ has label $(3,5)$. The following proposition is easy to verify.

Proposition 4.6.2. *If the tableau T has label (a,b), then the collection of labels of the children of T is exactly the set*

$$\{(x,y) \mid 3 \leq x \leq a+1 \text{ and } a+3 \leq y \leq b+2\},$$

each pair occurring with multiplicity one.

This labeling is not identical to the labeling of Proposition 4.6.1, but the two labelings are easy to relate to each other.

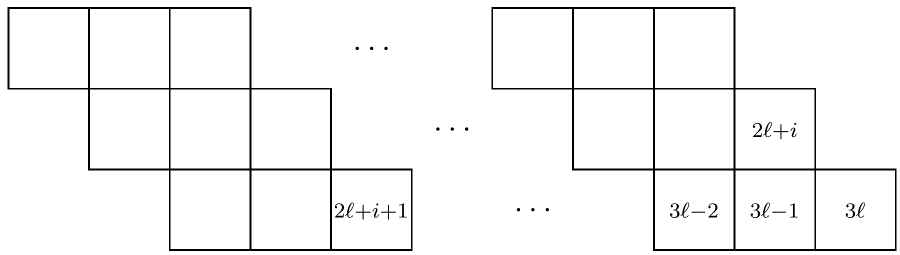
Proposition 4.6.3. *The map $(x,y) \mapsto (x+2, y+2)$ is a generating tree isomorphism between the tree given by the labeling of Proposition 4.6.1 and the tree of Proposition 4.6.2. In particular, the set $A_{2n}(3412)$ of alternating permutations of length $2n$ avoiding the pattern 3412 is in bijection with the set of standard three-row truncated shifted tableaux with $n+1$ boxes in each row.*

Finally, by an observation of Greta Panova, these tableaux (and thus the permutations in $A_{2n}(3412)$) are easy to enumerate.

Proposition 4.6.4 (Panova, private communication)**.** *The number of standard truncated shifted tableaux with three rows and $n+1$ boxes in each row is*

$$\sum_{i=0}^{n-1} \frac{(2n+i+1)! \cdot (n-i+1)(n-i)}{(n+1)! \cdot n! \cdot i! \cdot (2n+1)(n+i+1)(n+i)}.$$

Proof. Every standard truncated shifted tableaux with three rows and ℓ boxes in each row and $(2, \ell+1)$-entry equal to $2\ell+i$ has the following form:



With $\ell = n+1$, the number of tableaux of this form is precisely the number $g^{\langle n+1,n,i\rangle}$ of SHSYT of shape $\langle n+1, n, i\rangle$. In addition, every standard truncated shifted tableaux T with three rows and $n+1$ boxes in each row satisfies $T(2, n+2) = 2n+2+i$ for some $i \in [0, n-1]$. Thus the total number of such tableaux is

$$\sum_{i=0}^{n-1} g^{\langle n+1,n,i\rangle},$$

and the result follows immediately from the hook-length formula for SHSYT. □

The associated sequence in the OEIS [12] is A181197.

4.7 Open problems

In this section we pose a number of open enumerative problems related to pattern avoidance in alternating permutations. Some of these problems have been at least partially resolved by Nihal Gowravaram and Ravi Jagadeesan (unpublished).

If permutations p and q satisfy $|A_{2n}(p)| = |A_{2n}(q)|$ for all $n \geq 1$, we say that p and q are **equivalent** for even-length alternating permutations. Note that if $p = p_1 \cdots p_k$ and $q = (k+1-p_k)(k+1-p_{k-1}) \cdots (k+1-p_1)$ (i.e., p and q are reverse-complements) then p and q are equivalent for even-length alternating permutations: for every n, the operation of reverse-complementation is a bijection between $A_{2n}(p)$ and $A_{2n}(q)$. Pairs of patterns that are equivalent for this reason are said to be **trivially equivalent**. Similarly, if $|A_{2n+1}(p)| = |A_{2n+1}(q)|$ for all $n \geq 0$, we say that p and q are equivalent for odd-length alternating permutations, and if p is the reverse of q then we say they are trivially equivalent.

Numerical data (see Appendix A) suggest the following conjecture.

Conjecture 4.7.1. *We have* $|A_{2n}(p)| = |A_{2n}(1234)| = |A_{2n}(2143)|$ *for all* $n \geq 1$ *and every* $p \in \{1243, 2134, 1432, 3214, 2341, 4123, 3421, 4312\}$.

Observe that these eight patterns come in four pairs of trivially equivalent patterns. The results of West [40] and computer investigations of short permutations suggest that some of these equivalences may be susceptible to a generating tree attack. In particular, the generating trees for alternating permutations avoiding 1243 or 2134 may be isomorphic to the generating tree discussed in Sections 4.2, 4.3 and 4.4.

For alternating permutations of even length, the only other possible equivalences not ruled out by numerical data are captured by the following conjecture.

Conjecture 4.7.2. *We have the equalities* $|A_{2n}(3142)| = |A_{2n}(3241)| = |A_{2n}(4132)|$ *and* $|A_{2n}(2413)| = |A_{2n}(1423)| = |A_{2n}(2314)|$ *for all* $n \geq 1$.

In both cases, the second of the two equalities is a trivial equivalence. It is interesting to note that the patterns in the first set of equalities are the complements (and also the reverses) of the patterns in the second set of equalities; does this have any significance?

For odd-length alternating permutations, computational data suggest the following conjectures.

Conjecture 4.7.3. *We have* $|A_{2n+1}(p)| = |A_{2n+1}(1234)|$ *for all* $n \geq 0$ *and every* $p \in \{2134, 4312, 3214, 4123\}$.

We also have the trivial equivalence $|A_{2n+1}(1234)| = |A_{2n+1}(4321)|$. The equivalence between 4321 and 4312 may be amenable to generating tree methods.

Conjecture 4.7.4. *We have* $|A_{2n+1}(p)| = |A_{2n+1}(2143)|$ *for all* $n \geq 0$ *and every* $p \in \{1243, 3421, 1432, 2341\}$.

We also have the trivial equivalence $|A_{2n+1}(2143)| = |A_{2n+1}(3412)|$. The equivalence between 3412 and 3421 may be amenable to generating tree methods.

The only other possible equivalence for odd-length alternating permutations not ruled out by data is captured by the following conjecture.

Conjecture 4.7.5. *The permutations* 2314, 4132, 2413, 3142, 1423 *and* 3241 *are equivalent for odd-length alternating permutations.*

Other than the cases covered by Conjectures 4.7.1, 4.7.3 and 4.7.4, none of the sequences $\{|A_{2n}(p)|\}_n$ or $\{|A_{2n+1}(p)|\}_n$ for $p \in S_4$ are recognizable to the present author (and in particular they do not appear in the OEIS [12]). Numerical data (see Appendix A) rule out simple product formulas similar to those of Theorem 4.4.5 and Proposition 4.5.1. However, in some cases it may be possible to give a generating tree and so perhaps to adapt the method of [8] to find generating functions or even closed formulas.

The preceding conjectures concern two equivalence relations on patterns of length four: equivalence for even-length alternating permutations and equivalence for odd-length alternating permutations. It happens that these relations are both refinements of the usual (Wilf-)equivalence relation for all permutations (see, for example, [33]). (Note that this is true on account of numerical data, regardless of the truth of any of the preceding conjectures.) This suggests the following conjecture.

Conjecture 4.7.6. *If permutations* p *and* q *are equivalent for alternating permutations of either parity then* p *and* q *are Wilf-equivalent for all permutations.*

More broadly, we can ask the following question.

Question 4.7.7. *Are there any large families of patterns that can be shown to be equivalent for alternating permutations (of either parity)? (Tables A.3 and A.4 suggest some possible first directions.)*

The work in Chapter 3 suggests that $\mathcal{L}_{n,k}$ is a good generalization of alternating permutations for purposes of pattern avoidance; unfortunately, it does not seem to give rise to nontrivial equivalences. However, as Note 1 suggests, an alternative generalization is the set $\mathrm{Des}_{n,k}$ of permutations of length n with descent set $\{k, 2k, \ldots\}$.

Question 4.7.8. *Is* $\mathrm{Des}_{n,k}$ *a "good" context to study pattern avoidance? In particular, are there any pairs or families of patterns that can be shown to be Wilf-equivalent for these permutations? (Table A.5 shows some possible equivalences.) Is Conjecture 4.7.6 valid in this context?*

In analogy with the case of alternating permutations, it is natural to consider separate cases depending on the congruence class of n modulo k. Finally, we note that it may also be fruitful to consider permutations whose descent set is contained in (rather than "is equal to") $\{k, 2k, \ldots\}$, a case on which our work here does not touch.

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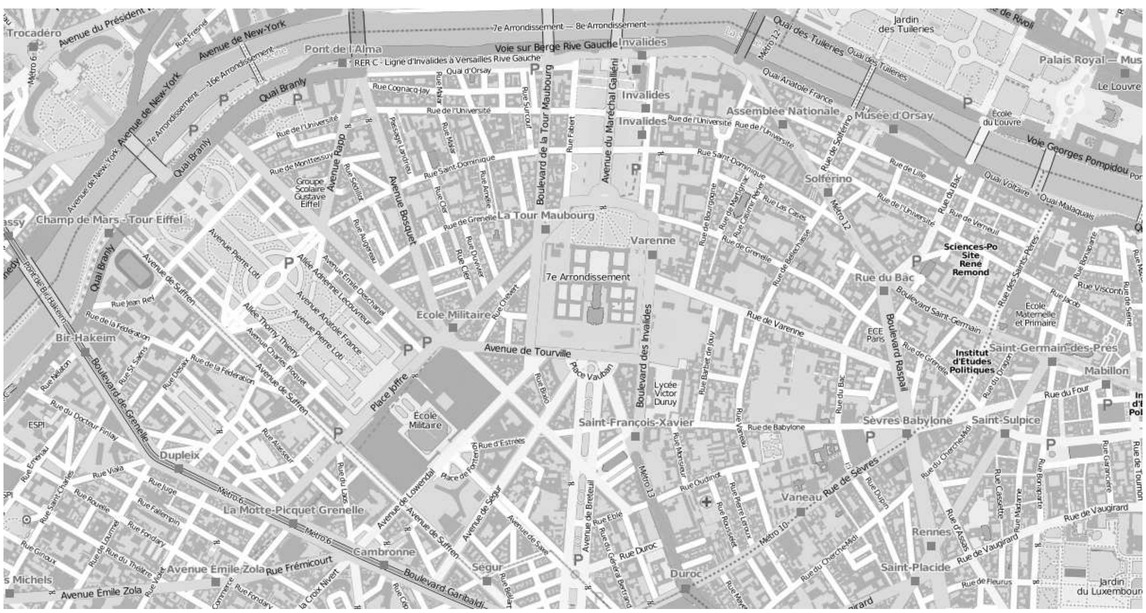

Chapter 5

Pattern avoidance in reading words of Young tableaux of arbitrary skew shape

So far, we have considered permutations that arise as the reading words of standard skew Young tableaux of particular nice shapes. In this section, we expand our study to include pattern avoidance in the reading words of standard skew Young tableaux of *any* shape. As is the case for pattern avoidance in other settings, it is relatively simple to handle the case of small patterns (in our case, patterns of length three or less), but it is quite difficult to prove exact results for larger patterns.

In addition to encompassing pattern avoidance for permutations of length n (via the shape $\langle n, n-1, \ldots, 2, 1\rangle/\langle n-1, n-2, \ldots, 1\rangle$), alternating permutations (via the shape $\langle n+1, n, \ldots, 3, 2\rangle/\langle n-1, n-2, \ldots, 1\rangle$ and three other similar shapes), and more generally $\mathcal{L}_{n,k}$ for any k, this type of pattern avoidance also encompasses the enumeration of pattern-avoiding permutations by descent set (when the skew shape is a ribbon) or with certain prescribed descents (when the shape is a vertical strip). Thus, on one hand the strength of our results is constrained by what is tractable to prove in these circumstances, while on the other hand any result we are able to prove in this context applies quite broadly.

Note 6. We place the following restriction on all Young diagrams in this chapter: we will only consider diagrams λ/μ such that the inner (northwestern) boundary of λ/μ contains the entire outer (southeastern) boundary of μ. For example, the shape $\langle 4, 2, 1\rangle/\langle 2, 1\rangle$ meets this condition, while the shape $\langle 5, 2, 2, 1\rangle/\langle 3, 2, 1\rangle$ does not. ■

Note that imposing this restriction does not affect the universe of possible results, since for a shape λ/μ failing this condition we can find a new shape λ'/μ' that passes it and has an identical set of reading words by moving the various disconnected components of λ/μ on the plane. For example, for $\lambda/\mu = \langle 5, 2, 2, 1\rangle/\langle 3, 2, 1\rangle$ we have $\lambda'/\mu' = \langle 4, 2, 1\rangle/\langle 2, 1\rangle$ – just slide disconnected sections of the tableau together until they share a corner. This is illustrated in Figure 5-1.

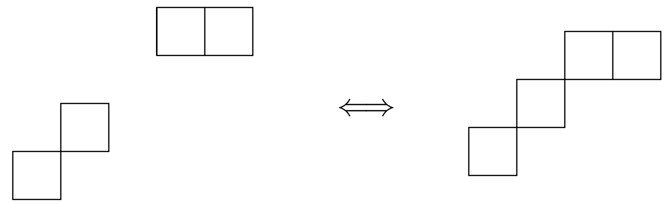

Figure 5-1: Moving separated components gives a new shape but leaves the set of reading words of tableaux unchanged.

5.1 The patterns 213 and 132

In Chapter 2, we proved the known results $|A_{2n}(132)| = |A_{2n+1}(132)| = C_n$ and $|S_n(132)| = C_n$ via a novel bijection and a simple recursion, respectively. Taking reverse-complements, we have also $|S_n(213)| = |A_{2n}(213)| = C_n$. In this section, we extend these results to the reading words of tableaux of any fixed shape.

Theorem 5.1.1. *The number of tableaux of skew shape λ/μ whose reading words avoid the pattern* 213 *is equal to the number of partitions whose Young diagram is contained in that of μ (subject to Note 6).*

Note that this is a very natural μ-generalization of the Catalan numbers: the outer boundaries of shapes contained in $\langle n-1, n-2, \ldots, 2, 1\rangle$ are essentially Dyck paths of length $2n$.

Proof. We begin with a warm-up and demonstrate the claim in the case that μ is empty. In this case, the Proposition states that there is a unique standard Young tableau of a given shape $\lambda = \langle \lambda_1, \lambda_2, \ldots\rangle$ whose reading word avoids the pattern 213. In order to see this, we note that the reading word of every straight (i.e., non-skew) tableau ends with an increasing run of length λ_1 and that the first entry of this run is 1. Since this permutation is 213-avoiding, each entry following the 1 must be smaller than every entry preceding the 1 and so this run consists of the values from 1 to λ_1. Applying the same argument to the remainder of the tableau (now with the minimal element $\lambda_1 + 1$), we see that the only possible filling is the one we get by filling the first row of the tableau with the smallest possible entries, then the second row with the smallest remaining entries, and so on. On the other hand, the reading word of the tableau just described is easily seen to be 213-avoiding, so we have our result in this case.

For the general case we give a recursive bijection. This bijection is heavily geometric in nature, and we recommend that the reader consult Figures 5-2 and 5-3 to most easily understand what follows. Suppose we have a pair $(\lambda/\mu, \tau)$ such that τ fits inside μ.

Let i be the largest index such that $\tau_{i-1} > \mu_i$, or let $i = 1$ if no such index exists. We divide τ and λ/μ into two pieces. To split τ, we delete the boxes that belong to the rectangle of shape $\langle(\mu_i + 1)^{i-1}\rangle$, leaving a partition of shape $\nu = \langle \tau_1 - \mu_i - 1, \tau_2 - \mu_i - 1, \ldots, \tau_{i-1} - \mu_i - 1\rangle$ to the right of the rectangle and a partition

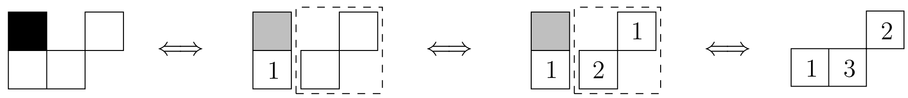

Figure 5-2: Our bijection applied to the pair $(\langle 3,2\rangle/\langle 2\rangle, \langle 1\rangle)$ to generate a standard Young tableau.

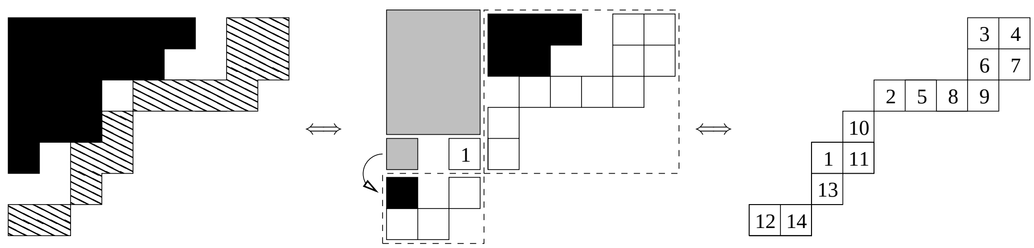

Figure 5-3: A partial example: the standard Young tableau at right corresponds to the pair $(\langle 9,9,8,4,4,3,2\rangle/\langle 7,7,4,3,2,2\rangle, \langle 6,5,3,3,1\rangle)$ under our bijection.

of shape $\iota = \langle \tau_i, \tau_{i+1}, \ldots\rangle$ below the rectangle. To split λ/μ, we begin by filling the box $(i, \mu_i + 1)$ with the entry 1. Then we take the boxes to the right of this entry as one skew shape $\alpha_1/\beta_1 = \langle \lambda_1 - \mu_i - 1, \lambda_2 - \mu_i - 1, \ldots, \lambda_i - \mu_i - 1\rangle/\langle \mu_1 - \mu_i - 1, \mu_2 - \mu_i - 1, \ldots, \mu_{i-1} - \mu_i - 1\rangle$ and the boxes below it as our second skew shape $\alpha_2/\beta_2 = \langle \lambda_{i+1}, \lambda_{i+2}, \ldots\rangle/\langle \mu_{i+1}, \mu_{i+2}, \ldots\rangle$.

Note that by the choice of i, ν fits inside β_1 while ι fits inside β_2. Thus, we can apply the construction recursively with the pairs $(\alpha_1/\beta_1, \nu)$ and $(\alpha_2/\beta_2, \iota)$, but we fill α_1/β_1 with the values $2, \ldots, |\alpha_1/\beta_1| + 1$ and we fill α_2/β_2 with the values $|\alpha_1/\beta_1| + 2, \ldots, |\lambda/\mu| = |\alpha_1/\beta_1| + |\alpha_2/\beta_2| + 1$. (Observe that this coincides with what we did in the first paragraph for $\mu = \emptyset$.) By induction, this gives us a standard skew Young tableau of shape λ/μ. Note that the reading word of this tableau can be decomposed as the reading word of α_2/β_2 followed by the entry 1 followed by the reading word of α_1/β_1. By the recursive nature of the construction we may assume that the reading words of α_1/β_1 and α_2/β_2 are 213-avoiding, and we chose the entries of α_2/β_2 to be all larger than the entries of α_1/β_1, so it follows that the reading word of the resulting tableau is 213-avoiding.

To invert this process, begin with a tableau T of shape λ/μ with entry 1 in position (i, j), an inner corner.

Divide T into two pieces, one consisting of rows 1 through i with the box (i, j) removed, the other consisting of rows numbered $i+1, i+2$, etc. Let T_1 be the tableau order-isomorphic to the first part and let T_2 be the tableau order-isomorphic to the second part. Working recursively, suppose we have defined our map for smaller tableaux: let $\nu = \langle \nu_1, \ldots, \nu_i\rangle$ be the image of T_1 and let $\iota = \langle \iota_1, \iota_2, \ldots\rangle$ be the image of T_2. Then the partition τ associated to T is given by $\tau = \langle \nu_1 + j, \ldots, \nu_i + j, \iota_1, \iota_2, \ldots\rangle$. Geometrically, τ consists of all boxes (k, l) with $k < i$ and $l \leq j$ together with the result of applying our process to the right of this rectangle and the result of applying it below the rectangle, shifted up one row. By construction, τ is a partition whose

Young diagram fits inside μ.

Since the two maps we've described are clearly inverse to each other, we have that they are the desired bijections. $\square$

In the case that our tableaux is of one of the shapes of Figure 3-2, this implies the following corollary.

Corollary 5.1.2. *We have* $|\mathcal{L}_{n,k;r}(213)| = C_n$ *for all* $n \geq 1$ *and* $k - 1 \geq r \geq 0$.

In particular, this establishes the results mentioned at the beginning of this section.

Note that knowing the number of tableaux of each shape whose reading words avoid 213 automatically allows us to calculate the number of tableaux of a given shape whose reading words avoid 132: if $\lambda = \langle \lambda_1, \ldots, \lambda_k \rangle$ and μ is contained in λ, the rotated complement operation $T \mapsto T^*$ defined in Section 1.1 is a bijection between tableaux of shape λ/μ and tableaux of shape $\langle \lambda_1 - \mu_k, \lambda_1 - \mu_{k-1}, \ldots, \lambda_1 - \mu_1 \rangle / \langle \lambda_1 - \lambda_k, \lambda_1 - \lambda_{k-1}, \ldots, \lambda_1 - \lambda_2 \rangle$. Moreover, the reading word of T^* is exactly the reversed-complement of the reading word of T. It follows that the reading word of T avoids 132 if and only if the reading word of T^* avoids 213. This argument establishes the following corollary of Theorem 5.1.1.

Corollary 5.1.3. *The number of tableaux of skew shape* λ/μ *whose reading words avoid the pattern* 132 *is equal to the number of partitions whose Young diagram is contained in that of the partition* $\langle \lambda_1 - \lambda_k, \lambda_1 - \lambda_{k-1}, \ldots, \lambda_1 - \lambda_2 \rangle$ *(subject to Note 6).*

We can use this result to extract a simple formula for the size of $\mathcal{L}_{n,k;r}(132)$.

Corollary 5.1.4. *We have* $|\mathcal{L}_{n,k}(132)| = C_n$ *for all* $n, k \geq 1$ *and* $|\mathcal{L}_{n,k;r}(132)| = C_{n+1} + (k - r - 1)C_n$ *for all* $n \geq 1$ *and* $k - 1 \geq r > 0$.

Proof. Both halves of this result follow from Corollary 5.1.3: in the case of $\mathcal{L}_{n,k}$, we are counting Young diagrams contained in $\langle n-1, n-2, \ldots, 2, 1 \rangle$ while in the case of $\mathcal{L}_{n,k;r}$ for $r > 0$ we are counting shapes contained in $\langle n+k-r-1, n-1, n-2, \ldots, 1 \rangle$. As we noted before, the former diagrams are naturally in bijection with Dyck paths and so are a Catalan object; for the latter, consider separately two cases. First, if the shape λ has first row of length at most n then it is one of the C_{n+1} shapes contained in $\langle n, n-1, \ldots, 1 \rangle$. Second, if λ has first row of length longer than n then we may independently choose any length between $n + 1$ and $n + k - r - 1$ for the first row and any shape contained in $\langle n-1, n-2, \ldots, 1 \rangle$ for the lower rows and so we have $(k - r - 1)C_n$ additional shapes. $\square$

5.2 The patterns 312 and 231

If a shape λ/μ contains a square, every tableau of that shape will have as a sub-tableau four entries

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

with $a < b < d$ and $a < c < d$, and the reading word of every such tableau will be of the form $\cdots cd \cdots ab \cdots$. This permutation contains both an instance dab of the pattern 312 an instance cda of the pattern 231. Thus, the number of tableaux of shape λ/μ whose reading words avoid 312 or 231 is zero unless λ/μ contains no square, i.e., unless λ/μ is contained in a ribbon. In this case, choose a tableau T of shape λ/μ with reading word w. Since λ/μ is a ribbon, the reading word of the conjugate tableau T' is exactly the reverse w^r of w. Since w avoids 312 if and only if w^r avoids 213, we can apply Theorem 5.1.1 to deduce the following result.

Proposition 5.2.1. *If skew shape λ/μ is contained in a ribbon then the number of tableaux of shape λ/μ whose reading words avoid the pattern* 312 *is equal to the number of partitions whose Young diagram is contained in that of μ. Otherwise, the number of such tableaux is* 0.

Either using the same argument that gave us Corollary 5.1.3 and applying it to Proposition 5.2.1 or using the same argument that gave us Proposition 5.2.1 but using Corollary 5.1.3 in place of Theorem 5.1.1 gives the following result.

Corollary 5.2.2. *If skew shape λ/μ is contained in a ribbon then the number of tableaux of shape λ/μ whose reading words avoid the pattern* 231 *is equal to the number of partitions whose Young diagram is contained in that of the partition $\langle \lambda_1 - \lambda_k, \lambda_1 - \lambda_{k-1}, \ldots, \lambda_1 - \lambda_2 \rangle$. Otherwise, the number of such tableaux is* 0.

In the special cases of $\mathcal{L}_{n,k;r}$ it follows that for $k \geq 3$ and $n \geq 2$ we have $\mathcal{L}_{n,k;r}(231) = \mathcal{L}_{n,k;r}(312) = \emptyset$ while for $1 \leq k \leq 2$ we have that $|\mathcal{L}_{n,k;r}(231)|$ and $|\mathcal{L}_{n,k;r}(312)|$ are Catalan numbers [15, 38].

5.3 The patterns 123 and 321

For the patterns 123 and 321, our results are not as nice as those in the preceding sections. We show that the enumeration of tableaux of given shapes whose reading words avoid these patterns is equivalent to the enumeration of permutations with certain prescribed ascents and descents avoiding these patterns. This latter problem can easily be solved in any particular case by recursive methods, though the author knows of no closed formula for the resulting values. As we mentioned in the introduction to this chapter, the problem of enumerating pattern-avoiding permutations with certain fixed ascents and descents is the special case of our problem in which we consider tableaux contained in a ribbon.

Note that if a shape λ/μ has any rows of length three or more, the reading word of any tableau of shape λ/μ contains a three-term increasing subsequence. Consequently, in order to study 123-avoidance it suffices to consider tableaux with all rows of length one or two. The following result allows us to reduce even further the set of skew shapes we need to consider in order to completely deal with the case of 123-avoidance.

Proposition 5.3.1. *If λ/μ is a skew shape, all of whose rows have length one or two, and $\lambda_i - \mu_i = 2$ then the number of tableaux of shape λ/μ whose reading words*

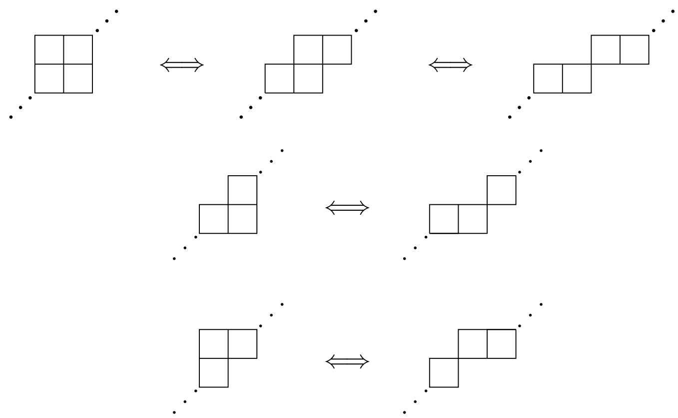

Figure 5-4: Moves that do not change the set of 123-avoiding reading words of tableaux.

avoid 123 *is the same as the number of tableaux of the following shapes whose reading words avoid* 123*:*

- $\langle \lambda_1 + 1, \ldots, \lambda_i + 1, \lambda_{i+1}, \lambda_{i+2}, \ldots \rangle / \langle \mu_1 + 1, \ldots, \mu_i + 1, \mu_{i+1}, \mu_{i+2}, \ldots \rangle$
- $\langle \lambda_1 + 1, \ldots, \lambda_{i-1} + 1, \lambda_i, \lambda_{i+1}, \ldots \rangle / \langle \mu_1 + 1, \ldots, \mu_{i-1} + 1, \mu_i, \mu_{i+1}, \ldots \rangle$

In other words, the proposition states that we can locally slide rows of length two without changing the number of tableaux with 123-avoiding reading words that result. Figure 5-4 illustrates these moves.

Proof. Given λ and μ, let $\alpha = \langle \lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_i + 1, \lambda_{i+1}, \lambda_{i+2}, \ldots \rangle$ and $\beta = \langle \mu_1 + 1, \mu_2 + 1, \ldots, \mu_i + 1, \mu_{i+1}, \mu_{i+2}, \ldots \rangle$ and let $n = |\lambda/\mu| = |\alpha/\beta|$. There is a natural map between fillings of the shape λ/μ and fillings of the shape α/β: fill each row of α/β with the same numbers in the same order as the corresponding row of λ/μ. We must show that when restricted to standard tableaux whose reading words avoid 123, this is a bijection.

Given a tableau T of shape λ/μ, shifting each of the first i rows one square to the right gives a filling U of α/β with $[n]$. Under this operation, every box is adjacent to the same boxes except for boxes in rows i and $i+1$. Thus, the filling U is strictly increasing along rows, and along columns except possibly between rows i and $i+1$. By transforming T into U, we've moved boxes in row i to the right, so each entry in row i is above a larger entry in U than it was in the tableau T, or is above an entry in T but above an empty space in U. It follows that U is also increasing down columns and so is a standard Young tableau. Thus, the map $T \mapsto U$ sends standard Young tableaux to standard Young tableaux and is injective. We must show that it is surjective.

Suppose we have a tableau U of shape α/β whose reading word avoids 123. We show that this tableau is the image under the above-described map of some tableau of shape λ/μ, i.e., that shifting the first i rows of U one unit to the left results in a standard Young tableau. Let T be the filling of λ/μ that results from this shift. Again, the only possible obstruction is that T might fail to increase along columns between rows i and $i+1$. In particular, there are two ways in which this could happen. Either the rightmost of the two entries in the ith row of U (the entry $U(i, \mu_i + 2) = U(i, \lambda_i)$) is larger than the entry $U(i, \mu_i + 1)$ immediately below it and to its left or the leftmost of the two entries in the ith row of U (the entry $U(i, \mu_i + 1) = U(i, \lambda_i - 1)$) is larger than the entry of $U(i + 1, \mu_i)$ immediately below it and to its left. Suppose for sake of contradiction that one of these two situations occurs.

In the former case, since α/β is the image of a skew shape λ/μ under a shift to the first i rows, row $i + 1$ must have length two. Thus, the reading word of U has the form $\cdots U(i + 1, \mu_i)U(i + 1, \mu_i + 1) \cdots U(i, \mu_i + 2) \cdots$ and the subsequence $U(i+1, \mu_i)U(i+1, \mu_i+1)U(i, \mu_i+2)$ is a 123 pattern, a contradiction with the choice of U. In the latter case, the reading word of U has the form $\cdots U(i+1, \mu_i) \cdots U(i, \mu_i + 1)U(i, \mu_i + 2) \cdots$, and the subsequence $U(i + 1, \mu_i)U(i, \mu_i + 1)U(i, \mu_i + 2)$ is a 123 pattern, again a contradiction. Thus neither potential obstruction can occur, so U is the image of a standard Young tableau T and the map is surjective, as desired.

The other case is extremely similar, and we omit it here. □

It is an immediate consequence of this proposition that the problem of counting tableaux of a given skew shape whose reading words avoid 123 reduces to the case of shapes contained in ribbons. This is equivalent to counting 123-avoiding permutations with certain prescribed ascents and descents, and this enumeration can be carried out by straightforward recursive methods.

A similar analysis can be applied to the pattern 321. Here we shift the first i or $i - 1$ columns down (instead of shifting the first i or $i - 1$ rows to the right). This results in a modest increase in the care needed to make the argument work. In particular, shifting columns does not preserve reading words. Luckily, in the 321-avoiding case all of our columns have length two, so the reading words change in a relatively controlled way; for example, a 2×2 square with reading word 3412 will, after shifting, have reading word 3142. The interested reader is invited to work out the details for herself.

Appendix A

The tables of data that follow form the basis for several of the conjectures in Section 4.7. They were generated by brute-force computer enumeration.

Tables A.1 and A.2 give the number of alternating permutations avoiding patterns of length four, grouped by conjectural equivalence.

Patterns	1	3	5	7	9	11
$(1234, 4321)$, $(2134, 4312)$, $(3214, 4123)$	1	2	16	168	2112	30030
$(2143, 3412)$, $(1243, 3421)$, $(1432, 2341)$	1	2	12	110	1274	17136
$(2314, 4132)$, $(2413, 3142)$, $(1423, 3241)$	1	2	12	106	1138	13734
$(1324, 4231)$	1	2	12	110	1285	17653
$(1342, 2431)$	1	2	12	108	1202	15234
$(3124, 4213)$	1	2	16	168	2072	28298

Table A.1: Values of $|A_n(p)|$ for odd n and $p \in S_4$. Parentheses indicate trivial equivalences.

Patterns	2	4	6	8	10	12
1234, $(1243, 2134)$, $(1432, 3214)$, 2143, $(2341, 4123)$, $(3421, 4312)$	1	5	42	462	6006	87516
3142, $(3241, 4132)$	1	5	42	444	5337	69657
$(1423, 2314)$, 2413	1	4	28	260	2844	34564
3412	1	4	29	290	3532	49100
1324	1	4	29	292	3620	51866
$(1342, 3124)$	1	5	42	453	5651	77498
$(2431, 4213)$	1	5	42	454	5680	78129
4231	1	5	42	462	6070	90686
4321	1	5	61	744	10329	157586

Table A.2: Values of $|A_n(p)|$ for even n and $p \in S_4$. Parentheses indicate trivial equivalences.

Tables A.3 and A.4 give the number of alternating permutations avoiding certain patterns of length five. Only those patterns that might potentially have nontrivial equivalences are included, and they are grouped by these potential equivalences.

Patterns	1	3	5	7	9	11
(12534, 43521), (21534, 43512)	1	2	16	243	5291	144430
(12453, 35421), (21453, 35412)	1	2	16	243	5307	146013
(12354, 45321), (12543, 34521), (15432, 23451), (21354, 45312), (21543, 34512), (32154, 45123)	1	2	16	243	5330	148575
(12435, 53421), (21435, 53412)	1	2	16	243	5330	148764
(12345, 54321), (21345, 54312), (32145, 54123), (43215, 51234)	1	2	16	272	6531	194062

Table A.3: Selected values of $|A_n(p)|$ for odd n and $p \in S_5$. Parentheses indicate trivial equivalences. All possible nontrivial equivalences are among the permutations in this table.

Patterns	2	4	6	8	10	12
(12534, 23145), (21534, 23154)	1	5	56	997	23653	679810
(34512, 45123), 45312	1	5	56	1004	24310	724379
(12435, 13245), (13254, 21435)	1	5	56	1004	24336	727807
(12453, 31245), (21453, 31254)	1	5	61	1194	30802	953088
12345, 21354, (12354, 21345), (12543, 32145), (15432, 43215), (21543, 32154), (23451, 51234), (34521, 54123), (45321, 54312)	1	5	61	1194	30945	970717

Table A.4: Selected values of $|A_n(p)|$ for even n and $p \in S_5$. Parentheses indicate trivial equivalences. All possible nontrivial equivalences are among the permutations in this table.

Table A.5 gives the number of permutations of length $3n$ with descent set $\{3, 6, \ldots\}$ that avoid certain patterns of length four and five. Only those patterns that might potentially have nontrivial equivalences are included.

Patterns	3	6	9	12
2413, (1423, 2314)	1	9	153	3465
(1243, 2134), (2341, 4123)	1	9	153	3579
3142, (3241, 4132)	1	19	642	27453
2143, 4231, (1432, 3214), (3421, 4312)	1	19	642	29777
(12354, 21345), (23451, 51234)	1	19	887	66816
(15243, 32415), (35241, 52413)	1	19	1077	102051
(12543, 32154), (34521, 54123)	1	19	1134	114621
21354, 52341	1	19	1134	115515
(15432, 43215), (21543, 32154), (25431, 53214), (31542, 42153), (32541, 52143), (35421, 54213), (41532, 43152), (42531, 53142), (43251, 51432), (43521, 54132), (45321, 54312), (52431, 53241), (53421, 54231)	1	19	1513	211425

Table A.5: Selected values of $|\operatorname{Des}_{n,3}(p)|$ for n divisible by 3 and $p \in S_4$ or S_5. Parentheses indicate trivial equivalences. All possible nontrivial equivalences are among the permutations in this table.

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