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School Admissions Reform in Chicago and England: Comparing Mechanisms by their Vulnerability to Manipulation

School Admissions Reform in Chicago and England: Comparing Mechanisms by their Vulnerability to Manipulation[*]

Parag A. Pathak[†]   Tayfun Sönmez[‡]

This version: December 2011

Abstract

In Fall 2009, officials from Chicago Public Schools abandoned their assignment mechanism for coveted spots at selective college preparatory high schools midstream. After asking about 14,000 applicants to submit their preferences for schools under one mechanism, the district asked them re-submit preferences under a new mechanism. Officials were concerned that "high-scoring kids were being rejected simply because of the order in which they listed their college prep preferences" under the abandoned mechanism. What is somewhat puzzling is that the new mechanism is also manipulable. This paper introduces a method to compare mechanisms based on their vulnerability to manipulation. Under our notion, the old mechanism is more manipulable than the new Chicago mechanism. Indeed, the old Chicago mechanism is at least as manipulable as any other plausible mechanism. A number of similar transitions between mechanisms took place in England after the widely popular Boston mechanism was ruled illegal in 2007. Our approach provides support for these and other recent policy changes involving allocation mechanisms.

KEYWORDS: student assignment, Boston mechanism, matching, strategy-proofness

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1 Introduction

In the last few years, policymakers at several school districts have sought to simplify strategic aspects of school admissions in their open enrollment or school choice plans. A first change occurred to Boston's grade K-12 assignment system, known as the *Boston mechanism*, in place since 1999. Abdulkadiroğlu and Sönmez (2003) show that this mechanism is vulnerable to strategic manipulation, and suggest two alternatives which are not. Following a newspaper article describing these issues (Cook 2003), leadership at Boston Public Schools invited a team of economists to conduct an empirical evaluation of the mechanism. In June 2005, the Boston school committee voted to replace the Boston mechanism with the *student-optimal stable mechanism* (Gale and Shapley 1962), a mechanism where participants can do no better than report their preferences truthfully. The strategic complexity of the Boston mechanism along with its adverse effects on less sophisticated families were key factors in Boston's decision (Abdulkadiroğlu, Pathak, Roth, and Sönmez 2006, Pathak and Sönmez 2008). Another factor was the potential to use unmanipulated preference data generated by the student assignment mechanism in various policy-related issues including the evaluation of schools.[1]

The Boston episode challenges a paradigm in traditional mechanism design that treats incentive compatibility only as a constraint and not as a direct design objective, at least for the specific context of school choice. Given economists' advocacy efforts, it is possible that this incident is isolated, and the Boston events do not adequately represent the desirability of non-consequentialist objectives as design goals. In this paper, we provide further, and perhaps more striking, evidence that excessive vulnerability to "gaming" is considered highly undesirable in the context of school choice. Officials in England and Chicago have taken drastic measures to attempt to reduce it, and remarkably the Boston mechanism plays a central role in both incidents.

In England, forms of school choice have been available for at least three decades. The nationwide 2003 School Admissions Code mandated that Local Authorities, an operating body much like a U.S. school district, coordinate their admissions practices. This reform provided families with a single application form and established a common admissions timeline, leading to a March announcement of placements for anxious 10 and 11 year-olds on "National Offer Day." The next nationwide reform came with the 2007 School Admissions Code. While strengthening

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[1]The use of data generated by manipulable mechanisms presents challenges for empirical research and evaluation. For example, Hastings, Kane, and Staiger (2006) utilize preference data from Charlotte-Mecklenburg, which uses the Boston mechanism, to estimate preferences for school characteristics and examine implications for the local educational market. They argue that the vagueness of the description of the mechanism in the first year of implementation makes strategic manipulation less of an issue. Similarly, Lim et al. (2009) tie the limited presence of minorities at senior Army ranks to racial differences between cadet preferences over Army branches, but they are unable to offer an explanation of these differences since the ROTC mechanism used to generate their data is highly manipulable. They indicate that the policy recommendation to increase diversity would depend on the extent of manipulation in the data.

the enforcement of admissions rules, this legal code also prohibited authorities from using what they refer to as "unfair oversubscription criteria" in Section 2.13:

> In setting oversubscription criteria the admission authorities for all maintained schools must not:
>
> > give priority to children according to the order of other schools named as preferences by their parents, including 'first preference first' arrangements.

A *first preference first* system is any "oversubscription criterion that gives priority to children according to the order of other schools named as a preference by their parents, or only considers applications stated as a first preference" (School Admissions Code, 2007, Glossary, p. 118). The 2007 Admissions Code outlaws use of this system at more than 150 Local Authorities across the country, and this ban continues with the 2010 Code. The best known first preference first system is the Boston mechanism, and since 2007 it is banned in England.[2] The rationale for this ban, as stated by England's Department for Education and Skills, is that "the 'first preference first' criterion made the system unnecessarily complex to parents" (School Code 2007, Foreword, p. 7). Moreover, Education Secretary Alan Johnson remarked that the first preference first system "forces many parents to play an 'admissions game' with their children's future."

While Local Authorities had some time to adjust their admissions rules in England, the adoption of a new mechanism was considerably more abrupt in Chicago. The district abandoned their selective high school mechanism halfway through running it in 2009. That is, after participants had submitted preferences under one mechanism, but before announcing placements, Chicago Public Schools asked the same participants to resubmit their preferences under another mechanism a few months later. This is the only case of a midstream change of an assignment mechanism we are aware of, and in our view it is stunning given the potentially high-stakes involved. The abandoned mechanism prioritized applicants based on how schools were ranked and is the most basic form of the Boston mechanism. Under it, Chicago authorities argued that "high-scoring kids were being rejected simply because of the order in which they listed their college prep preferences." The vulnerability of the Boston mechanism to strategic manipulation led to its elimination in yet another district.

These new case studies from England and Chicago provide additional evidence that the use of strategically complex assignment mechanisms is considered undesirable in the context of school choice. Unlike the case of Boston, the reforms in England and Chicago developed without the guidance of economists (to the best of our knowledge). Not only were the Boston mechanism and its variants abandoned in both cases, but extreme measures were taken in the process. Given these circumstances, one would expect local authorities in England and Chicago to adopt strategy-proof mechanisms, which are immune to manipulation. And yet, several local authorities in England as well as Chicago adopted alternative mechanisms that

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[2]A formal definition of these mechanisms is presented in Section 3.

are also vulnerable to manipulation. Therefore, the new mechanisms must be perceived to be "less manipulable" than the abandoned mechanisms. This motivates our goal to develop a rigorous methodology to compare mechanisms based on their vulnerability to manipulation. In this paper we propose a method to compare manipulable mechanisms by examining three increasingly more demanding notions and relating our notion to these policy changes.

Our most basic notion is based on the following simple idea. Given an economic environment, there are often cases where this environment is *vulnerable to manipulation* under a mechanism ψ but not under an alternative mechanism φ. This may not mean much unless environments are systematically inclined to be more vulnerable under one of the mechanisms. This approach can be formalized as follows: A mechanism ψ is *at least as manipulable* as mechanism φ if any environment that is vulnerable under φ is also vulnerable under ψ, and it is *more manipulable* if in addition there is at least one environment that is vulnerable under ψ but not under φ. This notion justifies a number of recent school choice reforms including those in England and Chicago. While we focus on these recent reforms, our framework is also useful to formalize other policy debates that have so far remained informal. For instance, one application involves changes in the auction mechanism for U.S. Treasury bonds from discriminatory to uniform-price format and informal arguments dating back to Milton Friedman (1960). Not only is the discriminatory auction is more manipulable than the uniform-price auction (providing a formalization of Friedman's position), but we can establish an even stronger comparison taking into account the intensity of manipulation.

Related Literature

One approach to studying a mechanism's vulnerability to manipulation is to characterize domains under which the mechanism is not manipulable (see, e.g., Barberá (2010) for a survey of strategy-proof social choice rules). However, strategy-proof mechanisms may not exist, may not be practical, or even if they do exist, they may not be desirable for reasons other than their incentive properties. In such cases, our paper argues that reducing the vulnerability to manipulation is desirable. Several recent papers, many motivated by the school choice reforms, argue that strategy-proofness can also be thought of as a design objective (see, e.g., Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006), Abdulkadiroğlu, Pathak, and Roth (2009), Pathak and Sönmez (2008), and Roth (2008)).

Azevedo and Budish (2011) is the closest paper in the spirit of our methodological contribution. Like us, they are concerned about vulnerability to manipulation when mechanisms are not strategy-proof. They take an entirely different, but equally plausible, approach and propose a relaxation of strategy-proofness based on the idea that vulnerability to manipulation disappears in large economies for some mechanisms, but not others. Their complementary approach can also be used to formulate the Friedman position in the context of U.S. Treasury Auctions. The advantage of their approach is that they offer an explicit design desideratum, namely *strategy-*

proofness in large. The advantage of our approach is its ability to compare two mechanisms each of which fail strategy-proofness even in large. Indeed, an evaluation based on strategy-proofness in large is not possible for our main applications in school choice. Focusing on voting applications, Carroll (2011) proposes another criteria to evaluate mechanisms based on extent to which they encourage manipulation. Other papers that relate to our methodological contribution include Parkes et al. (2001), Day and Milgrom (2008), and Erdil and Klemperer (2011) who each seek to design a combinatorial auction that minimizes manipulability, and to a lesser extent Kesten (2006) and Dasgupta and Maskin (2008) who make comparisons across allocation rules based on inclusion of environments focusing on non-strategic properties of student assignment and voting mechanisms, respectively.

Our paper also contributes to an on-going debate on the features of the Boston mechanism, still the most widely used U.S. school choice mechanism. While efficiency considerations have not been central during policy deliberations at Boston Public Schools, experimental evidence from Chen and Sönmez (2006) and theoretical results from Ergin and Sönmez (2006) show that the student-optimal stable mechanism is more efficient than the Boston mechanism in complete information environments. Ergin and Sönmez (2006) further observe that the efficiency advantage of the student-optimal stable mechanism may not persist in incomplete information environments, whereas Pathak and Sönmez (2008) show that strategic students are better off under the Boston mechanism in the presence of non-strategic students in complete information environments. In a recent series of papers, Abdulkadiroğlu, Che, and Yasuda (2011), Featherstone and Niederle (2011), and Miralles (2008) argue that the earlier literature might be too quick to dismiss the Boston mechanism in favor of the student-optimal stable mechanism. They all provide examples of specific environments where the symmetric Bayes-Nash equilibria of the Boston mechanism dominates the dominant-strategy equilibria of the student-optimal stable mechanism. In our view these papers promote the point of view that the efficiency comparison between these two mechanisms is highly non-robust, but the lack of robustness stems from the Boston mechanism.

2 General Framework

There is a finite set I of players with a generic member i, and a finite set of outcomes A. Each player has a preference relation R_i defined over the set of outcomes, where P_i is the strict counterpart of R_i. Let $R = (R_i)_{i \in I}$ and $P = (P_i)_{i \in I}$ denote the profile of weak and strict preferences, respectively. The set of possible types for player i is T_i with generic element t_i. We adopt the convention that t_{-i} denotes the type profile of players other than player i, and define R_{-i} and P_{-i} accordingly. We sometimes refer to a type profile $t = (t_i)_{i \in I}$ as a **problem**. Let $T = \prod_{i \in I} T_i$.

A direct mechanism is a function $\varphi : T \to A$, a single-valued mapping of a type profile to

an element in A. Let $\varphi(t)$ denote the outcome produced by mechanism φ under t. We do not always expect players to be truthful when reporting their types. This motivates the following definition.

Definition 1. *A mechanism φ is **manipulable by player i** at problem t if there exists a type t'_i such that $\varphi(t'_i, t_{-i}) \, P_i \, \varphi(t)$.*

We will say that **profile t is vulnerable under mechanism φ** if φ is manipulable by some player at t.

A mechanism is manipulable by a player at a problem if he can profit by misrepresenting his type. Observe that each mechanism induces a natural game form where the strategy space is the set of types for each player and the outcome is determined by the mechanism. A mechanism is **strategy-proof** if truthful type revelation is a dominant strategy of this game for any player. Equivalently, a mechanism is strategy-proof if it is not manipulable by any player at any problem.

We next present a notion to compare mechanisms by their vulnerability to manipulation.

Definition 2. *A mechanism ψ is **at least as manipulable** as mechanism φ if any profile that is vulnerable under mechanism φ is also vulnerable under ψ.*

Two mechanisms can be equally manipulable if they are manipulable for exactly the same set of problems. Our next definition rules out this possibility.

Definition 3. *A mechanism ψ is **more manipulable** than mechanism φ if*

1. *ψ is at least as manipulable as φ, and*

2. *there is a set of players I, a set of outcomes A, and a profile t where t is vulnerable under ψ but not under φ.*

If mechanism φ is strategy-proof while mechanism ψ is not, then mechanism ψ is more manipulable than mechanism φ. Our main interest is the case where neither ψ nor φ are strategy-proof. Our notion is somewhat conservative in the sense that we deem a mechanism to be more manipulable than another only if there is strict inclusion of profiles where they can be manipulated. For example, it is more demanding to compare mechanism with this notion than an alternative notion that simply counts the number of profiles where the mechanisms are manipulable. However, this fact also means that any comparison we can make under our notion provides a stronger result.

Although our notion makes no explicit reference to an equilibrium concept, it is possible to provide it with an equilibrium interpretation. Consider the type revelation game induced by a direct mechanism. The contrapositive of the first part of the definition implies that for a problem, if ψ is not manipulable, then φ is not manipulable. This means that if at any

problem, truth-telling is a Nash equilibrium of the type revelation game induced by mechanism φ, it is also a Nash equilibrium of the type revelation game induced by mechanism ψ (even though the converse does not hold). Recall that if truth-telling is a Nash equilibrium of the type revelation game induced by mechanism φ for all problems, then φ is strategy-proof (see, e.g., Austen-Smith and Banks 2005).

While these definitions are general, in the applications in this paper, we mostly focus on assignment or matching problems. In such problems, A is the set of possible assignments, each player has strict preferences, and we assume that each only cares about her own assignment. We let $\varphi_i(t)$ denote the assignment obtained by player i under type profile t.

3 Applications in School Choice

Throughout this section and the next, the type space of each agent is the set of his preferences. Hence the focus of Sections 3 and 4 is preference revelation mechanisms.

3.1 Reform at Chicago's Public Schools in 2009

To describe the assignment problem for Chicago's selective high schools, we begin by introducing some notation. There is a finite set I of students and a finite set S of schools. School s has capacity q_s, so the total capacity is $Q = \sum_{s \in S} q_s$. We assume that $|I| > Q$ so the seats are in short supply. In 2009, there were over 14,000 applicants for the 9 selective Chicago Public Schools (CPS) high schools, consisting of 3,040 seats.[3]

Each student i has a strict preference ordering P_i over schools and being unassigned. Since each student must take an admissions test as part of their application, each student also has a composite score. We assume that no two students have the same composite score. In practice, if two students have the same test scores, the younger student is coded by CPS as having a higher composite score. The outcome of the admissions process is a matching μ, a function which maps each student either to her assigned school or to being unassigned.[4] Let $\mu(i)$ denote the assignment of student i.

The mechanism that was abandoned in Fall 2009 works as follows:

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[3]In practice, Chicago Public Schools splits selective high schools into five parts. The first 'unrestricted' part is reserved for all applicants. The other four groups are reserved for students from particular neighborhoods, where students are ordered by their test scores within their neighborhood group. To implement this the district simply modifies the rank order list of participants to accommodate this neighborhood constraint. That is, a student who ranks a school is interpreted by the assignment algorithm to rank both the 'unrestricted' part and the part in their neighborhood tier in that order. We abstract away from this modification because it does not affect our analysis.

[4]If a student is not assigned a seat at one of Chicago's selective high schools, she typically later enrolls in a neighborhood school, pursues other public school options such as charter and magnet schools, or leaves the public school system for either private or parochial schools.

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Round 1: In the first round, only the first choices of students are considered. At each school, students who rank the school as their first choice are assigned one at a time according to their composite score until either there are no students who have ranked the school as their first choice left or there are no additional seats at the school.

Round ℓ: In round ℓ, each student who is not yet assigned is considered at her ℓ^{th} choice school. At each school with remaining seats, these students are assigned one at a time according to their composite score until either there are no students who have ranked the school as their ℓ^{th} choice left or there are no additional seats at the school.

Let CHI^k be the version of this mechanism that stops after k rounds. At CPS in Fall 2009, the district employed CHI^4, with only 4 rounds. After eliciting preferences from applicants throughout the city, CPS officials computed assignments internally for discussion. The Chicago Sun-Times reported on November 12, 2009:

> Poring over data about eighth-graders who applied to the city's elite college preps, Chicago Public Schools officials discovered an alarming pattern.
>
> High-scoring kids were being rejected simply because of the order in which they listed their college prep preferences.
>
> "I couldn't believe it," schools CEO Ron Huberman said. "It's terrible."
>
> CPS officials said Wednesday they have decided to let any eighth-grader who applied to a college prep for fall 2010 admission re-rank their preferences to better conform with a new selection system.

To help understand this quote, let us consider the situation for an applicant who is interested in applying to both Northside and Whitney Young, two of Chicago's most competitive college preps. Under CHI^k, it is possible that a student who ranks Northside and Whitney Young in that order ends up unassigned, while had she only ranked Whitney Young, she would have been assigned. If the student does not have a high enough composite score to obtain a placement at Northside, then when she ranks Northside and Whitney Young, she will only obtain a seat at Whitney Young if there are seats left over after the first round. This scenario is unlikely given the popularity of that school, so the student ends up unassigned. Had the student only ranked Whitney Young, she would be considered alongside first choice applicants and her score may be high enough to obtain an offer of admissions there. Hence, it is possible for a high-scoring applicant to be rejected from a school because of the order in which preferences are listed.

The Chicago Sun-Times article continues:

Previously, some eighth-graders were listing the most competitive college preps as their top choice, forgoing their chances of getting into other schools that would have accepted them if they had ranked those schools higher, an official said.

Under the new policy, Huberman said, a computer will assign applicants to the highest-ranked school they qualify for on their new list.

"It's the fairest way to do it." Huberman told the Chicago Sun-Times editorial board Wednesday.

After eliciting preferences under mechanism CHI^4 but not reporting assignments to applicants, CPS officials announced new selection system that works as follows:

> The student with the highest composite score is placed into her top choice. The student with the next highest score obtains her top choice among those she ranked with remaining capacity. If there are no schools left with remaining capacity, then the student is unassigned. The mechanism continues with the student with the next highest composite score until either all schools are filled or each student is processed.

Let SD^k be the version of the mechanism where only the first k choices of a student's rank order list are considered. When all choices on a student's rank order list are considered, it is well known that this **serial-dictatorship** mechanism is strategy-proof. Indeed, in the letter sent from CPS to all students who submitted an application under CHI^4, the district explains:

> ... the original application deadline is being extended to allow applicants an opportunity to review and re-rank their Selection Enrollment High School choices, if they wish. It is recommended that applicants rank their school choices honestly, listing schools in the order of their preference, while also identifying schools where they have a reasonable chance of acceptance.

It would be unnecessary for students to consider what schools they have a reasonable chance of acceptance at if all choices were considered in this mechanism because the serial-dictatorship is strategy-proof. But when only a subset of choices are considered, a student's likelihood of acceptance becomes an important consideration, and a student may obtain a more preferred assignment by manipulating her preferences. Just as the old Chicago mechanism, SD^k is also manipulable.

These two mechanisms are versions of widely studied assignment mechanisms for assigning students to schools. As we have already mentioned the new mechanism adopted in Chicago is a variant of a serial-dictatorship, where only the first four choices are considered. The old Chicago mechanism is a variant of the Boston mechanism that was used by Boston Public Schools until

June 2005, with two important differences. First, although there are nine selective high schools in Chicago, the mechanism considers only the top four choices on a student's application form. This was not a feature of Boston's old school choice system, where all of a student's choices were potentially considered. Second, in Chicago the **priority ranking** of applicants is the same at all schools and it is based on student composite scores. Under the Boston mechanism priority rankings of applicants potentially differ across schools. (In the case of Boston Public Schools, these rankings depend on sibling and walk zone priority.)

Any version of the Boston mechanism, including the version that is abandoned in Chicago, is manipulable. This shortcoming evidently played a role in its elimination in Chicago. However, the new mechanism in Chicago is also manipulable and the school district appears to be aware of this fact since it explicitly suggests that applicants list schools where they have a reasonable chance of acceptance. CPS officials must have felt that the old mechanism is more vulnerable to manipulation. Our first result justifies this point of view.

Proposition 1. *Suppose there are at least k schools and let $k > 1$. The old Chicago mechanism (CHI^k) is more manipulable than truncated serial-dictatorship (SD^k) CPS adopted in 2009.*

We find it remarkable that one of the largest public school districts in the US abandoned a mechanism after about 14,000 participants submitted their preferences citing reasons like those in the newspaper article.[5] The outrage expressed in the quotes from the Chicago Sun-Times suggests that the old mechanism was considered quite undesirable. Our next result allows to formalize the sense in which the old mechanism stands out among other reasonable mechanisms.

A desirable goal of a student assignment mechanism is to produce a "fair" outcome. One basic fairness notion in the context of priority-based student placement was proposed by Balinski and Sönmez (1999) and it is based on the well-known stability notion for two-sided matching markets: If student i prefers school s to her assignment $\mu(i)$ and under matching μ, either school s has a vacant seat or is assigned another student with lower composite score, then student i may have a legitimate objection to her assignment. An individually rational matching that cannot be blocked by such a pair (i, s) is a **stable** matching.

The notion of stability has long been studied in the literature on two-sided matching problems for both normative and positive reasons (see Roth and Sotomayor 1990). In the operations research literature, the stability condition is often treated a sort of feasibility requirement and two-sided matching problems are often described as the "stable matching problem." And yet many school choice mechanisms do not produce stable outcomes. That is perhaps why there is a long gap between the introduction of two-sided matching problems by Gale and Shapley (1962) and formal analysis of school choice mechanisms by Abdulkadiroğlu and Sönmez (2003). The old CPS mechanism (CHI^k) is one of those mechanisms that is not stable. A key reason why

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[5]We only because aware of the policy change in Chicago after this newspaper article. Since then, we have corresponded with CPS officials.

so many school districts use mechanisms that fail stability is that many school districts wish to pay special attention to the first choices of applicants. For instance, the class of mechanisms recently banned in England are known as "first preference first" mechanisms. This observation motivates the following definition.

Define matching μ to be **strongly unstable** if there is a student i and school s such that student i is not assigned to s under μ, student i's top choice is school s, and either school s has a vacancy or there is another student assigned there with lower composite score. A matching is **weakly stable** if it is not strongly unstable. This notion is a relaxation of stability because a student is allowed to block a matching only with its top choice school. While there are quite a few school districts that use unstable mechanisms, we are unaware of any school district which prioritizes students at schools with some criteria and yet uses a mechanism that fails weak stability. In that sense weak stability is a natural requirement in the context of priority-based student admissions. In particular, both the old abandoned CPS mechanism in 2009 and its replacement are weakly stable.

We are ready to present our next result which may explain why CPS CEO Ron Huberman was frustrated enough with the mechanism in 2009 to abandon it in the middle of the assignment process.

Theorem 1. *Suppose each student has a complete rank ordering and $k > 1$. The old CPS mechanism (CHI^k) is at least as manipulable as any weakly stable mechanism.*

We assume that students have complete rank orderings to keep the proof relatively simple. It is possible to state a version of this result without this assumption, but at the expense of significant expositional complexity. This and all other proofs are contained in the appendix.

Based on Proposition 1 and Theorem 1, the new mechanism in Chicago is an improvement in terms of discouraging manipulation. That being said, the lack of efficiency in the 2009 mechanism is due to constraining choices. Any mechanism that restricts reported student preferences to only 4 choices suffers a potential efficiency loss. Moreover, it is possible to have a completely non-manipulable system (a strategy-proof one) by not constraining the choices of applicants. These observations beg the question of what Chicago Public Schools should do in future years. For the 2010-2011 school year, Chicago Public Schools decided to consider up to 6 (out of a total of 9 choices) from applicants.

In the next section, we demonstrate that even though the new 2010 mechanism is still manipulable, its incentive properties are an improvement over the 2009 mechanism under our notion.

3.2 Comparing Constrained Versions of Student-Optimal Stable Mechanism

Understanding the properties of constrained school choice mechanisms is relevant for districts other than Chicago. To describe these issues, it is necessary to present a richer model of student assignment where students may be ordered in different ways across schools.

Vulnerability of school choice mechanisms to manipulation played a role in the adoption of new student assignment mechanisms not only in Chicago, but also in Boston and New York City.[6] An important difference between Chicago and these two cities is that the priority rankings of students are not the same at all schools. To handle this situation, both cities currently employ versions of the student-optimal stable mechanism.[7] For given student preferences and list of priority rankings at schools, the outcome of this mechanism can be obtained with the following **student-proposing deferred acceptance** algorithm:

Round 1: Each student applies to her first choice school. Each school rejects the lowest-ranking students in excess of its capacity and all unacceptable students among those who applied to it, keeping the rest of students temporarily (so students not rejected at this step may be rejected in later steps.)

In general, at

Round ℓ: Each student who was rejected in Round ℓ-1 applies to her next highest choice (if any). Each school considers these students *and* students who are temporarily held from the previous step together, and rejects the lowest-ranking students in excess of its capacity and all unacceptable students, keeping the rest of students temporarily (so students not rejected at this step may be rejected in later steps.)

The algorithm terminates either when every student is matched to a school or every unmatched student has been rejected by every acceptable school. Since there are a finite number of students and schools, the algorithm terminates in a finite number of steps. Gale and Shapley (1962) show that this algorithm results in a stable matching that each student weakly prefers to any other stable matching. Moreover, Dubins and Freedman (1981) and Roth (1982) show that truth-telling is a dominant strategy for each student under this mechanism. Their result implies that student-optimal stable mechanism is strategy-proof in the context of school choice where only students are potentially strategic agents.

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[6]More details on these cases is presented in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2005).

[7]Abdulkadiroğlu and Sönmez (2003) first advocated using the celebrated student-optimal stable mechanism (Gale and Shapley 1962) when priorities differ across schools. Earlier, Balinski and Sönmez (1999) advocated the use of student-optimal stable mechanism in a centralized student placement model where priorities at schools are determined by standardized exams.

Interaction of matching theorists with officials at New York City and Boston lead to adoption of versions of student-optimal stable mechanism by these school districts in 2003 and 2005, respectively. In New York City, however, the version of the mechanism adopted only allows students to submit a rank order list of 12 choices. Based on the strategy-proofness of the student-optimal stable mechanism, the following advice was given to students:

> You must now rank your 12 choices according to your true preferences.

For a student with more than 12 acceptable schools, truth-telling is no longer a dominant strategy under this version of the mechanism. In practice, between 20 to 30 percent of students rank 12 schools, even though there are over 500 choice options in New York City.[8] This issue was first theoretically investigated by Haeringer and Klijn (2009) and experimentally by Calsamiglia, Haeringer, and Klijn (2010).

Some authorities using the student-optimal stable mechanism have increased the number of choices participants can express. For instance, Ajayi (2011) reports that the secondary school admission system in Ghana moved from GS^3 to GS^4 in 2007, and then to GS^6 in 2008. Newcastle England switched from GS^3 to GS^4 by 2010. We next show that the greater the number of choices a student can make, the less vulnerable the constrained version of student-optimal stable mechanism is to manipulation. Let GS be the student-optimal stable mechanism, and GS^k be the constrained version of the student-optimal stable mechanism where only the top k choices are considered.

Proposition 2. *Let $\ell > k > 0$ and suppose there are at least ℓ schools. Then GS^k is more manipulable than GS^ℓ.*

When there is a unique priority ranking across all schools (as in the case of Chicago), mechanism GS^k reduces to mechanism $\textsc{Sd}^k$. Hence the following corollary to Proposition 2 is immediate:

Corollary 1. *Let $\ell > k > 0$. Mechanism $\textsc{Sd}^l$ is more manipulable than mechanism $\textsc{Sd}^k$.*

Just like the change in length list in Newcastle England, Chicago switched from $\textsc{Sd}^4$ to $\textsc{Sd}^6$ in 2010. In terms of promoting truth-telling, this is a further improvement although the unconstrained version of the mechanism would completely eliminate the possibility of manipulation.

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[8]These details together with the entire description of the new assignment procedure is contained in Abdulkadiroğlu, Pathak and Roth (2009).

3.3   The Ban of the Boston Mechanism in England

The mechanism that was abandoned in Chicago midstream in 2009 is a special case of the widely studied Boston mechanism. For given student preferences and school priorities, the outcome of the **Boston mechanism** is determined with the following procedure:

Round 1: Only the first choices of students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until there are no seats left or there is no student left who has listed it as her first choice.

In general, at

Round ℓ: Consider the remaining students. In Round ℓ, only the ℓth choices of these students are considered. For each school with still available seats, consider the students who have listed it as their ℓ^{th} choice and assign the remaining seats to these students one at a time following their priority order until there are no seats left or there is no student left who has listed it as her ℓ^{th} choice.

The procedure terminates when each student is assigned a seat at a school.

Aside from Boston, variants of the mechanism have been used in several U.S. school districts including: Cambridge MA, Charlotte-Mecklensburg NC, Denver CO, Miami-Dade FL, Minneapolis MN, Providence RI, and Tampa-St. Petersburg FL.[9] However, the U.S. is not the only country where versions of the Boston mechanism are used to assign students to public schools. As we discussed in the Introduction, a large number of English Local Authorities had been using what they referred to as "first preference first" systems until it became illegal in 2007. Formally, a **first preference first (FPF)** mechanism is a hybrid between the student-optimal stable mechanism and the Boston mechanism: under this mechanism, a school is either a **first preference first school** or an **equal preference school**, and the outcome is determined by the student-proposing deferred acceptance algorithm, where

1) the base priorities for each student are used for each equal preference school, whereas

2) the base priorities of students are adjusted so that

   - any student who ranks school s as his first choice has higher priority than any student who ranks school s as his second choice,

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[9]Many school districts using variants of the Boston mechanism limit the number of schools that participants may rank. In Providence Rhode Island, students may only list four schools (out of 28 schools), while in Cambridge Massachusetts, students may only list three schools (out of 9 schools).

- any student who ranks school s as his second choice has higher priority than any student who ranks school s as his third choice,

- ...

for each first preference first school.[10]

Observe that the Boston mechanism is a special case of this mechanism when all schools are first preference first schools and the student-optimal stable mechanism is a special case when all schools are equal preference schools.

For given fixed sets of first preference first schools and equal preference schools, let FPF be the first preference first mechanism and FPF^k be the version that only considers the top k student choices. Let β be the Boston mechanism and β^k be the Boston mechanism when only the top k student choices are considered. It will be convenient to let a matching in this and the next section indicate not only which school a student is assigned, but also which students are assigned to a school. Let $\text{FPF}_s(P)$ denote the set of students assigned to school s by the FPF mechanism under profile P, and similarly $\beta_s(P)$ denote the set of students assigned to school s by the Boston mechanism under profile P.

One of the key reasons for the ban of the first preference first mechanism (and hence the Boston mechanism as well) was the strong incentives it gives parents to distort their submitted preferences. Even before the 2007 ban, this issue was central in several debates comparing the first preference first mechanism with the student-optimal stable mechanism (known as equal preference system in England). The following statement from the Coldron, et. al (2008) report prepared for Department for Children, Schools and Families summarizes what is at the heart of the debate:

> Further, the difference between the two systems in the numbers of parents gaining their first preferences should not be interpreted as necessarily meaning that equal preference systems lead to less parental satisfaction overall. In a first preference first area, if the schools a parent puts as first, second or third are oversubscribed they risk not getting in to their first preference school and are also likely not to get their second or third choice because they do not fit the first preference over-subscription criterion of those schools. This means that the first preference system to some extent restricts parents' room for manoeuvre, reduces their options and constrains them to put preferences for schools that are not their real preferred choice.

According to the report, a large number of Local Authorities in England abandoned the first preference first mechanism as a result of the 2007 ban. Table 1 provides a list of districts where we have been able to obtain documentation on systems, building on a large list due to

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[10]The relative priority ranking of two students do not change, unless one ranks the first priority first school higher than the other.

Coldron (2006). The list shows that Local Authorities switched from a constrained version of the first preference first mechanism to a constrained version of the student-optimal stable mechanism, where the constraint is typically greater in more populated areas like London. As in the case of Chicago, the vulnerability of the Boston mechanism to manipulation resulted in its removal throughout England along with its first preference first generalizations, while several Local Authorities adopted a constrained version of the student-optimal stable mechanism.

Our next result shows that not only is the FPF mechanism more manipulable than the student-optimal stable mechanism, its constrained version is more manipulable than the constrained version of the student-optimal stable mechanism. This result suggests that recent reforms throughout the England involve adoption of less manipulable mechanisms.

Proposition 3. *Suppose there are at least k schools where $k > 1$. Then* FPF^k *is more manipulable than* GS^k.

The following result is immediate.

Corollary 2. *Suppose there are at least k schools where $k > 1$. Then β^k is more manipulable than GS^k.*

Another corollary that immediately follows from Proposition 2 and Proposition 3 is of interest based on the reforms in Newcastle and Kent, which both moved from β^3 to GS^4.

Corollary 3. *Let $\ell > k > 0$ and suppose there are at least ℓ schools. Then* FPF^k *is more manipulable than* GS^ℓ.

When each school orders applicants using the same criteria, the old Chicago mechanism CHI^k is a special case of the β^k and the new Chicago mechanism SD^k is a special case of GS^k. As a result, Proposition 1 is a corollary of Proposition 3.

3.4 Seattle's Unusual Experience

Chicago and England are the only places we know about where the Boston mechanism has stopped being used, aside from Boston itself. The fact that these changes occurred without economists' prompting might suggest that the debate over the Boston mechanism has now been resolved. Nevertheless, the majority of U.S. school districts continue to employ versions of the Boston mechanism, and in some districts the debate about its merits rages on.[11]

Seattle Public Schools has undertaken a series of important changes to their student assignment system. After the first draft of this paper, we learned that Seattle switched from the Boston mechanism to the student-optimal stable mechanism in 1999, though it was called the

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[11]For another example, see Abebe (2009) describing the debate in Cambridge Public Schools.

Barnhart-Waldman (BW) amendment in honor of two school board members who proposed the modification. But, it seems that this change was not advertised well, if at all, or well understood by participants.

There are many symptoms that the BW amendment was not well-understood, even though strategy-proof mechanisms allow for straightforward advice. For instance, in a court challenge to the Seattle choice plan by *Parents Involved in Community Schools*, a case eventually decided by the U.S. Supreme Court, confusion surrounding the BW amendment came up in the school board president's deposition.[12] Eventually, in 2007, a parent obtained the computer code and verified that the BW amendment actually corresponds to the student-optimal stable mechanism (MacGregor 2007). Interestingly, researchers did not learn about the change until Seattle returned to the Boston mechanism in 2009.[13]

At first glance, the return to the Boston mechanism may seem to contradict the desirability of reducing a school choice mechanism's vulnerability to manipulation. However, there are other factors at play and the recent Seattle decision generated considerable controversy. Opponents of going back to the Boston mechanism raised points like those discussed in Boston, Chicago, and England. For instance, in her parent guide to the Seattle choice system, Walkup (2009) writes:

> The new choice algorithm can punish naive players. The best strategy for listing school choices for the old algorithm has been to list them in your true order of preference. You did not need to know how likely you were to get into a school to know the right order to list them.

The Seattle situation highlights the importance of considering communication and guidance as key parts of a mechanism's design. When a mechanism is vulnerable to manipulation, it is not easy to provide advice. Walkup (2009) continues:

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[12]Page 58 of the U.S. Court of Appeals for the Ninth Circuit, No. 01-35450, Parents Involved in Community Schools vs. Seattle School District, No.1, 2001 states:

Q: Can you explain for me what the Barnhart/Waldman Amendment is and how it works?

A: If I could I'd be the first. The Barnhart/Waldman – this is my understanding. The Barnhart/Waldman Amendment affects the way that choices are processed. Before we adopted that amendment, all the first choices were processed in one batch and assignments made. If you did not get your first choice, it is my understanding that all the students who did not get the first choice fell to the bottom of the batch processing line, and then they would process the second choices, et cetra. Barnhart/Waldman says that after all the first choices are processed, in the next batch, if you don't get your first choice, you don't fall to the bottom of the list but you are then processed, your second choice, with all the other second choices together. The result is that instead of a high degree of certainty placed - or of value placed on first choice, people can list authentically their first, second and third choices and have a higher degree of getting their second and third choice if they do not get their first choice. Now, was that clear as mud?

[13]The first reference to Seattle is in Abdulkadiroglu, Che and Yasuda (2011), who describe the episode as the "clock turning back."

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During the first few years of the new plan it is likely that many families will still be repeating the previously correct advice to list schools in the actual order you prefer them.

The Seattle case also makes clear that additional benefits of a strategy-proof mechanism are lost with a manipulable mechanism. Walkup continues:

Since the algorithm is no longer blind to strategy, many people will use strategy when listing schools. The district will therefore no longer have accurate information about which programs families prefer.

The fact that family groups were lobbying against the elimination of the BW amendment suggests that the Seattle change was deliberate. Intrigued by this episode, we corresponded with some of the school committee members involved.[14] While they mentioned a few hard-to-square reasons (such as computer implementation costs), one suggested that the BW amendment encouraged mobility among students since parents could freely express their choices. By forcing families to adopt more conservative strategies such as ranking their neighborhood schools, the policy change could discourage student movement and therefore reduce transportation costs.

As a result, we do not believe that Seattle's return to the Boston mechanism is a counter-example to the desirability of a less manipulable mechanism, but rather illustrates that mechanisms can change for multiple reasons. Surely, the original motivation for the BW amendment involved limiting manipulation, and the district may not have reaped the benefits of the earlier policy change given the ongoing confusion. It appears that some policymakers were uncomfortable with the idea of school choice in the first place, but they did not succeed in bringing back a neighborhood school system. Returning to the Boston mechanism was a politically attractive alternative to entirely giving up school choice since it could decrease mobility across neighborhood zones. Whether Seattle continues with the Boston mechanism in future years remains to be seen.

4 Agent-by-Agent Comparisons

4.1 Definitions

So far we compared real-life mechanisms based on set inclusion of their associated vulnerable profiles. Our next application involves the following stronger comparison of mechanisms.

Definition 4. *A mechanism ψ is* **as strongly manipulable as** *mechanism φ if for any profile mechanism φ is vulnerable, ψ is (not only vulnerable but also) manipulable by any player who can manipulate φ.*

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[14]Not all committee members were willing and available to discuss the issue with us.

Definition 5. *A mechanism ψ is* **strongly more manipulable** *than mechanism φ if*

1. *ψ is as strongly manipulable as mechanism φ, and*

2. *there is a set of players I, a set of outcomes A, and a profile t where t is vulnerable under ψ but not under φ.*

Clearly if mechanism ψ is strongly more manipulable than mechanism φ, then mechanism ψ is also more manipulable than mechanism φ.

4.2 Two-Sided Matching Markets

Our next application pertains to college admissions model of Gale and Shapley (1962). Let J be the set of students with generic element j, C be the set of colleges with generic element c, and the set of players are $I = J \cup C$. Here, both sides of the market are active players, in that both submit preference lists over the other side of the market. Following most of the literature, we assume that colleges have responsive preferences (Roth 1985). That is, the ranking of a student is independent of her colleagues, and any set of students exceeding the quota is unacceptable. Given this assumption, we sometimes abuse notation and let P_c denote the preferences of college c defined over singleton student sets and the empty set.

As we have discussed in the context of school choice, truth-telling is a dominant strategy for each student under the student-optimal stable mechanism. We denote this mechanism as $GS^{\mathcal{J}}$. Gale and Shapley (1962) show that there exists an analogous stable matching that favors colleges. We refer to this mechanism as the college-optimal stable mechanism, and denote it as $GS^{\mathcal{C}}$.

While truth-telling is a dominant strategy for each student under $GS^{\mathcal{J}}$, an analogous result does not hold for colleges under $GS^{\mathcal{C}}$. Indeed, there is no stable mechanism where truth-telling is a dominant strategy for colleges in the college admissions model (Roth 1985). Our next result allows us to compare stable mechanisms by their vulnerability to manipulation for colleges. We need to extend definitions 4 and 5 before we present our next result. Fix a subset of agents $I' \subset I$.

A mechanism **ψ is as strongly manipulable as mechanism φ for members of I'** if for any profile $t \in T$, and any agent $i \in I'$,

$$\exists t_i' \in T_i \ \ \text{s.t.} \ \ \varphi(t_i', t_{-i}) \, P_i \, \varphi(t) \quad \implies \quad \exists t_i^* \in T_i \ \ \text{s.t.} \ \ \psi(t_i^*, t_{-i}) \, P_i \, \psi(t).$$

A mechanism **ψ is strongly more manipulable than mechanism φ for members of I'** if

1. ψ is as strongly manipulable as φ for members of I', and

2. there is a set of players I, a set of outcomes A, and a profile t where ψ can be manipulated by an agent in $I' \subseteq I$ although φ cannot.

Results in this section easily follow from the following Lemma.

Lemma 1. *Fix a set of agents $I' \subset J \cup C$. Let φ, ψ be two stable mechanisms such that, for any preference profile P, and any agent $i \in I'$,*

$$\varphi_i(P) \; R_i \; \psi_i(P).$$

Then mechanism ψ is as strongly manipulable as mechanism φ for members of I'.

We are ready to present our next result.

Proposition 4. *$GS^{\mathcal{J}}$ is strongly more manipulable than $GS^{\mathcal{C}}$ for colleges.*

Another natural question is whether it is possible to compare vulnerability of stable mechanisms to manipulation when both students and colleges are able to manipulate. Unfortunately, no comparison is possible because of the well-known conflict of interest between the two sides of the market. This tension is apparent in the following result.

Theorem 2. *Let φ be an arbitrary stable mechanism. Then*

a) *φ is as strongly manipulable as $GS^{\mathcal{C}}$ for colleges,*

b) *$GS^{\mathcal{J}}$ is as strongly manipulable as φ for colleges, and*

c) *$GS^{\mathcal{C}}$ is as strongly manipulable as φ for students.*

These results are related to discussions about the National Resident Matching Program (NRMP), the job market clearinghouse that annually fills more than 25,000 jobs for new physicians in the United States. Prior to 1998, its mechanism of choice was the college-optimal stable mechanism under which truth-telling is not a dominant strategy for students or colleges. In the mid-1990s, the NRMP came under increased scrutiny by students and their advisors who believed that the NRMP did not function in the best interest of students and was open to the possibility of different kinds of strategic behavior (Roth and Rothblum 1999). The mechanism was changed to one based on the student-optimal stable mechanism (Roth and Peranson 1999). One rationale was that truth-telling is a dominant strategy for students. For instance, the minutes of the Committee of the American Medical Student Association (AMSA) and the Public Citizen Health Research Group (cited in Ma 2010) state:

> ...Since it is impossible to remove all incentives for hospitals to misrepresent, it would be best to choose the student-optimal algorithm to remove incentives, at least for students. In other words, within the set of stable algorithms, you either have incentives for both the hospitals and the students to misrepresent their true preferences or only for the hospitals.

Theorem 2 and Proposition 4 imply that an unavoidable consequence of selecting a stable mechanism that removes incentives for manipulation among students is that the mechanism is the most vulnerable to manipulation for colleges.

5 Comparisons based on Intensity of Manipulation

5.1 Definitions

All of our applications up to this point are for models where agents are endowed with ordinal preferences. The magnitude of gain from a manipulation is not well-defined in these models without a cardinal utility representations. In many models, however, agents are endowed with cardinal preferences and one may want to compare magnitude of gain from potential manipulations when comparing two competing mechanisms for such models. We make one observation before proposing such a notion. To avoid interpersonal utility comparisons, a notion that incorporates the magnitude of manipulation will have to be even more demanding than the stronger of our two notions. That is, to deem mechanism ψ more vulnerable to such a notion of manipulation than mechanism φ,

1. any agent who can manipulate φ will need to manipulate ψ as well, and

2. the benefit from the latter manipulation will have to be at least as large.[15]

Hence one can compare fewer mechanisms with this notion (even compared to our stronger notion). Notwithstanding, we present two important applications of this demanding notion later in this section.

For each agent $i \in I$, let $u_i : A \to \mathbb{R}$ be a utility function that represents preferences of agent i over the set of allocations. Having defined these cardinal preferences, we can present the next definition:

Definition 6. *A mechanism ψ is **as intensely and strongly manipulable as** mechanism φ if for any agent i, problem t, type t'_i, and arbitrarily small $\epsilon > 0$,*

$$u_i\big(\varphi(t'_i, t_{-i})\big) - u_i\big(\varphi(t)\big) > 0 \implies \exists t^*_i \text{ s.t. } u_i\big(\psi(t^*_i, t_{-i})\big) - u_i\big(\psi(t)\big) > u_i\big(\varphi(t'_i, t_{-i})\big) - u_i\big(\varphi(t)\big) - \epsilon.$$

It is worth noting that we allow the benefit from the manipulation of mechanism ψ to be marginally smaller than the benefit from the manipulation of mechanism φ. This minor adjustment help us avoid complications associated with choice of tie-breakers in applications and thus significantly increases the scope of our most demanding comparison.

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[15]To avoid potential reversals associated with choice of tie-breakers, we will require the benefit from latter manipulation to be either more or at least arbitrarily close to the benefit from the original manipulation.

Definition 7. *A mechanism ψ is **intensely and strongly more manipulable than** mechanism φ if*

1. *ψ is as intensely and strongly manipulable as mechanism φ, and*

2. *there is a set of players I, a set of outcomes A, and a profile t where t is vulnerable under ψ but not under φ.*

5.2 Multi-Unit Auctions

Our next application involves the auctioning of multiple units of identical objects. The U.S. Treasury's bond issue auctions, auctions for electricity and other commodities, and financial market auctions such as the opening batch auctions at the NYSE, Paris, and Amsterdam exchanges are examples of auctions involving multiple identical objects.[16] We are interested in comparing two sealed-bid auction formats. In each format, a bidder is asked to submit bids for each of the k units indicating how much she is willing to pay for each unit.

In the discriminatory format, also known as the pay-your-bid auction, each bidder pays an amount equal to the sum of her bids that are winning bids. The discriminatory auction is a natural multi-unit extension of the first-price sealed bid auction. Milton Friedman (1960) initially proposed a uniform-price auction, where all k units are sold at a "market-clearing" price such that the total amount demanded is equal to the total amount supplied.

Formally, a seller wishes to sell k units of identical items to a set I of bidders, where $|I| \geq 2$. The bidders, who are the agents in our framework, are asked to report their valuations for the k objects, where v_i^ℓ is bidder i's valuation for the ℓth unit. The vector $v_i = (v_i^1, \ldots, v_i^k) \in \mathbb{R}_+^k$ is the type of bidder i in our framework.

In both auctions we consider, each bidder submits a vector $b_i = (b_i^1, \ldots, b_i^k) \in \mathbb{R}_+^k$, and the k units are awarded to the bidders with the k highest reported valuations.[17]

The utility of bidder i who wins ℓ objects at a total cost of c is:

$$u_i = v_i^1 + \cdots + v_i^\ell - c.$$

We will assume that marginal values are declining for each bidder: $v_i^1 \geq v_i^2 \geq \ldots \geq v_i^k \geq 0$.

The two payment rules we consider are:

1. **Discriminatory auction**: For the units awarded, the bidder pays the value declared for each unit.

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[16]See Krishna (2002) for more examples and discussion.

[17]For both the discriminatory and uniform-price auction, we adopt the convention that when there is a tie, it is broken in favor of the bidder with the lower index i.

2. **Uniform-price auction**: For the units awarded, the bidder pays the $(k+1)^{\text{th}}$ highest declared value for each unit.[18]

The U.S. Treasury has employed a discriminatory format since 1929 for the sale of short-term treasury securities. In 1970s, the US Treasury also adopted a discriminatory format to auction Treasury bonds. In 1992, the US Treasury switched to a uniform-price auction for 2 and 5 year notes and since September 1998, all Treasury auctions use the uniform price format.

Throughout these policy changes, the Treasury has been influenced by a number of arguments. Milton Friedman's influential testimony to the Joint Economic Committee of the US Congress in 1959 argued that a uniform-price format levels the playing field by reducing the importance of specialized knowledge among dealers. According to Friedman, more bidders would be induced to bid directly in uniform-price auctions because the fear of being awarded securities at too high a price is eliminated. Merton Miller supported this argument stating, "All of that [gaming] is eliminated if you use the [uniform-price] auction. You just bid what you think it's worth." A US government report issued around that time jointly signed by the Treasury Department, SEC, and Federal Reserve Board states: "Moving to a uniform-price award method permits bidding at the auction to reflect the true nature of investor preferences."[19]

Neither the discriminatory nor the uniform-price auction is strategy-proof. In particular, in both formats, bidders have an incentive to shade their bids. In a discriminatory auction, bidders have an incentive to report that their bids are just above the lowest bid that wins a unit. In a uniform-price auction, a bidder has an incentive to shade her bid for the units other than the first one because these bids have the potential to influence the market-clearing price if she wins. This "demand-reduction" feature of the uniform-price auction prevents it from being strategy-proof.

The next proposition supports Milton Friedman's original argument about the incentive properties of the uniform-price auction relative to the discriminatory auction.

Proposition 5. *The discriminatory auction is intensely and strongly more manipulable than the uniform-price auction.*

An alternative and complementary formalization of Milton Friedman's argument is recently given by Azevedo and Budish (2011): While both auction formats are manipulable, the discriminatory auction persists to be manipulable even in large economies even though the uniform-price auction is no longer manipulable in large economies when agents are "price-takers."

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[18]It is possible to consider other "market clearing" rules such as paying the kth value or paying a value between the kth and $(k+1)^{\text{th}}$ value. The comparison between formats is not sensitive to this choice.

[19]For more discussion on the influence of Friedman's argument, see Malvey, Archibald and Flynn (1995) and Ausubel and Cramton (2002).

5.3   Keyword Auctions

Our final application involves the model for internet advertising pioneered by Edelman, Ostrovsky and Schwarz (2007) and Varian (2006). When an Internet user enters a search term into an online search engine, she obtains a webpage with search results and sponsored links. The advertisements are ordered on the webpage in different positions, with an advertisement shown at the top of the page more likely to be clicked than one at the bottom of the page. The process by which these advertisement slots are allocated to webpages is currently one of the largest auction markets: in 2005, Google generated more than 6 billion dollars in revenue via their auction mechanism (Edelman et. al 2007).

Our notation and model follow Edelman, Ostrovsky and Schwarz (2007). There is a set I of bidders, and $k < |I|$ ordered slots on a webpage. For any $\ell \in \{1, \dots, k\}$, slot ℓ has a **click-through rate** of α_ℓ, where $\alpha_1 > \alpha_2 > \dots > \alpha_k > 0$. The type t_i of bidder i is his valuation $v_i \in \mathbb{R}_+$ per click.

If bidder i wins the slot m at the cost of c, then his utility is:

$$u_i = \alpha_m v_i - c.$$

Edelman, Ostrovsky and Schwarz (2007) present a detailed historical overview of the origins of this market. In 1997, Overture introduced an auction for selling Internet advertising. In the original design, each advertiser simultaneously bids for a slot for a particular keyword. The highest bidder receives the first slot at a price of his bid times the click-through rate of slot 1, the second highest bidder receives the second slot at a price of his bid times the click-through rate of slot 2, and so on. Overture's search platform was adopted by major search engines including Yahoo! and MSN. This auction format is known as the **Generalized First Price (GFP)** auction.

In February 2002, Google introduced its own pay-per-click system, AdWords Select, based on a different payment rule. The highest bidder receives the first slot at a price of the second highest bid times the click-through rate of slot 1, the second highest bidder receives the second slot at a price of the third highest bid times the click-through rate of slot 2, and so on. This auction format has come to be known as the **Generalized Second Price (GSP)** auction. Once Google introduced this new format, many search engines including Yahoo!/Overture also switched to the GSP.

While neither mechanism is strategy-proof, Edelman, Ostrovsky, and Schwarz (2007) argue that

> The second-price structure makes the market more user friendly and less susceptible to gaming.

Our final result formalizes their insight.

Proposition 6. *The Generalized First Price Auction is intensely and strongly more manipulable than the Generalized Second Price Auction.*

6 Conclusion

Recent school admission reforms are motivated in part by the desire to reduce strategic considerations among participants, even though many new mechanisms are still not completely immune to manipulation. These changes motivate the methodology we propose to rank mechanisms by their vulnerability to manipulation. In Chicago, the abandoned mechanism is at least as manipulable as any other weakly stable mechanism. In England, the 2007 School Code outlawed first preference first mechanisms and numerous districts have adopted an equal preference system. According to our notion, numerous English districts have done away with more manipulable mechanisms.

The changes to school assignment systems in recent years are widespread. The list of reforms in Table 1 implies that hundreds of thousands of students have been impacted. Every listed change, except Seattle, involves a move towards a less manipulable mechanism. It is therefore clear that vulnerability to manipulation is perceived as an undesirable feature of school choice mechanisms.

While school choice reforms provide our main motivation, the methodology has applications in other matching and assignment models, including the college admissions model. We have also illustrated applications for auction settings, and examined manipulation definitions that take intensity of manipulation into account. Certainly, we haven't exhausted possible applications of these concepts. For instance, work in progress by Dasgupta and Maskin (2010) explores a similar idea in voting problems, comparing Condorcet and Borda rules, and similar ideas have been recently studied in problem of fair division with indivisible objects (see, e.g., Andersson, Ehlers, and Svensson (2010)).

The case studies we have explored all involve widespread condemnation of the Boston mechanism, and the participants themselves (and not matching theorists) advocated re-organizing market designs. In this respect, the school admissions reforms parallel changes in marketplace rules for the placement of medical residents in the early 1950s documented by Roth (1984). Following Boston Public Schools' abandonment of the mechanism in 2005, there has been a renewed interest in understanding its properties. Some researchers have cautioned against a hasty rejection of the Boston mechanism in favor of the student-optimal stable mechanism (Abdulkadiroğlu, Che and Yasuda 2011, Featherstone and Niederle 2011, Miralles 2008). When interpreted through the lens of the public and policymaker's revealed preferences, events in Chicago and England weigh against the desirability of the Boston mechanism.

It is worth emphasizing that vulnerability to manipulation is not the only criterion to consider when comparing mechanisms. That being said, manipulation seems to have been a

critical reason for the 2009 policy change in Chicago and changes throughout England. Of course, it is important to consider different properties of a mechanism and the alternatives (as well as political and practical issues) when deciding on the best mechanism. In situations where strategy-proof mechanisms do not have obvious drawbacks, as one might argue for eliminating restrictions on the number of choices allowed in school choice, an interesting question for future work is to understand the reasons they are not used.

Table 1. School Admissions Reforms

Allocation System	Year	From	To	Manipulable (More or Less?)	Source	Allocation System	Year	From	To	Manipulable (More or Less?)	Source
Boston Public Schools (K, 6, 9)	2005	Boston	GS	Less	A,B,E	Merton	2006	FPF[6]	GS[6]	Less	A,D
Chicago Selective High Schools	2009	Boston[4]	SD[4]	Less	A,B,C	Newcastle	2005	Boston[3]	GS[3]	Less	A
	2010	SD[4]	SD[6]	Less	A,B,C		2010	GS[3]	GS[4]	Less	A
Ghana - Secondary Schools	2007	GS[3]	GS[4]	Less	E	North Lincolnshire	2007*	FPF[3]	GS[3]	Less	A,D
	2008	GS[4]	GS[6]	Less	E	North Somerset	2007*	FPF[3]	GS[3]	Less	A,D
Seattle Public Schools	1999	Boston	GS	Less	A,B,C,E,F	North Tyneside	2007*	FPF[3]	GS[3]	Less	A,D
	2009	GS	Boston	More	A,B,C,F	Oldham	2007*	FPF[3]	GS[3]	Less	A,D
						Peterborough	2007*	FPF[3]	GS[3]	Less	A,D
England						Plymouth	2007*	FPF[3]	GS[3]	Less	A,D
Bath and North East Somerset	2007*	FPF[3]	GS[3]	Less	A, D	Poole	2007*	FPF[3]	GS[3]	Less	A,D
Bedford and Bedfordshire	2007*	FPF[3]	GS[3]	Less	A,D	Portsmouth	2007*	FPF[3]	GS[3]	Less	D
Blackburn with Darwen	2007*	FPF[3]	GS[3]	Less	A,D	Richmond	2005	FPF[6]	GS[6]	Less	D
Blackpool	2007*	FPF[3]	GS[3]	Less	D	Sefton primary	2007	Boston[3]	GS[3]	Less	A,D
Bolton	2007*	FPF[3]	GS[3]	Less	A,D	Sefton secondary	2007	FPF[3]	GS[3]	Less	A,D
Bradford	2007*	FPF[3]	GS[3]	Less	A,D	Slough	2006*	FPF[3]	GS[3]	Less	D
Brighton and Hove	2007	Boston[3]	GS[3]	Less	A,C,D,E	Somerset	2007*	FPF[3]	GS[3]	Less	A,D
Calderdale	2006	FPF[3]	GS[3]	Less	A,C	South Gloucestershire	2007*	FPF[3]	GS[3]	Less	A,D
Cornwall	2007*	FPF[3]	GS[3]	Less	D	South Tyneside	2007*	FPF[3]	GS[3]	Less	D
Cumbria	2007*	FPF[3]	GS[3]	Less	D	Southhampton	2007*	FPF[3]	GS[3]	Less	D
Darlington	2007*	FPF[3]	GS[3]	Less	D	Stockton	2007*	FPF[3]	GS[3]	Less	A, D
Derby	2005*	FPF[4]	GS[4]	Less	A,D	Stoke-on-Trent	2007*	FPF[3]	GS[3]	Less	D
Devon	2006*	FPF[3]	GS[3]	Less	A,D	Suffolk	2007*	FPF[3]	GS[3]	Less	D
Durham	2007	FPF[3]	GS[3]	Less	A,D	Sunderland	2007*	FPF[3]	GS[3]	Less	D
Ealing	2006*	FPF[6]	GS[6]	Less	A,D	Surrey	2007	FPF[3]	GS[3]	Less	A,D
East Sussex	2007	Boston[3]	GS[3]	Less	A,D		2010	GS[3]	GS[6]	Less	A
Gateshead	2007*	FPF[3]	GS[3]	Less	D	Sutton	2006	FPF[6]	GS[6]	Less	A,D
Halton	2007*	FPF[3]	GS[3]	Less	A,D	Swindon	2007*	FPF[3]	GS[3]	Less	D
Hampshire	2007	FPF[3]	GS[3]	Less	A,D	Tameside	2007*	FPF[3]	GS[3]	Less	D
Hartlepool	2007	FPF[3]	GS[3]	Less	A,D	Telford and Wrekin	2007*	FPF[3]	GS[3]	Less	D
Isle of Wright	2007*	FPF[3]	GS[3]	Less	D	Torbay	2007*	FPF[3]	GS[3]	Less	D
Kent	2007	Boston[3]	GS[4]	Less	A,D	Warrington	2007*	FPF[3]	GS[3]	Less	D
Kingston upon Thames	2007*	FPF[3]	GS[4]	Less	A	Warwickshire	2007*	FPF[7]	GS[7]	Less	D
Knowsley	2007*	FPF[3]	GS[3]	Less	A,D	Wilgan	2007*	FPF[3]	GS[3]	Less	D
Lancashire	2007*	FPF[3]	GS[3]	Less	A,D						
Lincolnshire	2007*	FPF[3]	GS[3]	Less	A,D	Wales					
Luton	2007*	FPF[3]	GS[3]	Less	D	Wrexham County Borough	2011	FPF[3]	GS[3]	Less	A
Manchester	2007*	FPF[3]	GS[3]	Less	A,D						

Notes. * For changes in the 2007 code, an asterisk indicates that we assume that the number of choices allowed has not changed. A - Documentation from schools (brochures) or official policy minutes; B - Direct communication with school officials; C - Documentation from press clippings; D - Coldron report; E - Other academic papers; F - Other online materials. In some cases, we do not know the exact year the mechanism changed, the years correspond to the last possible year. The appendix includes sourcing for all mechanism changes.

A   Proofs [For Online Publication]

Theorem 1. *Suppose each student has a complete rank ordering and $k > 1$. The old Chicago Public Schools mechanism (CHI^k) is at least as manipulable as any weakly stable mechanism.*

Proof. Fix a problem P and let φ be an arbitrary mechanism that is weakly stable. Suppose that CHI^k is not manipulable for problem P.

   Claim 1: Any student assigned under $\mathrm{CHI}^k(P)$ receives her top choice.
   Proof. If not, since each student has a complete rank order list, $|I| > Q$, $k > 1$, there must be a student that is assigned to a school s he has not ranked first. Consider the highest composite score student i who is unassigned. Student i can rank school s first and will be assigned a seat there in the first round of CHI^k mechanism instead of some student who has not ranked school s first. That contradicts CHI^k is not manipulable for problem P.

   Claim 2. The set of students who are assigned a seat under $\mathrm{CHI}^k(P)$ is equal to the set of top Q composite score students.
   Proof. If not, there is a school seat assigned to a student j who does not have a top Q score. Let student i be the highest scoring top Q student who is not assigned. Since student i has a complete rank order list, she can manipulate CHI^k by ranking student j's assignment as her top choice again contradicting CHI^k is not manipulable for problem P.

   Since each of the top Q students is matched to her top choice in matching $\mathrm{CHI}^k(P)$, all other students are unassigned.

   Claim 3. In problem P, matching $\mathrm{CHI}^k(P)$ is the unique weakly stable matching.
   Proof. By Claims 1 and 2 it is possible to assign each one of the top Q students a seat at their top choice school under P and $\mathrm{CHI}^k(P)$ picks that matching. Let $\mu \neq \mathrm{CHI}^k(P)$. That means under μ there exists a top Q student i who is not assigned to her top choice s. Pick the highest composite score such student i. Since all higher score students are assigned to their top choices, either there is a vacant seat at her top choice s or it admitted a student with lower composite score. In either case the pair (i, s) strongly blocks matching μ. Hence $\mathrm{CHI}^k(P)$ is the unique weakly stable matching under P.

   We are now ready to complete the proof. By Claim 3, $\varphi(P) = \mathrm{CHI}^k(P)$ and hence mechanism φ assigns all top Q students a seat at their top choices. None of the top Q students has an incentive to manipulate φ since each receives her top choice. Moreover no other student can manipulate φ because regardless of their stated preferences, $\varphi(P) = \mathrm{CHI}^k(P)$ remains the unique weakly stable matching and hence φ picks the same matching for the manipulated economy. Hence, any other weakly stable mechanism is also not manipulable under P.   □

Proposition 1. *Suppose there are at least k schools and let $k > 1$. The old Chicago mechanism (CHI^k) is more manipulable than truncated serial-dictatorship (SD^k) CPS adopted in 2009.*

Proof. CHI^k is a special case of the FPF^k mechanism where all schools are first preference first schools with an identical priority ranking. Similarly SD^k is a special case of GS^k where all schools have an identical priority ranking. Therefore CHI^k being *as manipulable as* SD^k directly follows from Proposition 3. We complete the proof by giving an example where CHI^k is manipulable even though SD^k is not.

There are three students and three schools each with one seat. The student preferences and the uniform school priorities are:

$$
\begin{aligned}
R_{i_1} &: s_1, s_2, s_3, i_1 & \pi_{s_1} &: i_1, i_2, i_3 \\
R_{i_2} &: s_1, s_2, s_3, i_2 & \pi_{s_2} &: i_1, i_2, i_3 \\
R_{i_3} &: s_2, s_3, s_1, i_3 & \pi_{s_3} &: i_1, i_2, i_3
\end{aligned}
$$

The outcomes of CHI^2 and SD^2 are:

$$
\mathrm{CHI}^2(R) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & i_2 & s_2 \end{pmatrix} \quad \text{and} \quad \mathrm{SD}^2(R) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.
$$

Since no student remains unmatched under SD^2, strategy-proofness of SD implies that no student can manipulate SD^2 under profile R. In contrast

$$
\mathrm{CHI}^2(R_{-i_2}, R'_{i_2}) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}
$$

where R'_{i_2} is any preference relation student i_2 ranks school s_2 as his first choice, and therefore

$$
\mathrm{CHI}^2_{i_2}(R_{-i_2}, R'_{i_2}) \, P_{i_2} \, \mathrm{CHI}^2_{i_2}(R)
$$

implies that CHI^2 is vulnerable under profile R. Hence CHI^2 is more manipulable than SD^2. It is straightforward to extend this example to show that CHI^k is more manipulable than SD^k for $k > 2$. $\qquad\square$

Proposition 2. *Let $\ell > k > 0$ and suppose there are at least ℓ schools. Then GS^k is more manipulable than GS^ℓ.*

Proof. Suppose there is a student i and preference $\hat{P}_i$ such that

$$GS_i^\ell(\hat{P}_i, P_{-i}) \ P_i \ GS_i^\ell(P). \tag{1}$$

For any student j, let P_j^ℓ be the truncation of P_j after the ℓth choice. This means that in P_j^ℓ any choice after the top ℓ in P_j are unacceptable, and choices among the top ℓ are ordered according to P_j. Observe that relation (1) implies that

$$GS_i(\hat{P}_i^\ell, P_{-i}^\ell) \ P_i \ GS_i(P^\ell). \tag{2}$$

Since GS is strategy-proof, relation (2) implies that student i does not receive one of her top ℓ choices from the GS mechanism under profile P^ℓ. Hence, $GS_i(P^\ell) = GS_i^\ell(P) = i$.

For $k < \ell$, there are two cases to consider.

Case 1: $GS_i^k(P) = i$.

Let $GS_i^\ell(\hat{P}_i, P_{-i}) = s$ and let $\tilde{P}_i$ be such that s is the only acceptable school.

Claim: $GS_i^k(\tilde{P}_i, P_{-i}) = s$.
Proof: First note that $GS_i^\ell(\tilde{P}_i, P_{-i}) = s$. Moreover, by definition

$$GS^\ell(\tilde{P}_i, P_{-i}) = GS(\tilde{P}_i, P_{-i}^\ell) \quad \text{and} \quad GS^k(\tilde{P}_i, P_{-i}) = GS(\tilde{P}_i, P_{-i}^k).$$

Gale and Sotomayor (1985) (see also Theorem 5.34 of Roth and Sotomayor 1990) implies that

$$GS_i(\tilde{P}_i, P_{-i}^k) \ R_i \ GS_i(\tilde{P}_i, P_{-i}^\ell).$$

Substituting the definitions,

$$GS_i^k(\tilde{P}_i, P_{-i}) \ R_i \ \underbrace{GS_i^\ell(\tilde{P}_i, P_{-i})}_{=s}.$$

Since c is the only acceptable school in $\tilde{P}_i$, the claim follows. $\diamond$

Thus, in the first case, student i can manipulate GS^k:

$$\underbrace{GS_i^k(\tilde{P}_i, P_{-i})}_{=s} \ P_i \ \underbrace{GS_i^k(P)}_{=i}.$$

Case 2: $GS_i^k(P) \neq i$.

Claim 1: $\exists j \in I$ such that $GS_j^k(P) = j$ although $GS_j^\ell(P) \neq j$.
Proof: Suppose not. Then, since $GS_i^\ell(P) = i$ and $GS_i^k(P) \neq i$, there is a school that is assigned strictly more students under $GS^k(P)$ than $GS^\ell(P)$. This is a contradiction to Gale

and Sotomayor (1985), which requires that each school is weakly worse off under GS^k (since profile P^k is a truncation of profile P^ℓ). ◇

Pick any $j \in I$ such that $GS_j^k(P) = j$ although $GS_j^\ell(P) \neq j$. Let $GS_j^\ell(P) = s$ and let $\tilde{P}_j$ be such that s is the only acceptable school.

Claim 2: $GS_j^k(\tilde{P}_j, P_{-j}) = s$.

Proof: Since $GS_j^\ell(P) = s$, we have $GS_j^\ell(\tilde{P}_j, P_{-j}) = c$ as well. Moreover, by definition

$$GS^\ell(\tilde{P}_j, P_{-j}) = GS(\tilde{P}_j, P_{-j}^\ell) \quad \text{and} \quad GS^k(\tilde{P}_j, P_{-j}) = GS(\tilde{P}_j, P_{-j}^k).$$

Gale and Sotomayor (1985) implies that

$$GS_j(\tilde{P}_j, P_{-j}^k) \ R_j \ GS_j(\tilde{P}_j, P_{-j}^\ell).$$

Substituting the definitions,

$$GS_j^k(\tilde{P}_j, P_{-j}) \ R_j \ \underbrace{GS_j^\ell(\tilde{P}_j, P_{-j})}_{=s}.$$

Since s is the only acceptable school in $\tilde{P}_j$,

$$GS_j^k(\tilde{P}_j, P_{-j}) = s,$$

which establishes the claim. ◇

Thus, for the second case, student j can manipulate GS^k:

$$\underbrace{GS_j^k(\tilde{P}_j, P_{-j})}_{=s} \ P_j \ \underbrace{GS_j^k(P)}_{=j}.$$

Finally, we describe a problem where GS^ℓ is not manipulable by any student, but GS^k is manipulable by some student for $\ell > k > 0$. Suppose there are two students, i_1 and i_2, and two schools, s_1 and s_2, each with one seat. The students have identical preferences which rank s_1 ahead of s_2 and both schools have identical priority rankings where student i_1 has higher priority than student i_2. Under GS^2, no student can manipulate because each one is assigned a school and GS is strategy-proof. In contrast, student i_2 is unassigned under GS^1, and he can benefit from ranking s_2 as his top choice. This example can be generalized to the case of GS^k and GS^ℓ for any $\ell > k > 0$. Since all schools have the same priority ranking in this example, it also proves that SD^k is more manipulable than SD^ℓ for any $\ell > k > 0$. This completes the proof.[20] □

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[20]It is also possible to provide an alternative, indirect proof of this result using the equilibrium interpretation of the definition of weakly more manipulable than together with the characterization of the set of Nash equilibria in the preference revelation game induced by GS^k in Theorem 6.5 of Haeringer and Klijn (2009).

Proposition 3. *Suppose there are at least k schools where $k > 1$. Then $\textsc{Fpf}^k$ is more manipulable than GS^k.*

Proof. For any student j, let P_j^k be the truncation of P_j after the kth choice. By definition,

$$\textsc{Fpf}^k(P) = \textsc{Fpf}(P^k) \qquad \text{and} \qquad GS^k(P) = GS(P^k).$$

Suppose that no student can manipulate $\textsc{Fpf}^k$. We will show that no student can manipulate GS^k either. Consider two cases:

Case 1: $\textsc{Fpf}^k(P) = \textsc{Fpf}(P^k)$ is stable under profile P.

Since $\textsc{Fpf}(P^k)$ is stable under P, it is stable under P^k as well. Moreover, $GS(P^k)$ is stable for P^k by definition. Since the set of unmatched students across stable matchings is the same (McVitie and Wilson 1970), for all students i,

$$GS_i(P^k) = i \quad \Leftrightarrow \quad \textsc{Fpf}_i(P^k) = i. \tag{3}$$

Pick some student i. If $GS_i^k(P^k) \neq i$, then student i receives one of her top k choices. This implies that i receives one of her top k choices under GS. Since GS is strategy-proof, student i cannot manipulate GS^k.

Suppose $GS_i^k(P^k) = i$ and student i can manipulate. We derive a contradiction. Since i can manipulate, there exists some school s and preference $\hat{P}_i$ such that

$$\underbrace{GS_i^k(\hat{P}_i, P_{-i}^k)}_{=s} \ P_i \ i.$$

Observe that s is not one of the top k choices of student i under P_i for otherwise student i could manipulate GS. Construct $\tilde{P}_i$ which lists s as the only acceptable school.

Matching $GS^k(\hat{P}_i, P_{-i}^k)$ remains stable under $(\tilde{P}_i, P_{-i}^k)$ and therefore

$$GS_i^k(\tilde{P}_i, P_{-i}^k) = s.$$

Since $GS(P^k)$ is stable under P^k and $GS_i^k(P^k) = i$ by assumption, relation (3) implies

$$\textsc{Fpf}_i(P^k) = i.$$

By Roth (1984), matching $\textsc{Fpf}(P^k)$ is not stable under $(\tilde{P}_i, P_{-i}^k)$ since student i remains single under $\textsc{Fpf}(P^k)$ although not under stable matching $GS^k(\hat{P}_i, P_{-i}^k)$. Since matching $\textsc{Fpf}(P^k)$ is not stable under $(\tilde{P}_i, P_{-i}^k)$, but it is stable for P^k, the only possible blocking pair of $\textsc{Fpf}(P^k)$ in $(\tilde{P}_i, P_{-i}^k)$ is (i, s). But since $\textsc{Fpf}_i(P^k) = i$, this implies that (i, s) also blocks $\textsc{Fpf}(P^k)$ under P^k, which is the desired contradiction. Thus, in case 1, no student can manipulate GS^k.

Case 2: $\text{FPF}(P^k)$ is not stable for profile P.

In this case, a student i along with a *first preference first school* s block $\text{FPF}(P^k)$: That is, there exists $j \in \text{FPF}_s(P^k)$ such that not only i has higher base priority than j at school s, but also $s \, P_i \, \text{FPF}_i(P^k)$.

Construct $\tilde{P}_i$ so that school s is the only acceptable school for student i. Since $j \in \text{FPF}_s(P^k)$ and student i has higher base priority than student j at school s, we must have $i \in \text{FPF}_s(\tilde{P}_i, P^k_{-i})$. But this means that

$$\underbrace{\text{FPF}_i(\tilde{P}_i, P^k_{-i})}_{=s} \, P_i \, \text{FPF}_i(P^k),$$

contradicting the assumption that no student can manipulate FPF at P^k.

Finally we give an example where FPF^2 is manipulable but GS^2 is not. It is straightforward to extend the example for any $k > 2$. There are three students and three *first preference first* schools each with one seat. Since all schools are first preference first, FPF mechanism reduces to the special case of the Boston mechanism in this example. The student preferences and the (uniform) school priorities are:

$$\begin{aligned}
R_{i_1} &: s_1, s_2, s_3, i_1 &\qquad \pi_{s_1} &: i_1, i_2, i_3 \\
R_{i_2} &: s_1, s_2, s_3, i_2 &\qquad \pi_{s_2} &: i_1, i_2, i_3 \\
R_{i_3} &: s_2, s_3, s_1, i_3 &\qquad \pi_{s_3} &: i_1, i_2, i_3
\end{aligned}$$

The outcomes of FPF^2 and GS^2 are:

$$\text{FPF}^2(R) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & i_2 & s_2 \end{pmatrix} \quad \text{and} \quad GS^2(R) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

Since no student remains unmatched under GS^2, strategy-proofness of GS implies that no student can manipulate GS^2 under profile R. In contrast

$$\text{FPF}^2(R_{-i_2}, R'_{i_2}) = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

where R'_{i_2} is any preference relation student i_2 ranks school s_2 as his first choice, and therefore

$$\text{FPF}^2_{i_2}(R_{-i_2}, R'_{i_2}) \, P_{i_2} \, \text{FPF}^2_{i_2}(R)$$

implies that FPF^2 is vulnerable under profile R. Hence FPF^2 is more manipulable than GS^2. $\qquad \square$

Lemma 1. *Fix a set of agents $I' \subset J \cup C$. Let φ, ψ be two stable mechanisms such that, for any preference profile P, and any agent $i \in I'$,*

$$\varphi_i(P) \ R_i \ \psi_i(P).$$

Then mechanism ψ is as strongly manipulable as mechanism φ for members of I'.

Proof. Let $I' \subset J \cup C$ and mechanisms φ, ψ be as in the statement of the Lemma. Let preference profile P, agent $i \in I'$, and preference relation $\hat{P}_i$ be such that

$$\varphi_i(\hat{P}_i, P_{-i}) \ P_i \ \varphi_i(P). \tag{4}$$

We want to show that there exists a preference relation $\tilde{P}_i$ such that

$$\psi_i(\tilde{P}_i, P_{-i}) \ P_i \ \psi_i(P).$$

By assumption

$$\varphi_i(P) \ R_i \ \psi_i(P). \tag{5}$$

Let the preference relation $\tilde{P}_i$ be such that only agents in $\varphi_i(\hat{P}_i, P_{-i})$ are acceptable to agent i under $\tilde{P}_i$. Since matching $\varphi(\hat{P}_i, P_{-i})$ is stable under profile $(\hat{P}_i, P_{-i})$, it is also stable under profile $(\tilde{P}_i, P_{-i})$. Moreover by Roth (1984), agent i is matched with the same number of agents on the other side of the market at any stable matching under any given preference profile, and in particular under profile $(\tilde{P}_i, P_{-i})$. Therefore, since only agents in $\varphi_i(\hat{P}_i, P_{-i})$ are acceptable to agent i under $\tilde{P}_i$, stability of matching $\varphi(\hat{P}_i, P_{-i})$ under $(\tilde{P}_i, P_{-i})$ implies

$$\psi_i(\tilde{P}_i, P_{-i}) = \varphi_i(\hat{P}_i, P_{-i}). \tag{6}$$

Hence, by (4), (5), and (6), we have

$$\underbrace{\psi_i(\tilde{P}_i, P_{-i})}_{=\varphi_i(\hat{P}_i, P_{-i})} \ P_i \ \varphi_i(P) \ R_i \ \psi_i(P),$$

which shows that agent i can manipulate mechanism ψ by reporting $\tilde{P}_i$. This completes the proof. $\qquad\square$

Proposition 4. *$GS^{\mathcal{J}}$ is strongly more manipulable than $GS^{\mathcal{C}}$ for colleges.*

Proof. Given any problem, the college-optimal stable matching is weakly preferred to student-optimal stable matching by any college (Gale and Shapley 1962). Therefore, Lemma 1 implies $GS^{\mathcal{J}}$ is as strongly manipulable as $GS^{\mathcal{C}}$ for colleges.

Next, we give a problem where $GS^{\mathcal{C}}$ is not manipulable by any college, while some college can manipulate $GS^{\mathcal{J}}$. Suppose there are two students, j_1 and j_2, and two colleges, c_1 and c_2, each with one seat. The student and college preferences are

$$R_{j_1} : c_1, c_2, j_1 \qquad R_{c_1} : \{j_2\}, \{j_1\}, \emptyset$$
$$R_{j_2} : c_2, c_1, j_2 \qquad R_{c_2} : \{j_1\}, \{j_2\}, \emptyset.$$

The outcomes of $GS^{\mathcal{C}}$ and $GS^{\mathcal{J}}$ are:

$$GS^{\mathcal{C}}(R) = \begin{pmatrix} j_1 & j_2 \\ c_2 & c_1 \end{pmatrix} \quad \text{and} \quad GS^{\mathcal{J}}(R) = \begin{pmatrix} j_1 & j_2 \\ c_1 & c_2 \end{pmatrix}.$$

Since each college obtains its top choice under $GS^{\mathcal{C}}$, no college can manipulate. However, if college c_1 declares that only j_2 is acceptable, it can manipulate $GS^{\mathcal{J}}$. This completes the proof. $\qquad\square$

Theorem 2. *Let φ be an arbitrary stable mechanism. Then*

   a) φ is as strongly manipulable as $GS^{\mathcal{C}}$ for colleges,

   b) $GS^{\mathcal{J}}$ is as strongly manipulable as φ for colleges, and

   c) $GS^{\mathcal{C}}$ is as strongly manipulable as φ for students.

Proof. Let φ be any stable mechanism and P be any preference profile. Then

   a) $GS_c^{\mathcal{C}}(P) \, R_c \, \varphi(P)$ for any $c \in C$,

   b) $\varphi_c(P) \, R_c \, GS_c^{\mathcal{J}}(P)$ for any $c \in C$, and

   c) $\varphi_j(P) \, R_j \, GS_j^{\mathcal{C}}(P)$ for any $j \in J$

by Gale and Shapley (1962). Therefore Lemma 1 implies the desired result. $\qquad\square$

Proposition 5. *The discriminatory auction is intensely and strongly more manipulable than the uniform-price auction.*

Proof. Let δ denote the discriminatory auction and Υ denote the uniform-price auction. Fix $\epsilon > 0$ and a bidder i. Let t_{-i} be the type profile of all other bidders. Recall that the type of each bidder is the vector of his valuations. Given t_{-i}, order the $k(|I| - 1)$ valuations of all bidders in $I \setminus \{i\}$ from highest to lowest. Let b_1 be the highest valuation, b_2 be the next highest valuation, and so on. That is, $b_1 \geq b_2 \geq \cdots \geq b_{k(|I|-1)} > 0$.

Let $t_i = (v_i^1, \dots, v_i^k)$ be the type of bidder i. We will consider two cases. For the first case bidder i will not be able to manipulate the uniform-price auction. For the second case he potentially can but whenever that happens he will have an at least as profitable deviation under the discriminatory auction.

Case 1: $v_i^1 < b_k$. For this case bidder i's highest valuation is less than b_k. Therefore if he reports his true values under the uniform-price auction, he will not receive any object and will not make any payment. Hence $u_i\big(\Upsilon(t)\big) = 0$. In order to have a profitable manipulation, bidder i will need to receive an object. However, since $v_i^1 < b_k$, that will require bidder i to pay a unit price that is higher than his highest valuation. Hence $u_i\big(\Upsilon(t_i', t_{-i})\big) - u_i\big(\Upsilon(t)\big) \leq 0$ for any $t_i' \in T_i$, showing there exists no profitable manipulation of the uniform-price auction for Case 1.

Case 2: $v_i^1 \geq b_k$. Let bidder i receive m units under the uniform price auction when he reports his true type $t_i = (v_i^1, \dots, v_i^k)$. That means $v_i^m \geq b_{k-m+1}$ and the market clearing-price for profile t is

$$p^* = \begin{cases} \max\{v_i^{m+1}, b_{k-m+1}\} & \text{if } m < k \\ b_{k-m+1} & \text{if } m = k \end{cases}$$

which in turn implies

$$u_i\big(\Upsilon(t)\big) = (v_i^1 + \cdots + v_i^m) - mp^* \geq 0. \tag{7}$$

Let the potential manipulation $\hat{t}_i = (\hat{v}_i^1, \dots, \hat{v}_i^k)$ be such that bidder i receives n units under $\Upsilon(\hat{t}_i, t_{-i})$. Then the market-clearing price for profile $(\hat{t}_i, t_{-i})$ is

$$\hat{p} = \begin{cases} \max\{\hat{v}_i^{n+1}, b_{k-n+1}\} & \text{if } n < k \\ b_{k-n+1} & \text{if } n = k \end{cases}$$

and hence

$$u_i\big(\Upsilon(\hat{t}_i, t_{-i})\big) = (v_i^1 + \cdots + v_i^n) - n\hat{p}.$$

Observe that,

$$\hat{p} \geq b_{k-n+1}. \tag{8}$$

Suppose

$$u_i\big(\Upsilon(\hat{t}_i, t_{-i})\big) - u_i\big(\Upsilon(t)\big) = (v_i^1 + \cdots + v_i^n - n\hat{p}) - (v_i^1 + \cdots + v_i^m - mp^*) > 0$$

and thus bidder i can manipulate the uniform-price auction at profile t. We will construct $\tilde{t}_i \in T_i$ such that

$$u_i\big(\delta(\tilde{t}_i, t_{-i})\big) - u_i\big(\delta(t)\big) > u_i\big(\Upsilon(\hat{t}_i, t_{-i})\big) - u_i\big(\Upsilon(t)\big) - \epsilon.$$

First observe that $u_i\big(\delta(t)\big) = 0$, since bidder i pays her reported valuation for each unit she wins under the discriminatory auction. Let $\tilde{t}_i = (\tilde{v}_i^1 \ldots \tilde{v}_i^k)$ be such that

$$\tilde{v}_i^\ell = \begin{cases} b_{k-n+1} + \frac{\epsilon}{2n} & \text{if } \ell \leq n \\ 0.5 b_{k-n+1} & \text{if } \ell > n \end{cases}$$

Given t_{-i}, bidder i wins n units and pays $b_{k-n+1} + \frac{\epsilon}{2n}$ for each unit upon reporting $\tilde{t}_i$. Therefore inequalities 7 and 8 imply

$$
\begin{aligned}
u_i\big(\delta(\tilde{t}_i, t_{-i})\big) - u_i\big(\delta(t)\big) &= \left(v_i^1 + \cdots + v_i^n - n(b_{k-n+1} + \frac{\epsilon}{2n})\right) - 0 \\
&= (v_i^1 + \cdots + v_i^n - n b_{k-n+1}) - \frac{\epsilon}{2} \\
&> (v_i^1 + \cdots + v_i^n - n\hat{p}) - (v_i^1 + \cdots + v_i^m - m p^*) - \epsilon \\
&= u_i\big(\Upsilon(\hat{t}_i, t_{-i})\big) - u_i\big(\Upsilon(t)\big) - \epsilon
\end{aligned}
$$

showing that bidder i has an at least as profitable manipulation, subject to an upper bound of ϵ deviation, under the discriminatory auction for Case 2.

This covers all cases, so to complete the proof, we describe an example where some bidders can manipulate δ, but not Υ. Suppose that all bidders other than bidder 1 have the same value $\bar{v}$ for all of the units. Bidder 1's value for the first unit is strictly greater than $\bar{v}$, while her value for each of the remaining units is strictly less than $\bar{v}$. Under the uniform price auction, when bidders are truthful, every bidder wins one unit. Bidder 1 cannot manipulate to win more units because she would have to pay $\bar{v}$ for the additional units. She does not want to manipulate to win fewer units because she obtains strictly positive utility by reporting the truth and she cannot manipulate to change the price she pays. No other bidder would find it strictly profitable to manipulate because each would still have to pay at least $\bar{v}$ for that unit, and none can change the price paid. Hence, no bidder can manipulate the uniform-price auction. Under the discriminatory price auction, when each bidder reports truthfully, every bidder wins one unit. However, bidder 1 would prefer to under-report her valuation for the first unit to pay less for it. Hence, for this example, bidder 1 can manipulate δ, but not Υ. □

Proposition 6. *The Generalized First Price Auction is intensely and strongly more manipulable than the Generalized Second Price Auction.*

Proof. Given a type profile t, let $\text{GSP}(t)$ denote the outcome of GSP auction and $\text{GFP}(t)$ denote the outcome of GFP auction. Fix $\epsilon > 0$ and a bidder i. Let t_{-i} be the type profile of all other bidders. Recall that the type of each bidder is his valuation per click. Given t_{-i}, order the $|I|-1$ valuations of all bidders in $I \setminus \{i\}$ from highest to lowest. Let b_1 be the highest valuation, b_2 be the next highest valuation, and so on. That is, $b_1 \geq b_2 \geq \cdots \geq b_{|I|-1} > 0$.

Let $t_i = v_i$ be the type of bidder i. We will consider two cases with four sub-cases for the second case. For all cases except Case 2d, bidder i will not be able to manipulate the GSP auction. For Case 2d, he potentially can but whenever that happens he will have an at least as profitable deviation under the GFP auction.

Case 1: $v_i \leq b_k$.

In this case $u_i\big(\text{GSP}(t)\big) = 0$ either because bidder i does not receive a slot, or because she receives a slot at 0 utility.[21] Let $t'_i = v'_i$ be a potential manipulation. For this manipulation to be profitable, bidder i shall receive a slot. Let this slot be slot ℓ. Then

$$b_{\ell-1} \geq v'_i \geq b_\ell \geq b_k \geq v_i$$

and therefore,
$$u_i\big(\text{GSP}(t'_i, t_{-i})\big) = \alpha_\ell v_i - \alpha_\ell b_\ell = \alpha_\ell(v_i - b_\ell) \leq 0.$$

Hence bidder i does not have a profitable manipulation of GSP for Case 1.

Case 2: $v_i > b_k$.

Let bidder i receive slot m under GSP when he reveals his type truthfully. Then $b_{m-1} \geq v_i \geq b_m$ and
$$u_i\big(\text{GSP}(t)\big) = \alpha_m v_i - \alpha_m b_m \geq 0. \tag{9}$$

Let $t'_i = v'_i$ be a potential manipulation and suppose bidder i receives slot ℓ under $t'_i = v'_i$. This implies $v'_i \geq b_\ell$. We have four sub-cases to consider.

Case 2a: $v'_i > b_{m-1}$.

For this case, $\ell \leq m - 1$ and hence $b_\ell \geq b_{m-1} \geq v_i$. Therefore

$$u_i\big(\text{GSP}(t'_i, t_{-i})\big) = \alpha_\ell v_i - \alpha_\ell b_\ell = \alpha_\ell(v_i - b_\ell) \leq 0$$

and thus, bidder i does not have a profitable manipulation of GSP for Case 2a.

Case 2b: $v'_i = b_{m-1}$.

For this case there is a tie and bidder i either receives slot $m - 1$ at a cost of $\alpha_{m-1} b_{m-1}$ or slot m at a cost of $\alpha_m b_m$. If the former happens,

$$u_i\big(\text{GSP}(t'_i, t_{-i})\big) = \alpha_{m-1} v_i - \alpha_{m-1} b_{m-1} = \alpha_{m-1}(v_i - b_{m-1}) \leq 0.$$

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[21] The latter can happen only if $v_i = b_k$.

If the latter happens,

$$u_i\big(\textsc{Gsp}(t_i', t_{-i})\big) = \alpha_m v_i - \alpha_m b_m = u_i\big(\textsc{Gsp}(t)\big).$$

In either case, bidder i does not have a profitable manipulation of GSP.

Case 2c: Either $b_{m-1} > v_i' > b_m$ or $v_i' = b_m$ and bidder i receives slot m with tie-breaker. In this case bidder i receives slot m at a cost of $\alpha_m b_m$. Therefore,

$$u_i\big(\textsc{Gsp}(t_i', t_{-i})\big) = \alpha_m v_i - \alpha_m b_m = u_i\big(\textsc{Gsp}(t)\big),$$

and hence bidder i does not have a profitable manipulation of GSP.

Case 2d: $v_i' \leq b_m$ and bidder i receives a slot ℓ with $\ell > m$.
In this case

$$v_i \geq b_m \geq b_\ell \tag{10}$$

and

$$u_i\big(\textsc{Gsp}(t_i', t_{-i})\big) = \alpha_\ell v_i - \alpha_\ell b_\ell = \alpha_\ell(v_i - b_\ell) \geq 0. \tag{11}$$

Suppose $u_i\big(\textsc{Gsp}(t_i', t_{-i})\big) > u_i\big(\textsc{Gsp}(t)\big)$ so that bidder i can manipulate GSP at profile t. We will construct $\tilde{t}_i \in T_i$ such that,

$$u_i\big(\textsc{Gfp}(\tilde{t}_i, t_{-i})\big) - u_i\big(\textsc{Gfp}(t)\big) > u_i\big(\textsc{Gsp}(t_i', t_{-i})\big) - u_i\big(\textsc{Gsp}(t)\big) - \epsilon.$$

First observe that,

$$u_i\big(\textsc{Gfp}(t)\big) = 0. \tag{12}$$

Let $\tilde{t}_i = \tilde{v}_i = b_\ell + \frac{\epsilon}{2\alpha_\ell}$. Given t_{-i}, bidder i either wins slot ℓ at a cost of $\alpha_\ell\big(b_\ell + \frac{\epsilon}{2\alpha_\ell}\big)$ or a better slot n (with $\alpha_n > \alpha_\ell$) at a cost of $\alpha_n\big(b_\ell + \frac{\epsilon}{2\alpha_n}\big)$. If the former happens,

$$u_i\big(\textsc{Gfp}(\tilde{t}_i, t_{-i})\big) = \alpha_\ell v_i - \alpha_\ell\big(b_\ell + \frac{\epsilon}{2\alpha_\ell}\big) = \alpha_\ell(v_i - b_\ell) - \frac{\epsilon}{2}$$

and if the latter happens,

$$u_i\big(\textsc{Gfp}(\tilde{t}_i, t_{-i})\big) = \alpha_n v_i - \alpha_n\big(b_\ell + \frac{\epsilon}{2\alpha_\ell}\big) = \alpha_n(v_i - b_\ell) - \frac{\alpha_n}{\alpha_\ell}\frac{\epsilon}{2} > \alpha_\ell(v_i - b_\ell) - \frac{\epsilon}{2}$$

where the last inequality holds by inequality 10 and $\alpha_n > \alpha_\ell$. Therefore,

$$u_i\big(\textsc{Gfp}(\tilde{t}_i, t_{-i})\big) \geq \alpha_\ell(v_i - b_\ell) - \frac{\epsilon}{2}. \tag{13}$$

We are ready to finalize Case 2d. Relations 9, 11, 12, and 13 imply

$$\underbrace{u_i\big(\textsc{Gfp}(\tilde{t}_i, t_{-i})\big)}_{\geq \alpha_\ell(v_i - b_\ell) - \frac{\epsilon}{2}} - \underbrace{u_i\big(\textsc{Gfp}(t)\big)}_{=0} > \underbrace{u_i\big(\textsc{Gsp}(t_i', t_{-i})\big)}_{=\alpha_\ell(v_i - b_\ell)} - \underbrace{u_i\big(\textsc{Gsp}(t)\big)}_{=\alpha_m v_i - \alpha_m b_m \geq 0} - \epsilon$$

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showing that bidder i has an at least profitable manipulation, subject to an upper bound of ϵ deviation, under GFP auction for Case 2d.

This covers all cases, so to complete the proof, we describe an example where some bidder can manipulate the GFP, but no bidder can manipulate the GSP. Suppose that $v_1 > v_2 = \ldots = v_S = v_{S+1} > v_{S+2} > \ldots > v_N$. Under the GSP, when all bidders are truthful, the highest value bidder's payoff is $\alpha_1(v_1 - v_2) > 0$. She cannot change her payoff unless she reports a bid of v_2 or lower. If she reports her value to be v_2, she obtains a zero payoff. If she reports her value to be less than v_2, she does not win a slot and obtains a zero payoff. Hence, she cannot manipulate. Any bidder with value equal to v_2 who obtains a slot cannot manipulate. Reporting a value greater than v_1 will give the first slot, but this is not profitable. Reporting a value between v_1 and v_2 does not change her payoff. Reporting a value below v_2 prevents her from obtain a slot. Finally, no bidder with value less than v_2 can manipulate because the only way to change the outcome is to report a value greater than or equal to v_2, which is unprofitable. Hence, with this value distribution, no bidders can manipulate the GSP. In the GFP, if every bidder reports the truth, the outcome is the same as the GSP, but each bidder obtains a zero payoff. If bidder 1 reports a value less than v_1, but greater than v_2, she wins the first slot, but pays a lower price than had she reported the truth. This shows that bidder 1 can manipulate the GFP, but not GSP.

$\square$

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