Nonlinear Device Noise Models: Thermodynamic Requirements

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Abstract

This paper proposes three tests to determine whether a given nonlinear device noise model is in agreement with accepted thermodynamic principles. These tests are applied to several models. One conclusion is that every Gaussian noise model for any nonlinear device predicts thermodynamically impossible circuit behavior: these models should be abandoned. But the nonlinear shot-noise model predicts thermodynamically acceptable behavior under a constraint derived here. Further, this constraint specifies the current noise amplitude at each operating point from knowledge of the device $v - i$ curve alone. For the Gaussian and shot-noise models, this paper shows how the thermodynamic requirements can be reduced to concise mathematical tests involving no approximations.

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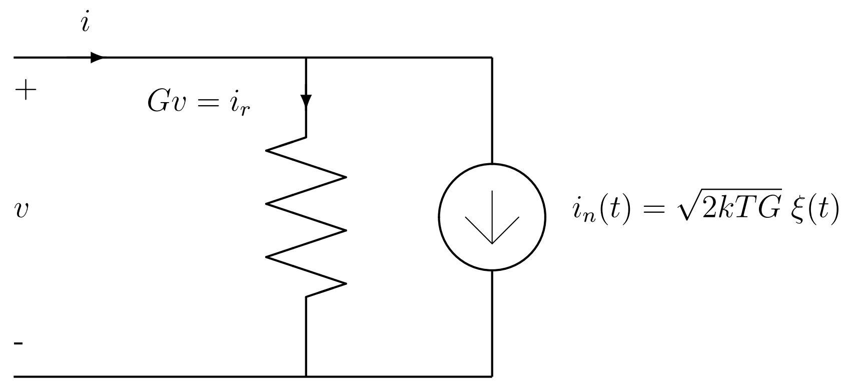

Figure 1: The Norton equivalent Nyquist-Johnson noise model for a linear conductor.

1 Introduction

Unlike idealized capacitors and inductors, dissipative devices such as resistors, diodes, and transistors degrade electrical energy to thermal energy. This pathway is bidirectional: they also convert thermal energy to electrical noise.

The Nyquist-Johnson thermal noise model asserts that the behavior of a linear conductor G at thermal equilibrium at a temperature T Kelvin is accurately modelled by the Norton representation in Fig. 1, where (ignoring the high-frequency roll-off in the infrared) the current noise source is zero-mean and white with power spectral density

$$S_{ii}(\omega) = 2kTG, \quad \forall \omega.^{\dagger} \tag{1}$$

Equation (1) involves only the conductance and the temperature: it is independent of the physical construction of the conductor [Johnson; Nyquist]. Nyquist's theoretical derivation was based on fundamental thermodynamic principles.

The aptly-named fluctuation-dissipation theorem [Pathria, Sect. 13.7; Callen and Welton] governs the conversion of thermal energy to noisy fluctuations in macroscopic variables. It generalizes Johnson's and Nyquist's resistor noise model to mechanical, chemical, hydraulic, and other domains. But the classical fluctuation-dissipation theorem is limited to linear devices. This paper will show how thermodynamics also constrains the behavior of nonlinear devices.

The physical idea in this paper is similar to that in [Nyquist], where resistors were connected to a transmission line. In Fig. 2, a nonlinear 2-terminal device at constant temperature is connected to an arbitrary lossless network, which contains, in general, nonlinear multiterminal inductors and capacitors plus ideal gyrators and transformers, as in [Tan and Wyatt], though most results can be obtained with a simple linear capacitor.

†The power spectral density expression is $4kTG$ when only positive frequencies are considered.

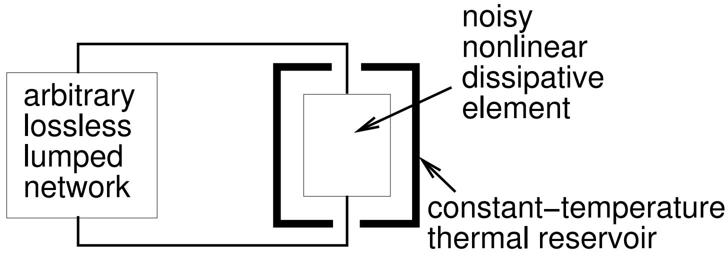

Figure 2: Test circuit with nonlinear device in thermal equilibrium with an isothermal reservoir, connected to a lossless network.

At thermal equilibrium, the voltage and current fluctuations are generally small and the nonlinear device behavior could be approximated by linearizing about the origin of the $v - i$ curve. But on rare occasions, the fluctuations will be large enough to briefly drive the device into the nonlinear regime. *Its behavior during such large equilibrium fluctuations is also constrained by thermodynamic principles.* This requirement serves as a pruning mechanism for rejecting many noise models *ab initio* and tentatively accepting others: models that predict non-thermodynamic behavior during large fluctuations (however rare) are *non-physical* and should be abandoned.

One universally accepted nonequilibrium requirement is nondecreasing entropy: the total entropy of the entire network in Fig. 2, including the thermal reservoir, cannot decrease as the system converges to equilibrium. This requirement governs circuits with arbitrary initial conditions and provides a second means of pruning out unacceptable models.

These equilibrium and nonequilibrium requirements greatly restrict and simplify the class of acceptable models. One contribution of this paper is to reduce these requirements to simple mathematical and circuit-theoretical tests for some noise models. In the literature on nonlinear noise modeling, approximations and limiting assumptions often introduce confusion over the domain where results apply. This paper avoids such approximations by treating the nonlinear problems *exactly*, using stochastic differential equation and master equation methods. But we restrict consideration to two-terminal, voltage-controlled resistive elements for simplicity.

Section 2 lists specific tests a model must pass. Sections 3 and 4 introduce the Gaussian noise model for linear and nonlinear elements. Section 5 develops the shot-noise model and can be read independently of Sects. 3 and 4. Appendices A and B contain certain mathematical derivations. An explanatory list of symbols appears as Appendix C.

2 Thermodynamic Requirements on Resistor Noise Models

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

One elementary consequence of the second law of thermodynamics is that no isothermal system can have as its sole effect the conversion of some amount of heat into work [Huang, p. 9]. A noisy dissipative device at a fixed temperature T, biased at a voltage V with the resulting average current $\overline{i_T(V)}$, must not supply power, on average, to the external circuit. Thus the I-V curve must lie in the first and third quadrants, *i.e.*,

$$\overline{i_T(V)}\, V \geq 0, \quad \forall T > 0, \forall V.$$

Since the average current is assumed to be a continuous function of the applied voltage, this also implies that the average short-circuit current for a dissipative device must be zero.‡

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

For any lossless lumped network in thermal equilibrium with a dissipative device at constant temperature, the equilibrium distribution for inductor fluxes $\boldsymbol{\phi}$ and capacitor charges $\mathbf{q}$ must have the Gibbs (or Maxwell-Boltzmann) form [Landau and Lifshitz, Sect. 28; Pathria, p. 66; Huang, p. 144],

$$\rho(\boldsymbol{\phi}, \mathbf{q})^o = A \exp\left[-E(\boldsymbol{\phi}, \mathbf{q})/kT\right], \tag{2}$$

where $E(\boldsymbol{\phi}, \mathbf{q})$ is the sum of all inductor and capacitor stored energies and A serves to normalize the distribution.

Thermodynamic Requirement #3: Increasing Entropy During Transients

The second law of thermodynamics must be satisfied during nonequilibrium transient behavior of any circuit driven by the fluctuations of the dissipative device. The total entropy of a circuit, *i.e.*, the sum of the entropies of the lossless elements and the reservoir, must be a nondecreasing function of time, with a maximum value corresponding to the equilibrium distribution [Huang, p. 17].

These requirements are all consequences of the second law of thermodynamics. The first requirement under short-circuit conditions and the second requirement in general govern equilibrium behavior. The first requirement with nonzero d.c. voltage limits nonequilibrium steady-state behavior. The third governs transient nonequilibrium operation.

‡Since $\overline{i_T(V)} \geq 0$ for all $V > 0$ and $\overline{i_T(V)} \leq 0$ for all $V < 0$, the average current cannot be strictly positive or negative for $V = 0$ by continuity.

3 Linear Gaussian Model

The Extended Nyquist-Johnson Model

This section considers an extended version of the Nyquist-Johnson model in which the noise source $\xi(t)$ is Gaussian and the circuit model in Fig. 1 holds for all equilibrium and nonequilibrium voltages. More specifically, we assume that $\xi(t)$ is unit-amplitude, stationary, zero-mean Gaussian white noise [Wong and Hajek, Sect. 4.5] and

$$i_n(t) = \sqrt{2kTG}\,\xi(t), \tag{3}$$

for *all* time-varying voltages v. This extends the model far beyond the thermodynamic equilibrium regime for which it was originally proposed [Nyquist; Johnson].

Compliance of the extended Nyquist-Johnson linear Gaussian model with the thermodynamic requirements was exhaustively addressed in [Tan and Wyatt], which describes the behavior of general *nonlinear* LC circuits driven by this model. However, as an introduction to stochastic differential equation methods and the Fokker-Planck equation used later, the tests are applied here to simple first-order RC networks.

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

For the linear Gaussian model, the average noise current is zero and is independent of the applied voltage. Thus the average electric power dissipated in the element is always nonnegative for $G \geq 0$, and of course the short-circuit average current is automatically zero. This requirement is met.

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

A (possibly nonlinear) capacitor with charge q on the upper plate and constitutive relation

$$v = f(q)$$

is attached to the left side of the noise model in Fig. 1. The differential equation for the resulting circuit,

$$\dot{q} = -G\, f(q) + \sqrt{2kTG}\,\xi(t), \tag{4}$$

is of the Langevin form [Schuss, Sect. 2.1; van Kampen, Chap. 9]. The link between stochastic differential equations of this sort and thermodynamic variables is provided by the Fokker-Planck equation (FPE, also known as the forward Kolmogorov equation)[§] a differential equa-

[§]A brief introduction is found in [Papoulis, pp. 650-54]; more mathematical rigor is found in [Wong and Hajek, p. 172]; more physical intuition is found in [van Kampen, Chap. 8]; and the authors found [Schuss, Sect. 5.2] to be generally helpful.

tion for the probability density $\rho(q,t)$ of solutions to stochastic differential equations. For the capacitor charge random process $q(t)$ in (4), the FPE takes the form

$$\dot{\rho} = \frac{\partial}{\partial q}\left[G f(q)\,\rho + kTG\,\frac{\partial \rho}{\partial q}\right] = -\frac{\partial}{\partial q}\left[J(q)\right], \tag{5}$$

where $J(q)$ is the "probability flux" [van Kampen, p. 193]. Using the stored capacitor energy $E_C(q) = \int_0^q f(q')\,dq'$, the Gibbs distribution (2) can be immediately written:

$$\rho^o(q) = A\exp\left[-E_C(q)/kT\right]. \tag{6}$$

A simple differentiation shows that this density does in fact satisfy the equilibrium condition $\dot{\rho}^o = 0$ in (5). Thus the second requirement is also met.

Note that furthermore J itself vanishes at ρ^o. (It need only be constant for an equilibrium density of (5)). Thus the equilibrium is "detail balanced" in the language of statistical physics [van Kampen, Sect. 5.6] or, equivalently, "reversible" in the language of random processes. Detailed balance is an additional physical requirement for reciprocal RC circuits that does *not* hold for general RLC circuits [van Kampen, Sect. 5.6; Anderson].

Thermodynamic Requirement #3: Increasing Entropy During Transients

The entropy S_C of the capacitor charge distribution is given by the traditional formula [Landau and Lifshitz, p. 26; Stratonovich, p. 30]:

$$S_C \stackrel{\triangle}{=} -k\int \rho\ln\rho\,dq, \tag{7}$$

where k is Boltzmann's constant. (Some authors differ by an additive or multiplicative constant of no interest here.) The capacitor entropy rate is then

$$\begin{aligned}\dot{S}_C &= \frac{d}{dt}\left(-k\int \rho\ln\rho\,dq\right) = -k\int_{-\infty}^{+\infty}\frac{d}{dt}(\rho\ln\rho)\,dq \\ &= -k\int_{-\infty}^{+\infty}\left(\dot{\rho}\,\ln\rho + \rho\,\frac{1}{\rho}\,\dot{\rho}\right)dq = -k\int_{-\infty}^{+\infty}\dot{\rho}\,\ln\rho\,dq - k\int_{-\infty}^{+\infty}\dot{\rho}\,dq.\end{aligned} \tag{8}$$

The second term must integrate to zero, since the total probability must remain equal to one. Before attempting to compute the first integral, we also seek an expression for the rate of change of thermal reservoir entropy. The thermodynamic identity $d\overline{E} = T\,dS$ and its time-dependent form $\frac{d\overline{E}}{dt} = T\frac{dS}{dt}$ relate the heat flow into the reservoir to its entropy. By conservation of energy, this heat flow is equal to the energy flow out of the capacitor. Thus we obtain for the time derivative of the reservoir entropy

$$\dot{S}_R = \frac{1}{T}\frac{d}{dt}\left(-\overline{E}_C\right) = -\frac{1}{T}\frac{d}{dt}\int_{-\infty}^{+\infty}E_C(q)\,\rho\,dq = -\frac{1}{T}\int_{-\infty}^{+\infty}E_C(q)\,\dot{\rho}\,dq. \tag{9}$$

Combining the two entropy rate terms yields the total entropy rate $\dot{S}_{tot}$:

$$\dot{S}_{tot} = \dot{S}_C + \dot{S}_R = -k \int_{-\infty}^{+\infty} \dot{\rho} \ln \rho \, dq - \frac{1}{T} \int_{-\infty}^{+\infty} E_C(q) \, \dot{\rho} \, dq, \tag{10}$$

which, using (5), becomes

$$\dot{S}_{tot} = \int_{-\infty}^{+\infty} \left[-k \ln \rho - \frac{1}{T} E_C(q) \right] \frac{\partial}{\partial q} \left(G \, f \rho + kTG \frac{\partial \rho}{\partial q} \right) dq. \tag{11}$$

Integrating by parts, noting that ρ and its derivative fall off to zero very quickly at infinity so that the product term vanishes there, and recalling that $dE_C/dq = f$, we have

$$\begin{aligned} \dot{S}_{tot} &= \int_{-\infty}^{+\infty} \left[k \frac{1}{\rho} \frac{\partial \rho}{\partial q} + \frac{1}{T} f \right] \left(G \, f \rho + kTG \, \frac{\partial \rho}{\partial q} \right) dq \\ &= \int_{-\infty}^{+\infty} \frac{1}{\rho \, GT} \left(G \, f \rho + kTG \, \frac{\partial \rho}{\partial q} \right)^2 dq \ \geq 0. \end{aligned} \tag{12}$$

As we hoped, the entropy rate will always be non-negative, since it is the integral of a squared quantity.

Thus the Gaussian Nyquist-Johnson noise model for a linear resistor satisfies the equilibrium thermodynamic requirements, and the extended Nyquist-Johnson model satisfies the nonequilibrium requirements.

4 Nonlinear Gaussian Models

The total current through any nonlinear resistor at any fixed voltage V and temperature T can be written as the sum of an average current $g_T(V)$ and a zero-mean noise current with some power spectral density $S_{ii}(\omega)$. In many models, *e.g.*, [Gupta; van der Ziel], the noise is white, and thus the current can be written in the form

$$i(t) = g_T(V) + h_T(V) \, \xi(t), \tag{13}$$

where $\xi(t)$ is unit-amplitude, stationary, zero-mean white noise. It follows from (13) that at each fixed V and T,

$$\begin{aligned} \bar{i} &= g_T(V) \\ S_{ii}(\omega) &= h_T^2(V), \quad \forall \omega. \end{aligned}$$

This section considers the analytically simplest special class of such models where $\xi(t)$ is Gaussian and (13) holds for time-varying voltages at each instant, *i.e.*,

$$i = g_T(v(t)) + h_T(v(t)) \, \xi(t).$$

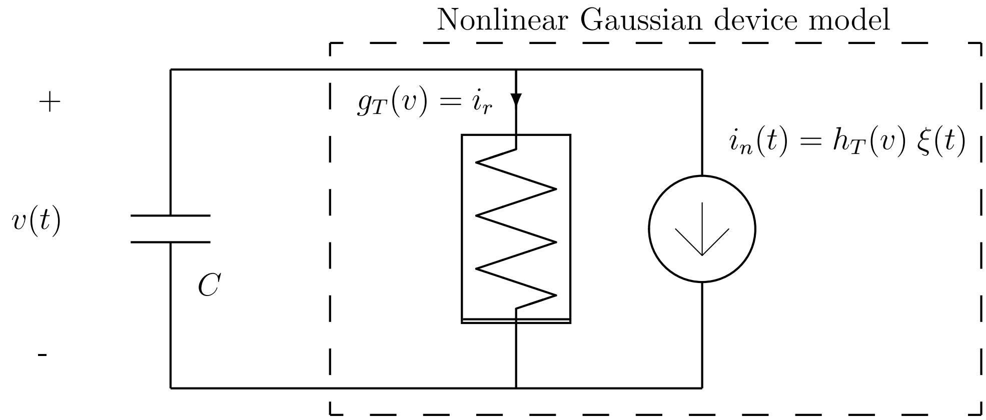

Figure 3: Nonlinear Gaussian device model connected to a linear capacitor. The noise current amplitude varies instantaneously with the applied voltage in this model.

(See Fig. 3.) These models are a natural extension of the linear Gaussian model in Sect. 3.

We will show a somewhat surprising result: no nonlinear device can be described by a model in this class that meets the equilibrium thermodynamic requirements (Requirement #2), regardless of the choice of $h_T(v)$. We only need a linear capacitor to illustrate the problem. This lets us focus on the voltage rather than the charge, since $\rho_v(v,t)dv = \rho_q(q,t)dq$, *i.e.*,

$$\rho_v(v,t) = \rho_q(q,t)\frac{dq}{dv} = C\rho_q(Cv,t) \tag{14}$$

where ρ_q is the probability density for charge and ρ_v is the probability density for voltage.

The stochastic differential equation (15), a nonlinear variant of the Langevin equation, describes the dynamics of the capacitor voltage in Fig. 3:

$$\dot{v}(t) = -\frac{g_T(v)}{C} - \frac{h_T(v)}{C}\xi(t). \tag{15}$$

Certain technical problems arise with this equation because white noise of unlimited bandwidth is a mathematical fiction. These problems become especially severe in (15) because $h_T(v)$ can vary with v, in contrast to the usual Langevin equation. The literature focuses on two interpretations for the integral of (15), the Itô and the Stratonovich integrals [van Kampen, Sect. 9.5; Wong and Zakai]. (See Appendix A.) The interpretations lead to different densities ρ^I and ρ^S for the capacitor voltage.

The Fokker-Planck equation (FPE) for the Itô interpretation of (15) is:

$$\frac{\partial \rho^I}{\partial t} = \frac{\partial}{\partial v}\left\{\frac{g_T(v)}{C}\rho^I(v,t) + \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{h_T^2(v)}{C^2}\rho^I(v,t)\right]\right\} = -\frac{\partial}{\partial v}\left[J^I(v)\right], \tag{16}$$

where $J^I(v)$ is the probability flux, as in (5). The Stratonovich FPE for ρ^S contains one additional term:

$$\frac{\partial \rho^S}{\partial t} = \frac{\partial}{\partial v}\left\{\frac{g_T(v)}{C}\rho^S(v,t) - \frac{h_T(v)p^S(v,t)}{2\,C^2}\frac{\partial}{\partial v}h_T(v) + \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{h_T^2(v)}{C^2}\rho^S(v,t)\right]\right\} = -\frac{\partial}{\partial v}\left[J^S(v)\right]. \tag{17}$$

Whichever interpretation is used, the equilibrium solution for charge must fit the Gibbs form (2). Equivalently, using (14), we require

$$\rho_v^o(v) = \frac{\exp\left(-Cv^2/2kT\right)}{\sqrt{2\pi kT/C}}, \tag{18}$$

which happens to be Gaussian only because the capacitor is linear with energy $E = \frac{1}{2}Cv^2$. Noting that

$$\frac{\partial \rho_v^o(v)}{\partial t} = 0$$

and recalling from Sect. 3 that $J^I(v)$ and $J^S(v)$ must vanish identically for RC circuits, we substitute $\rho_v^o(v)$ from (18) into (16) and arrive at the differential equation

$$\frac{\partial h_T^2(v)}{\partial v} = C\left[\frac{v}{kT}\,h_T^2(v) - 2\,g_T(v)\right], \tag{19}$$

or into (17) to arrive at

$$\frac{\partial h_T^2(v)}{\partial v} = 2C\left[\frac{v}{kT}\,h_T^2(v) - 2\,g_T(v)\right]. \tag{20}$$

Since $h_T^2(v)$ is a characteristic of the device model, it cannot depend on the value of C. The only solutions of (19) and (20) that do not vary with C are those for which the term in brackets vanishes, *i.e.*,

$$h_T^2(v) = 2kT\,\frac{g_T(v)}{v}.$$

On the left side, this implies that

$$\frac{\partial h_T^2(v)}{\partial v} \equiv 0. \tag{21}$$

Together, these last two equations imply that $g_T(v)/v$ is constant, *i.e.*,

$$\frac{g_T(v)}{v} = G, \quad \forall v. \tag{22}$$

Thus, we have concluded that for both the Itô and Stratonovich interpretations for (15), in order to have the correct equilibrium distribution, the resistor *must* be a linear resistor with

$$i = Gv,$$

and the resulting noise amplitude,

$$h_T^2(v) = 2kTG,$$

is precisely that from the the traditional Nyquist-Johnson model for the linear case.

This calculation has shown that no resistor with a nonlinear constitutive relation $\bar{i} = g_T(v)$ has a Gaussian white noise-current model of the form shown in Fig. 3, even within the special domain of thermal equilibrium. This calculation also gives an independent derivation of the Gaussian Nyquist-Johnson model for a linear resistor at thermal equilibrium.

Nyquist's derivation used two resistors connected to a transmission line (a distributed LC circuit) and required the equipartition theorem to be satisfied by the energy in the modes of the transmission line. Our derivation uses a simpler circuit, consisting of only one resistor and one capacitor. However, the Gibbs distribution is a more stringent requirement than the equipartition theorem, since other non-thermodynamic distributions satisfy the equipartition theorem.

5 Shot-Noise Models

5.1 Poisson Models for Shot Noise

The shot-noise model for a current of electrons or holes describes the arrival of each charged particle as a Dirac delta function of current

$$\pm e\, \delta(t - t_n),$$

where t_n is the n-th arrival time, $e > 0$ is the magnitude of the electron charge, and the sign is chosen positive for a hole and negative for an electron. The arrival times are randomly distributed. If we further require that the distribution of the arrival times be *memoryless*, that is,

$$Pr(t_n - t_{n-1} > t + h \,|\, t_n - t_{n-1} > t) = Pr(t_n - t_{n-1} > h),$$

we obtain the *Poisson point process* (PPP), which is a Markov process [Gallager, Chap. 2]. A *homogeneous* Poisson point process is stationary, and the *average arrival rate* λ is a constant. In a shot-noise model, this would mean that the expected number of arrivals in any time interval of length T is λT, and the average current is $\pm e\lambda$.

However, λ need not be constant, in which case we obtain an *inhomogeneous* Poisson point process, which is not stationary. The expected number of arrivals in any interval $[t, t + T]$ is

$$\int_t^{t+T} \lambda(\tau)\, d\tau.$$

If we connect our shot-noise source to a capacitor, the charge on the capacitor will be given by the familiar Poisson counting process (PCP), the integral of the PPP with respect to time, as seen in Fig. 4.

For this paper, it will be useful to note that one can reparameterize the time axis such that an inhomogeneous PCP can be expressed as a homogeneous PCP on a non-uniform time axis. Let $N(t)$ be a PCP with rate 1. Then to generate an inhomogeneous PCP N_{inhom} with the rate $\lambda(t)$, let

$$N_{inhom}(t) = N\left(\int_0^t \lambda(\tau) d\tau\right). \tag{23}$$

The random process N_{inhom} is still Markovian, with independent increments. [Pfeiffer; Gallager, p. 44]

5.2 Poisson Device Models

A two-terminal Poisson device model (*i.e.*, a shot-noise model) consists simply of two independent forward and reverse current random processes. (See Fig. 5.)

Each current is a Poisson counting process with an average rate that is a function of the instantaneous applied voltage v and the temperature T, *i.e.*,

$$i(t) = \frac{d}{dt}\left\{eN_f\left(\int_0^t f_T(v(\tau))d\tau\right) - eN_r\left(\int_0^t r_T(v(\tau))d\tau\right)\right\}, \tag{24}$$

where N_f and N_r are the independent homogeneous forward and reverse counting processes, and $f_T(v)$ and $r_T(v)$ the forward and reverse rates, respectively. The rates must be positive for all applied voltages at all temperatures:

$$\begin{aligned} f_T(v) &> 0, \qquad \forall v,\ T \\ r_T(v) &> 0, \qquad \forall v,\ T. \end{aligned} \tag{25}$$

Note that the Poisson device model incorporates both the deterministic constitutive relation for the device as well as the stochastic noise behavior.

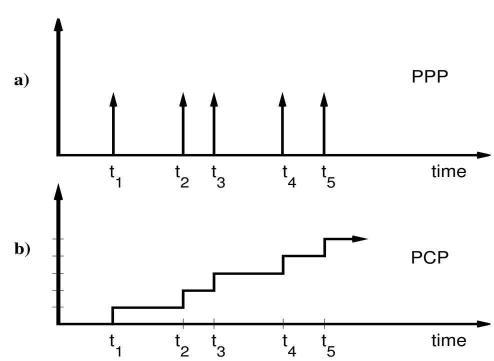

Figure 4: A sample realization of the Poisson point process (a) and the corresponding counting process (b).

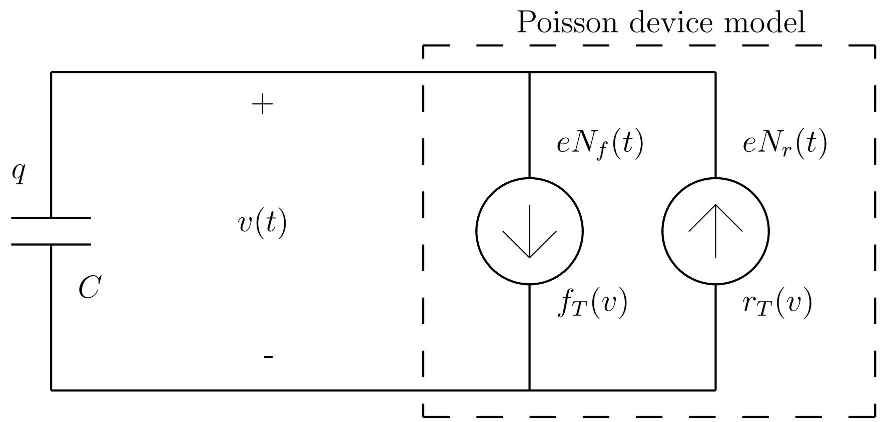

Figure 5: Poisson device model connected to a capacitor. The forward current source, $eN_f(t)$, has a voltage-dependent average arrival rate $f_T(v)$; similarly for the reverse current.

The average current is

$$\overline{i(t)} = e\left[f_T(v(t)) - r_T(v(t))\right],$$

and the constitutive relation for the device (*i.e.*, the $v - i$ curve) is

$$\overline{i(v)} = e\left[f_T(v) - r_T(v)\right]. \tag{26}$$

Under d.c. bias conditions with constant V, the current random process $i(t)$ becomes stationary and hence has a power spectral density. The spectrum is white, apart from the d.c component [Papoulis, p. 321], with magnitude

$$S_{ii}(\omega) = e^2\left[f_T(V) + r_T(V)\right], \quad \forall\omega \neq 0. \tag{27}$$

The analytical simplicity of this model comes from the three *very strong* assumptions that 1) the electron arrival is instantaneous and can therefore be modeled as a δ-function, 2) the two random processes are mutually independent and memoryless, and 3) the expected arrival rate changes instantaneously with v.

For some devices, this model is reasonably accurate over a wide enough range of d.c. bias voltages to include substantially nonlinear portions of the $v - i$ curve. The *pn* junction and the MOSFET in the subthreshold regime are two interesting examples. One would expect this model also applies to other devices under nonequilibrium bias conditions, provided a) the lattice remains at a uniform constant temperature during such operation, and b) the carrier population remains locally in thermal equilibrium with the lattice, *i.e.*, retains approximately the Gibbs distribution at a constant temperature T, throughout the device during such operation.

Since the noise statistics are determined by the *sum* of the average currents (27) while the constitutive relation is determined by the *difference* (26), the development so far does not imply any unique relation between the constitutive relation and the noise. We will show

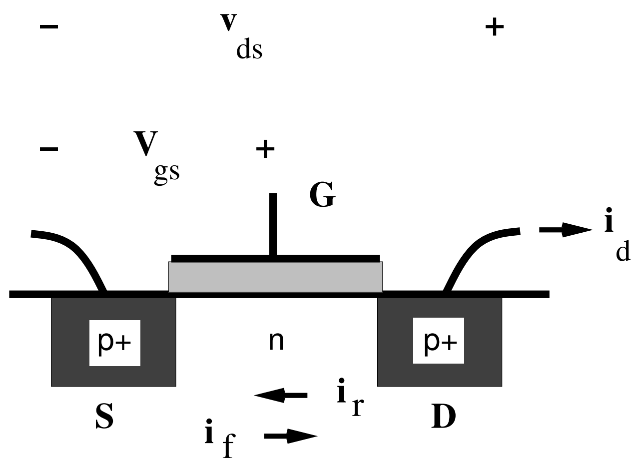

Figure 6: MOSFET cross-section

that with the thermodynamic requirements, the constitutive relation and the temperature *uniquely specify* the current noise at each operating voltage V.

5.2.1 Subthreshold MOSFET

The subthreshold p-channel MOSFET with fixed gate-to-source voltage V_{gs} is a two-terminal device that is well-described by a Poisson model. The derivation of this model and a comparison with experimental results is given in [Sarpeshkar, *et al.*]. There are only two currents, i_f and i_r, and both are hole diffusion currents in the n-region shown in Fig. 6. The separation of the total currents into forward and reverse currents in this model is done as follows: given the hole concentration at both ends, the current from each end is calculated as the diffusion that would occur if the concentration at the far end were zero. In this model,

$$\begin{aligned} \overline{i_f} &= ef_T(v) = I_{sat}(V_{gs}) \\ \overline{i_r} &= er_T(v) = I_{sat}(V_{gs}) \exp(-ev_{ds}/kT), \end{aligned}$$

so that

$$\overline{i_d} = \overline{i_f} - \overline{i_r} = I_{sat}(V_{gs})[1 - \exp(-ev_{ds}/kT)],$$

and the shot-noise amplitude is given by the sum

$$S_{ii}(\omega) = e(\overline{i_f} + \overline{i_r}), \quad \forall \omega \neq 0.$$

5.2.2 PN Junction

To develop a shot-noise model for the pn junction in Fig. 7, we need expressions for the forward and reverse currents. The dominant currents are the electron and hole diffusion

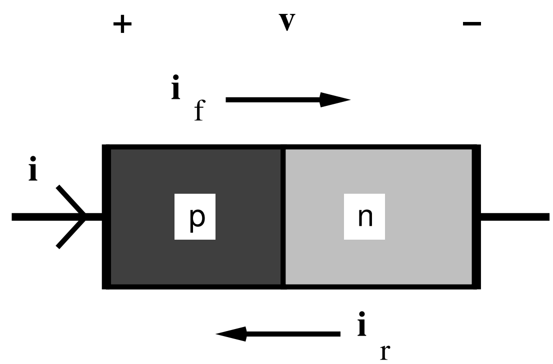

Figure 7: A pn junction

currents.

Diffusion currents result from the differences in carrier concentrations on opposite sides of the junction. At the edge of the space charge region on the p side, the electron concentration is $n_{po}\exp(eV/kT)$, but deep in the bulk p region, the electron concentration is n_{po}. The electron diffusion current, therefore, is proportional to

$$n_{po}\left[\exp(eV/kT) - 1\right].$$

The hole diffusion concentration is proportional to a similar factor $p_{no}\left[\exp(eV/kT) - 1\right]$. Although electrons and holes diffuse in opposite directions, the currents are in the same direction, yielding a net average current

$$\bar{i} = I_S\left[\exp(ev/kT) - 1\right],$$

where I_S, called the saturation current, incorporates all the constants, such as the bulk carrier concentrations and diffusion coefficients.

Dividing the current into forward and reverse currents in this model is not as clearly justified as it was in the MOSFET case. Nevertheless, following the philosophy of the alternate derivation of noise for the linear resistor in [Sarpeshkar, *et al.*], we take the concentration near the electrode, in this case the electron concentration deep in the bulk p region, to determine the concentration for the reverse current of electrons. Correspondingly, we get a reverse current of holes from their concentration deep in the n region. This results in a total reverse current (of holes and electrons)

$$\overline{i_r} = I_S = er_T(v)$$

and a forward current

$$\overline{i_f} = I_S\exp(ev/kT) = ef_T(v).$$

Shot noise is generated by both currents, and for fixed V, the power spectral density is

$$S_{ii}(\omega) = e(\overline{i_f} + \overline{i_r}), \quad \forall\omega \neq 0.$$

More physical detail can be found in most semiconductor device textbooks. For more details on the noise model, the reader is referred to [van der Ziel].

5.3 Thermodynamic Tests on Poisson Models

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

The requirement is that

$$Ve\left[f_T(V) - r_T(V)\right] \geq 0, \quad \forall T > 0, \forall V. \tag{28}$$

It is satisfied for both the subthreshold MOSFET and the *pn* junction shot-noise models.

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

For this second test, we consider our noisy device in a circuit with a single linear capacitor, as in Fig. 5. The equilibrium distribution of charge on this capacitor must have the Gibbs form.

Integrating the circuit differential equation $\frac{dq}{dt} = -i$ and using the device current from equation (24), we find

$$q(t) = -e\left\{N_f\left(\int_0^t f_T(q(\tau)/C)d\tau\right) - N_r\left(\int_0^t r_T(q(\tau)/C)d\tau\right)\right\}, \tag{29}$$

and we choose the initial condition $q(0) = 0$. (For mnemonics, recall that in defining the device model, f_T was used for "forward" current and r_T for "reverse" current, with respect to the sign conventions for the device. But in the circuit, it is better to think of f_T as standing for "falling" charge and r_T for "rising" charge on the capacitor.) Note that the rates $f_T(q(t)/C)$ and $r_T(q(t)/C)$ are discontinuous functions of time, since the capacitor can only have integer numbers of electrons on its plates. This raises a question about interpreting the transition rates correctly. Should we use the charge value before the jump, the value afterwards, or the average?

It turns out that using the charge values before the jump mishandles the discontinuities in $f_T(v(t))$ and $r_T(v(t))$: in the subthreshold MOSFET and *pn* junction examples, it results in an equilibrium charge distribution that is not Gibbsian and has a mean value of $-\frac{1}{2}e$, contrary to the requirement. More fundamentally, if one requires a Gibbsian equilibrium distribution, the resulting constraint on f_T and r_T (a variant on (35)) depends on the value of the capacitor to which the device happens to be connected, contrary to the concept of a device model that is valid in a variety of circuits.

For this reason we let the transition rate be governed by the average of the capacitor

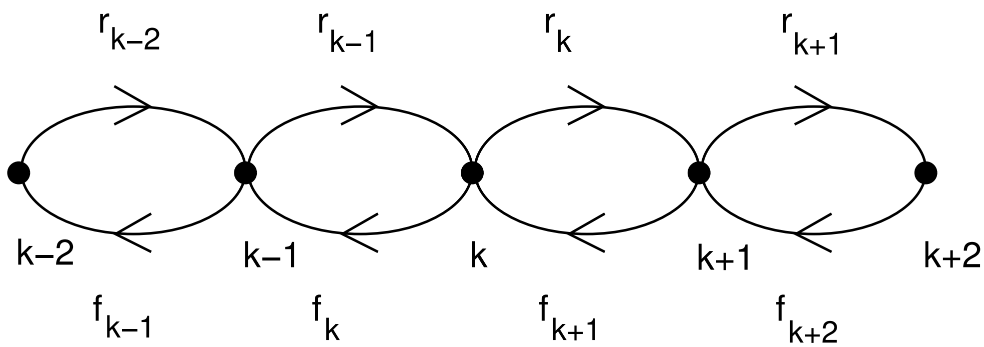

Figure 8: Section of the Infinite Markov Chain (30). Node k represents the state with charge $+ke$ on the upper capacitor plate.

voltages before and after the jump. Using simplified notation for the transition rates

$$\begin{aligned} r_n &\stackrel{\triangle}{=} r_T\left(\frac{(n+1/2)e}{C}\right) \\ f_n &\stackrel{\triangle}{=} f_T\left(\frac{(n-1/2)e}{C}\right), \end{aligned}$$

and for the conditional probabilities

$$p(n,t\,|\,m,s) \stackrel{\triangle}{=} Pr\{q(t)=ne\,|\,q(s)=me\},$$

one arrives at the forward evolution equation for the probability distribution (*i.e.*, the master equation [van Kampen, Sect. 5.1])

$$\frac{d}{dt}p(n,t) \;=\; r_{n-1}\,p(n-1,t)+f_{n+1}\,p(n+1,t)-[r_n+f_n]\,p(n,t), \quad \forall n. \tag{30}$$

For more detail, see [Wyatt and Coram, Sect. 3]. These transition probabilities describe the infinite Markov chain in Fig. 8.

The equilibrium distribution p_n^o satisfies (30) with the left hand side set to zero. Again requiring detailed balance, the total flow between adjacent nodes must vanish, *i.e.*,

$$r_n\,p_n^o = f_{n+1}\,p_{n+1}^o, \quad \forall n,$$

or

$$\frac{p_{n+1}^o}{p_n^o} = \frac{r_n}{f_{n+1}}, \quad \forall n. \tag{31}$$

The equilibrium solution can quickly be found in closed form (except, perhaps, for normalization):

$$\frac{p_n^o}{p_0^o} = \begin{cases} \displaystyle\prod_{j=0}^{n-1} \frac{r_j}{f_{j+1}}, & n > 0 \\ \displaystyle\prod_{j=0}^{n+1} \frac{f_j}{r_{j-1}}, & n < 0. \end{cases} \tag{32}$$

We may now test this distribution for consistency with the Gibbs form. Gibbs statistics for our circuit require that the ratio of probabilities of neighboring states satisfy

$$\begin{aligned} \frac{p_{n+1}^o}{p_n^o} &= \frac{\exp\left[-\left(\frac{(n+1)^2e^2}{2CkT}\right)\right]}{\exp\left[-\left(\frac{n^2e^2}{2CkT}\right)\right]} &=& \exp\left[-\left(\frac{(2n+1)e^2}{2CkT}\right)\right] \\ &= \exp\left[-\left(\frac{(2n+1)e}{2Cv_T}\right)\right] &=& \exp\left[-\left(\frac{ne}{Cv_T}\right)\right]\exp\left[-\left(\frac{e}{2Cv_T}\right)\right]. \end{aligned} \tag{33}$$

The Markov chain for the circuit also gives an expression (31) for the ratios of the neighboring equilibrium probabilities:

$$\frac{p_{n+1}^o}{p_n^o} = \frac{r_n}{f_{n+1}} = \frac{r_T((n+1/2)e/C)}{f_T((n+1/2)e/C)} = \frac{r_T(v_n + e/2C)}{f_T(v_n + e/2C)}, \tag{34}$$

where v_n is the voltage on the capacitor when the upper plate stores n positive charges. Equation (34) agrees with the thermodynamic requirement (33) for all capacitors if and only if

$$\frac{r_T(v)}{f_T(v)} = \exp\left(-v/v_T\right), \quad \forall\, v. \tag{35}$$

The probability ratio (34) from the Markov chain becomes

$$\begin{aligned} \frac{p_{n+1}^o}{p_n^o} &= \frac{r_T(v_n + e/2C)}{f_T(v_n + e/2C)} = \exp\left(-\frac{v_n + e/2C}{v_T}\right) \\ &= \exp\left(-ne/Cv_T\right)\exp\left(-e/2Cv_T\right), \end{aligned}$$

which agrees precisely with (33).

Thus the constraint (35) is both necessary and sufficient to guarantee that every shot-noise model leads to a Gibbsian equilibrium distribution of charge on a linear capacitor, as required by thermodynamics. We have also shown that this conclusion continues to hold even when the capacitor is nonlinear [Wyatt and Coram, Appendix I]. Both the *pn* junction and the subthreshold MOSFET shot noise models satisfy (35).

Furthermore, given the positivity restrictions (25), the constraint (35) guarantees that Thermodynamic Requirement #1 is satisfied as well.

Thermodynamic Requirement #3: Increasing Entropy During Transients

We show in Appendix B that the total entropy of the circuit in Fig. 5 increases monotonically with time, given an arbitrary initial probability distribution, provided the constraint (35) is met.

In summary, the shot-noise model satisfies all the thermodynamic requirements presented here if and only if the forward and reverse rates are related by (35), which applies to both time-varying and d.c. voltages. After developing this model, we discovered a distinct but related treatment in [Stratonovich]. His derivation is based on a complicated "kinetic potential" argument. It uses an approximation [Stratonovich, eq. (3.3.43)] not used or needed here and handles the discontinuities in $v(t)$ differently. In addition, we have explicitly verified that the Poisson model satisfies the increasing entropy requirement.

For d.c. voltages V, the constraint (35) leads to a prediction of a unique current noise amplitude at each operating point. If we define

$$\bar{\imath} = g(V) = e\,[f_T(V) - r_T(V)] = e\,[\exp\left(V/v_T\right) - 1]\,r_T(V),$$

then for all $\omega \neq 0$,

$$\begin{aligned} S_{ii}(\omega) &= e^2\,[f_T(V) + r_T(V)] = e^2\,[\exp\left(V/v_T\right) + 1]\,r_T(V) \\ &= \frac{e\,[\exp\left(V/v_T\right) + 1]}{[\exp\left(V/v_T\right) - 1]}\,g(V) = \frac{e\,g(V)}{\tanh\left(V/2v_T\right)}, \end{aligned} \tag{36}$$

at each d.c. voltage V.

6 Comparison Between Shot-Noise and Extended Nyquist-Johnson Models

The two thermodynamically acceptable models, Nyquist-Johnson and shot, are fundamentally distinct since the former is Gaussian and the latter is not. But their power spectra are both white and can be compared. For a device with average current given by $g_T(V)$ at a fixed operating voltage V and temperature T, we compare the Poisson model power spectral density (36) with the value S_{ii}^{NJ} predicted by the Nyquist-Johnson model for the linearized conductance $g'_T(V)$,

$$S_{ii}^{NJ} = 2kTg'_T(V). \tag{37}$$

It is reassuring to note that the Poisson (36) and Nyquist-Johnson (37) power spectral densities agree in the short-circuit case. This can be seen by expanding (36) about $V = 0$ using l'Hôpital's rule. But they do not agree elsewhere in general. We note that there is no reason to believe (37) gives a correct prediction for any device with $V \neq 0$, despite its occasional use in the literature.

To push the comparison further, we apply both models to a linear conductor G. The Poisson model (36) reduces to

$$S_{ii}^P = \frac{e\,G\,V}{\tanh(V/2v_T)}, \tag{38}$$

while the Nyquist-Johnson model, of course, gives

$$S_{ii}^{NJ} = 2kTG. \tag{39}$$

It is interesting that two noise models with *different* power spectral densities are *both* thermodynamically acceptable. The Poisson model predicts a larger current noise than the Nyquist-Johnson model at each nonzero bias point, since

$$\frac{S_{ii}^P}{S_{ii}^{NJ}} = \frac{V/2v_T}{\tanh(V/2v_T)} > 1, \quad \forall V \neq 0.$$

The shot-noise model is “noisier” than the extended Nyquist-Johnson model for $V \neq 0$. This is a direct result of the finite size of the electron charge. To see this, consider a hypothetical family of linear conductors, all having the same conductance G and temperature T, but in which the charge quantum e comes in various sizes. (These are rare or nonexistent in electronics, but the Ca^{++} channel in nerve membrane is one example of a non-unity charge quantum.) The limiting behavior is

$$S_{ii}^P \to \lim_{e \to 0} \frac{eGV}{\tanh(\frac{eV}{2kT})} = 2kTG = S_{ii}^{NJ}, \tag{40}$$

i.e., the shot noise magnitude converges to the extended Nyquist-Johnson noise amplitude as the charge quantum vanishes.

A closer analysis shows that for any nonzero V, S_{ii}^P grows monotonically with e as e increases from zero: the larger the charge quantum, the larger the noise.

The following table summarizes the hypotheses and results of the two approaches:

Model	**Shot-noise**	**Extended Nyquist-Johnson**
State space	discrete	continuous
Stochastic process statistics	Poisson	Gaussian
Equilibrium condition (detailed balance)	equal forward and reverse flows in Master Equation (31)	probability flux J (5) vanishes in Fokker-Planck Equation
Power spectral density	$S_{ii}^P = \frac{eg(V)}{\tanh(V/2v_T)}$ (36)	$S_{ii}^{NJ} = 2kTG$ (39)
Gibbs distribution requirement	forward and reverse rates exponentially related (35)	resistor must be linear (22)

7 Conclusion

This paper has presented three specific requirements that determine whether a noise model is acceptable. All are based on the second law of thermodynamics. They provide guidelines for developing physically correct device noise models to correspond with experimental data.

The Nyquist-Johnson Gaussian thermal noise model for linear resistors, extended to include nonequilibrium operating conditions, satisfies all three of these thermodynamic requirements. In contrast, even the equilibrium requirements *cannot* be met by the Gaussian model for *any* nonlinear element with *any* choice of (operating-point dependent) noise amplitude. In particular, the Gaussian noise model obtained by applying the Nyquist-Johnson formula to the linearized conductance, *e.g.* (37), is physically incorrect except in the short-circuit case, though it occasionally appears in the literature.

We have derived a constraint (35) under which the shot-noise model satisfies all thermodynamic requirements presented here, when connected to a capacitor. This constraint allows one to predict the current-noise amplitude at every operating point *from knowledge of the device's* $v - i$ *curve alone.* The familiar subthreshold MOSFET and *pn* junction models satisfy this constraint.

The comparison in Sect. 6 showed that one cannot determine whether a noise model is thermodynamically acceptable by examining its power spectral density alone. Further information on the underlying probability distribution is required.

Appendices

A Interpretations of the Stochastic Differential Equation

Gaussian white noise of unlimited bandwidth is an idealization of the derivative of Brownian motion $W(t)$. Though $\frac{dW}{d\tau}$ does not exist, (15) is really shorthand for

$$\begin{aligned} v(t) &= v(0) - \frac{1}{C}\int_0^t g_T\left(v(\tau)\right) d\tau + \frac{1}{C}\int_0^t h_T\left(v(\tau)\right) \frac{dW}{d\tau} d\tau \\ &= v(0) - \frac{1}{C}\int_0^t g_T\left(v(\tau)\right) d\tau + \frac{1}{C}\int_0^t h_T\left(v(\tau)\right) dW(\tau). \end{aligned}$$

The second line, in which $\frac{dW}{d\tau}$ does not appear, is almost a rigorous statement of the meaning of (15), since $v(t)$ and $W(t)$ are continuous functions. But one ambiguity remains.

For any λ, $0 \le \lambda \le 1$, we can define the parameterized integral

$$(\lambda) \int_0^t h_T(v(\tau))\, dW(\tau)$$

$$\triangleq \lim_{(P)\ N\to\infty} \sum_{n=0}^{N-1} \left[(1-\lambda)\, h_T\left(v\left(\frac{nt}{N}\right)\right) + \lambda\, h_T\left(v\left(\frac{(n+1)t}{N}\right)\right)\right] \times \left[W\left(\frac{(n+1)t}{N}\right) - W\left(\frac{nt}{N}\right)\right], \tag{41}$$

where $\lim_{(P)\ N\to\infty}$ means that the summation converges in probability as $N \to \infty$. The fact that $W(t)$ and $v(t)$ (and hence $h_T(v(t))$) are not of bounded variation implies that the summation converges only in probability but unfortunately not along sample paths. Furthermore, this same lack of smoothness causes the random process to which this function converges in probability to depend on the particular value of λ, contrary to the more familiar case when $W(t)$ is of bounded variation.

The literature is primarily concerned with two interpretations for the above equation: the Itô or stochastic integral ($\lambda = 0$) and the Stratonovich integral ($\lambda = 1/2$). The Itô approach yields a non-anticipating martingale [Schuss, Sect. 3.2]. The Stratonovich approach is obtained by considering mathematical limits of idealized physical systems. Rationalization for the other values of λ is not clear.

Wong and Zakai have shown [Wong and Zakai] that, when considering equations of the form (15), one needs to include the "Zakai-Wong correction term," corresponding to the additional term in (17), to convert between approximate solutions obtained from bounded-variation-approximations to Brownian motion, *i.e.*, to convert between the Stratonovich and Itô forms [Schuss, p. 95].

When the stochastic differential equation (15) is interpreted in a more general sense for any λ as an integral equation

$$v(t) = v(0) - \frac{1}{C}\int_0^t g_T\left(v(\tau)\right) d\tau + \frac{1}{C}(\lambda)\int_0^t h_T\left(v(\tau)\right) dW(\tau),$$

and when the functions g_T and h_T and their derivatives are continuous and satisfy Lipschitz conditions (found in [Wong and Zakai]), it can be shown [Çinlar] that the corresponding Fokker-Planck equation is

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial v}\left\{\frac{g_T(v)}{C}\rho(v,t) - \frac{\lambda h_T(v)\rho(v,t)}{C^2}\frac{\partial}{\partial v}h_T(v) + \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{h_T^2(v)}{C^2}\rho(v,t)\right]\right\}, \tag{42}$$

which simplifies to (16) or (17), respectively, in the Itô and Stratonovich cases.

Although the rational and physical justification for other values of λ are unclear, one specific value weakens the conclusion in Section 4. For $\lambda = 1$, which yields a backwards equation [Schuss, p. 68], there is a second form of solution to (19):

$$h_T^2(v) = 2kT\,\frac{g_T(v)}{v}, \tag{43}$$

a unique noise amplitude determined solely by the resistor constitutive relation, again independent of C. Note that it reduces to the Nyquist-Johnson model in the case of a linear resistor.

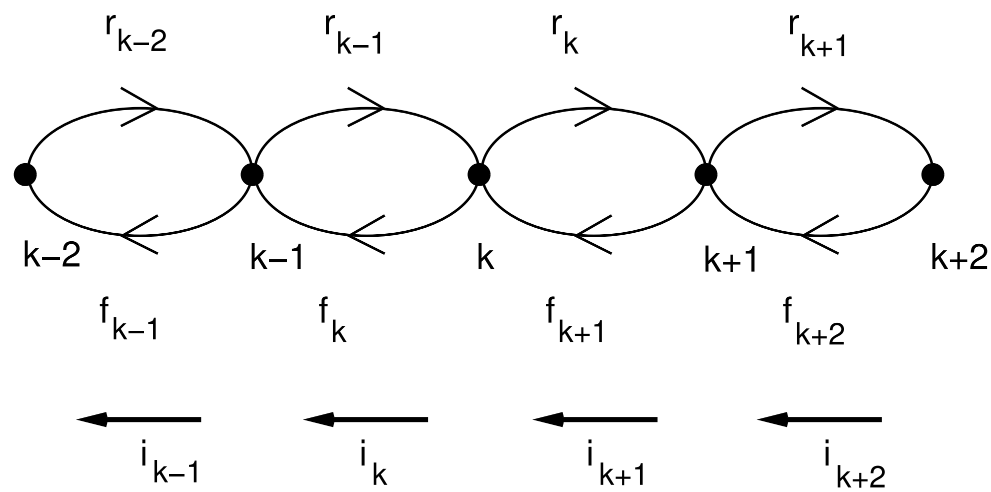

Figure 9: Markov chain with probability currents

B Elementary Proof of the Second Law for Reversible Markov Chains

In this appendix, we first give an elementary proof that the relative entropy (relative to equilibrium) increases for reversible Markov chains. (This is also true for non-reversible chains [Cover and Thomas], but the proof is more intricate.) Then we show that this implies the Second Law of Thermodynamics. Consider the diagram in Fig. 9. We define the probability currents as

$$i_j \triangleq f_j\, p_j - r_{j-1}\, p_{j-1}. \tag{44}$$

Then the change of probability at each node is given by

$$\begin{aligned} \dot{p}_j &= (f_{j+1}\, p_{j+1} - r_j\, p_j) - (f_j\, p_j - r_{j-1}\, p_{j-1}) \\ &= i_{i+j} - i_j. \end{aligned}$$

The chain is reversible [Gallager, p. 163], so

$$\begin{aligned} f_{j+1}\, p^o_{j+1} &= r_j\, p^o_j \\ f_j\, p^o_j &= r_{j-1}\, p^o_{j-1}, \end{aligned} \tag{45}$$

where p^o_j is the equilibrium value of p_j. This implies that $i^o_j = 0$ for all j, *i.e.*, there is no current at equilibrium. We will define the relative entropy S_{rel} as

$$S_{rel} \triangleq -\sum_{j=-\infty}^{\infty} p_j \ln\left(\frac{p_j}{p^o_j}\right). \tag{46}$$

We first calculate $\dot{S}_{rel}$:

$$\begin{aligned}
\frac{d}{dt}\left\{-\sum_{j=-\infty}^{\infty} p_j \ln\left(\frac{p_j}{p_j^o}\right)\right\} &= -\sum_{j=-\infty}^{\infty}\left[\dot{p}_j \ln\left(\frac{p_j}{p_j^o}\right) + \dot{p}_j\right] \\
&= -\sum_{j=-\infty}^{\infty} \dot{p}_j \ln\left(\frac{p_j}{p_j^o}\right) \\
&= \sum_{j=-\infty}^{\infty} (i_j - i_{j+1}) \ln\left(\frac{p_j}{p_j^o}\right) \\
&= \sum_{j=-\infty}^{\infty} i_j \ln\left(\frac{p_j}{p_j^o}\right) - \sum_{j=-\infty}^{\infty} i_{j+1} \ln\left(\frac{p_j}{p_j^o}\right) \\
&= \sum_{j=-\infty}^{\infty} i_j \ln\left(\frac{p_j}{p_j^o}\right) - \sum_{j=-\infty}^{\infty} i_j \ln\left(\frac{p_{j-1}}{p_{j-1}^o}\right) \\
&= \sum_{j=-\infty}^{\infty} i_j \left[\ln\left(\frac{p_j}{p_j^o}\right) - \ln\left(\frac{p_{j-1}}{p_{j-1}^o}\right)\right].
\end{aligned} \tag{47}$$

But from (44), we can express the probability currents as

$$i_j = f_j\, p_j - r_{j-1}\, p_{j-1} = \left(f_j\, p_j^o\right)\frac{p_j}{p_j^o} - \left(r_{j-1}\, p_{j-1}^o\right)\frac{p_{j-1}}{p_{j-1}^o}.$$

Equation (45) allows us to re-write the second term

$$i_j = \left(f_j\, p_j^o\right)\left[\frac{p_j}{p_j^o} - \frac{p_{j-1}}{p_{j-1}^o}\right]. \tag{48}$$

Substituting (48) into (47) yields

$$\begin{aligned}
&\frac{d}{dt}\left\{-\sum_{j=-\infty}^{\infty} p_j \ln\left(\frac{p_j}{p_j^o}\right)\right\} \\
&= \sum_{j=-\infty}^{\infty} \left(f_j\, p_j^o\right)\left[\frac{p_j}{p_j^o} - \frac{p_{j-1}}{p_{j-1}^o}\right]\left[\ln\left(\frac{p_j}{p_j^o}\right) - \ln\left(\frac{p_{j-1}}{p_{j-1}^o}\right)\right] \geq 0,
\end{aligned} \tag{49}$$

with equality only at equilibrium.

The question remains as to the relation between (46) and the physical entropy of the circuit in Fig. 5 . The entropy of the capacitor distribution is

$$S_C = -k \sum_{j=-\infty}^{\infty} p_j \ln p_j. \tag{50}$$

Using properties of the logarithm, we find this term in S_{rel}:

$$kS_{rel} = -k \sum_{j=-\infty}^{\infty} p_j \left(\ln p_j - \ln p_j^o\right), \tag{51}$$

so we must interpret the remaining term,

$$k \sum_{j=-\infty}^{\infty} p_j \ln p_j^o. \tag{52}$$

From (32)

$$p_j^o = p_0^o \prod_{n=0}^{j-1} \frac{r_n}{f_{n+1}} = p_0^o \prod_{n=0}^{j-1} \frac{r_T(v_n + e/2C)}{f_T(v_n + e/2C)}, \quad j > 0.$$

On account of our constraint (35),

$$\begin{aligned}
p_j^o &= p_0^o \prod_{n=0}^{j-1} \exp\left[-\frac{(v_n + e/2C)}{v_T}\right] = p_0^o \prod_{n=0}^{j-1} \exp\left[\frac{-ne}{Cv_T}\right] \exp\left[\frac{-e}{2Cv_T}\right] \\
&= p_0^o \exp\left[\frac{-e}{Cv_T} \sum_{n=0}^{j-1} n\right] \exp\left[\frac{-je}{2Cv_T}\right] = p_0^o \exp\left[\frac{-e}{Cv_T}\frac{j^2 - j}{2}\right] \exp\left[\frac{-je}{2Cv_T}\right] \\
&= p_0^o \exp\left[\frac{-j^2 e}{2Cv_T}\right] = p_0^o \exp\left[\frac{-j^2 e^2}{2CkT}\right], \quad j > 0,
\end{aligned}$$

and a similar derivation gives the same result for $j \leq 0$. Substituting into (52),

$$\begin{aligned}
k \sum_{j=-\infty}^{\infty} p_j \ln p_j^o &= k \sum_{j=-\infty}^{\infty} p_j \ln\left[p_0^o \exp\left(-\frac{j^2 e^2}{2CkT}\right)\right] \\
&= k \sum_{j=-\infty}^{\infty} p_j \ln p_0^o + k \sum_{j=-\infty}^{\infty} p_j \left(-\frac{j^2 e^2}{2CkT}\right) \\
&= k \ln p_0^o - \sum_{j=-\infty}^{\infty} p_j \frac{E_j}{T} \\
&= k \ln p_0^o - \frac{\overline{E}_C}{T}.
\end{aligned} \tag{53}$$

From (9) the thermal reservoir entropy rate is

$$\dot{S}_R = -\frac{\dot{\overline{E}}_C}{T}.$$

Thus, from (50), (51), and (53),

$$k\dot{S}_{rel} = \dot{S}_R + \dot{S}_C.$$

Therefore, (49) shows that $\dot{S}_R + \dot{S}_C \geq 0$, *i.e.*, the Second Law holds for the circuit in Fig. 5 for an arbitrary initial probability distribution of capacitor charge. Thus the Poisson model satisfies Requirement #3, at least when connected to a linear capacitor.

C List of Symbols

Variables:

ρ	probability density
ρ^o	equilibrium probability density
ρ_q	probability density for charge
ρ_v	probability density for voltage
p	discrete probability distribution
p^o	equilibrium probability distribution
Pr	probability
q	charge
E	energy
v, V	voltage (time-varying or constant)
i, I	current (time-varying or constant)
$S_{ii}(\omega)$	power spectral density (of current noise)
$\xi(t)$	random process; generally unit-amplitude, white, and Gaussian

Constants:

e	electron charge, 1.602×10^{-19} C
k	Boltzmann's constant, 8.617×10^{-5} eV/K
v_T	thermal voltage kT/e, 0.02585 V at T=300 K
C	capacitance
G	conductance
T	absolute temperature, in Kelvin (K)
I_S	saturation current
n_{po}	electron concentration in the bulk p-region
p_{no}	hole concentration in the bulk n-region

Notation:

$\overline{x}$	expectation of the random variable x
$\dot{x}$	time derivative of the variable x
$\triangleq$	defined equal to

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