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Some exact properties of the effective slip over surfaces with hydrophobic patternings

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Shear flows of viscous fluid layers over nonuniformly hydrophobic surfaces are characterized in the far-field by an effective slip velocity, which relates to the applied stress through some mobility tensor characterizing the surface. Here, we identify two methods to determine the mobility tensor for flat surfaces with arbitrary slip-length variations. A family of “Cross Flow Identities” is then analyzed, which equate mobility components of different unidirectional patternings. We also calculate an analytical mobility solution for a family of continuously varying patterns. We validate the results numerically and discuss implications in various limits. © 2013 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4790536>]

Driven primarily by developing technology in microfluidics, there has been continuing interest in the potential of using surface texture to influence, direct, or reduce drag on fluid flows. When the scale of surface fluctuations is small compared to the size of macroscopic lengths, there are advantages to using “effective boundary conditions,”^{1–6} which replicate the far-field effects of a fluctuating boundary surface, but are applied instead on the smooth, mean surface. Effective boundary conditions are generally tensorial in three-dimensions,^{7–14} taking the form of a tensorial Navier slip boundary condition,¹² or similarly, as a “surface mobility tensor” relating slip velocity to applied shear traction.¹⁵ In this letter, we provide a sequence of analytical relations for the surface mobility and related flow features of fluid motion over hydrophobically varying flat surfaces. These results can be used as an expeditious toolset in the design and optimization of surface textures that may otherwise require multiple experiments or continuum simulation. One such relation provides a new take on a recent result independently arrived at in Ref. 16.

Throughout we assume creeping flows so that the Stokes equations apply. The equations can be non-dimensionalized to absorb the viscosity into the stress units, allowing us to write

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

for velocity $\mathbf{u} = (u, v, w)$ and pressure p . We focus on a geometry appropriate for the study of effective boundary conditions. Suppose an infinite, rigid, flat surface (at $z = 0$) with spatially varying hydrophobicity that is governed by some Navier slip-length distribution $\lambda(x, y)$. Fluid flow along such a surface obeys

$$\mathbf{u}(x, y, 0) = \lambda(x, y)(\mathbf{1} - \hat{\mathbf{z}}\hat{\mathbf{z}}) \cdot \left. \frac{\partial \mathbf{u}}{\partial z} \right|_{z=0}. \quad (2)$$

Let $\lambda(x, y)$ be periodic in x and y with periods ℓ_x and ℓ_y , respectively. Suppose a tall fluid layer sits atop this surface and is set into motion by a lateral shear stress $\boldsymbol{\tau} = (\tau_x, \tau_y, 0)$ applied far above the surface. We define the *effective slip* $\mathbf{u}_s = (u_s, v_s, 0)$ as the vector by which the flow field differs from simple shearing far above the surface; i.e.,

$$\mathbf{u}(x, y, z \gg 1) = \boldsymbol{\tau} z + \mathbf{u}_s. \quad (3)$$

It is useful to consider a surface mobility tensor \mathbf{M} to characterize the effective slip.¹⁵ Specifically, it has been demonstrated¹⁴ that for each $\lambda(x, y)$ a symmetric mobility tensor \mathbf{M} can be found

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that relates \mathbf{u}_s to the shear traction $\boldsymbol{\tau}$ such that

$$\mathbf{u}_s = \mathbf{M} \cdot \boldsymbol{\tau}. \quad (4)$$

We represent the mapping from a surface pattern $\lambda(x, y)$ to the corresponding mobility tensor \mathbf{M} by a functional $\mathbf{M} = \tilde{\mathbf{M}}[\lambda(x, y)]$. We abuse convention slightly by neglecting the trivial z components of \mathbf{M} such that the \mathbf{M} matrix is 2×2 .

The results discussed in this letter all pertain to analytical solutions or solution methods to calculate mobility and other behaviors in this family of geometries. We can divide the results into four categories, which are presented in a more-or-less chronological sequence: (i) Relations to determine the mobility tensor and surface-level flow over any hydrophobic surface $\lambda(x, y)$ [Eqs. (5) and (16)]; (ii) an infinite set of ‘‘Cross Flow Identities’’ relating surface flows and mobility components within the family of unidirectional surfaces $\lambda(x)$ [Eqs. (11) and (12)]; (iii) an exact solution for the mobility tensor of sinusoidally varying surfaces, a paradigm case [Eq. (23)]; and (iv) asymptotics on the exact solution to reveal generic behaviors of small vs large amplitude surface fluctuations. Numerical validations are discussed throughout. Topics (i) and (ii) are presented two different ways, first using Fourier analysis, and then using Green’s functions. We recently discovered that one of our notable results, a corollary to Eq. (12), was independently obtained in a very recent paper¹⁶ utilizing an analogous Fourier argument. The Green function approach, which we also provide, gives perhaps a more straightforward way to see this and other results.

First, we utilize Fourier analysis to discuss general relationships for surfaces with arbitrary slip-length distributions. The supplementary material¹⁷ displays a general solution of Eqs. (1) under the given boundary conditions, expressed as a Fourier series of the form

$$\mathbf{u}(x, y, z) = \boldsymbol{\tau}z + \mathbf{u}_s + \sum_{m,n \neq 0} e^{i(k_m x + k_n y)} \mathbf{F}(k_m, k_n, z) \cdot \mathbf{c}(m, n), \quad (5)$$

where $k_m = m\pi/\ell_x$, $k_n = n\pi/\ell_y$, and the 3×2 matrix-valued function $\mathbf{F}(k_m, k_n, z)$ is known. The unknown constants, which are used to match the patterned lower boundary condition, are the two coefficient sets within $\mathbf{c}(m, n) = (B(m, n), C(m, n))$ for all non-zero pairs (m, n) , and the two non-trivial scalar components of the effective slip \mathbf{u}_s . Hence, by fitting the above to obey Eq. (2) for some given periodic $\lambda(x, y)$ —requiring solving an infinite linear system—we obtain \mathbf{u}_s as a (small) part of the full flow solution, and from this can determine \mathbf{M} . One can truncate the system at large enough values of $|m|$ and $|n|$ to obtain approximate numerical solutions. Due to the nature of Fourier methods, functions λ lacking certain smoothness properties may require more terms to achieve the desired accuracy.

Let us simplify the problem by only considering unidirectional patterns of the form $\lambda = \lambda(x)$. Due to geometric symmetry, this constraint ensures the mobility matrix is diagonal in the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ basis since an x -directed shear stress induces no flow in the y -direction, and vice versa for a y -directed shear stress. Choosing $\boldsymbol{\tau} = (1, 0, 0)$, a unit shear stress oriented *across* the striping, the u component of the general flow solution reduces to

$$u(x, y, z) = z + u_s + \frac{2}{3} \sum_{m \neq 0} e^{ik_m x - zk|m|} (1 - zk|m|) B(m, 0), \quad (6)$$

for $k = \pi/\ell_x$. Substituting the above into Eq. (2), $u(x, y, z)$ can be obtained from the solution to the following infinite matrix equation for the unknown coefficients:

$$\begin{aligned} (\dots \hat{\lambda}(m) \dots \hat{\lambda}(0) \dots \hat{\lambda}(-m) \dots)^T &= (\mathbf{I} + 2k\mathbf{Q}) \\ &\times \left(\dots \frac{2B(m,0)}{3} \dots u_s \dots \frac{2B(-m,0)}{3} \dots \right)^T. \end{aligned} \quad (7)$$

In the above $\lambda(x) = \sum \hat{\lambda}(m) e^{ik_m x}$ is the Fourier series of the pattern. \mathbf{I} is the infinite identity matrix, and the matrix \mathbf{Q} is defined component-wise by $Q_{\alpha\beta} \equiv |\beta| \hat{\lambda}(\alpha - \beta)$, for all integer indices α, β . The mobility component $M_{xx} = \tilde{M}_{xx}[\lambda(x)]$ is equal to u_s above.

Suppose we now consider a flow over the same surface, but shearing with a unit stress in the y -direction, *along* the pattern (Fig. 1). Here, the general solution gives the following expression for

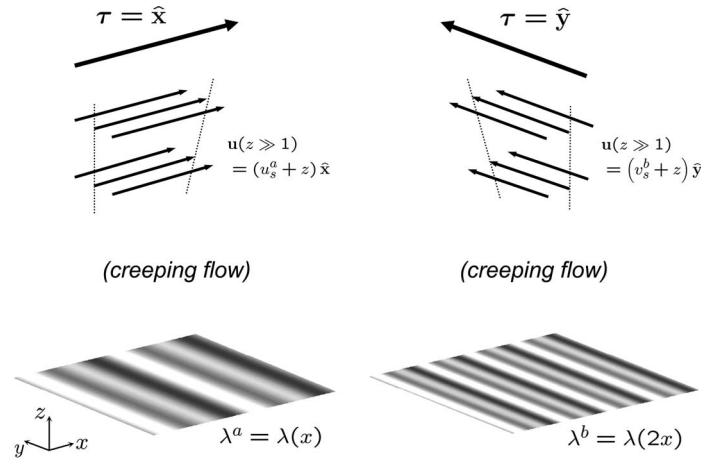


FIG. 1. Basic Cross Flow Identity: The effective slips u_s^a and v_s^b above are always equal. The function λ describing the surface patterns can be arbitrary.

the y -directed flow component:

$$v(x, y, z) = z + v_s + \sum_{m \neq 0} e^{ikmx - zk|m|} C(m, 0). \quad (8)$$

Proceeding as before, we substitute this result into Eq. (2) and we obtain the following matrix relation for the flow coefficients,

$$\begin{aligned} (\dots \hat{\lambda}(m) \dots \hat{\lambda}(0) \dots \hat{\lambda}(-m) \dots)^T &= (\mathbf{I} + k\mathbf{Q}) \\ \times (\dots C(m, 0) \dots v_s \dots C(-m, 0) \dots)^T. \end{aligned} \quad (9)$$

Solving for v_s gives the other component of \mathbf{M} ; i.e., $v_s = M_{yy} = \tilde{M}_{yy}[\lambda(x)]$. Altogether, by solving the two linear systems described above, we determine the entire mobility matrix for $\lambda(x)$.

However, without having to solve anything directly, we can make the following major observation in view of Eqs. (7) and (9). Suppose we consider two surfaces that relate to our arbitrary $\lambda(x)$. Let $\lambda^a(x) = \lambda(x)$ and $\lambda^b(x) = \lambda(2x)$, respectively. Hence, b 's surface pattern is the same as a 's except compressed to a twice smaller period. Likewise, we can write $\lambda^a(x) = \sum \hat{\lambda}(m)e^{ik^a mx}$ and $\lambda^b(x) = \sum \hat{\lambda}(m)e^{ik^b mx}$ where $k^b = 2k^a = 2k$. Consider the flow $\mathbf{u}^a(x, y, z)$ induced by $\boldsymbol{\tau} = \hat{\mathbf{x}}$ applied over surface a and the flow $\mathbf{u}^b(x, y, z)$ over surface b that is induced by $\boldsymbol{\tau} = \hat{\mathbf{y}}$. Applying Eq. (7) to surface a and Eq. (9) to surface b , we obtain now the same infinite matrix in both cases, $\mathbf{I} + 2k\mathbf{Q}$, and the same vectors on the left-side. Consequently, the solution for the unknown vector of coefficients in these two problems is the same, and thus $u_s^a = v_s^b$. We call this fact the Cross Flow Identity (CFI). Written explicitly:

Theorem: For any periodic function $\lambda(x)$, it holds that $\tilde{M}_{xx}[\lambda(x)] = \tilde{M}_{yy}[\lambda(2x)]$. In words, *the mobility of flow across any arbitrary, unidirectional pattern is always equal to the mobility along the same pattern compressed by a factor of 2.*

This result relating flow in different directions on different but related surface patterns is met with a certain level of physical intuition. Pushing across a pattern is more resistive than pushing along, and smaller periods impede flow more than wider periods. The CFI says, in effect, that these two influences cancel exactly when the period is shrunk by a factor of 2. The emergence of “2” in relating behaviors in orthogonal directions in Stokesian dynamics is not uncommon; it occurs, for example, in studies of flows through striped pipes¹⁸ as well as the classical sedimentation work of Batchelor for rods of arbitrary cross-section.¹⁹ Unlike other general theorems regarding the mobility tensor,¹⁴ we have not been able to prove the CFI “quickly” using global identities such as Lorentz reciprocity. That said, shortly, we shall provide what we feel is a more straightforward derivation of our results using Green’s functions. We also note it might be possible to derive the CFI from a

conformal mapping method specialized to handle Navier-slip conditions, in the spirit of recent work on perfect-slip stripings.²⁰ The Fourier series approach used above will be called on again later in deriving exact mobility over sinusoidal surfaces.

Note that the CFI is only one consequence of the invariance apparent in the matrix equations corresponding to our a and b cases. In fact, the entire $B(m, 0)$ series for flow across surface a implies the entire $C(m, 0)$ series for flow along surface b . Consequently, the two flow fields are actually identical in the $z = 0$ plane, modulo the compressed period. That is,

$$u^a(2x, y, 0) = v^b(x, y, 0). \quad (10)$$

This result is made more interesting by noting it does not arise from any standard scaling argument—due to the differing z dependences in Eqs. (6) and (8) and the fact that flow across surface a also has a non-trivial w component (which we have not yet needed to consider) the full flow fields for $z > 0$ are in fact very different from each other.

The CFI can be merged with more obvious identities to form a family of equivalences implied by a single mobility measurement. If $\mathbf{u}(x, y, z)$ solves the Stokes equations, $\mathbf{u}(\beta x, \beta y, \beta z)$ does as well, which yields the basic scaling relation $\beta \tilde{\mathbf{M}}[\lambda(x)] = \tilde{\mathbf{M}}[\beta \lambda(x/\beta)]$ for any scalar β . Combined with CFI, we obtain the generalized identity,

$$\beta \tilde{M}_{xx}[\lambda(x)] = \tilde{M}_{yy}[\beta \lambda(2x/\beta)], \quad (11)$$

for all scalars β . Similarly, the generalization of Eq. (10) is

$$\lambda^b(x) = \beta \lambda^a(2x/\beta) \rightarrow v^b(x, y, 0) = \beta u^a(2x/\beta, y, 0). \quad (12)$$

An interesting case is $\beta = 2$, for which $2\tilde{M}_{xx}[\lambda(x)] = \tilde{M}_{yy}[2\lambda(x)]$, $v^b(x, y, 0) = 2u^a(x, y, 0)$. This result was independently obtained in Ref. 16. It agrees with past mobility approximations for small perturbations to a no-slip surface.¹³ Furthermore, if λ is a piecewise function that takes on values of 0 or ∞ only, as in superhydrophobic surfaces, then $2\lambda(x) = \lambda(x)$ and thus $2\tilde{M}_{xx}[\lambda(x)] = \tilde{M}_{yy}[\lambda(x)]$, a result obtained through other means in Ref. 20.

To check these identities further, we conducted a series of tests where the Stokes equations were solved numerically using a finite-difference method in the geometry at hand, using various $\lambda(x)$. The geometry was box-shaped (for one period cell) with periodic boundary conditions at side walls. A variety of functions for $\lambda(x)$ were used on the bottom surface, chosen to represent a broad range of smoothness properties, namely a sine wave, sawtooth wave, triangular wave, and square wave. A unit shear stress boundary condition was applied along the top surface, which is chosen high enough that a strong linear flow profile appears in the top half of the flow domain. For each $\lambda(x)$, by refining the finite-difference grid, we observed convergence to the CFI—the effective slip due to an x -directed shear stress (using Eq. (3) in the high- z flow), approached that obtained when the same surface was compressed by a factor of 2 and flow was driven by a y -directed shear stress. We stopped refining when 0.1% relative error between the effective slips for each surface pair was observed.

While our above solution approach as well as that of Ref. 19 used a Fourier series argument, it begs the broader question of whether a more straightforward derivation exists to explain these very “simple looking” results. Here we shall provide another approach for the CFI and its related results using a more direct argument involving the fundamental stokeslet solution to the Stokes equations. In a sense, what we do is to build the solution from stokeslets distributed along the surface rather than an *ad hoc* Fourier set for this particular geometry. We point out that the stokeslet is also responsible for the factor of 2 arising in slender-body sedimentation,¹⁹ so the ensuing derivation serves to connect these two problems to the same core principle.

Let us return to the basic setup, with $\lambda = \lambda(x, y)$ and arbitrary horizontal $\boldsymbol{\tau}$. Define the disturbance flow field $\mathbf{u}^* = \mathbf{u} - \boldsymbol{\tau} z - \mathbf{u}_s$, which vanishes at $z = \infty$. Since the disturbance flow also solves the Stokes equations, the flow at any surface point $\mathbf{x} = (x, y, 0)$ must satisfy the fundamental boundary integral equation for Stokes flow,²¹ which for our geometry takes the form

$$\mathbf{u}^*(\mathbf{x}) = \frac{1}{4\pi} \int_A [\mathbf{u}^*(\mathbf{x} - \mathbf{x}') \cdot \mathbf{T}(\mathbf{x}') - \mathbf{G}(\mathbf{x}') \cdot \boldsymbol{\sigma}^*(\mathbf{x} - \mathbf{x}')] \cdot \hat{\mathbf{z}} dS', \quad (13)$$

where A is the entire xy -plane and

$$\mathbf{G}(\mathbf{x}) = \frac{\mathbf{1}}{(x^2 + y^2)^{1/2}} + \frac{\mathbf{xx}}{(x^2 + y^2)^{3/2}} \quad (14)$$

is the stokeslet (i.e., Oseen tensor) giving the free-space Green's function for a point-force, and $\mathbf{T}(\mathbf{x}) = -6\frac{\mathbf{xx}\mathbf{x}}{(x^2+y^2)^3}$. Since the surface is at $z = 0$, we have $\mathbf{T} \cdot \hat{\mathbf{z}} = \mathbf{0}$, which cancels the first term in brackets in Eq. (13). Regarding the surface stress, note that on the surface

$$\int_A \boldsymbol{\sigma}^*(\mathbf{x} - \mathbf{x}') \cdot \hat{\mathbf{z}} dS' = \int_A \left[\frac{\mathbf{u}(\mathbf{x} - \mathbf{x}')}{\lambda(\mathbf{x} - \mathbf{x}')} - \boldsymbol{\tau} - p(\mathbf{x} - \mathbf{x}')\hat{\mathbf{z}} \right] dS' = \mathbf{0} \quad (15)$$

to satisfy global force balance on the fluid body. By Lorentz reciprocity of $\mathbf{u}^*(\mathbf{x})$ and $\boldsymbol{\tau}z$, we find $\mathbf{u}_s = \langle \mathbf{u}(\mathbf{x}) \rangle$, where $\langle \cdot \rangle$ is the average over A . Applying all these results, we can rewrite Eq. (13) as a completely self-contained integral equation uniquely defining the surface flow:

$$\langle \mathbf{u}(\mathbf{x}) \rangle - \mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int_A \mathbf{G}(\mathbf{x}') \cdot \left(\frac{\mathbf{u}(\mathbf{x} - \mathbf{x}')}{\lambda(\mathbf{x} - \mathbf{x}')} - \boldsymbol{\tau} \right) dS'. \quad (16)$$

Note the pressure term vanishes in the above, because $\mathbf{G} \cdot \hat{\mathbf{z}} \propto \hat{\mathbf{z}}$ on the surface and the surface flow has no z component.

Equation (16) can be thought of as the most general surface relationship underlying our study, and its solution determines the surface flow for any $\lambda(x, y)$. It is worth noting that never in the derivation do we compute the flow in the bulk; the Navier-slip condition permits us to reduce immediately to a self-contained surface flow equation because it expresses the surface traction directly in terms of the surface flow. This is interesting especially since the flow is generated from a stress far above the surface. Also note that no periodicity constraints need to be assumed at any point.

Let us consider, as before, two unidirectional patterns $\lambda^a(x)$ and $\lambda^b(x)$ and suppose over surface a that $\boldsymbol{\tau} = \hat{\mathbf{x}}$ induces a flow with surface velocity $u^a(x, y, 0) = U^a(x)$, and $\boldsymbol{\tau} = \hat{\mathbf{y}}$ drives a surface flow $v^b(x, y, 0) = V^b(x)$ over b . We can apply Eq. (16) to obtain

$$\langle U^a(x) \rangle - U^a(x) = \frac{1}{4\pi} \int_A \frac{2x'^2 + y'^2}{(x'^2 + y'^2)^{3/2}} \left(\frac{U^a(x - x')}{\lambda^a(x - x')} - 1 \right) dx' dy', \quad (17)$$

$$\langle V^b(x) \rangle - V^b(x) = \frac{1}{4\pi} \int_A \frac{x'^2 + 2y'^2}{(x'^2 + y'^2)^{3/2}} \left(\frac{V^b(x - x')}{\lambda^b(x - x')} - 1 \right) dx' dy'. \quad (18)$$

Observe the direction-dependent factor of 2 arising above, which is brought on by the factor of 2 strength difference within the stokeslet in the direction of force vs orthogonal to the force. We can reduce the integrals above by noting that $\int_{-\infty}^{\infty} \frac{x'^2 dy'}{(x'^2 + y'^2)^{3/2}} = 2$, which, when combined with Eq. (15), annihilates the x'^2 terms in the numerators of each of the integrals above, leaving

$$\langle U^a(x) \rangle - U^a(x) = \frac{1}{4\pi} \int_A \frac{y'^2}{(x'^2 + y'^2)^{3/2}} \left(\frac{U^a(x - x')}{\lambda^a(x - x')} - 1 \right) dx' dy', \quad (19)$$

$$\langle V^b(x) \rangle - V^b(x) = \frac{1}{4\pi} \int_A \frac{2y'^2}{(x'^2 + y'^2)^{3/2}} \left(\frac{V^b(x - x')}{\lambda^b(x - x')} - 1 \right) dx' dy'. \quad (20)$$

The lingering factor of 2 in Eq. (20) is the root of our cross-flow results. By direct substitution, we see that if $U^a(x)$ is the solution of Eq. (19) over some $\lambda^a(x) = \lambda(x)$, then $V^b(x) = \beta U^a(2x/\beta)$ identically solves Eq. (20) for $\lambda^b(x) = \beta\lambda(2x/\beta)$. This provides another proof of Eq. (12). And combining this result with $\mathbf{u}_s = \langle \mathbf{u}(\mathbf{x}) \rangle$ from before implies the generalized CFI, Eq. (11).

This derivation provides qualitative intuition for why the CFI arises. For some $\lambda(x)$, consider the disturbance flow field when $\boldsymbol{\tau} = \hat{\mathbf{x}}$. The surface shear tractions that emerge can be visualized as an array of point forces along the surface, and the flow can be seen as a superposition of the stokeslet fields about each point force. Suppose we produce a new flow by rotating each of these point-forces in place 90° so that they now are all parallel to $\hat{\mathbf{y}}$. Due to the direction-sensitivity of the stokeslet, the influence of each force is now precisely two times larger in the y direction than previously. Upon

superposing the stokeslet fields, subtle cancellation occurs in the x -superposition (due to global equilibrium), so that the doubling effect in y causes the entire surface flow field to double. However, the surface traction distribution is the same (only rotated), implying that the boundary condition the new flow satisfies is $2\lambda(x)$, and that $\boldsymbol{\tau} = \hat{\mathbf{y}}$. Thus we see that by switching to pushing along the pattern, and doubling the Navier slip everywhere, the surface flow field (and hence the effective slip) doubles, in accord with Eq. (12), $\beta = 2$.

We now switch gears, and turn attention to finding an exact solution to Eqs. (7) and (9) for some fluctuating surface $\lambda(x)$ in order to determine an analytical, non-perturbative solution for the mobility matrix for a non-trivial smoothly varying surface pattern. Since physical slip-lengths are necessarily non-negative, the simplest surface to consider is $\lambda(x) = 2s(1 + \cos(kx))$, where s determines the amplitude of slip-length variation. The behavior for this surface could serve as a model for characterizing general “striped” surfaces that oscillate between no-slip and a finite slip-length. The only non-zero Fourier coefficients of the surface are $\hat{\lambda}(0) = 2s$, $\hat{\lambda}(1) = s$, and $\hat{\lambda}(-1) = s$. We first consider Eq. (7), which must be solved to determine M_{xx} . Define $x_0 = 3u_s/2$ and for $m \neq 0$, $x_m = B(m, 0)$. Then the system represented by Eq. (7) can be written algebraically as

$$\begin{aligned} x_0 + 4skx_1 - 3s &= 0, & (1 + 4sk)x_1 + 4skx_2 - 3s/2 &= 0, \\ 2(m-1)skx_{m-1} + (1 + 4msk)x_m + 2(m+1)skx_{m+1} &= 0 \text{ for } m \geq 2. \end{aligned} \quad (21)$$

To aid in solving the above, we define a generating function $f(t) = \sum_{m=0}^{\infty} x_m t^m$. Observe that the above algebraic system emerges in the coefficients of the powers of t when the following differential equation is expanded, $2sk(1+t)^2 f'(t) + f(t) - \frac{3}{2}st - \frac{3}{4}(a+2s) = 0$. Hence, our set $\{x_m\}$ arises by solving for f , where, by definition, $f(0) = 3u_s/2$ provides the initial condition. The analytical solution is

$$f(t) = \frac{3}{4}u_s + \frac{3}{4}e^{\frac{1}{2sk(1+t)}} \left(u_s e^{-\frac{1}{2sk}} - \frac{E_1\left(\frac{1}{2sk}\right)}{k} \right) + \frac{3}{4k} e^{\frac{1}{2sk(1+t)}} E_1\left(\frac{1}{2sk(1+t)}\right), \quad (22)$$

where $E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral for positive real numbers x . However, the problem is not finished because the solution still depends on the unknown u_s . We have yet to use the information that because the x_m are coefficients of a convergent Fourier series for the surface flow $u(x, y, 0)$, then $x_{m \rightarrow \infty} \rightarrow 0$. Upon Taylor expanding $f(t)$ above, the only way to prevent $x_m = f^{(m)}(0)/m!$ from diverging as $m \rightarrow \infty$ is to ensure the second term above vanishes. This requirement necessitates that $u_s = \frac{e^{\frac{1}{2sk}}}{k} E_1\left(\frac{1}{2sk}\right)$ giving us our solution for M_{xx} . We may follow a completely analogous procedure to solve Eq. (9) for v_s , and once done arrive at the analytical solution for the mobility matrix of our surface:

$$\tilde{\mathbf{M}}[2s(1 + \cos(kx))] = \begin{pmatrix} \frac{e^{\frac{1}{2sk}}}{k} E_1\left(\frac{1}{2sk}\right) & 0 \\ 0 & \frac{2e^{\frac{1}{sk}}}{k} E_1\left(\frac{1}{sk}\right) \end{pmatrix}. \quad (23)$$

It is interesting to observe that the analytical solution clearly satisfies the generalized CFI, Eq. (11). Figure 2 evidences the correctness of this result by comparing it to numerical solutions of the Stokes equations, obtained using a finite-difference method.

Now that we have an exact formula for the mobility tensor for a family of surfaces, we can study directly the influence of the two coefficients s and k on the mobility tensor. It is a well-known asymptotic result that $e^z E_1(z) \sim 1/z$ when $z \gg 1$, and $e^z E_1(z) \sim -\log(z)$ when $z \ll 1$. We deduce that when $sk \ll 1$, i.e., small-amplitude slip fluctuations and/or wide period, the mobility tensor approaches that of a uniform surface of the same mean slip-length, $\mathbf{M} \sim 2s\mathbf{I}$. Using the other asymptotic relation we arrive at the behavior for large-variation surfaces, characterized by a large value of sk we obtain $M_{xx} \sim -\log(sk)/k$ and $M_{yy} \sim -2\log(sk)/k$. Since s is proportional to the mean slip of a surface, this result indicates there are diminishing returns when attempting to increase the mobility of a fluctuating surface by increasing the amplitude alone. The mobility increases only logarithmically in s , which can be attributed to the fact that the surface returns to no-slip once per period, which constricts the flow rather globally making it difficult to achieve large effective slips.

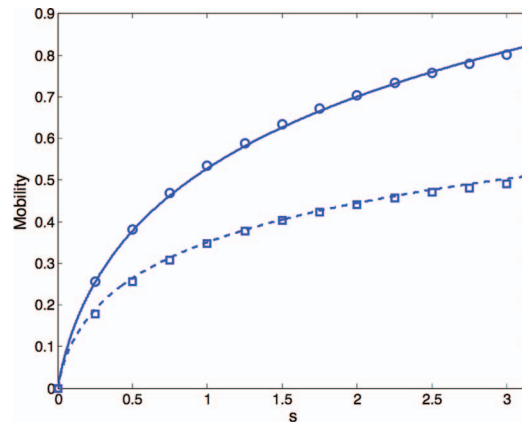


FIG. 2. Analytical results for M_{xx} (- -) and M_{yy} (—) using Eq. (23) for the pattern $\lambda(x) = 2s(1 + \cos(2\pi x))$. Symbols show the same mobility components measured from finite-difference solutions of the Stokes equations.

In this work we have exploited general solutions for shear flow over flat, hydrophobically varying surfaces, Eq. (5) and Eq. (16), to produce several simple analytical rules governing the mobility of flows over such surfaces. We point out that many of the analytical deductions made herein were possible because the surface shape was flat, which simplified the application of the lower boundary condition in the general solution. Based on analytical results discussed in Ref. 22 it is unclear whether similar identities hold in the presence of surfaces with height fluctuations.

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