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Factorizations and partial contraction of nonlinear systems

M.-A. Belabbas and J.-J. E. Slotine

Abstract—In this paper, we introduce new results in the analysis of convergence of nonlinear systems. The point of view we take is the one of contraction theory and we focus in particular on convergence to smooth manifolds. A main characteristic of contraction theory is that it does not require nor use any knowledge about the asymptotic trajectory of the system. Our contribution is to extend the core body of contraction results to include such knowledge in the analysis. As a result, this approach naturally leads to the definition of a new type of commutator for vector fields. We will show that the vanishing of this commutator, together with a contraction assumption, yields a sufficient condition for convergence and we will illustrate the results on the Andronov-Hopf oscillator.

I. INTRODUCTION

The study of stabilization and convergence phenomena in dynamical systems is a centerpiece of control theory, from the early classical work of Nyquist [1] and its applications to circuit analysis, to work addressing more recent questions raised by neuroscience [2], to the study of flocks and networks [3], [4]. The tools used in this context are varied, from the nowadays classical linear analysis, to Lyapunov theory [5], to group theoretic methods [6].

In recent years, contraction theory has been shown in recent years to be rather effective in the investigation of stabilization and synchronization of systems [7]. The main difference between the Lyapunov and contraction points of view is that, while stabilization will occur at the minima of an appropriately defined Lyapunov function, contraction theory emphasizes a differential approach, giving conditions under which trajectories of the system tend to coalesce. As a consequence, it *does not require to know beforehand* along which trajectory the system will converge. This characteristic is enviable at times, as it allows to draw conclusions that would be quite more difficult to obtain otherwise, but can be restrictive in other contexts where more information about a limit cycle or attracting manifold is available. For example, one can obtain [8] sufficient conditions under which certain auxiliary systems describing the synchronization behavior of

networks of oscillators with diffusive connections are contracting, hence proving that such networks will robustly synchronize. If on the one hand, it would be quite cumbersome to exhibit a closed-form description of the periodic cycle for the case of Fitzhugh-Nagumo oscillators, on the other much more is known about the Andronov-Hopf oscillator—for which this approach yields sufficient conditions that are not optimal [9], [10]—but this information is not used in the convergence analysis.

The objective of this paper is to extend the classical results of contraction theory to the case of convergence to smooth manifolds and as a consequence *to allow the inclusion of knowledge* about the limit cycle or attracting manifold in the analysis. Specifically, the new framework for analysis we propose relies on three steps: the definition of an inner factorization of a vector field (see Definition 1), the contraction analysis of an appropriately defined virtual system (see Equation (5)) and lastly the evaluation of a factorization-dependent bracket between functions (see Definition 2). As such, this approach also allows one to bypass the search for metrics of partial rank [11].

The paper is organized as follows. We provide in the next section a brief review of the basics of contraction theory, but refer the reader to [7] for a complete exposition. The following section contains the main definitions and results of this paper. We will in particular spend some time on the important special case of *parallel factorizations* and revisit the results of [9]. The reader will also notice that many concepts introduced are geometric in their nature, but we will work in coordinates and leave most of the more geometric considerations to future research.

II. REVIEW OF CONTRACTION

A dynamical system is said to be contracting if, roughly speaking, it forgets its initial conditions as time passes. From this simple characterization, it follows that all trajectories of a contracting system will asymptotically converge to a *unique trajectory* independently of its initial conditions. Contraction analysis builds around this circle of ideas, aiming to derive practical criteria for the study of stabilization and synchronization of systems.

Consider the autonomous dynamical system

$$\dot{x} = f(x). \quad (1)$$

This system is said to be *contracting* if there exists an

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invertible matrix $\Theta(t, x)$ such that the symmetric part¹ of

$$(\dot{\Theta} + \Theta \frac{\partial f}{\partial x}) \Theta^{-1}$$

is uniformly negative definite. Equivalently, if I denotes the identity matrix of appropriate dimension, (1) is contracting if there exists a symmetric positive definite matrix M and a positive real number β such that

$$M \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} M + \dot{M} \leq -\beta M$$

Setting $M = \Theta^T \Theta$, it is easy to see that the two conditions above are equivalent, the former requiring the existence of coordinates for which the eigenvalues of the Jacobian of f are negative, and the latter emphasizing the existence of a metric for which the same conclusion holds.

Contraction analysis can thus be reduced to the spectral analysis of an appropriately defined operator. As a consequence, and modulo some mild assumptions, the contracting behavior is preserved through series or parallel connections as well as certain type of feedback [7].

III. MAIN RESULTS

Consider the set

$$N_c^q \triangleq \{x \in \mathbb{R}^n | q(x) = c\},$$

and observe that observe that for $q(x), r(x)$ the set of points $x \in \mathbb{R}^n$ such that $q(x) = r(x)$ can be written as

$$\bigcup_{c \in \mathbb{R}^{n_1}} N_c^q \cap N_c^r = N_0^{q-r}. \quad (2)$$

We will derive in this section conditions for the convergence towards sets of these types. The main tools used in our approach are the inner factorization of f and a commutator between q and r that we define below.

A. Inner factorizations

Consider the system of equation (1) where we let $f(x)$ be a smooth vector field. We define

Definition 1 (Inner factorization). *Given $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $q : \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$ and $z : \mathbb{R}^n \rightarrow \mathbb{R}^{n_2}$, we say that f factors through (q, z) if there exists a differentiable vector field $\bar{f} : \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{R}^n$, such that $f(x) = \bar{f}(q(x), z(x))$, $\forall x \in \mathbb{R}^n$.*

We illustrate this definition in Figure 1.

If we take either q or z to be the identity on $\mathbb{R}^n$, f will always admit a factorization through (q, z) , but this factorization may not be very informative if either $\frac{\partial \bar{f}}{\partial y}(y, z) = 0$ or $\frac{\partial \bar{f}}{\partial z}(y, z) = 0$ as we will see below. Hence, though it is easy to see that factorizations always exist, it is not clear that *non-trivial factorizations* do, i.e. factorizations not involving the identity function in the diagram of Figure 1.

Furthermore, under the assumption that non-trivial factorizations exist, it would be of great use to have *systematic*

Fig. 1. The vector field f factors through (q, z) if there exists a $\bar{f}$ such that the above diagram is commutative.

methods to find them. Remarkably, one can readily find partial answers to these questions. Indeed, problems of a similar nature have a long history in mathematics, and have been put in the forefront by Hilbert in his thirteenth problem, though they were mostly cast as relating to approximability of functions. In order to show how these results apply, we define a particular class of factorizations: the ones of additive type, i.e. $\bar{f}(q(x), z(x)) = \bar{f}(q(x) + z(x))$. The following result, which is a consequence of a Theorem of Kolmogorov, Lorentz, Arnold et al., states that under certain conditions, one can always find such factorizations:

Lemma 1. *Let $f : [0, 1]^n \rightarrow \mathbb{R}$ There exists $2n + 1$ functions $\phi_i : [0, 1] \rightarrow \mathbb{R}$ which are strictly increasing and Lipschitz with exponent c such that for every continuous function $f : [0, 1]^n \rightarrow \mathbb{R}$, one can find n continuous functions $g_i : [0, 1] \rightarrow \mathbb{R}$ such that if*

- 1) $0 < r < n$ is an integer
- 2) $\bar{f}(q, z) = \sum_{l=1}^{2n+1} [g_l(q_l + z_l), \dots, g_n(q_l + z_l)]^T$
- 3)

$$q(x) = \begin{bmatrix} \sum_{i=1}^r \phi_1(x_i) \\ \sum_{i=1}^r \phi_2(x_i) \\ \vdots \\ \sum_{i=1}^r \phi_{2n+1}(x_i) \end{bmatrix}$$

and

$$z = \begin{bmatrix} \sum_{i=r+1}^n \phi_1(x_i) \\ \sum_{i=r+1}^n \phi_2(x_i) \\ \vdots \\ \sum_{i=r+1}^n \phi_{2n+1}(x_i) \end{bmatrix}$$

then

$$f(x) = \bar{f}(q(x), z(x))$$

Proof. This is a corollary of the results in [12], [13] ■

Note that only the g_i 's depend on f , the functions ϕ_i 's are defined independently from it. For example, every real function of two variables x and y over the unit square in $\mathbb{R}^2$ can be written as

$$\sum_{l=1}^5 g(\phi_l(x) + \phi_l(y))$$

¹The symmetric part of A is given by $\frac{A+A^T}{2}$

This result is rather encouraging: it shows the existence of non-trivial factorizations for nonlinear systems and moreover it has a constructive proof, thus yielding a principled way to obtain the factorization. It is not an entirely satisfying answer to the factorization problem though. Indeed, the factorizations obtained via this route are often not practical due to the complexity of the functions ϕ_i it yields which is in turn a consequence of the rather strong restrictions that the ϕ_i be real-valued functions of a single real variable and that they be independent of f . In this context, it would be of great interest to *relax the methods used by Kolmogorov et al to allow for ϕ_i 's mapping $\mathbb{R}^p$ to $\mathbb{R}^q$ with p and q not necessarily equal to one.*

We also mention here the work of [14] on stable mappings, which investigates related, though not identical, questions. We do not expand on this due to the lack of space. Even though we have given a coordinate dependent description of the problem, it can also be stated in a coordinate free manner. Besides the works mentioned above, similar questions have been investigated under a geometric light [15].

Remark 1. *The results below extend with only minor modifications to the non-autonomous case $\dot{x} = f(t, x)$ by taking $f(t, x) = \bar{f}(t, q(x), z(x))$.*

B. Commutator of q and r

We first recall the following definition: for a vector field $f(x)$ on $\mathbb{R}^n$ and a real-valued function q , the *Lie derivative* of q along f is given by

$$L_f q = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} q$$

This is readily extended to $\mathbb{R}^{n_1}$ -valued functions $q : \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$ as

$$L_f q = \sum_{i=1}^n \begin{bmatrix} f_i(x) \frac{\partial}{\partial x_i} q_1 \\ f_i(x) \frac{\partial}{\partial x_i} q_2 \\ \vdots \\ f_i(x) \frac{\partial}{\partial x_i} q_{n_1} \end{bmatrix}.$$

Given a vector field $g(x)$, we introduce the following commutator

Definition 2. For $q, r : \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$ and $g : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^n$, we define

$$c(q, r) \triangleq L_{g \circ r} q - L_{g \circ q} r.$$

We are mostly interested in the case where $g = \bar{f}$ is a factorization of f through q, z . The commutator then can be written in coordinates as

$$c(q, r) = \frac{\partial q}{\partial x} \bar{f}(r(x), z(x)) - \frac{\partial r}{\partial x} \bar{f}(q(x), z(x)).$$

Due to the factorization requirement, the latter term in the equation above is nothing more than the time-derivative of $r(x)$; the former is the Lie derivative of r along a nonlinear transformation of the vector field $f(x)$.

Remark 2. If q and r are two vector fields in $\mathbb{R}^n$, the Lie bracket of q and r is given by $[q, r] = \frac{\partial r}{\partial x} q - \frac{\partial q}{\partial x} r$. If $\bar{f}$ is the identity for its first variable and z is a constant, we recover the bracket of vector fields: $c(q, r) = [q, r]$.

Example 1. Let $x = (x, y)$ and consider the vector field

$$f(x) = \begin{bmatrix} -x^3 - x^2 y \\ -2x^2 y \end{bmatrix}.$$

We take $q(x) = x^2 y$ and $z(x) = [x, y]^T$. A factorization is then given by

$$\bar{f} = \begin{bmatrix} -z_1^3 - z_1^2 z_2 \\ -2z_1 z_2 \end{bmatrix}.$$

Take $r(x) = x^2 y + y$; we have after a short calculation that

$$c(q, r) = 2xy(-x^3 - x^2 y) - 2x^4 y - 2x^2 y - 2xy(-x^3 - x^2 y) + 2(x^2 + 1)x^2 y = 0, \quad (3)$$

hence q and r commute. More generally, $c(q, x^2 y + \alpha y) = 0$ for $\alpha \in \mathbb{R}$.

Example 2 (The linear case). We evaluate the commutator of Definition 2 in the case of a linear vector field and linear functions. Let $f(x) = Ax$ and $q(x) = Qx$, $r(x) = Rx$ for $A, B, R \in \mathbb{R}^{n \times n}$. We have

$$L_{f(q)} r = L_{AQx} Rx = RAQx$$

and similarly $L_{f(r)} q = QAR$. We conclude that $c(q, r) = QAR - RAQ$, which reduces to the usual commutator of matrices if A is the identity matrix.

Example 3 (q is the identity). The authors of [8] have studied systems of the type $\dot{x} = f(x, x)$ and the associated virtual system $\dot{y} = f(y, x)$. This corresponds, in the setting we have introduced, to taking q and z to be the identity function.

We have the following result:

Theorem 1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth vector field admitting a factorization $\bar{f}$ through (q, z) with $q : \mathbb{R}^m \rightarrow \mathbb{R}^{n_1}$, $z : \mathbb{R}^m \rightarrow \mathbb{R}^{n_2}$. Consider the system

$$\dot{x} = f(x) \quad (4)$$

and assume that

- the auxiliary virtual system

$$\dot{y} = \frac{\partial q}{\partial x} \bar{f}(y, z) = \begin{bmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \cdots & \frac{\partial q_1}{\partial x_{n-1}} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_2}{\partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_{n_1}}{\partial x_1} & \frac{\partial q_{n_1}}{\partial x_2} & \cdots & \frac{\partial q_{n_1}}{\partial x_{n-1}} \end{bmatrix} \bar{f}(y, z) \quad (5)$$

is contracting with respect to y , for all x .

- the commutator $c(q, r) = 0$.

then all trajectories of (4) converges to $\{x \in \mathbb{R}^n | q(x) = r(x)\}$.

The proof is rather straightforward:

Proof. Because the virtual system of Equation (5) is contracting, all of the trajectories converge to a single trajectory [7]. On the one hand, observe that because $f(q, z)$ is a factorization of f , we have

$$\frac{d}{dt}q(x) = \frac{\partial q}{\partial x}f(x) = \frac{\partial q}{\partial x}\bar{f}(q, z)$$

and q is a solution of the virtual system. On the other hand, because the commutator of q and r is zero, we have

$$\frac{d}{dt}r(x) = \frac{\partial r}{\partial x}\bar{f}(q, z) = \frac{\partial q}{\partial x}\bar{f}(r, z).$$

Hence r is another solution of the virtual system, and the result is a consequence of the contraction assumption. ■

C. Parallel factorization

Let $M = \{x \in \mathbb{R}^n \text{ s.t. } q(x) = 0\}$. With the objective of directly addressing the problem of convergence to towards M , we introduce in this section a particular kind of factorization which we term *parallel factorizations*:

Definition 3. We call a factorization $\bar{f}$ of $f(x)$ through (q, z) parallel if $\bar{f}(0, z(x))$ is in the kernel of $\frac{\partial q}{\partial x}$ for all $x \in \mathbb{R}^n$:

$$\frac{\partial q}{\partial x}\bar{f}(0, z(x)) = 0.$$

Remark 3. When $q(x) = 0$, if $\frac{\partial q}{\partial x}f(x) = 0$ then $f(x)$ is parallel to the tangent space of M . We say that the factorization is parallel to refer to the fact that the above is true even when $q(x)$ is not zero.

Parallel factorizations are important in light of the following immediate but noteworthy corollary:

Corollary 1. Let M be a subset dimension m of $\mathbb{R}^n$ and $q : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$ smooth and such that $q(x) = 0 \Leftrightarrow x \in M$. Assume that system (4) admits M as an invariant subspace, i.e. if $x(0) \in M$, then the solution of Equation (4) belongs to M for $t > 0$.

If the virtual system

$$\dot{y} = \frac{\partial q}{\partial x}\bar{f}(y, z(x)) \quad (6)$$

is contracting with respect to y for all x , and if the factorization of f through (q, z) is parallel, then x tends asymptotically to M .

Proof. This is a consequence of Theorem 1 with $r = 0$. ■

We illustrate the above Corollary on the Ellipsoidal Andronov-Hopf oscillator. The Andronov-Hopf oscillator [2], which has widespread use in neuroscience and other fields, is a canonical example of nonlinear system with a limit cycle. We will in particular show that the approach of this paper allows to improve on classical contraction analysis.

(a)

(b)

(c)

Fig. 2. **Panel (a):**The virtual vector field of Equation (9). **Panel (b):**The original vector field of Equation (7). **Panel (c):** The surface $(q(x), z(x))$ and the virtual vector field restricted to this surface. This vector field agrees with the original vector field of Panel (b)

Example 4 (Ellipsoidal Andronov-Hopf). We let a, b, ρ be strictly positive real constants. The equations of motion are given by

$$\dot{x} = f(x) = \begin{bmatrix} -\frac{b}{a}x_2 - a^2x_1^3 - b^2x_2^2x_1 + \rho^2x_1 \\ \frac{a}{b}x_1 - a^2x_1^2x_2 - b^2x_2^3 + \rho^2x_2 \end{bmatrix}. \quad (7)$$

We will show that this system admits the ellipse of equation

$$a^2x_1^2 + b^2x_2^2 = \rho^2$$

as a limit cycle. We set

$$q(x) = a^2x_1^2 + b^2x_2^2 - \rho^2 \quad (8)$$

and $z(x) = (x_1, x_2)$. The virtual vector field is given by

$$\bar{f}(y, z) = \begin{bmatrix} -\frac{b}{a}z_2 - yz_1 \\ \frac{a}{b}z_1 - yz_2 \end{bmatrix}. \quad (9)$$

A short calculation yields that $\bar{f}(q(x), z(x)) = f(x)$; furthermore we have

$$\frac{\partial q}{\partial x} = (2a^2x_1, 2b^2x_2)$$

and

$$\bar{f}(0, z(x)) = \begin{bmatrix} -\frac{b}{a}x_2 \\ \frac{a}{b}x_1 \end{bmatrix}.$$

Hence

$$\frac{\partial q}{\partial x} \bar{f}(0, z(x)) = (2a^2 x_1, 2b^2 x_2) \begin{bmatrix} -\frac{b}{a} x_2 \\ \frac{a}{b} x_1 \end{bmatrix} = 0 \quad (10)$$

and the factorization is parallel.

The virtual system

$$\begin{aligned} \dot{y} &= \frac{\partial q}{\partial x} \bar{f}(y, z(x)) = (2a^2 x_1, 2b^2 x_2) \begin{bmatrix} -\frac{b}{a} x_2 - y x_1 \\ \frac{a}{b} x_1 - y x_2 \end{bmatrix} \\ &= -2y(a^2 x_1^2 + b^2 x_2^2), \end{aligned}$$

is clearly contracting for $x_1, x_2 \neq 0$. Thus Corollary 1 tells us that every trajectory is such that $q(x) \rightarrow 0$ or equivalently that every trajectory tends to the ellipse of equation $q(x) = 0$.

Remark 4. Example 4 shows the power of the approach introduced here. Indeed, the limit cycle of the oscillator escapes a classical contraction analysis, as trajectories along the ellipsoidal cycle will remain out of phase. A more subtle analysis, extending the linear point of view of Section III-D below to hypersurfaces, was made in [9], but required that the initial conditions lie outside of the disk of radius $1/3$ to obtain convergence — an unnecessary restriction from the point of parallel factorizations.

The notion of parallel factorization has a distinguished role in the investigation of stabilization problems using contraction theory. Since when combined with a contraction hypothesis and non-degeneracy conditions on the Jacobian of q it implies stabilization to a manifold [16], it should not come as a surprise that there are obstructions to the existence of such factorizations. We derive here such a necessary condition.

Consider two points $x_1, x_2 \in \mathbb{R}^m$ such that $z(x_1) = z(x_2)$, and consequently $\bar{f}(0, z(x_1)) = \bar{f}(0, z(x_2))$. The requirement that $\bar{f}$ be a parallel factorization of f through (q, z) implies that $\bar{f}(0, z(x_1))$ belongs to the kernel of both $\frac{\partial q}{\partial x}|_{x_1}$ and $\frac{\partial q}{\partial x}|_{x_2}$. This observation immediately yields a necessary condition for $\bar{f}$ to be a parallel factorization, as described in the following Proposition

Proposition 1. Let $\bar{f}(q, z)$ be a parallel factorization of f . Then

$$\forall x \in N_c^z \text{ we have } \bar{f}(0, c) \in \ker \frac{\partial q}{\partial x}|_x. \quad (11)$$

In other words the family of linear operators $\frac{\partial q}{\partial x}|_{x \in N_c^z}$ has a common eigenvector with eigenvalue 0.

In the case of Example 4, since $z(x)$ has been taken to be the identity, $N_c^z = \{c\}$ and Equation (11) reduces to Equation (10). However, Equation (11) does become informative if one tries to establish the existence of a 'simpler' parallel factorization (q, z) for the system, simpler in the sense that $z(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ instead of z mapping $\mathbb{R}^2$ to $\mathbb{R}^2$. In this case, the sets N_c^z are generically one-dimensional and because we are in $\mathbb{R}^2$, the condition of Equation (11) fully determines the direction of $\frac{\partial q}{\partial x}$ along N_c^z : if s represents a local coordinate

for the set N_c^z , we have that $\frac{\partial q}{\partial x}(s) \bar{f}(0, c) = 0$ or similarly there exists a real-valued $\alpha(s)$ such that $q(s) = \alpha(s)q(0)$ where, without loss of generality, $q(0) \neq 0$.

When combined with topological properties of M , Proposition 1 can yield precise topological obstructions to the existence of parallel factorizations. We illustrate this further below by pursuing our investigation of whether a simpler parallel factorization for the Andronov-Hopf oscillator exists.

Example 5 (Parallel factorizations for the Andronov-Hopf oscillator go through $\mathbb{R}^n$ with $n \geq 3$). As observed above, $\ker \frac{\partial q}{\partial x}$ is one dimensional, the requirement that $\bar{f}$ imposes the direction of $\bar{f}(0, z(x))$ modulo a $\mathbb{Z}_2$ action, which changes the orientation of $\bar{f}(0, z(x))$. Since $f(x)$ is never zero on the unit circle, neither is $\bar{f}(0, z(x))$. Clearly, if $x(t)$ goes around the unit circle once and $x(0) = x(1)$, then $z(x(0)) = z(x(1))$ and because z is continuous, there exists $a, b \in (0, 1)$ with $z(a) = z(b)$. Hence $\bar{f}(0, z(a)) = \bar{f}(a) = \bar{f}(b) = \bar{f}(0, z(b))$ and the unit circle is thus periodic of period less than one, which is a contradiction.

In this example, the two key ingredients are the topology of the attractor, and the nature of the flow on the attractor — the above conclusion would not have held had the Andronov-Hopf oscillator had a fixed point of the unit circle.

D. Linear subspaces

The case of M a linear subspace of $\mathbb{R}^n$ was extensively dealt with in [9]. To illustrate the use of the methods introduced above, we briefly show how to recover their main result. We refer to the original paper for a more in-depth analysis and many applications of this result.

We need the following simple Lemma

Lemma 2. Let V, W be two subspaces of $\mathbb{R}^n$ of dimensions n_1 and n_2 respectively and such that $\mathbb{R}^n = V \oplus W$. Let $\pi_V : \mathbb{R}^n \rightarrow V$ be the projection onto V parallel to W and similarly π_W the projection onto W parallel to V . Then any f factors through π_V, π_W .

Proof. Since we can write the identity on $\mathbb{R}^m$ as $x \rightarrow \pi_V x + \pi_W x$, then we immediately have

$$f(x) = f(\pi_V x + \pi_W x) = \bar{f}(\pi_V x, \pi_W x)$$

where $\bar{f}(y, z) = f(y + z)$ ■

We have the following theorem for linear invariant subspaces:

Theorem 2. Let M be a linear subspace of $\mathbb{R}^m$ of dimension n that is invariant under the flow $f(x)$, i.e. $f(M) \subset M$. Let the columns of V^T contain a basis of $M^\perp$, where the orthogonal is taken with respect the canonical inner product. If the eigenvalues of the symmetric part of $V \frac{\partial f}{\partial x} V^T$ are negative, then $\dot{x} = f(x)$ converges exponentially to M .

Proof. Let the columns of U^T contain orthonormal basis of $M^\perp$. We can clearly choose U^T, V^T such that their columns are pairwise orthogonal. Now set $q(x) = Vx$, $z(x) = Ux$

and $\bar{f}(y, z) = f(V^T y + U^T z)$. According to Lemma 2, $\bar{f}(q(x), z(x))$ is a factorization of f .

Consider the virtual system

$$\dot{y} = \frac{\partial q}{\partial x} \bar{f}(y, z(x)) = V \bar{f}(y, z(x)). \quad (12)$$

Because $f(M) \subset M$, by definition of $\bar{f}$ we have that $\bar{f}(0, z) \in M$. Hence $V \bar{f}(0, x) = 0$ and $\bar{f}(q, z)$ is a parallel factorization.

We check that the virtual system (12) is contracting; indeed, by hypothesis we have that the eigenvalues of the symmetric part of

$$\frac{\partial q}{\partial x} \frac{\partial \bar{f}}{\partial y}(y, x) = V \frac{\partial \bar{f}}{\partial y}(y, x) = V \frac{\partial f}{\partial y} V^T$$

are negative. The result is now follows from Corollary 1. ■

Remark 5. The results above relied on finding a function r such that the commutator $c(q, r)$ vanishes. This requirement is in many instances too strong. Indeed, because we are interested in asymptotic stabilization, it is enough to satisfy the condition that

$$\lim_{t \rightarrow \infty} c(q, r) = 0$$

in order to obtain the conclusion of Theorem 1. We will say in this case that q and r eventually commute. Similarly, if

$$\lim_{t \rightarrow \infty} \frac{\partial q(x)}{\partial x} \bar{f}(0, z(x)) = 0$$

then the conclusion of Corollary 1 holds. These conditions can in some cases significantly weaken the constraints put by the requirement that two functions commute, and allow one to make use of Lyapunov techniques to show that $c(x) \rightarrow 0$.

Remark 6. The results of this section on the convergence to linear subspaces of $\mathbb{R}^n$ can easily be extended to certain hypersurfaces N of $\mathbb{R}^n$, namely the ones for which there exists a globally defined coordinate system on the hypersurface. In more detail, let $u(x), p(x) \in \mathbb{R}^n$ and $r(x) \in \mathbb{R}$ be such that

$$u(x)r(x) + p(x) = x.$$

In addition, we require that $r(x) = 0$ if and only if $x \in N$ and that where $x \in N$, $u(x)$ and $p(x)$ are orthogonal. For example, we could take $r(x)$ to be the distance between x and N along $u(x)$. Note that this last requirement of equivalent to saying that $p(x)$ is a coordinate system on N . We can then define the auxiliary system

$$\dot{y} = \frac{\partial r}{\partial x} f(u(x)y + p(x))$$

as a means to study convergence to N . We refer to [10] for details, but point out that $f(u(x)r(x) + p(x))$ is a factorization of f of the additive type — actually, obtained through a factorization of the identity — and this approach falls naturally into the broader framework introduced in this paper.

IV. CONCLUSION AND OUTLOOK

In many settings in which one seeks to understand the asymptotic stabilization or synchronization of systems, there is some additional knowledge about the asymptotic behavior to be exploited. We have developed in this paper a theory that allows to incorporate this knowledge in a contraction analysis and illustrated it on several examples, demonstrating its use by improving on the classical contraction theory treatment of the Andronov-Hopf oscillator.

Perhaps more importantly, this paper introduced some concepts that are new, as far as the authors are aware, and deserve a further analysis; namely, the inner factorization of a system and the commutator of Definition 2, both of which can be revisited from a geometric perspective. In addition, though the authors have some preliminary results on what information is to be gained from a non zero commutator $c(q, r)$, this is still an open problem.

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