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On Some Aspects of Random Walks for Modelling Mobility in a Communication Network*

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This note examines a two-dimensional symmetric random walk model of mobility for terminals in a communication network. A stochastic process associated with the location of a terminal is defined. For a certain location finding scheme, the mean time between terminal transmissions is derived. We also give a first-order analysis of the trade-off between the amount of location related data to be transmitted per unit time and the accuracy about the terminals' positions.

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1. Introduction

This paper examines some aspects of a stochastic process which might be used to model the mobility of users in a mobile communication network. Specifically, we model the motion of a user as a two-dimensional symmetric random walk. This might be an appropriate model for a patrol car in a city. We assume that there is a controller who needs to be kept informed of the locations of the various users to within some tolerance. Here we will require that each user notify the controller of its exact location whenever it hits a square boundary of a certain size centered on the point it occupied (Point C in figure 1) the last time it communicated its location to the controller.

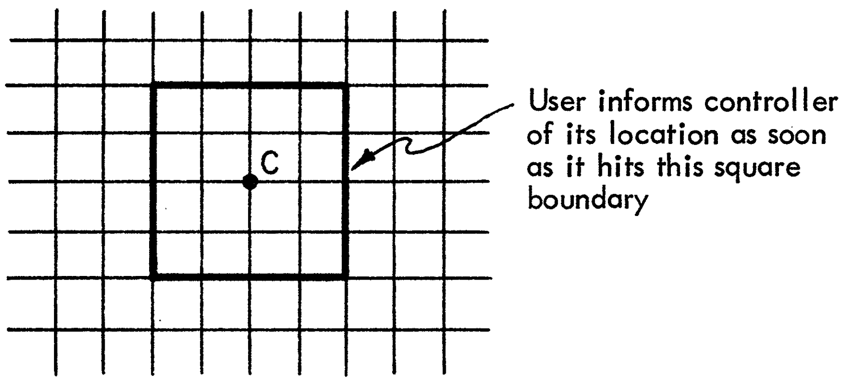

Figure 1

Since the controller knows point C, the user need only transmit an index which indicates the point at which it hits the square boundary. In the following sections, we will derive expression for (1) the expected number of steps or time a user takes to reach the boundary starting from the center C and (2) the probabilities with which the user hits the different point on the boundary and the corresponding entropy. It is clear that the smaller the size of the square boundary, the better informed is the controller of the location of the user. On the other hand, one would surmise that the user will have to transmit more data back. This trade-off is discussed in section IV.

II. Expected time between user transmissions.

In this section, we derive an expression for the expected time for a two-dimensional symmetric random walk starting from the center of a square boundary to reach some point on the boundary. For convenience, we consider the square to have length π and subdivide it into n^2 small squares as shown in figure 2 for the case n=4. We assume n to be an even integer.

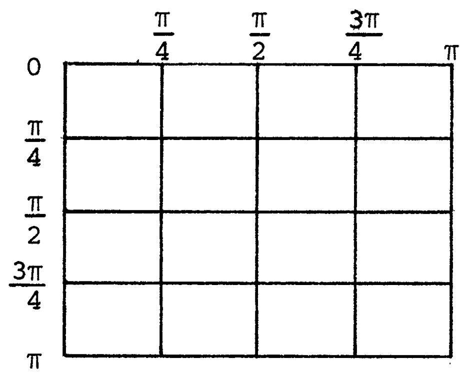

Figure 2

Let us denote the expected time to hit the boundary starting from the point (x,y) by $t_{x,y}$. Of course x and y take on values in the set $\{0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(n-1)\pi}{n}, \pi\}$. We are primarily interested in $t_{\pi/2,\, \pi/2}$ even though it will be fairly easy to write expressions for $t_{x,y}$ for arbitrary x and y.

Starting from (x,y), the user can move to (x+h,y), (x-h,y), (x,y+h) or (x,y-h) where $h = \pi/n$ and each motion occurs with equal probability 1/4. The conditional expected time to hit the boundary assuming the first step takes the user to (x',y') is $t_{x',y'}+1$. This argument shows that $t_{x,y}$ satisfies the difference equation

$$t_{x,y} = \frac{1}{4} t_{x-h,y} + \frac{1}{4} t_{x+h,y} + \frac{1}{4} t_{x,y-h} + \frac{1}{4} t_{x,y+h} + 1. \tag{1a}$$

where $0 < x < \pi$, $0 < y < \pi$ with the boundary conditions

$$t_{0,y} = t_{\pi,y} = t_{x,0} = t_{x,\pi} = 0. \tag{1b}$$

In order to solve this system of equations, we first consider the eigenvalue problem

$$4\lambda t_{x,y} = t_{x-h,y} + t_{x+h,y} + t_{x,y-h} + t_{x,y+h} \tag{3}$$

associated with the homogeneous part of (1). It should be observed that as x and y range over the set $\{\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(n-1)\pi}{n}\}$ (3) represents an $(n-1)^2 \times (n-1)^2$ matrix eigenvalue problem. The eigenvalues and eigenvectors of this system are given in [1, p.289]. The eigenvalues are

$$\lambda_{p,q} = \frac{1}{2} (\cos ph + \cos qh), \quad p,q = 1,2,\ldots,n-1 \tag{4}$$

and the corresponding eigenvectors are

$$U_{p,q} = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,n-1} \\ u_{2,1} \\ \vdots \\ u_{2,n-1} \\ \vdots \\ u_{n-1,n-1} \end{bmatrix}$$

where $u_{r,s}$ = sin prh sin qsh, p,q = 1,2,...,n-1. This can be verified by setting $t_{x,y}$=sin px sin qy and noting that

$$2 \sin px \cos ph = \sin p(x+h) + \sin p(x-h). \tag{6}$$

We now show that the eigenvectors defined by (5) are mutually orthogonal, that is,

$$D \triangleq \sum_{r,s=1}^{n-1} \sin prh \sin qsh \sin p'rh \sin q'sh = 0 \tag{7}$$

if p' ≠ p <u>or</u> q' ≠ q.

$$D = \sum_{r=1}^{n-1} \sin pr \frac{\pi}{n} \sin p'r \frac{\pi}{n} \cdot \sum_{s=1}^{n-1} \sin qs \frac{\pi}{n} \sin q's \frac{\pi}{n} \tag{8}$$

Thus to show that the eigenvectors are orthogonal it suffices to show that if p' ≠ p,

$$\sum_{r=1}^{n-1} \sin pr \frac{\pi}{n} \sin p'r \frac{\pi}{n} = 0. \tag{9}$$

$$\sum_{r=1}^{n-1} \sin pr \frac{\pi}{n} \sin p'r \frac{\pi}{n} = \sum_{r=1}^{n-1} \frac{1}{2} [\cos(p-p')r \frac{\pi}{n} - \cos(p+p')r \frac{\pi}{n}] \tag{10}$$

From [2, p.78, Eq. (418)],

$$\sum_{r=1}^{n-1} \cos r(p-p') \frac{\pi}{n} = \cos (p-p') \frac{\pi}{2} \sin (n-1)(p-p') \frac{\pi}{2n} \operatorname{cosec} (p-p') \frac{\pi}{2n} \tag{11}$$

$$=\cos (p-p') \frac{\pi}{2} [\sin (p-p') \frac{\pi}{2} \cos (p-p') \frac{\pi}{2n} - \cos (p-p') \frac{\pi}{2} \sin (p-p') \frac{\pi}{2n}] \tag{12}$$

$$\cdot \operatorname{cosec} (p-p') \frac{\pi}{2}.$$

$$= - \cos^2 (p-p') \frac{\pi}{2}.$$

In (13) we have used the fact that $p' \neq p$. Similarly,

$$\sum_{r=1}^{n-1} \cos r(p+p') \frac{\pi}{n} = - \cos^2 (p+p') \frac{\pi}{2} . \tag{14}$$

Finally, substituting (13) and (14) in (10) yields

$$\sum_{r=1}^{n-1} \sin pr \frac{\pi}{n} \sin p'r \frac{\pi}{n} = \frac{1}{2} [-\cos^2 (p-p') \frac{\pi}{2} + \cos^2 (p+p') \frac{\pi}{2}] \tag{15}$$

$$= \frac{1}{4} [\{-1- \cos(p-p')\pi\} + \{1 + \cos(p+p')\pi\}] \tag{16}$$

$$= \frac{1}{4} [- 2 \sin p\pi \sin p'\pi] = 0 \tag{17}$$

It is now fairly easy to obtain the solution of (1) in terms of the eigenvectors $U_{p,q}$. Let T denote the $(n-1)^2$ dimensional vector

$$T = \begin{bmatrix} t_{h,h} \\ t_{h,2h} \\ \vdots \\ t_{h,(n-1)h} \\ t_{2h,h} \\ \vdots \\ t_{2h,(n-1)h} \\ \vdots \\ t_{(n-1)h,(n-1)h} \end{bmatrix}$$

Since the $U_{p,q}$ are linearly independent, we can write

$$T = \sum_{p,q=1}^{n-1} \beta_{p,q} U_{p,q} \tag{18}$$

for some constants $\beta_{p,q}$ to be determined. Similarly we can write the $(n-1)^2$ vector $\vec{1}$, all of whose components are ones, as

$$\vec{1} = \sum_{p,q=1}^{n-1} \alpha_{p,q} U_{p,q}. \tag{19}$$

Taking the dot product of both sides of (19) with $U_{i,j}$ and recalling the fact that the eigenvectors $U_{p,q}$ are orthogonal, we obtain

$$\vec{1} \cdot U_{i,j} = \alpha_{i,j} |U_{i,j}|^2 \tag{20}$$

$$\text{where} \quad |U_{i,j}|^2 = \sum_{r,s=1}^{n-1} \sin^2 irh \sin^2 jsh. \tag{21}$$

It can be verified [2, p.82] that $|U_{i,j}|^2 = n^2/4$. Also,

$$\vec{1} \cdot U_{i,j} = \sum_{r,s=1}^{n-1} \sin irh \sin jsh \tag{22}$$

$$= \frac{1}{4} [1-(-1)^i][1-(-1)^j] \cot \frac{i\pi}{2n} \cot \frac{j\pi}{2n} . \tag{23}$$

In (23) we have used the fact [2, p.78] that

$$\sum_{r=1}^{n-1} \sin irh = \frac{1}{2} [1-(-1)^i] \cot \frac{i\pi}{2n} . \tag{24}$$

Thus from (20) we can write

$$\alpha_{i,j} = \frac{[1-(-1)^i][1-(-1)^j] \cot \frac{i\pi}{2n} \cot \frac{j\pi}{2n}}{n^2} . \tag{25}$$

Now, if we denote the matrix associated with the homogeneous part of (1) by A, then solving (1) is equivalent to solving $AT = IT - \vec{1}$, i.e.

$$BT = -\vec{1} \tag{26}$$

where $B = A - I$ and I is the $(n-1)^2 \times (n-1)^2$ identity matrix. Since

$$BU_{p,q} = AU_{p,q} - U_{p,q} = (\lambda_{p,q} - 1)U_{p,q} \tag{27}$$

the eigenvalues of B are $(\lambda_{p,q}-1)$ and its eigenvectors are $U_{p,q}$. From (26) and (18) we obtain

$$BT = \sum_{p,q=1}^{n-1} \beta_{p,q}(\lambda_{p,q}-1)U_{p,q} = -\vec{1} = \sum_{p,q=1}^{n-1} - \alpha_{p,q}U_{p,q}. \tag{28}$$

Therefore

$$\beta_{p,q} = \frac{-\alpha_{p,q}}{\lambda_{p,q}-1} \tag{29}$$

$$= \frac{-2[1-(-1)^p][1-(-1)^q] \cot \frac{p\pi}{2n} \cot \frac{q\pi}{2n}}{n^2(\cos \frac{p\pi}{n} + \cos \frac{q\pi}{n} - n)} . \tag{30}$$

Finally, substituting (30) into (18) yields

$$T = \sum_{p,q=1}^{n-1} \frac{-2[1-(-1)^p][1-)-1)^q] \cot \frac{p\pi}{2n} \cot \frac{q\pi}{2n}}{n^2(\cos \frac{p\pi}{n} + \cos \frac{q\pi}{n} -2)} \begin{bmatrix} \sin \frac{p\pi}{n} \sin \frac{q\pi}{n} \\ \sin \frac{p\pi}{n} \sin \frac{q2\pi}{n} \\ \vdots \\ \sin \frac{p\pi}{n} \sin \frac{q(n-1)\pi}{n} \\ \sin \frac{p2\pi}{n} \sin \frac{q\pi}{n} \\ \sin \frac{p2\pi}{n} \sin \frac{q2\pi}{n} \\ \vdots \\ \sin p\frac{(n-1)\pi}{n} \sin \frac{q(n-1)\pi}{n} \end{bmatrix} \tag{31}$$

Equation (31) gives us an expression for the expected number of steps a symmetric two-dimensional random walk takes to hit a square boundary of size n starting from some point inside the boundary. Let us denote by t_n^* the expected time $t_{\pi/2,\pi/2}$ to hit the boundary starting from the center. Then

$$t_n^* = \sum_{p,q=1}^{n-1} \frac{-8 \cot \frac{p}{2n} \cot \frac{q}{2n} \sin \frac{p}{2} \sin \frac{q}{2}}{n^2(\cos \frac{p\pi}{n} + \cos \frac{q\pi}{n} - 2)} \tag{32}$$

Figure 3 shows t_n^* for values of n ranging from 2 to 20.

III. Probabilities of boundary points

Let $p_{x,y}^*$ denote the probability that the random walk will first hit the square boundary at point (x^*,Y^*) given that it starts from the point (x,y). In this section we will derive an expression for $p_{\pi/2,\ \pi/2}^*$. Using an argument similar to that leading to (1), we see that $p_{x,y}^*$ satisfies the difference equation

$$p_{x,y}^* = \frac{1}{4} p_{x-h,y}^* + \frac{1}{4} p_{x+h,y}^* + \frac{1}{4} p_{x,y-h}^* + \frac{1}{4} p_{x,y+h}^* \tag{33a}$$

where $0 < x < \pi$, $0 < y < \pi$

with the boundary conditions

$$p_{0,y}^* = p_{\pi,y}^* = p_{x,0}^* = p_{x,\pi}^* = 0 \quad \text{except that } p_{x^*,y^*}^* = 1. \tag{33b}$$

For notational convenience, let us denote the point inside the boundary immediately adjacent to (x^*, y^*) by (x', y'). Also let P denote the $(n-1)^2$ dimensional vector

$$P = \begin{bmatrix} p^*_{h,h} \\ p^*_{h,2h} \\ \vdots \\ p^*_{h,(n-1)h} \\ p^*_{2h,h} \\ \vdots \\ p^*_{(n-1)h,(n-1)h} \end{bmatrix}$$

Then we can write (33) as

$$AP = IP - \frac{1}{4} e_{x',y'} \tag{34}$$

where A is the same matrix as in the previous section and $e_{x',y'}$ is the $(n-1)^2$ dimensional vector with 1 in the position corresponding to x',y' and 0's elsewhere. Because of the symmetry, there is no loss of generality in assuming that $x^* = \pi$ and $\frac{\pi}{2} \leq y^* \leq \pi$. In this case, $x' = (n-1)\frac{\pi}{n}$, $y' = y^*$ and the non-zero entry in $e_{x',y'}$ is the $[(n-2)(n-1) + \ell]$-th entry where $\ell = \frac{n}{\pi} y^*$.

Proceeding in a manner completely analogous to the method of the previous section, and setting

$$\frac{1}{4} e_{x',y'} = \sum_{p,q=1}^{n-1} \alpha_{p,q} U_{p,q} \tag{35}$$

we find that

$$\alpha_{i,j} = \frac{\sin [i(n-1)h] \sin j\ell h}{n^2} \tag{36}$$

Also letting

$$P = \sum_{p,q=1}^{n-1} \beta_{p,q} U_{p,q} \tag{37}$$

we find that $\beta_{p,q} = \frac{-\alpha_{p,q}}{\lambda_{p,q}-1} = - \frac{2 \sin[p(n-1)h] \sin q\ell h}{n^2(\cos ph + \cos qh -2)}$ (38)

Thus,

$$P = \sum_{p,q=1}^{n-1} \frac{2 \sin p(n-1) \frac{\pi}{n} \sin q\ell \frac{\pi}{n}}{n^2[2 - (\cos p\frac{\pi}{n} + \cos q \frac{\pi}{n})]} \begin{bmatrix} \sin \frac{p\pi}{n} \sin \frac{q\pi}{n} \\ \sin \frac{p\pi}{n} \sin \frac{q2\pi}{n} \\ \vdots \\ \sin \frac{p\pi}{n} \sin q(n-1) \frac{\pi}{n} \\ \sin \frac{p2\pi}{n} \sin \frac{q\pi}{n} \\ \vdots \\ \sin \frac{p(n-1)\pi}{n} \sin \frac{q(n-1)\pi}{n} \end{bmatrix} \tag{39}$$

From (39) we can deduce that

$$P^*_{\pi/2,\pi/2} = \sum_{p,q=1}^{n-1} \frac{2 \sin p(n-1) \frac{\pi}{n} \sin q\ell \frac{\pi}{n} \sin \frac{p\pi}{2} \sin \frac{q\pi}{2}}{n^2[2 -(\cos \frac{p\pi}{n} + \cos \frac{q\pi}{n})]}$$

Using (40), the entropy H(n) in bits associated with the random variable indicating the boundary point the random walk first hits was plotted as a function of n in figure 4. For a square boundary of size n, there are 4(n-1) boundary points which could first be hit. If we assume that all these points are equi-probable, the resulting entropy $H_{max}(n) = 2 + \log_2(n-1)$ is clearly an upperbound on H(n). It was found numerically that for even positive integers n less than or equal to 20, the difference between $H_{max}(n)$ and H(n) was monotomically increasing but did not exceed 0.18. The maximum percentage error was below 3%.

IV. Discussion

In order to give a rough indication† of the trade-off between the size, n, of the square and the amount of information to be transmitted back to the controller every time unit, a plot of $H(n)/t_n^*$ is shown in figure 5. From a practical viewpoint, the trade-off between $[2 + \log_2(n-1)]/t_n^*$ and n should be examined.

Finally, we note that the symmetric model assumed here may be refined in many ways. For example, a more appropriate model of a patrol car might exclude the possibility of the car making a U-turn. Unfortunately, such models seem to be hard to analyze.

† Note that we are neglecting the fact that there is dependence between the time of hitting the boundary and the boundary point which is first hit. A more accurate analysis should take this into account.

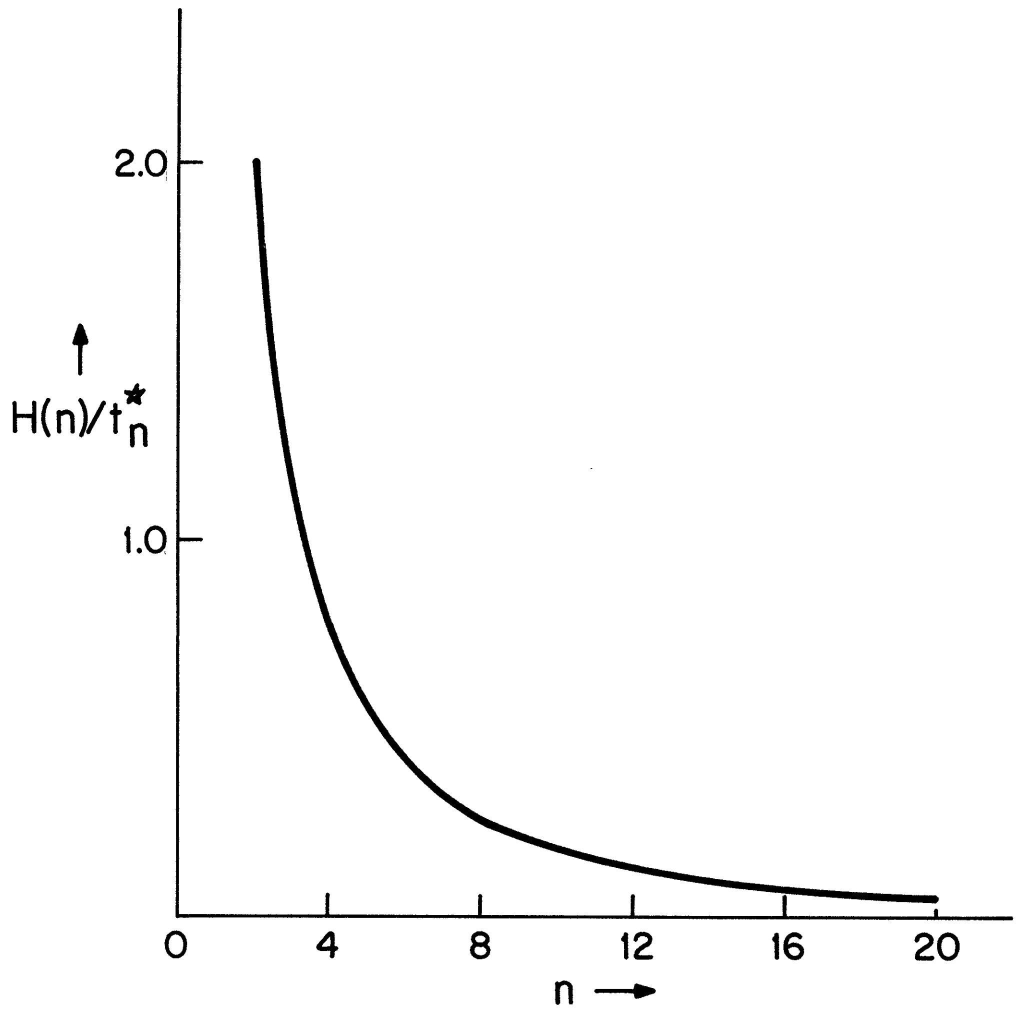

Figure 5. Data rate against boundary size.

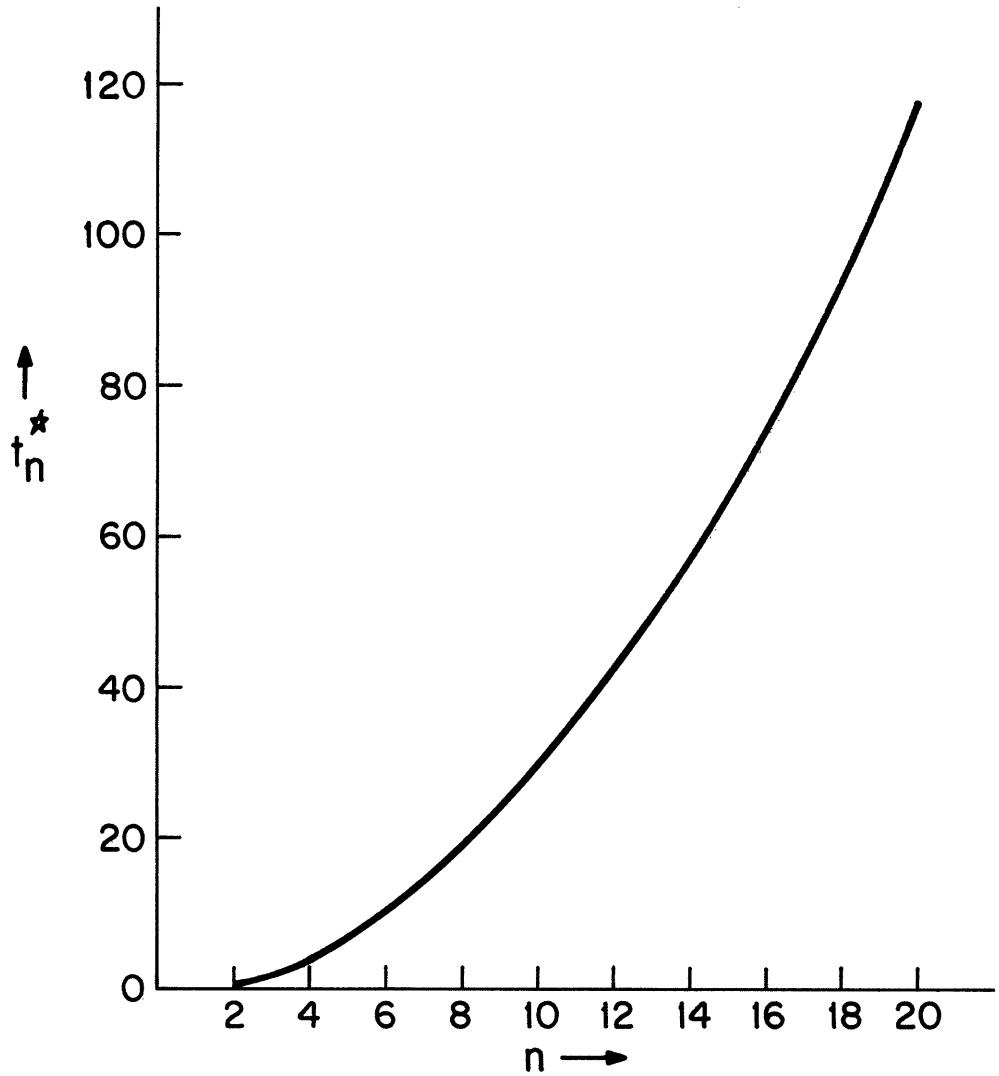

Figure 3. Expected time to hit boundary against size of boundary.

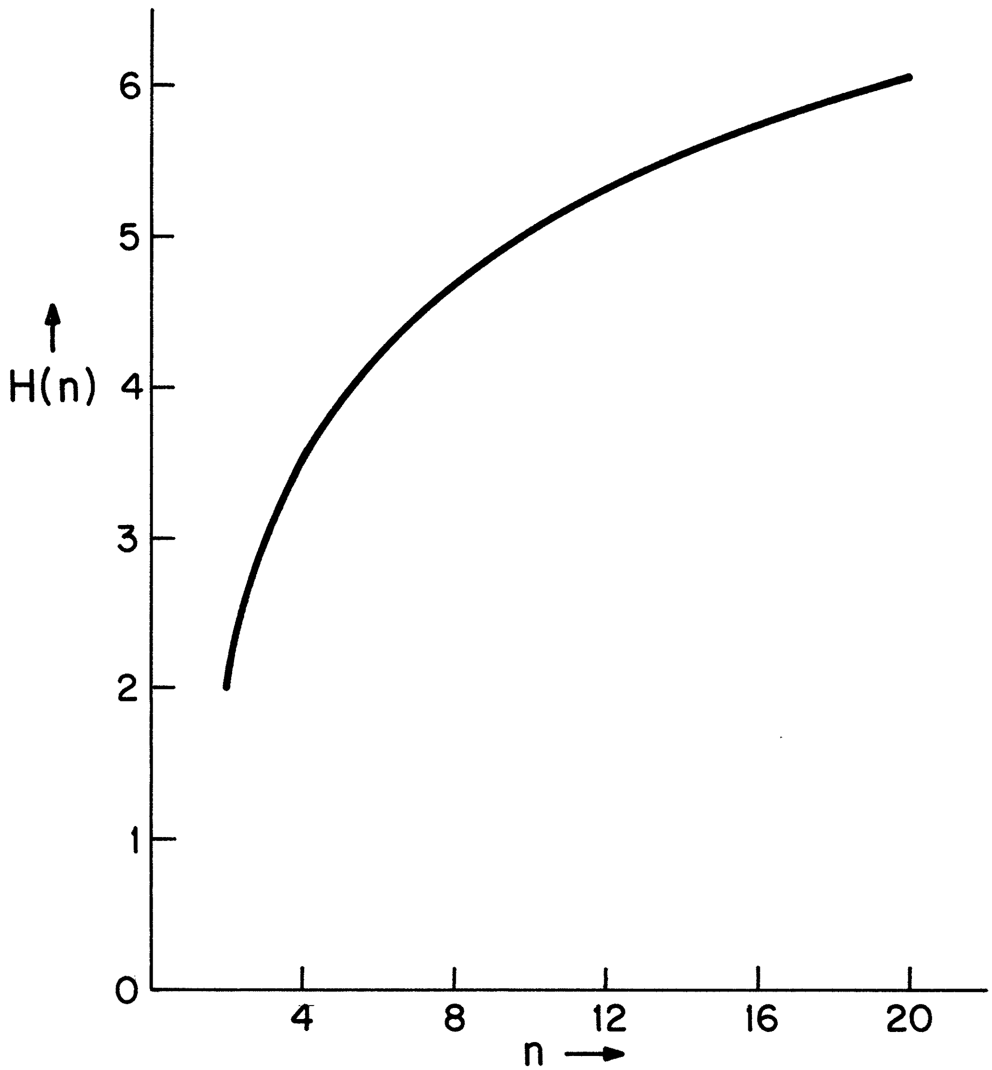

Figure 4. H(n) against boundary size.

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