

OPTIMAL PROGRAMME OF CAPITAL ACCUMULATION IN A  
MULTI-SECTOR ECONOMY\*

S. Chakravarty

Section 1.

Discussion of optimal programmes of capital accumulation has so far been almost exclusively aggregative in nature. The purpose of this paper is to generalize the results obtained for one sector models to a multi-sectoral economy, where the number of sectors is completely unrestricted. The word optimal is used here in the sense of maximizing a utility functional, where the argument function involves consumption of very different types of commodities. The formal structure for such a model was discussed by Samuelson and Solow in a paper in QJE, 1956.<sup>1</sup> Their purpose was to establish certain qualitative results in the theory of capital, particularly with deriving certain intricate transversality conditions for an infinite horizon model connected with the postulation of a least upper bound on utility, a device introduced by F. Ramsey in his well known paper.<sup>2</sup> From the economic point of view, the chief contribution of the Samuelson-Solow paper consisted in showing that even in a multiple capital goods model, there was a unique terminal composition of stocks that a society acting as a maximizer of undiscounted utilities over an infinite horizon would aim at. This is true no matter what initial conditions are assumed. This result, therefore, gives one more justification for using a single capital good model for heuristic purposes.

Our purpose in this paper is somewhat different from that of the paper of Samuelson and Solow cited earlier. In contrast with the rather general

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assumption on the side of technology that they adopt, e.g. homothetic production functions with the right curvatures, we shall assume that the production processes are described by two Leontief matrices of stock and flow coefficients. Our instantaneous efficiency frontiers would now be of the degenerate type consisting of several flat segments. This simplified treatment of technology enables us to obtain sharper formulation of the necessary and sufficient conditions that constitute a maximum for the case of strictly concave utility functions. In particular, we shall obtain an exact generalization of the Ramsey condition for the one sector case. This condition can be interpreted as a 'zero profit' condition if the price of a commodity at any point of time is measured by its marginal utility at the same instant. Thus, we establish a certain correspondence between the concave programming problems in a finite dimensional case with maximization of functionals in suitably chosen function spaces.<sup>3</sup> Further study of this correspondence is not, however, pursued in this paper, partly for the magnitude of the mathematical difficulties, and partly from the fact that the characterization obtained through our procedure based on more traditional methods of variational calculus is all that we need for our immediate objective.

While from the technological point of view, our assumption is considerably simpler than that of Samuelson-Solow, we adopt a more general viewpoint with respect to the treatment of inequality relationships. In general, we seek to ensure that all our relevant variables are nonnegative. Further, we also assume that different sectors may have different amounts of excess capacity. The introduction of inequalities is, however, much more important in our case than in the more neoclassical versions considered by Samuelson

and Solow. In the latter case, the inequality conditions may be automatically satisfied because of the strict convexity assumptions on both preference and technology.

The necessary and sufficient conditions obtained in this case have the desirable property that in some important special cases they can be explicitly integrated through suitable choice of variables.

These explicit solutions have the virtue of giving us all relevant information about an optimizing policy in an easily understandable fashion. Further, sensitiveness of optimal paths to changes in parameters is also very easily calculated when such solutions are available. The importance of this latter point arises from two sources--one economic and the other computational. The economic reason is that our estimates of parameter values are never firm when they refer to behavioral or technological characteristics. They are even more shaky when they directly concern some ethical considerations, involving shapes of relevant utility functions. Thus, it is useful to know what happens when some assumptions are relaxed. An extra economic reason is that the extremizing equations for such problems are non-linear differential equations with boundary conditions pertaining to initial and terminal configurations of the economic system. Numerical analysis of such solutions is very complicated in itself, not to speak of any perturbation analysis that we may like to carry out.

While we have pointed out the virtues of explicit solutions in so many lines, we should point out that the explicit solutions obtained here are purchased at the cost of considerable simplifications. This reflects our inability to devise a suitable algorithm for constructing suitable functions satisfying the necessary and sufficient conditions for optimality when the

admissible functions are allowed to have piecewise continuous derivatives. For simple cases, one can hope to do something through simple trial and error, although the problem is by no means trivial even for a model with few sectors.

Our explicit solutions are obtained on the assumption that no excess capacity exists and that the utility function is a linear logarithmic one. The latter assumption is much more easily relaxable than the first. In fact, we can indicate some more utility functions for which explicit solutions may be obtained. The absence of excess capacity is, however, a much stronger assumption in this context. While there is no general reason why an optimizing solution should necessarily satisfy this condition, it is worthwhile making this assumption just to see how much it entails, and because of also, the assumption that an optimizing solution may in general be expected to satisfy this condition for at least part of the planning horizon.

In our subsequent discussion we assume the planning horizon to be a finite one with appropriate initial and terminal conditions. Thus, we have a variational problem of the fixed end point variety. It should, however, be possible to extend parts of our analysis to an infinite horizon model, if the utility functions are bounded, or suitable discount factors are used to ensure convergence of the functional. What<sup>is</sup> a 'suitable' discount factor

will depend on the nature of technologically admissible growth parts which are available to society. If the utility function is assumed to be bounded, it would be useful to make the additional assumption that the least upper bound on utility is achieved for a finite vector of consumption. This will rule out possible pathologies in the undiscounted infinite horizon problem.<sup>4</sup>

There is no question that, from a fundamental conceptual point of view, adoption of a finite horizon for a planning model introduces an element of discontinuity in the treatment of preferences for consumption within the planning period and consumption beyond the planning period. Further, it is also true that there is an element of arbitrariness in the choice of terminal capital stock. This argument, however, loses part of its force in certain cases, where the optimal consumption path is shown to be relatively insensitive to the choice of terminal conditions.<sup>5</sup> Despite these limitations, the use of a finite horizon model has a considerable amount of attractiveness from the point of view of constructing planning models, as many planners tend to think along such lines.

## Section 2.

We have the following <sup>general</sup> problem: maximize  $\int_0^T U(c_1, c_2, \dots, c_n) dt$  subject to the requirement that (i)  $S \geq Bx$  and (ii)  $\dot{x} \geq Ax + c + \dot{S}$ . Here  $U$  represents total utility of consumption,  $c = (c_1, \dots, c_n)$  represent consumption levels of 'n' different commodities,  $c$  is a vector,  $x$  denotes a vector of output levels,  $S$  is the vector of capital stocks,  $\dot{S}$  is a vector of investment levels. Further, the matrices  $A$  and  $B$  represent the two Leontief matrices on current and capital account. Inequality implies that the vector of capital stock existing must be greater than or equal to the demand for capital coming from the side of production. Thus, we can have excess capacities. Inequality two implies that goods produced may be in excess of demand. Thus, we assume disposal to be costless.

Further, we should stipulate that  $S \geq 0$ ,  $\dot{S} \geq 0$ ,  $c \geq 0$ ,  $x \geq 0$ . Now, since it is highly unlikely that one will want to throw away goods once they are produced, we can write  $x = Ax + c + \dot{S}$  or,  $x = (I-A)^{-1} [c + \dot{S}]$ . Now, because  $(I-A)^{-1} \geq 0$ ,  $x \geq 0$  if  $c + \dot{S} \geq 0$ . Since we require  $c$  and  $\dot{S}$  to be separately nonnegative, we do have sufficient conditions for ensuring nonnegativity of  $x$ 's. Further, if  $S$ 's are initially greater than zero, we have  $S > 0$  for all future time because  $\dot{S} \geq 0$  for all  $t$ . Thus, we have three inequalities to be considered: (1)  $S \geq Bx$ , (2)  $c \geq 0$ , (3)  $\dot{S} \geq 0$ . We may generalize the last inequality by allowing  $\dot{S} < 0$  up to the amount permitted by the depreciation of capital, but this will require that  $x$ 's should be separately stipulated to be nonnegative. We shall consider this generalization subsequently. We now form the Lagrangian function and use the technique employed by Valentine in his Chicago dissertation in 1937.<sup>6,6a</sup> For this purpose all the inequalities are replaced by equalities:

Thus: (2)  $c_j - n_j^2 = 0$

(1)  $s_j - \sum b_{1j} x_j - \epsilon_j^2 = 0$

(3)  $\dot{s}_j - u_j^2 = 0$

where  $n_j$ ,  $\epsilon_j$ ,  $u_j$  are all real valued functions. It is obvious that if these equalities hold all the nonnegativity, conditions are automatically preserved.

We have then to form the expression

$$U(c_1, c_2, \dots, c_n) + \sum \lambda_j^c (c_j - n_j^2) + \sum \lambda_j^k (x_j - \sum a_{1j} x_j - c_j - \dot{s}_j) \\ + \sum \gamma_j (s_j - \sum b_{1j} x_j - \epsilon_j^2) + \sum \gamma_j' (\dot{s}_j - u_j^2) \quad \text{where } \lambda_j^c, \lambda_j^k, \gamma_j, \gamma_j' \text{ refer}$$

to four sets of Lagrange multipliers corresponding to each constraint. These Lagrange multipliers have their usual interpretation as prices. In particular,  $\lambda_j^k$  and  $\gamma_j$  represent flow prices of  $j^{\text{th}}$  commodity and the rental associated with  $j^{\text{th}}$  commodity if it is used as capital.

We have the following first order extremizing conditions:

(1)  $\frac{\partial F}{\partial c_1} = 0$  (4)  $\partial F / \partial n_j = 0$

(2)  $\frac{\partial F}{\partial x_j} = 0$  (5)  $\partial F / \partial \epsilon_j = 0$

(3)  $\frac{\partial F}{\partial s_j} - \frac{d}{dt} (\partial F / \partial \dot{s}_j) = 0$  (6)  $\partial F / \partial u_j = 0$

In addition, we have to assume that the Lagrange multipliers are nonnegative.

Because our utility function is strictly concave, one would expect the above first order conditions to be sufficient for a maximum. Further, this will also rule out possible pathologies connected with the existence of conjugate points, since, if a maximum exists locally, it will also be a global maximum.<sup>7, 8</sup>

We can now write out the equivalent of the above conditions for our problem:

$$(1) \quad u_j + \lambda_j^c - \lambda_j^k = 0$$

$$(2) \quad (I - A)^j, \lambda^k - (b^j, \gamma) = 0$$

$$(3) \quad x_j + \lambda_j^k - \dot{x}_j' = 0$$

$$(4) \quad n_j \lambda_j^c = 0$$

$$(5) \quad \epsilon_j \delta_j = 0$$

$$(6) \quad u_j x_j' = 0$$

The last three conditions are the Valentine conditions. For each  $j$ , we have the requirement that either  $\lambda_j = 0$  or  $\eta_j = 0$  which implies  $c_j = 0$ . Thus we have  $\lambda_j c_j = 0$ ,  $x_j' S_j = 0$ , or  $(x, S - Bx) = 0$  for each component.

Thus, the interpretation of the Valentine conditions would be that either the multipliers associated with the inequality constraints are zero, or the corresponding variables are zero.

Now assume that the Lagrange multipliers associated with the inequality constraints on  $c_j$  and  $\dot{S}_j$  are zero, but that the "no excess capacity" condition holds. This is a subcase of our general case considered in the preceding set of equations.

Then we have the following conditions:

$$(1)' \quad u_j = \lambda_j^k$$

$$(2)' \quad ((I - A)^j, \lambda^k) - (b^j, \gamma) = 0$$

$$(3)' \quad x_j + \lambda_j^k = 0$$



$$(4) \lambda_j^c = 0$$

$$(5) \epsilon_j = 0$$

$$(6) \pi_j' = 0$$

Now taking (1)' and (3)' together, we get (7)  $-\dot{u}_j = \pi_j$ . In other words, absolute value of the rate of change in marginal utility with respect to  $j^{\text{th}}$  item equals the rental associated with the  $j^{\text{th}}$  type of capital. Now we can interpret values of marginal utilities of consumption of different commodities along the optimizing paths as their respective prices, provided they are suitably normalized. In that case, we can interpret the above condition as the 'zero profit' condition of programming theory. If marginal utility of any particular commodity is used as a numeraire: this restates the well known proposition of capital theory that along an intertemporally efficient path of capital accumulation, the rate of interest (which is the own rate of interest in terms of the numeraire commodity) must equal the own rate of interest for the  $j^{\text{th}}$  commodity plus the relative rate of change of price of  $j^{\text{th}}$  commodity in terms of the numeraire.<sup>9</sup>

Substituting (7) in (2)', we get

$$((I - A)^j, u) - (b^j, \dot{u}) = 0, \text{ for } j = 1, 2, \dots, n \text{ or}$$

$$u(I - A) = -\dot{u}B \text{ where } u = (u_1, \dots, u_n). \text{ This condition is seen to be}$$

none other than the multisector generalization for the well known Ramsey condition that the relative rate of change in the marginal utility of consumption is equal to the absolute value of marginal productivity of capital.<sup>10</sup> The striking thing about a linear technology is that this condition gives essentially a system of first order linear differential equations with marginal utilities as variables. It is this linearity property which enables us to exhibit explicit solutions with special forms of utility functions. Before considering these

special cases, we should spell out the economic significance of this subcase which makes us operate at full capacity all the time.

To start with, the principal desirable feature is that all the economically relevant variables are strictly positive in this special case. Secondly, we do not leave any capital idle during any period. This second feature is the one that gives rise to difficulties in dynamic Leontief models if the initial conditions are arbitrary. However, if we have an open ended planning model with a finite time horizon, the usual objections lose part of their force. Moreover, for any specified economy, "initial conditions" are the result of decisions taken in the past and if these are based on competitive rules, one expects that these decisions need not be altogether contradictory in the sense that they cannot generate any viable path for the economy to follow. It is, nonetheless, true that these desirable features do not in any way preclude the possibility that the optimal path for the economy may deviate from this path at least for some stretch of the planning period. Hence, the desirability of having an extended treatment involving inequalities.<sup>11</sup>

This means that the optimal policies will have discontinuities. This requires that in addition to the necessary conditions stated above, they will have to satisfy the Weierstrass-Erdmann corner conditions. These conditions will imply that, in this time independent case, the function denoting the first integral of this system will have to assume the same value at all corners.<sup>12</sup> Of the various possible sequences of subarcs defining the W-E conditions and the first necessary conditions, we have to select the ones satisfying the Legendre-Clebsch conditions.<sup>13</sup>

Consider now another special case where  $c$  and  $\dot{S}$  may be allowed to assume zero values, but the no-excess capacity assumption is maintained throughout.

Then, we have  $u_j - \lambda_j^k < 0$ , as  $\lambda_j^c > 0$ . Thus, the marginal utility of the  $j^{\text{th}}$  consumption good is less than the price of the  $j^{\text{th}}$  good; hence, the good is not consumed. In this case, it is very unlikely that  $\dot{S}_j = 0$ ; hence, our guess will be to put  $\dot{S}_j = 0$ .

Since rentals are not allowed to fall to zero in any time period, we would get  $\dot{S}_j > 0$ , which implies  $\epsilon_j = 0$ . Since  $\dot{S}_j > 0$  we have  $\dot{S}_j = 0$ .

Thus, we have  $\dot{S}_j = -\dot{\lambda}_j^k$ . Since  $u_j < \lambda_j^k$ , we can no longer interpret  $\dot{S}_j$  as equal to the rate of change in the marginal utility of consumption. Our optimal path now has the following properties:

- (a) Consumption of  $j^{\text{th}}$  commodity is zero.
- (b) We let all  $j$  to be used as inputs, current and capital.
- (c) Rental associated with  $j$  equals the change in the price of the  $j^{\text{th}}$  commodity.
- (d)  $j^{\text{th}}$  commodity, if it is produced, (an indecomposable system will imply  $j^{\text{th}}$  commodity is necessarily produced) will have a position price.

As  $\dot{S}_j > 0$ , it implies that  $\lambda_j^k$  declines over time.

If  $u_j < \lambda_j^k$  to start with, we may get a situation where  $\lambda_j^k$  has declined enough to make it worthwhile to consume it. This may imply under certain assumptions on the technology and on the preference function that  $\dot{S}_j = 0$  in the new phase. This will be an illustration of the theorem which goes under the name of 'bang bang principle'.<sup>14</sup> For one or two sector models, the above proposition can be illustrated in much more detail.

Let us now relax the assumption that  $\dot{S}_j \geq 0$ . Instead we assume that  $\dot{S}$  can be negative up to an amount permitted by the decumulation of capital. Thus, we have  $\dot{S} \geq -h_j S_j$  or,  $\dot{S}_j + h_j S_j \geq 0$ . In this case, we have to add separately

that the  $x$ 's are nonnegative. Thus, our Lagrangian expression is now changed to the following one:

$$U(c_1, \dots, c_n) + \sum \lambda_j^c (c_j - n_j^2) + \sum \lambda_j^k (x_j - \sum a_{ij} x_j - c_j - \dot{s}_j) \\ + \sum \epsilon_j (s_j - \sum b_{ij} x_j - \epsilon_j^2) + \sum \gamma_j' (\dot{s}_j + h_j s_j - u_j^2) + \sum \gamma_j'' (x_j - m_j^2).$$

We have now the following first order condition:

- (1)  $u_j + \lambda_j^c - \lambda_j^k = 0$
- (2)  $((I - A^j), \lambda^k) - (b^j, \gamma) + \gamma_j'' = 0$
- (3)  $\gamma_j + h_j \gamma_j' + \dot{\lambda}_j^k - \dot{\gamma}_j' = 0$
- (4)  $n_j \lambda_j^c = 0$
- (5)  $\epsilon_j \gamma_j = 0$
- (6)  $u_j \gamma_j' = 0$
- (7)  $\gamma_j'' m_j = 0$

A comparison with the earlier models will indicate that the set of optimizing conditions is now somewhat changed. But the qualitative structure of the solutions is unchanged, as one would expect.

### Section 3.

We shall consider the special case of a linear logarithmic utility function.<sup>15</sup> Our utility functional can now be written as  $\int_0^T (\sum p_i \log c_i) dt$ . Since we assume that there is no excess capacity, we can write

$$c_i = x_i - \sum a_{ij} x_j - \sum b_{ij} \dot{x}_j.$$

For this case, we can derive the necessary Euler-Lagrange equations from the generalized Ramsey condition by putting  $u_j = p_j/c_j$ . Alternatively, we can derive the Euler-Lagrange equations through direct substitution of  $x_i$ 's for  $c_i$ 's. The latter procedure is the more straight forward one and the sufficiency of the Euler-Lagrange condition for a global maximum is directly proved in this case by the simple Legendre condition of diminishing marginal utility. We shall, therefore, use this straight forward derivation to indicate its simplified character.

Here our functional gets transformed into the following expression:

$$\begin{aligned} \int_0^T \sum p_i \log (x_i - \sum a_{ij} x_j - \sum b_{ij} \dot{x}_j) dt \\ = \int_0^T p_1 \log (x_1 - \sum a_{1j} x_j - \sum b_{1j} \dot{x}_j) \\ + \int_0^T p_2 \log (x_2 - \sum a_{2j} x_j - \sum b_{2j} \dot{x}_j) \\ + \int_0^T p_n \log (x_n - \sum a_{nj} x_j - \sum b_{nj} \dot{x}_j). \end{aligned}$$

This means

$$\int_0^T \sum_i (p_i \log c_i) dt = \int_0^T F(x_1, x_2, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) dt.$$

Here  $x_i$ 's represent output levels,  $(a_{ij})$  the matrix of current input requirements and  $(b_{ij})$  is the matrix of capital requirements per unit of output. The

Euler-Lagrange equation for Sector 1 is the following expression:

$$\frac{p_1(1-a_{11})}{x_1 - \sum a_{1j}x_j - \sum b_{1j}\dot{x}_j} - \frac{p_2 a_{21}}{x_2 - \sum a_{2j}x_j - \sum b_{2j}\dot{x}_j} \dots - \frac{p_n a_{n1}}{x_n - \sum a_{nj}x_j - \sum b_{nj}\dot{x}_j} \\ = \frac{d}{dt} \left( \frac{-p_1 b_{11}}{x_1 - \sum a_{1j}x_j - \sum b_{1j}\dot{x}_j} \dots \frac{-p_2 b_{21}}{x_2 - \sum a_{2j}x_j - \sum b_{2j}\dot{x}_j} - \frac{-p_n b_{n1}}{x_n - \sum a_{nj}x_j - \sum b_{nj}\dot{x}_j} \right)$$

$$\text{or, } \frac{p_1(1-a_{11})}{c_1} - \frac{p_2 a_{21}}{c_2} \dots - \frac{p_n a_{n1}}{c_n} = \frac{d}{dt} \left( \frac{-p_1 b_{11}}{c_1} + \frac{-p_2 b_{21}}{c_2} + \dots + \frac{-p_n b_{n1}}{c_n} \right)$$

$$\text{or, } \frac{p_1(1-a_{11})}{c_1} - \dots - \frac{p_n a_{n1}}{c_n} = \frac{p_1 b_{11}}{c_1^2} \frac{dc_1}{dt} + \frac{p_2 b_{21}}{c_2^2} \frac{dc_2}{dt} \dots + \frac{p_n b_{n1}}{c_n^2} \frac{dc_n}{dt}$$

Now, for Sector 2, we have similarly the following condition:

$$\frac{-p_1 a_{12}}{c_1} + \frac{p_2(1-a_{22})}{c_2} \dots - \frac{p_n a_{n2}}{c_n} = \frac{p_1 b_{12}}{c_1^2} \frac{dc_1}{dt} + \dots + \frac{p_n b_{n2}}{c_n^2} \frac{dc_n}{dt}$$

and similarly for all the other sectors,  $i = 3, \dots, n$ .

Now use as new variables  $\xi_i$ 's where  $\xi_i = \frac{1}{c_i}$

$$\text{Hence, } \frac{dc_1}{dt} \left( -\frac{1}{c_1^2} \right) = \frac{d\xi_1}{dt} \quad \text{or, } \frac{-d\xi_1}{dt} = \frac{1}{c_1^2} \frac{dc_1}{dt}$$

We have now the following system of linear differential equations with constant coefficients in  $\xi_i$ 's:

$$p_1(1-a_{11})\xi_1 - \dots - p_n a_{n1}\xi_n = - \left( p_1 b_{11} \frac{d\xi_1}{dt} + \dots + p_n b_{n1} \frac{d\xi_n}{dt} \right) \\ -p_1 a_{12}\xi_1 + p_2(1-a_{22})\xi_2 - \dots - p_n a_{n2}\xi_n = - \left( p_1 b_{12} \frac{d\xi_1}{dt} + \dots + p_n b_{n2} \frac{d\xi_n}{dt} \right) \\ -p_1 a_{1n}\xi_1 - p_2 a_{2n}\xi_2 \dots + p_n(1-a_{nn})\xi_n = - \left( p_1 b_{1n} \frac{d\xi_1}{dt} + \dots + p_n b_{nn} \frac{d\xi_n}{dt} \right)$$

This leads to the following system:

$$\begin{pmatrix} p_1 (1-a_{11}) & -p_2 a_{21} & -p_n a_{n1} \\ -p_1 a_{12} & +p_2 (1-a_{22}) & -p_n a_{n2} \\ -p_1 a_{1n} & -p_2 a_{2n} & +p_n (1-a_{nn}) \end{pmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_n \end{bmatrix} = \begin{pmatrix} -p_1 b_{11} & -p_n b_{n1} \\ -p_1 b_{1n} & -p_n b_{nn} \end{pmatrix} \begin{bmatrix} \frac{d\xi_1}{dt} \\ \frac{d\xi_n}{dt} \end{bmatrix}$$

$$\begin{pmatrix} 1-a_{11} & -a_{21} & -a_{n1} \\ -a_{1n} & -a_{2n} & 1-a_{nn} \end{pmatrix} \begin{pmatrix} p_1 \dots 0 \\ 0 \ p_2 \ 0 \\ 0 \dots p_n \end{pmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_n \end{bmatrix} = \begin{pmatrix} -b_{11} & -b_{n1} \\ -b_{1n} & -b_{nn} \end{pmatrix} \begin{pmatrix} p_1 \ 0 \dots 0 \\ 0 \ p_2 \dots 0 \\ 0 \dots p_n \end{pmatrix} \begin{bmatrix} \frac{d\xi_1}{dt} \\ \frac{d\xi_n}{dt} \end{bmatrix}$$

In condensed matrix notation, this may be written as

$$\{P(I-A)\}' = -(PB)' \frac{d\xi}{dt}$$

where P is a diagonal matrix of valuation coefficients  $\{\xi\}$  is a vector which represents reciprocals of consumption of different items.

$$-(B'P') \frac{d\xi}{dt} = \{(I-A)' P'\} \xi$$

$$\frac{d\xi}{dt} = -\{P'^{-1} B'^{-1} (I-A)' P'\} \xi$$

$$\frac{d\xi}{dt} = -[P'^{-1} \{(I-A)'^{-1} B'\}^{-1} P'] \xi$$

Hence the solution can be written as

$$Q(t) = e^{Mt} Q \text{ where } M = -[P'^{-1} \{((I-A)')^{-1} B'\}^{-1} P']$$

and a vector to be determined with the help of the necessary boundary conditions of the problem.

There are several interesting features about this solution. First, the optimal policy is given by the weighted combination of several exponential paths, which is represented by the matrix exponential. Secondly, we should note that

the expression  $\{((I-A)')^{-1} B'\}$  is a nonnegative square matrix because  $((I-A)')^{-1}$  is nonnegative and  $B'$  is nonnegative. Inverse of this matrix is, however, not nonnegative. However, we know that characteristic roots of inverse matrices are reciprocals of the characteristic roots of the original matrices whose inverses are being considered. Here, since the matrix  $\{((I-A)')^{-1} B'\}$  is a Frobenius matrix, we know that under rather general conditions this matrix has a maximal positive characteristic root with an associated positive characteristic vector. It would, therefore, follow that the reciprocal of this root will figure within the spectrum of the inverse of this matrix. Since the matrix  $M$  is similar to the matrix  $\{((I-A)')^{-1} B'\}^{-1}$ , we know that the positive root which is the reciprocal of the maximal root of the Frobenius matrix  $\{((I-A)')^{-1} B'\}$ , will figure as a characteristic root of the matrix  $M$ . Thus, we can assert that the  $(-\xi)$ 's will possess a steady growth solution. Hence, it follows that  $\xi$ 's will possess a steady decay solution. Since  $\xi$ 's are reciprocals of  $c$ 's, this means that  $c$ 's will possess a steady growth solution.

While  $\xi$ 's form a linear dynamic system, the  $c$ 's will, of course, be non-linear functions of time, and these will enter as nonhomogenous elements in the equation  $\hat{c}(t) = x(t) - Ax(t) - Bx(t)$  where  $\hat{c}(t)$  is the optimal solution. Computationally, if  $\xi$ 's are solved, we can find out  $c(t)$  at each point of time.

An important question in this context relates to the nonnegativity of  $\xi$ 's and hence  $c$ 's for every 't'. This is a problem which the traditional variational methods do not generally consider. In simpler cases, where an explicit solution is available, one can easily check whether the nonnegativity condition is satisfied. In our case, since we have an explicit solution, it should be numerically easy to check whether for given vectors  $k(0)$  and  $k(T)$ ,



we have nonnegative  $c(t)$  for  $0 \leq t \leq T$ . It should be added, however, that even if  $c$ 's were all to preserve nonnegativity for any ' $t$ ', a situation which we cannot guarantee in advance, there is no reason to think that  $x$ 's will preserve nonnegativity for all ' $t$ '. In fact, the discussion on the dynamic Leontief model following from the fundamental work of Dorfman, Samuelson and Solow<sup>16</sup> would lead one to infer that the nonnegativity condition is violated for an infinite  $T$ , save in special cases. For a finite horizon, we may be more lucky, but even in that case, there is no easily applicable list that we can rely upon to ensure nonnegativity in advance. To ensure nonnegativity, we have to fall back on the more general model discussed in Section 2 in this paper. As the discussion there indicates, computation of an optimal solution in that case raises problems which have not been tackled in the mathematical literature in its proper generality.

In addition to the linear logarithmic utility function, we can think of a few other utility functions for which the above technique will yield readily computable solutions. First, we can assume that  $U = c_1^{\alpha_1} c_2^{\alpha_2} \dots c_n^{\alpha_n}$  where  $\alpha$ 's are constants. Since we are maximizing an expression  $\int U dt$ , this utility function gives us a different optimizing solution from the one associated with  $\int \sum p_i \log c_i dt$ . This is obvious because  $\int U dt$  admits only of arbitrary linear transformations. The other case is a much simpler one, namely, the situation where the utility function is quadratic, with a negative definite quadratic form. Here, Euler equations are linear in the consumption levels, and hence the analysis of the resulting time paths is much easier.

Section 4.

Pontryagin's method for determining optimal controls can be applied to our problem in a straightforward fashion on simple terminological adaptations.<sup>17</sup>

His problem is to determine the vector function  $u(t)$  which maximizes an integral  $\int_0^T f^0(x, u) dt$  where the  $x$ 's and  $u$ 's are connected by differential equations such as  $\frac{dx}{dt} = f(x, u)$  where  $f$  is a vector function connected with vector variables  $x$  and  $u$ . This is the same problem as the one encountered in the ordinary calculus of variations of maximizing an integral subject to non-integrable constraints. As a special case, the solution to the simplest  $n$ -variable variational problem can be obtained from Pontryagin's maximum principle if  $\frac{dx_i}{dt} = u_i$  where  $u_i$ 's are our control variables. Pontryagin allows  $u$  to belong to a compact subset of a topological space.

Now Pontryagin's procedure requires the introduction of a number of auxiliary variables  $\psi_0, \psi_1 \dots \psi_n$  satisfying the condition

$$\frac{d\psi_1}{dt} = -\sum_{\alpha=0}^n \frac{\partial f^\alpha(x, u)}{\partial x_1} \psi_\alpha. \quad \text{These variables are conjugate with variables}$$

$x_1 \dots x_n$ . In fact, it can be proved that  $\sum \psi_\alpha x_\alpha = N$  for  $0 \leq t \leq T$  where  $N$  is constant. Thus, if  $\psi$ 's are interpreted as prices, this implies that the total value of output at any point on the optimal path is a constant. This is the same as the law of conservation of energy in classical mechanics when the Hamiltonian does not involve time explicitly.

Once the  $\psi$ 's are defined, Pontryagin's procedure consists of maximizing the function  $H = \sum_{\alpha=0}^n \psi_\alpha f_\alpha$  with respect to the control variables  $u$ 's.

In our case, the control variables are  $c$ 's. Thus, we have the following problem:

$$H(t) = \Psi_0 (p, \log c) + (\Psi, B^{-1} (I - A)x) - (\Psi, B^{-1} c).$$

$\Psi_0$  can be proved to be a constant for optimal solution to this class of problem.

The optimal values for  $c$ 's are obtained by putting  $\partial H / \partial c = 0$ .

On the other hand, we have

$$\begin{aligned} \frac{d\Psi_1}{dt} &= - \sum_{\alpha=1}^n \Psi_{\alpha} \frac{\partial f_{\alpha}}{\partial x_1} \\ &= - \Psi_1 B_{11} + \Psi_2 B_{21} + \dots + \Psi_n B_{n1} \end{aligned}$$

$$\text{Hence } \frac{d\Psi}{dt} = - \{B^{-1} (I - A)\}' \Psi.$$

Hence, we have  $\Psi(t) = e^{-\{B^{-1}(I-A)\}' t} Q$  where  $Q$  is a vector of initial conditions.

Once the  $\Psi$ 's are known, insertion of  $\Psi$ 's in the condition  $\partial H / \partial c = 0$  will give us back the equation for the optimal path obtained by classical methods.

The equations for  $\Psi$ 's indicate that they are independent of the utility function used. Therefore, so long as the technology is the same, the solution for  $\Psi$ 's will be the same. Moreover, since the equation for  $\Psi$ 's is always linear homogeneous, thus, even in the general case where technology is non-linear, the solution of  $\Psi$ 's is simpler than that of the state variables  $x$ 's. However, in the more general case this will be of only limited help because the relevant partial derivatives will not be constant and, hence, we would not be able to solve the  $\Psi$ 's quite independently of the state variables  $x$ 's.

We may interpret the  $\Psi$ 's as the values assumed by the marginal utilities of different commodities above the optimal path and this will be in perfect correspondence with the equations described in the earlier section. The correspondence is easily established if we note that  $p_i/c_i$ 's are the relevant marginal utilities of the different commodities for a linear logarithmic utility function.

## FOOTNOTES

1. Samuelson, P. A., and R. M. Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, 1956.
2. Ramsey, F. P., "A Mathematical Theory of Saving," Economic Journal, 1928.
3. Hurwicz, L., "Programming in Linear Spaces," in Arrow, Hurwicz, and Uzawa, Studies in Linear and Non-Linear Programming.
4. Chakravarty, S., "The Existence of an Optimum Savings Programme," Econometrica, 1962.
5. Chakravarty, S., "Optimal Savings with Finite Planning Horizon," International Economic Review, September, 1962.
6. Valentine, F. A., Contributions to the Calculus of Variations, 1933-37. The Problem of Lagrange with Differential Inequalities as Added Side Conditions. University of Chicago Press, 1937.
7. One should point out that the existence of a maximum is assured only if the relevant function space is compact. Proof of compactness is much more complicated for a function space than for the finite dimensional cases one encounters in calculus. We do not propose to enter into these discussions in this context. If compactness property does not hold, we have to replace the maximum of our functional by supremum. One should, however, expect on intuitive grounds that compactness property will carry through for the present problem.
8. Assuming compactness and upper semi-continuity of the functional, we know that there exists a maximum. From this we can conclude that there exists a solution of the differential equations satisfying the boundary conditions. Thus, we can prove global existence, even though the derivatives are not everywhere continuous. This is an application of Dirichlet's principle. The preceding assumptions on the function space and on the nature of the relevant functional are important in justifying the use of Dirichlet's principle. There are two principal reasons why one would expect such conditions to be satisfied for a model with finite planning horizon. First, the corresponding problem in discrete time has a maximum from the Kuhn-Tucker theorem. This is quite easy to verify as our preference function is strictly concave and the constraint condition is satisfied in this case. Secondly, for the case where there is no excess capacity, we can get explicitly integrable differential equations which will connect any two positions which are feasible from the point of view of technology. In the general case with excess capacities, we will get only a finite number of corners which will not change the nature of the problem. In fact, it is possible that even without the stipulation of no excess capacity condition, we can get a system such as described on Page 9. Hence, the existence of a solution is not affected by corners. Their nature, of course, is considerably modified.

Footnotes (continued)

9. Samuelson, P. A., "Efficient Paths of Capital Accumulation in Terms of Calculus of Variations," in Mathematical Methods in the Social Sciences, edited by Kenneth Arrow, Samuel Karlin, and Patrick Suppes, Stanford University Press, 1959.
10. This should be distinguished from the Keynes-Ramsey condition which states the saving rule:
 
$$U(x) \frac{dc}{dt} = B - U(x) \text{ where } U(x) \text{ denotes marginal utility of consumption, } \frac{dc}{dt} \text{ denotes the rates of capital formation, and } B, \text{ the bliss level of utility.}$$
 For a system of Euler equations that does not involve time explicitly, and makes the other Ramsey assumptions of an infinite horizon and a least upper bound on utility, the Ramsey condition can be shown to imply the Keynes-Ramsey condition quoted here.
11. Compare Solow's conjecture in Footnote 18, p. 42, of Solow, "Competitive Valuations in Dynamic Leontief Model," Econometrica, 1959, pp. 30-53.
12. Formally, the conditions can be written as follows:
 
$$(1) \left( \frac{\partial F}{\partial \dot{s}_k} \right)_- = \left( \frac{\partial F}{\partial \dot{s}_k} \right)_+ \quad (2) \left( F - \sum_j \dot{s}_j \frac{\partial F}{\partial \dot{s}_j} \right)_- = \left( F - \sum_j \dot{s}_j \frac{\partial F}{\partial \dot{s}_j} \right)_+$$
 If the system is time independent, the expressions within the parentheses in (2) are constant.
13. The Legendre-Clebsch condition is that  $F_{\dot{k}\dot{k}} < 0$  when we have a problem of maximizing
 
$$\int_0^T F(k, \dot{k}, t) dt.$$
 For the n variable case, there is a natural generalization in view of the negative definiteness of the relevant Hessian matrix.
14. Bellman, R. Adaptive Control Processes.
15. Radner, R., Notes on the Theory of Planning, Athens, 1963, has also used a linear logarithmic utility function in connection with a completely different model of technology. He may also adopt the convention from Radner that whenever some  $c_1 = 0$  by definition since the commodity cannot be used for consumption, the corresponding  $p_1 = 0$ , with the stipulation that  $0 \log 0 = 0$ .

Footnotes (continued)

16. Dorfman, R., Samuelson, P., and R. Solow, Linear Programming and Economic Analysis, Chapter 11.
17. Pontryagin, L. S., et al, The Mathematical Theory of Optimal Processes, Interscience Publishers, Inc., 1962.

- 6a. The Valentine procedure as used in this paper is the same as used by G. Leitmann and others in the volume "Optimization Techniques," Academic Press Inc., New York, 1962. On the other hand, the Valentine procedure can be more generally written as

$$g(x, x, t) - z(t)^2 = 0$$

where  $z(t)$  may now be allowed to be discontinuous. In this way, the class of admissible solutions is widened. This is also the procedure adopted by L. Berkowitz in his paper "Variational Methods in Problems of Control and Programming," Journal of Mathematical Analysis and Applications, 1961, pp. 145-169. The economic significance of these two interpretations was kindly pointed out to me by Professor L. Hurwicz of the University of Minnesota. However, from the point of view of economic planning, the procedure adopted in this paper seems to me to be the more natural one.