Rational Families of Vector Bundles on Curves

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2002

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Abstract

We find and describe the irreducible components of the space of rational curves on moduli spaces M of rank 2 stable vector bundles with odd determinant on curves C of genus $g \geq 2$. We prove that the maximally rationally connected quotient of such a component is either the Jacobian $J(C)$ or a direct sum of two copies of the Jacobian. We show that moduli spaces of rational curves on M are in one-to-one correspondence with moduli of rank 2 vector bundles on the surface $\mathbb{P}^1 \times C$.

Thesis Supervisor: Joseph Harris
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To my parents, Piroska and Valeriu Castravet

Acknowledgments

First and foremost, I would like to thank my advisor, Joe Harris. This thesis could not have been possible without his constant advice and support over the last five years. I am deeply grateful for having been introduced to many beautiful problems and ideas and I would like to thank him for being an inspiring mentor and teacher.

I would like to thank Johan de Jong for the idea of using moduli of vector bundles for solving my initial problem. I am grateful for the infinite time he spent discussing with me all the details in this thesis, for giving me many ideas and for making innumerable useful suggestions, which crucially helped this work. I would also like to thank him for believing in me.

I would like to thank Steven Kleiman for all his support and advice in the last five years. I am especially grateful for his careful and critical reading of my written notes over the years, as well as for the many discussions about my work that we had.

I would especially like to thank Jason Starr, for being there to explain patiently so many basic or advanced problems in algebraic geometry, as well as for sharing with me some of his research ideas. I am grateful for his friendship and for all the three-page e-mails about math that he sent me.

I would like to thank my undergraduate mentor, Paltin Ionescu, for introducing me to algebraic geometry. I would like to thank Ravi Vakil for being a wonderful teacher of algebraic geometry, as well as for his support and advice in the past years. I would like to thank Mihnea Popa for suggesting a way to generalize some of the results and for all the discussions that we had. I also thank Izzet Coskun for reading this material and Alina Marian for the comments about my writing and all our discussions in the past years.

I would like to thank all the people who made me able to do math and survive graduate school: my parents and my sister Isabela for their constant support, Ionut Chiose for his love, my friends Anda Degeratu, Anke Hildebrandt and Soosan Beheshti for being there for me and all the special people who helped me grow on a personal level.

Contents

Introduction

Our main goal is to give a complete description of the space of rational curves on moduli spaces of rank 2 vector bundles on curves of genus $g \geq 2$ over $\mathbb{C}$ with fixed determinant of odd degree.

Let C be a projective curve of genus $g \geq 2$ and x_0 a fixed point on C. Let M be the moduli space of rank 2 stable bundles with fixed determinant $O_C(x_0)$. Then M is a fine moduli space and it is a smooth projective scheme of dimension $3g-3$. There are some concrete descriptions for M. If C is a hyperelliptic curve of genus $g \geq 2$, then M is isomorphic to the Grassmanian of $(g-2)$-planes contained in the intersection of two quadrics in $\mathbb{P}^{2g+1}$. In particular, if $g = 2$, M is the intersection of two quadrics in $\mathbb{P}^5$.

The Picard group of M is $\mathbb{Z}$ and if we let Θ be the ample generator, then we say that a morphism $f : \mathbb{P}^1 \to M$ has degree k if $f^*\Theta \cong O(k)$. We find the irreducible components of the space of morphisms $\mathrm{Mor}_k(\mathbb{P}^1, M)$ parametrizing rational curves $\mathbb{P}^1 \to M$ of degree $k \geq 1$ and give a complete description of this space. In particular, we find that there are components of dimension bigger than expected. We also find the maximally rationally connected (MRC) fibrations of the irreducible components. The MRC quotient is given either by the Jacobian $J(C)$ or by a direct sum of two copies of the Jacobian $J(C)$.

We have two approaches to this problem.

The first approach

We use the classical idea of looking at spaces of extensions of line bundles:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}(x_0) \to 0 \qquad (*)$$

If we fix an integer $e \geq 0$, for every line bundle $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$, we have the $(2e+g)$ dimensional vector space parametrizing extensions of type (*):

$$V_{\mathcal{L}} = \mathrm{Ext}^1(\mathcal{L}^{-1}(x_0), \mathcal{L}).$$

We have a rational map

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \dashrightarrow M$$

which associates to an extension $(*)$ the isomorphism class of the bundle $\mathcal{E}$. It is defined outside the unstable locus, which has codimension at least 2. If we take lines in the projective spaces $\mathbb{P}(V_{\mathcal{L}})$ we obtain rational curves of degree $(2e+1)$ on M. Moreover, if we let $\mathcal{L}$ vary in $\mathrm{Pic}^{-e}(C)$, we obtain an irreducible component in the space of rational curves of degree k, which we call the *nice* irreducible component in the case when k is odd.

In this approach it is useful to mod out by the automorphisms of $\mathbb{P}^1$ and use rather the Kontsevich space $\overline{M}_0(M,k)$ to parametrize rational curves of degree k on M. For our purposes it does not really matter which space we are working with, since there is a one-to-one correspondence between the irreducible components of $\overline{M}_0(M,k)$ which are not contained in the boundary and irreducible components of the space of morphisms $\mathrm{Mor}_k(\mathbb{P}^1, M)$. Moreover, corresponding irreducible components have the same MRC fibration.

Theorem 0.1. *For any* odd *positive integer $k = 2e+1$ there is an irreducible component $\mathfrak{M}$ of the Kontsevich space $\overline{M}_0(M,k)$, of dimension $2k+3g-6$ such that $\overline{M}_0(M,k)$ is unobstructed at the general point of $\mathfrak{M}$, i.e., $H^1(\mathbb{P}^1, f^*T_M) = 0$.*

A general element $[f]$ of $\mathfrak{M}$ is obtained from a line in the projective space $\mathbb{P}(V_{\mathcal{L}})$, for some $\mathcal{L} \in Pic^{-e}(C)$. Moreover, the MRC fibration of the component $\mathfrak{M}$ is given by a rational map:

$$\mathfrak{M} \dashrightarrow Pic^{-e}(C)$$

which sends $[f]$ to the line bundle $\mathcal{L}$.

A key point that appears in the proof of Theorem 0.1 is that there is a *unique* line bundle $\mathcal{L}$ and a *unique* line in $\mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{k+g-2}$ corresponding to a general rational curve of degree $k = 2e+1$.

To obtain rational curves of even degree, we generalize the classical idea of looking at extensions

$$0 \to \mathcal{E} \to \mathcal{E}' \to O_y \to 0$$

where $y \in C$ is a point and $\mathcal{E}$ is a rank 2 stable bundle on C, with determinant $O_C(x_0 - y)$. We take $D \in \mathrm{Sym}^e(C)$ and consider extensions:

$$0 \to \mathcal{E} \to \mathcal{E}' \to O_D \to 0 \qquad (**)$$

If we fix an integer $e \geq 1$, for every $D \in \mathrm{Sym}^e(C)$ and $\mathcal{E}$ a rank 2 stable bundle with determinant $O_C(x_0 - D)$, we consider the $2e$ dimensional vector space of extensions

$$V_{D,\mathcal{E}} = \mathrm{Ext}^1(O_D, \mathcal{E}).$$

We have a rational map

$$\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \dashrightarrow M$$

which associates to the extension (**) the isomorphism class of the bundle $\mathcal{E}'$. It is defined outside the unstable locus, which has codimension 2. If we take lines in the projective spaces $\mathbb{P}(V_{D,\mathcal{E}})$ we obtain rational curves of degree $2e$ on M. Moreover, if we let D vary in $\mathrm{Sym}^e(C)$ and also let $\mathcal{E}$ vary in the moduli space of stable bundles of the appropriate determinant, we obtain an irreducible component in the space of rational curves of degree k, which we call the *nice* irreducible component in the case when k is even.

Theorem 0.2. *For any* even *positive integer $k = 2e$ there is an irreducible component $\mathfrak{M}$ of the Kontsevich space $\overline{M}_0(M,k)$, of dimension $2k+3g-6$ such that $\overline{M}_0(M,k)$ is unobstructed at the general point of $\mathfrak{M}$, i.e., $H^1(\mathbb{P}^1, f^*T_M) = 0$.*

A general element $[f]$ of $\mathfrak{M}$ is obtained from a line in the projective space $\mathbb{P}(V_{D,\mathcal{E}})$, for some $D \in Sym^e(C)$ and some stable rank 2 bundle $\mathcal{E}$ with determinant $O_C(x_0 - D)$. Moreover, the MRC fibration of the component $\mathfrak{M}$ is given by a rational map:

$$\mathfrak{M} \dashrightarrow Pic^e(C)$$

which sends $[f]$ to the image of $(\mathcal{E}, D)$ via the canonical morphism

$$Sym^e(C) \times_{Pic^{1-e}(C)} M(2, 1-e) \to Pic^{1-e}(C)$$

where $M(2,1-e)$ is the moduli space of stable rank 2 vector bundles with determinant of degree $(1-e)$ and $M(2,1-e) \to Pic^{1-e}(C)$ is the determinant map, while $Sym^e(C) \to Pic^{1-e}(C)$ is given by $D \mapsto O_C(x_0 - D)$.

Note that there is a unique D and $\mathcal{E}$, but an $(e-1)$ dimensional family of lines in $\mathbb{P}(V_{D,\mathcal{E}}) \cong \mathbb{P}^{k-1}$, corresponding to a general rational curve of degree $k = 2e$. The $(e-1)$ dimensional family comes from the automorphisms of the sheaf O_D acting on the space of extensions $V_{D,\mathcal{E}}$.

We expect that the arguments in this first approach extend to the case when M is a moduli space of stable vector bundles of rank 2 and fixed determinant of even degree. In fact, in Section 2.5 we make some generalizations in this sense, specifically for the case when k is odd. Interestingly, we need to use these generalizations to prove the result in Theorem 0.2 about rational curves of even degree.

The second approach

The second approach relies on a very basic fact: there is a *one-to-one correspondence* between morphisms $f : \mathbb{P}^1 \to M$ of some fixed degree k and rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ satisfying the stability condition on the fibers over $\mathbb{P}^1$ and having fixed Chern classes:

$$c_1(\mathcal{F}) = k\{pt\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{x_0\} \in A^1(\mathbb{P}^1 \times C)$$
$$c_2(\mathcal{F}) = k \in \mathbb{Z}$$

This is because on M we are fortunate enough to have a rigidified Poincaré bundle. Note that the main difficulty – though probably avoidable – of extending the arguments in the second approach to the case of moduli spaces with fixed determinant of even degree is the fact that there is no Poincaré bundle, not even on an open set.

Work of Brosius proves in [BR1], [BR2] that there exist moduli spaces for vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ with numerical Chern classes c_1 and c_2 and with the numerical first Chern class of the *canonical subbundle* fixed (note that there is no stability condition!). Using Brosius' work, one has moduli spaces $\mathfrak{B}$ for vector bundles $\mathcal{F}$ with the extra condition that the Chern class c_1 is fixed as an element in $A^1(\mathbb{P}^1 \times C)$.

In order to prove that one has moduli of vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ corresponding to morphisms f, the main technical point is to prove that once some clear necessary numerical conditions for stability on the fibers are satisfied, one has a *dense* open $\mathfrak{B}^0 \subset \mathfrak{B}$ corresponding to such bundles. To prove this we use results from the first approach, namely, the estimates about the codimension of the unstable locus.

Brosius' idea for constructing moduli of rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ is the following: every bundle $\mathcal{F}$, say with fiber degree k, determines an integer $a \geq \frac{k}{2}$ such that $\mathcal{F}$ has generic fiber type $(a, k-a)$ – meaning that for a general point $c \in C$, the bundle $\mathcal{F}_c = \mathcal{F}_{|\mathbb{P}^1 \times \{c\}}$ on $\mathbb{P}^1$ splits as $O(a) \oplus O(k-a)$. Moreover, $\mathcal{F}$ determines a *canonical sequence*

$$0 \to (p_{2*}p_2^*\mathcal{F}(-a))(a) \to \mathcal{F} \to \mathcal{J} \to 0$$

where $p_2 : \mathbb{P}^1 \times C \to C$ is the projection and $\mathcal{J}$ is by definition the cokernel of the canonical morphism

$$(p_{2*}p_2^*\mathcal{F}(-a))(a) \to \mathcal{F}$$

The bundle $\mathcal{F}' = (p_{2*}p_2^*\mathcal{F}(-a))(a)$ is called the *canonical subbundle* of $\mathcal{F}$.

The main idea about the canonical sequence is that such an extension determines and is determined by $\mathcal{F}$, so one could get moduli for $\mathcal{F}$ by looking at such extensions. The key point is that the canonical subbundle has the maximum possible fiber degree, which is a; hence, the quotient is forced to have fiber degree $k - a \leq a$. Therefore, if one fixes the bundles $\mathcal{F}'$ and $\mathcal{J}$, any two such extensions having the same middle term $\mathcal{F}$ will be in the same orbit of the action of the group of automorphisms of the sheaf $\mathcal{F}'$, and $\mathcal{J}$ respectively, on the space of such extensions. Conversely, any extension in such an orbit determines $\mathcal{F}$ as its middle term and the canonical sequence of $\mathcal{F}$ is contained in that orbit.

One has to distinguish between the case when $a > \frac{k}{2}$ and the case when $a = \frac{k}{2}$. In the case when $a > \frac{k}{2}$, we have that $\mathcal{L} = p_{2*}\mathcal{F}(-a)$ is a line bundle and the canonical sequence has the form:

$$0 \to p_1^* O(a) \otimes p_2^* \mathcal{L} \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{M} \to 0 \tag{0.1}$$

whith Z is a 0-cycle on $\mathbb{P}^1 \times C$ with ideal sheaf $\mathcal{I}_Z$ and $\mathcal{M}$ a line bundle on $\mathbb{P}^1 \times C$.

If we fix deg $\mathcal{L} = -e$, the length δ of the 0-cycle Z is given by:

$$\delta = (k - a) - e(2a - k).$$

Let $\mathrm{Hilb}^\delta(\mathbb{P}^1 \times C)$ be the Hilbert scheme of 0-cycles of length δ on $\mathbb{P}^1 \times C$.

Theorem 0.3. *For any pair of integers (a, e) in the range $(\star)$*

$$\{(a, e) \quad | \quad k \geq a > k/2, \quad \frac{k - a}{2a - k} \geq e > 0\} \cup \{(k, 0)\} \tag{$\star$}$$

there is an integral subscheme $\mathfrak{M}(a, e) \subset Mor_k(\mathbb{P}^1, M)$ and there is a morphism:

$$\pi : \mathfrak{M}(a, e) \to Pic^{-e}(C) \times Hilb^\delta(\mathbb{P}^1 \times C)$$

which sends an element $[f]$ to $(\mathcal{L}, Z)$, where $\mathcal{L}$ and Z are associated as in (0.1) to the bundle $\mathcal{F}$ (corresponding to f).

The MRC fibration of the scheme $\mathfrak{M}(a, e)$ is given by the morphism:

$$\rho : \mathfrak{M}(a, e) \to Pic^{-e}(C) \times Pic^\delta(\mathbb{P}^1 \times C)$$

which is the composition of π with the canonical morphism given by

$$Hilb^\delta(\mathbb{P}^1 \times C) \to Sym^\delta(\mathbb{P}^1 \times C) \to Sym^\delta C \to Pic^\delta(C)$$

The scheme $\mathfrak{M}(a, e)$ has dimension:

$$\dim \mathfrak{M}(a, e) = (2a - k + 2)g + (3k - 3a - 1) - e(2a - k - 2) \tag{0.2}$$

Theorem 0.4. *For a pair of integers (a, e) in the range $(\star)$, the closure of the scheme $\mathfrak{M}(a, e)$ is an irreducible component of $Mor_k(\mathbb{P}^1, M)$ if and only if its dimension is bigger or equal to the expected dimension $2k + 3g - 3$.*

If k is an odd integer, say $k = 2a + 1$, then the scheme $\mathfrak{M}(a + 1, a)$ corresponds to the nice irreducible component of Theorem 0.1. The line bundle $\mathcal{L}$ associated to the bundle $\mathcal{F}$ from (0.1) is the same as the line bundle $\mathcal{L}$ corresponding to $[f]$ by Theorem 0.1.

In the case when $k = 2a$ we have that $p_{2*}\mathcal{F}(-a)$ is a rank 2 bundle $\mathcal{E}$ and we are interested in vector bundles $\mathcal{F}$ such that the canonical sequence has the simplest form, which is:

$$0 \to p_1^*O(a) \otimes p_2^*\mathcal{E} \to \mathcal{F} \to p_1^*O(a-1) \otimes p_2^*O_D \to 0 \tag{0.3}$$

whith $D \in \mathrm{Sym}^a(C)$ and $\mathcal{E}$ is a *stable* rank 2 bundle with determinant $O_C(x_0 - D)$.

Theorem 0.5. *If $k = 2a$ is an even integer, there is an integral subscheme*

$$\mathfrak{M}_{even} \subset Mor_k(\mathbb{P}^1, M)$$

whose closure is an irreducible component of the space $Mor_k(\mathbb{P}^1, M)$ and its MRC fibration is given by a morphism

$$\rho : \mathfrak{M}_{even} \to Pic^{1-a}(C)$$

which sends an element $[f]$ to the image of $(D, \mathcal{E})$ via the canonical map

$$Sym^a(C) \times_{Pic^{1-a}(C)} M(2, 1-a) \to Pic^{1-a}(C),$$

where D and $\mathcal{E}$ are associated as in (0.3) to the bundle $\mathcal{F}$ (corresponding to f).

The component $\mathfrak{M}_{even}$ corresponds to the nice component of Theorem 0.2. The rank 2 bundle $\mathcal{E}$ and the element $D \in \mathrm{Sym}^a(C)$ corresponding to $\mathcal{F}$ by (0.3) are the same as $\mathcal{E}$ and D corresponding to $[f]$ from Theorem 0.2.

We prove that the subschemes $\mathfrak{M}(a, e)$, that do not correspond to irreducible components, are in fact contained in the nice component (if $k = 2a + 1$, it is $\mathfrak{M}(a + 1, a)$ and if $k = 2a$, it is $\mathfrak{M}_{even}$).

If $[f]$ is an element of the scheme $\mathfrak{M}(a, e)$ and $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ is the associated line bundle, then let $V_{\mathcal{L}}$ be the space of extensions from the first approach

$$V_{\mathcal{L}} = \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0), \mathcal{L}).$$

One could describe f as being given by a composition of rational maps:

$$\mathbb{P}^1 \dashrightarrow \mathbb{P}(V_{\mathcal{L}}) \dashrightarrow M$$

where the first map is sending a point $p \in \mathbb{P}^1$ to the class of the extension in $V_{\mathcal{L}}$ obtained by restriction to $\{p\} \times C$. The second map is the map $\kappa_{\mathcal{L}}$ from the first approach. If $\delta = 0$ then the rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}(V_{\mathcal{L}})$ is defined everywhere and it has degree $(2a - k)$; hence, an element in $\mathfrak{M}(a, e)$ is obtained from rational curves in spaces of extensions, therefore generalizing the first approach.

The material is organized as follows: in Chapter 1 we give background about our basic tools: the space of rational curves, maximally rationally connected fibrations and moduli spaces of stable vector bundles. In Chapter 2 we describe the first approach and in Chapters 3 and 4 we describe the second approach: in Chapter 3 we give the details of Brosius' construction of moduli of rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ and we find the locus of those vector bundles $\mathcal{F}$ which induce morphisms $\mathbb{P}^1 \to M$. In Chapter 4 we find and give descriptions of the irreducible components of the space $\mathrm{Mor}_k(\mathbb{P}^1, M)$.

Conventions and Notations

All schemes are considered over $\mathbb{C}$ and all products are over $\mathbb{C}$, unless we specify otherwise. All schemes are Noetherian and of finite type over $\mathbb{C}$.

Following the tradition, we use the word "vector bundle" for a locally free sheaf. Whenever $\mathcal{E}$ is a vector bundle on a scheme, by $\mathbb{P}(\mathcal{E})$ we mean the projective bundle $\mathrm{Proj}(Sym(\mathcal{E}^*))$.

If $\mathcal{F}$ is a sheaf on $X \times Y$ and $x \in X$, by $\mathcal{F}_x$ we denote the sheaf $\mathcal{F}_{|\{x\} \times Y}$ on $\{x\} \times Y$, unless we specify otherwise. In Chapter 3 and 4, if $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, we use the notation $\mathcal{F} \boxtimes \mathcal{G}$ for the sheaf $p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$ on $X \times Y$.

Chapter 1

Basic Tools

1.1 Parametrizing rational curves

Let X be a projective integral scheme over $\mathbb{C}$. By a rational curve on X we mean a non-constant morphism $f : \mathbb{P}^1 \to X$. Let's fix a closed immersion $X \subset \mathbb{P}^N$. We say that the rational curve has degree $k \geq 1$ if

$$f^* O_X(1) \cong O_{\mathbb{P}^1}(k).$$

We have two possibilities for a parameter space for rational curves of degree k on X:

1. The Kontsevich space $\overline{M}_0(X, k)$
2. The space of morphisms $\mathrm{Mor}_k(\mathbb{P}^1, X)$

For our purposes, we could work with either of the two spaces. Especially in Chapter 2, each space can be used as parameter space. In Chapter 3 and 4 however, the space of morphisms simplifies the arguments.

1.1.1 The Kontsevich space

We recall a few facts about Kontsevich moduli spaces of stable maps $\overline{M}_{0,r}(X, \gamma)$, where $\gamma \in \mathrm{H}_2(X; \mathbb{Z})$.

A *stable map* representing the class γ and having r marked points is a pair $(C, f, p_1, \ldots, p_r)$ such that:

i. C is a projective curve of arithmetic genus 0 and with only nodes as singularities

ii. $p_1, \ldots, p_r$ are smooth points of C

iii. there are finitely many automorphisms of C which commute with f

iv. $f_*[C] = \gamma$

The Kontsevich space $\overline{M}_{0,r}(X, \gamma)$ is the coarse moduli scheme representing the functor which associates to every scheme S over $\mathbb{C}$ families of stable maps representing the class γ with r marked points. It is a projective integral scheme over $\mathbb{C}$. We write $\overline{M}_0(X, \gamma)$ for the space $\overline{M}_{0,0}(X, \gamma)$ of stable maps with no marked points.

The *expected dimension* of $\overline{M}_{0,r}(X, \gamma)$ is defined as:

$$\int_\gamma c_1^{\text{top}}(T_X) + \dim\,(X) + r - 3.$$

It is a lower bound for the dimension of any irreducible component of $\overline{M}_{0,r}(X, \gamma)$.

A point $[f] \in \overline{M}_{0,r}(X, \gamma)$ given by $f : \mathbb{P}^1 \to X$ is called *unobstructed* if $\mathrm{H}^1(\mathbb{P}^1, f^*T_X) = 0$. If $[f]$ is an unobstructed point, then $\overline{M}_{0,r}(X, \gamma)$ is smooth at $[f]$ and it has the expected dimension.

An observation

Consider the space $\overline{M}_{0,3}(X, \gamma)$ and let π be the morphism that forgets the marked points:

$$\pi : \overline{M}_{0,3}(X, \gamma) \to \overline{M}_0(X, \gamma)$$

Let $\Delta \subset \overline{M}_0(X, \gamma)$ be the boundary, i.e., the locus of stable maps with reducible domain. Then Δ is a closed subscheme of $\overline{M}_0(X, \gamma)$. Let U be the complement of Δ. A closed point of U corresponds to a rational curve $f : \mathbb{P}^1 \to X$ of degree k. Then the fibers of the morphism

$$\pi^{-1}(U) \to U$$

are isomorphic to PGL(2). Note that $\pi^{-1}(U) \subset \overline{M}_{0,3}(X, \gamma)$ is the complement of the boundary of $\overline{M}_{0,3}(X, \gamma)$. The consequence is:

Observation 1.1. *The morphism π gives a one-to-one correspondence between the irreducible components of $\overline{M}_{0,3}(X, \gamma)$ which don't lie in the boundary and the irreducible components of $\overline{M}_0(X, \gamma)$ whith the same property. If $\mathfrak{M}$ is such an irreducible component of $\overline{M}_{0,3}(X, \gamma)$ then the morphism $\pi : \mathfrak{M} \to \pi(\mathfrak{M})$ has fibers isomorphic to PGL(2) over $U \cap \mathfrak{M}$.*

(Irreducible components are taken with the induced reduced structure.)

The space $\overline{M}_0(X,k)$

Assume that the singular cohomology group $\mathrm{H}^2(X;\mathbb{Z})$ is $\mathbb{Z}$ and the topological Chern class $c_1^{\mathrm{top}}(O_X(1))$ is a generator in $\mathrm{H}^2(X;\mathbb{Z})$. Equivalently, using Poincaré duality,

$$\mathrm{H}_{2n-2}(X;\mathbb{Z}) \cong \mathbb{Z}$$

and the fundamental class of a hyperplane section $h \in \mathrm{H}_{2n-2}(X;\mathbb{Z})$ is a generator.

Assume that $\mathrm{H}_2(X;\mathbb{Z})$ has no torsion. By Poincaré duality, there exists a class β generating $\mathrm{H}_2(X;\mathbb{Z})$ and such that the intersection of cycles $\beta.h$ is 1, or equivalently,

$$< \beta, c_1^{\mathrm{top}}(O_X(1)) >= 1.$$

For example, if X contains a line $l \subset \mathbb{P}^N$, then the class of l in $\mathrm{H}_2(X;\mathbb{Z})$ is β. More generally, if $f : \mathbb{P}^1 \to X$ is a rational curve of degree $k \geq 1$ and we denote by $[\mathbb{P}^1]$ the fundamental class of $\mathbb{P}^1$, then we have:

$$f_*[\mathbb{P}^1] = k\beta \in \mathrm{H}_2(X;\mathbb{Z}).$$

We will work only with schemes X satisfying the condition:

$$\mathrm{H}_2(X;\mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad c_1^{\mathrm{top}}(O_X(1))\mathbb{Z} = \mathrm{H}^2(X;\mathbb{Z}) \tag{1.1}$$

For such X, one way to parametrize rational curves of degree k on X is to consider the Kontsevich space of stable maps $\overline{M}_{0,r}(X,k\beta)$. For simplicity, we denote this by $\overline{M}_{0,r}(X,k)$.

1.1.2 The space of morphisms

The space of morphisms Mor(Y,X)

We recall a few facts about the scheme $\mathrm{Mor}(Y,X)$, when X and Y are projective schemes over $\mathbb{C}$. Consider the functor defined by:

$$\mathcal{M}or(Y,X) : \mathrm{Sch}_k \to \mathrm{Sets}$$
$$\mathcal{M}or(Y,X)(S) = \{S - \text{morphisms} : X \times S \to Y \times S\}$$

The graph of a morphism identifies the functor $\mathcal{M}or(Y,X)$ with a subfunctor of the Hilbert functor of subschemes of $Y \times X$. The functor $\mathcal{H}ilb(Y \times X)$ is representable by a projective scheme $\mathrm{Hilb}(Y \times X)$. It is proved in [K] that the functor $\mathcal{M}or(Y,X)$ is represented by an open subscheme

$$\mathrm{Mor}(Y,X) \subset \mathrm{Hilb}(Y \times X).$$

If we fix an ample line bundle $O(1)$ on $Y \times X$, say given by $p_1^*O_Y(1) \otimes p_2^*O_X(1)$, where $O_X(1)$ and $O_Y(1)$ are ample line bundles on X, respectively on Y, and p_1, p_2 are the projections from $Y \times X$ onto Y, respectively X. If P is a polynomial, the Hilbert functor $\mathcal{H}ilb_P(Y \times X)$, of subschemes of of $Y \times X$ with Hilbert polynomial P, is represented by a projective scheme $\mathrm{Hilb}_P(Y \times X)$. Then we have that

$$\mathrm{Hilb}(Y \times X) = \cup_P \mathrm{Hilb}(Y \times X)$$

where the union is a disjoint union over all polynomials P.

Let $\mathrm{Mor}_P(Y, X)$ be the corresponding open subscheme

$$\mathrm{Mor}_P(Y, X) \subset \mathrm{Hilb}_P(Y \times X).$$

parametrizing morphisms $f : Y \to X$ with Hilbert polynomial P. Here, by the Hilbert polynomial of f we mean the Hilbert polynomial of the closed subscheme Γ of $Y \times X$ given by the graph of f:

$$P(m) = \chi(\Gamma, O_\Gamma(m)) = \chi(Y, O_Y(m) \otimes f^*O_X(m)).$$

There is a universal morphism:

$$Y \times \mathrm{Mor}_P(Y, X) \to X \times \mathrm{Mor}_P(Y, X)$$

which sends the point (y, f), given by $[f] \in \mathrm{Mor}_P(Y, X)$ and $y \in Y$, to $(f(y), f)$. If we compose this universal morphism with the projection onto X, we obtain the sometimes called *evaluation morphism*:

$$\mathrm{ev} : Y \times \mathrm{Mor}_P(Y, X) \to X.$$

The scheme of morphisms $\mathbf{Mor}_k(\mathbb{P}^1, X)$

If we take $p_1^*O_{\mathbb{P}^1}(1) \otimes p_2^*O_X(1)$ as ample line bundle on $\mathbb{P}^1 \times X$, a morphism $f : \mathbb{P}^1 \to X$ will have Hilbert polynomial:

$$P(m) = \chi(\mathbb{P}^1, O(m) \otimes f^*O(m)) = \chi(\mathbb{P}^1, O(km + m)) = m(k + 1) + 1.$$

For the polynomial $P(t) = (k + 1)t + 1$, we denote:

$$\mathrm{Mor}_k(\mathbb{P}^1, X) = \mathrm{Mor}_P(\mathbb{P}^1, X).$$

A morphism $f : \mathbb{P}^1 \to X$ with Hilbert polynomial P has degree k.

The universality property of the evaluation map:

$$\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, X) \to X$$

says that if S is a scheme and we have a family of morphisms of degree k:

$$g : \mathbb{P}^1 \times S \to X$$

then there is a unique morphism $u : S \to \mathrm{Mor}_k(\mathbb{P}^1, X)$ such that $g = \mathrm{ev} \circ (\mathrm{id}_{\mathbb{P}^1} \times u)$.

If $f : \mathbb{P}^1 \to X$ is a morphism of degree k, then *the expected dimension* of $\mathrm{Mor}_k(\mathbb{P}^1, X)$ at the point $[f]$ is defined as the Euler characteristic of the pull-back of the tangent bundle of X :

$$\chi(\mathbb{P}^1, f^*T_X) = \mathrm{h}^0(\mathbb{P}^1, f^*T_X) - \mathrm{h}^1(\mathbb{P}^1, f^*T_X) = \deg f^*T_X + \dim (X).$$

If X is smooth, the expected dimension is a lower bound for any irreducible component of the scheme $\mathrm{Mor}_k(\mathbb{P}^1, X)$ containing the point $[f]$.

Assume that X is smooth. The Zariski tangent space to $\mathrm{Mor}_k(\mathbb{P}^1, X)$ at the point f is isomorphic to $\mathrm{H}^0(\mathbb{P}^1, f^*T_X)$.

A point $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, X)$ is called *unobstructed* if $\mathrm{H}^1(\mathbb{P}^1, f^*T_X) = 0$. If $[f]$ is an unobstructed point, then $\mathrm{Mor}_k(\mathbb{P}^1, X)$ is smooth at $[f]$ and it has the expected dimension.

Relation between $\mathrm{Mor}_k(\mathbb{P}^1, X)$ and $\overline{M}_0(X, k)$

Let X be such that $\mathrm{H}^2(X; \mathbb{Z})$ is generated by $c_1^{\mathrm{top}}(O_X(1))$ and that $\mathrm{H}_2(X; \mathbb{Z})$ has no torsion and consider the spaces $\overline{M}_{0,r}(X, k)$ defined in 1.1.1.

Lemma 1.2. *The scheme $Mor_k(\mathbb{P}^1, X)$ is isomorphic to the open in $\overline{M}_{0,3}(X, k)$ which is the complement of the boundary. In particular, any irreducible component of $Mor_k(\mathbb{P}^1, X)$ is isomorphic to an open of an irreducible component of $\overline{M}_{0,3}(X, k)$. This gives a one-to-one correspondence between the irreducible components of $Mor_k(\mathbb{P}^1, X)$ and the irreducible components of $\overline{M}_{0,3}(X, k)$ which are not contained in the boundary.*

Proof. Consider the evaluation map

$$\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, X) \to X.$$

Let π be the second projection:

$$\pi : \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, X) \to \mathrm{Mor}_k(\mathbb{P}^1, X).$$

The points 0, 1 and ∞ on $\mathbb{P}^1$ give sections of π. We have family of stable maps over $\mathrm{Mor}_k(\mathbb{P}^1, X)$ with three marked points. By the definition of $\overline{M}_{0,3}(X, k)$ as a coarse moduli scheme, it follows that there exist a morphism:

$$\phi : \mathrm{Mor}_k(\mathbb{P}^1, X) \to \overline{M}_{0,3}(X, k)$$

which sends a point $[f]$ corresponding to a rational curve $f : \mathbb{P}^1 \to X$ of degree k, to the point in $\overline{M}_{0,3}(X,k)$ corresponding to the stable map f, with 0, 1 and ∞ as marked points on $\mathbb{P}^1$.

Let $\Delta \subset \overline{M}_{0,3}(X,k)$ be the boundary. Let V be the open in $\overline{M}_{0,3}(X,k)$ given by the complement of Δ. The morphism ϕ maps to V:

$$\phi : \mathrm{Mor}_k(\mathbb{P}^1, X) \to V.$$

This gives a bijection between the closed points of $\mathrm{Mor}_k(\mathbb{P}^1,X)$ and the closed points of V.

Note that V is in the automorphism-free locus of $\overline{M}_{0,3}(X,k)$. Since on the automorphism-free locus there is a universal family of stable maps [FP], it follows that there exist

$$\pi : \mathcal{C} \to V; \qquad f : \mathcal{C} \to X; \qquad \sigma_1, \sigma_2, \sigma_3 : V \to \mathcal{C}$$

where $\sigma_1, \sigma_2, \sigma_3 : V \to \mathcal{C}$ are sections of π. As the fibers of π are all isomorphic to $\mathbb{P}^1$, the existence of the three sections of π implies that $\mathcal{C}$ is in fact a trivial bundle over V. Then by the universal property of the evaluation morphism, it follows that there is a morphism

$$\psi : V \to \mathrm{Mor}_k(\mathbb{P}^1, X)$$

sending the stable map $f : \mathbb{P}^1 \to X$ to the point $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, X)$. Then ψ is the inverse morphism of ϕ. Hence, we have an isomorphism:

$$\mathrm{Mor}_k(\mathbb{P}^1, X) \cong V.$$

□

The next Corollary follows from Observation 1.1 and Lemma 1.2.

Corollary 1.3. *There is a one-to-one correspondence between the irreducible components of* $\mathrm{Mor}_k(\mathbb{P}^1, X)$ *and those irreducible components of* $\overline{M}_0(X,k)$ *which are not contained in the boundary of* $\overline{M}_0(X,k)$.

1.2 Maximally rationally connected fibrations

Recall that all schemes that we are considering are over $\mathbb{C}$.

Definition 1.4. *A projective scheme* X *(not necessarily smooth) is* rationally connected *if there is a dense open* $X^0 \subset X$ *such that for every* $x_1, x_2 \in X^0$*, there is a rational curve through* x_1 *and* x_2.

Rational connectedness is a birational property. Note that a projective scheme X is rationally connected if and only if some desingularization is rationally connected.

We can extend the definition to quasi-projective schemes and say that a quasi-projective scheme X is rationally connected if some projective scheme $\overline{X}$ containing X as a dense open set is rationally connected.

Note that if X is a quasi-projective scheme which is rational, then it is also rationally connected. More generally, if X is unirational, i.e., dominated by a rational scheme, then it is rationally connected.

A *rationally connected fibration* is a rational map $\phi : X \dashrightarrow Y$ which restricts on a dense open $X^0 \subset X$ to a proper morphism $\phi^0 : X^0 \to Y$ with rationally connected fibers.

Definition 1.5. *A* maximally rationally connected fibration (MRC) *of a projective* smooth *integral scheme X is a rationally connected fibration*

$$\phi : X \dashrightarrow Z$$

whith Z a quasi-projective integral scheme and with the universal property that if $\psi : X \dashrightarrow Y$ is another rationally connected fibration, then there is a unique rational map $\rho : Y \to Z$ such that $\rho \circ \psi = \phi$.

Usually, the definition requires that a general fiber is *rationally chain connected*, i.e., a two general points can be connected by a chain of rational curves. Since for smooth schemes over $\mathbb{C}$ rational chain connectedness is the same as rational connectedness and since the general fiber is smooth, it follows that the two definitions are the same.

We call the scheme Z in the definition the *MRC quotient of* X and it is uniquely determined up to birational equivalence. Clearly, birational smooth projective schemes have the same MRC quotient. For example, the MRC quotient of a rationally connected scheme is just a point.

It is knwon that the maximally rationally connected fibration of a projective smooth integral scheme exists, see [K], p. 222. If X is a projective integral scheme which is not smooth, we define its MRC fibration to be the MRC fibration of its desingularization.

The "universality" property in Definition 1.5 is equivalent to the fact that almost all the rational curves in X lie in fibers of ϕ. To be precise, for a very general point $z \in Z$ any rational curves in X meeting the fiber of ϕ at z is contained in the fiber. For example, if Z is an abelian variety, this condition is satisfied, as there are no rational curves on an abelian variety.

One can characterize MRC fibrations using the following recent result:

Fact 1.6. *[GHS] If $X \to Y$ is a morphism of smooth projective schemes, with the general fiber rationally connected and with Y rationally connected, then X is rationally connected.*

The consequence is that MRC fibrations can be defined as follows.

Definition 1.7. *A* maximally rationally connected fibration (MRC) *of a projective* smooth *integral scheme X is a rationally connected fibration*

$$\phi : X \dashrightarrow Z$$

whith Z a quasi-projective integral scheme which is not uniruled, i.e., there is no rational curve through a general point.

We are going to use the following set-up: X and Z will be integral projective schemes and with Z not uniruled (Z will be an abelian variety or embedded in an abelian variety). We will have a rational map $\phi : X \dashrightarrow Z$ which defined on a dense open $X^0 \subset X$ and such that the restriction $\phi^0 : X^0 \to Z$ has the general fiber a quasi-projective scheme which is rationally connected. Then ϕ gives the MRC fibration of X.

We will also use Fact 1.6, in the slightly more general version, when the schemes are not smooth, and not necessarily projective, but rather open subschemes in some projective rationally connected schemes.

MRC quotients of spaces of rational curves

If $X \subset \mathbb{P}^N$ is a smooth projective intgeral scheme over $\mathbb{C}$, which satisfies (1.1), consider the Kontsevich moduli space $\overline{M}_0(X, k)$ for k some positive integer. By Observation 1.1 and Lemma 1.2, we have:

Observation 1.8. *There is a one-to-one correspondence between those irreducible components of $\overline{M}_0(X, k)$ which don't lie in the boundary and the irreducible components of $Mor_k(\mathbb{P}^1, X)$. Corresponding irreducible components have the same MRC fibration.*

1.3 Moduli spaces of stable vector bundles on a curve

1.3.1 Vector bundles of rank r and degree d

Let C be a genus g smooth projective curve over $\mathbb{C}$. A vector bundle $\mathcal{E}$ on C is called *stable* if for any proper subbundle $\mathcal{E}' \subset \mathcal{E}$, we have:

$$\frac{deg(\mathcal{E}')}{rk(\mathcal{E}')} < \frac{deg(\mathcal{E})}{rk(\mathcal{E})}.$$

$\mathcal{E}$ is called *semistable* if in the previous inequality we allow also equality.

Let $M(r,d)$ be the moduli scheme of isomorphism classes of semistable vector bundles $\mathcal{E}$ of rank r on C and degree d. More precisely, $M(r,d)$ is the coarse moduli scheme for the contravariant functor

$$F : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets}$$

which associates to the scheme S the set $F(S)$ of isomorphism classes of vector bundles $\mathcal{F}$ of rank r on $S \times C$ such that for all points $s \in S$ the bundle $\mathcal{F}_s$ is semistable and of degree d.

Recall that if F is a contravariant functor:

$$F : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets}$$

then M is a coarse moduli scheme for F, if there exists a transformation of functors

$$T : F \longrightarrow \mathrm{Hom}(-, M)$$

such that:

i. $T(\mathrm{Spec}(\mathbb{C})) : F(\mathrm{Spec}(\mathbb{C})) \to \mathrm{Hom}(\mathrm{Spec}\mathbb{C}), M)$ is a bijection of sets

ii. The pair (M,T) is unique in the sense that if there is a scheme N over $\mathbb{C}$ and a transformation $T' : F \longrightarrow \mathrm{Hom}(-,N)$ then there is a unique morphism $f : M \to N$ over $\mathbb{C}$ such that if we denote $F : \mathrm{Hom}(-,M) \longrightarrow \mathrm{Hom}(-,N)$ given by composition with f, then $T' = F \circ T$.

The scheme $M(r,d)$ is an integral projective scheme over $\mathbb{C}$ of and there is an open set $M^s(r,d)$ whose closed points correspond to stable bundles over C. Moreover, $M^s(r,d)$ is smooth over $\mathbb{C}$. When $(r,d) = 1$ semistability is equivalent to stability; hence, in this case, $M(r,d)$ is smooth. When $(r,d) \neq 1$ the open $M^s(r,d)$ is precisely the smooth locus of $M(r,d)$, except in the case when $g = 2, r = 2$, when the moduli space is isomorphic to $\mathbb{P}^3$.

Let $\mathrm{Pic}^d(C)$ be the moduli scheme $M(1,d)$ of line bundles of degree d. There is a canonical morphism given by taking the determinant:

$$\det\ : M(r,d) \to \mathrm{Pic}^d(C).$$

We have the following description of $M(r,d)$ according to the genus of C:

i. If $g \geq 2$ then $M(r,d)$ has dimension $r^2(g-1)+1$.

ii. If $g=1$ and if $(r,d)=1$ then the determinant map is an isomorphism. If $(r,d)=\nu$ then $M(r,d) \cong \mathrm{Sym}^{\nu}(C)$.

iii. If $g=0$, then the moduli space is just a point, since any vector bundle of rank r on $\mathbb{P}^1$ is direct sum of line bundles and such a vector bundle is not stable unless $r=1$, when it is just $O(d)$.

When $d \equiv d'(\mathrm{mod}\ r)$ one gets isomorphic moduli schemes:

$$M(r,d) \cong M(r,d').$$

Denote $M(r,\xi)$ the moduli space of stable vector bundles on C of rank r and determinant ξ. If ξ, ξ' are line bundles of the same degree, then

$$M(r,\xi) \cong M(r,\xi').$$

A Poincaré vector bundle on $M(r,d) \times C$ is a vector bundle with the property that for any closed point $\zeta \in M(r,d)$ the bundle $U_\zeta := \mathcal{U}_{|\{\zeta\}\times C}$ is in the isomorphism class corresponding to the point ζ in $M(r,d)$. Equivalently, the functorial morphism

$$F(S) \to \mathrm{Mor}\ (S, M(r,d))$$

surjects for any S. Note that $\mathcal{U}$ is not unique: $\mathcal{U} \otimes \pi_1^*(\mathcal{M})$ is also a Poincaré bundle for any $\mathcal{M} \in \mathrm{Pic}(M(r,d))$. Any two Poincaré bundles differ by such a twist.

If $(r,d)=1$ then there exists a Poincaré vector bundle on $M(r,d) \times C$. If $(r,d) \neq 1$ then there is no Poincaré vector bundle on $M(r,d) \times C$. In fact there is no Poincaré vector bundle on an open $U \times C$, where U is any open in $M(r,d)$ (see [R]).

We mention the following very useful classical result in the theory of stable vector bundles. For a reference, see [P] p.115.

Fact 1.9. *If T is an integral scheme and $\mathcal{F}$ is a rank 2 vector bundle on $C \times T$, the locus of $t \in T$ for which $\mathcal{F}_t$ is stable is open in T (possibly empty).*

1.3.2 Vector bundles of rank 2 and fixed determinant of odd degree

Assume $g \geq 2$ and $r = 2$. Then if d is an odd integer we noticed that

$$M(2, d) \cong M(2, 1).$$

We pick a point $x_0 \in C$ and let $M = M(2, O_C(x_0))$. Note again that if ξ is any line bundle on C of degree 1, we have:

$$M \cong M(2, \xi).$$

The moduli scheme M is a projective smooth scheme of dimension $3g-3$. If C is a hyperelliptic curve, Desale and Ramanan proved in [DR] that M can be realized as the space of $(g-2)$ planes contained in $Q_1 \cap Q_2 \subseteq \mathbb{P}^{2g+1}$, for some quadrics Q_1, Q_2 in $\mathbb{P}^{2g+1}$. In particular, when $g = 2$, M is isomorphic to the intersection of two quadrics in $\mathbb{P}^5$.

It is a result of Drezet and Narasimhan in [DN] that the Picard group $\mathrm{Pic}(M)$ is $\mathbb{Z}$. Let Θ be the ample generator. In fact, Θ is very ample.

It is also known that $\mathrm{H}^2(M; \mathbb{Z}) \cong \mathbb{Z}$ (see [N2]) and that one has in fact an isomorphism given by taking the topological Chern class:

$$c_1^{\mathrm{top}} : \mathrm{Pic}(M) \cong \mathrm{H}^2(M; \mathbb{Z}) \tag{1.2}$$

Let $\alpha \in \mathrm{H}^2(M; \mathbb{Z})$ be the image of Θ through the above map.

It is also known (see [N1]) that M is simply connected as a complex manifold; hence, $\mathrm{H}^1(M; \mathbb{Z}) = 0$. It is also helpful to know that $\mathrm{H}^3(M; \mathbb{Z})$ is torsion-free (see [N1]). It follows that:

$$\mathrm{H}_2(M; \mathbb{Z}) \cong \mathrm{H}^2(M; \mathbb{Z})^* \cong \mathbb{Z} \tag{1.3}$$

In particular, $\mathrm{H}_2(M; \mathbb{Z})$ is torsion-free.

The tangent bundle of M

Let $\mathcal{U}$ be a Poincaré bundle on $M \times C$. Let $\mathcal{E}nd^0(\mathcal{U})$ be the kernel subbundle of the trace morphism $\mathcal{E}nd(\mathcal{U}) \to O$. We have a split exact sequence:

$$0 \to \mathcal{E}nd^0(\mathcal{U}) \to \mathcal{E}nd(\mathcal{U}) \to O \to 0.$$

If p_1 is the projection of $M \times C$ into M and T_M is the tangent bundle of M, we have from [N2]

$$T_M \cong \mathrm{R}^1 p_{1*}(\mathcal{E}nd^0(\mathcal{U})) \quad \text{and} \quad \mathrm{R}^i p_{1*}(\mathcal{E}nd^0(\mathcal{U})) = 0 \quad \text{for} \quad i \neq 1 \tag{1.4}$$

From [N2], the canonical bundle of M is:

$$K_M = -2\Theta.$$

It follows from the constructions used in Chapter 2, that M is rational.

Rigidified Poincaré bundle

Based on the results mentioned above, one can prove that there is a unique Poincaré bundle $\mathcal{U}_0$ such that for $x \in C$:

$$c_1(\mathcal{U}_{0|M\times\{x\}}) = \Theta$$

This is called the rigidified Poincaré bundle.

We give a topological proof which follows the construction in [N2] of the class α, that generates $\mathrm{H}^2(M;\mathbb{Z})$.

Note that because of the (1.2), it is enough to prove that there is a Poincaré bundle $\mathcal{U}_0$ such that for $x \in C$:

$$c_1^{\mathrm{top}}(\mathcal{U}_{0|M\times\{x\}}) = \alpha.$$

Let $\mathcal{U}$ on $M \times C$ be a Poincaré bundle. Since M is simply-connected, the Kunneth components of the Chern classes $c_1^{\mathrm{top}}(\mathcal{U})$ and $c_2^{\mathrm{top}}(\mathcal{U})$ can be expressed in the form:

$$c_1^{\mathrm{top}}(\mathcal{U}) = \phi + f \qquad\qquad c_2^{\mathrm{top}}(\mathcal{U}) = \chi + \psi + \omega \otimes f, \tag{1.5}$$

where f is the positive generator of $\mathrm{H}^2(C;\mathbb{Z}) \cong \mathbb{Z}$ and

$$\phi, \omega \in \mathrm{H}^2(M;\mathbb{Z}), \quad \chi \in \mathrm{H}^4(M;\mathbb{Z}), \quad \psi \in \mathrm{H}^3(M;\mathbb{Z}) \otimes \mathrm{H}^1(C;\mathbb{Z}).$$

Define

$$\alpha = 2\omega - \phi \qquad \text{and} \qquad \beta = \phi^2 - 4\chi \tag{1.6}$$

One can easily check that if $\mathcal{U}'$ is another Poincaré bundle, say given by $\mathcal{U} \otimes \pi_1^*\mathcal{L}$ (where $\mathcal{L}$ is a line bundle over M), the classes α, β remain unchanged, as the classes ϕ, χ, ψ, ω change according to the following rule (denote by $\sigma = c_1^{\mathrm{top}}(\mathcal{L})$):

$$\phi' = \phi + 2\sigma \qquad \chi' = \chi + \phi.\sigma + \sigma^2$$
$$\psi' = \psi \qquad \omega' = \omega + \sigma$$

One should notice that the class ψ is invariant as well.

In [N2] it is proven that α generates $\mathrm{H}^2(M;\mathbb{Z})$.

If $\mathcal{L}$ is a line bundle on M such that $c_1^{\mathrm{top}}(\mathcal{L}) = \omega - \phi$, then if we let:

$$\mathcal{U}_0 = \mathcal{U} \otimes \pi_1^* \mathcal{L}$$

we have that $\phi = \alpha = \omega$ for this sheaf. (Here π_1 is the projection $M \times C$ onto M.) We have:

$$c_1^{\mathrm{top}}(\mathcal{U}_0) = \alpha + f \quad \text{and} \quad c_1^{\mathrm{top}}(\mathcal{U}_{0|M\times\{x\}}) = \alpha$$

This is the rigidified Poincaré bundle $\mathcal{U}_0$. It is unique, as any two Poincaré bundles differ by a twist with $\pi_1^*\mathcal{L}$. The topological Chern classes of $\mathcal{U}_0$ are:

$$c_1^{\mathrm{top}}(\mathcal{U}) = \alpha + f \qquad\qquad c_2^{\mathrm{top}}(\mathcal{U}) = \chi + \psi + \alpha \otimes f,$$

The degree of a morphism $f : \mathbb{P}^1 \to M$

Definition 1.10. *If $f : \mathbb{P}^1 \to M$ we define the degree deg (f) of f as the degree of the line bundle deg $(f^*\Theta)$.*

If $f : \mathbb{P}^1 \to M$ is such that deg $(f) = 1$, we say that f gives a *line* on M. Note that since Θ is very ample, if we consider the embedding given by Θ:

$$i : M \hookrightarrow \mathbb{P}^N,$$

then the morphism $i \circ f : \mathbb{P}^1 \to \mathbb{P}^N$ gives a line on $\mathbb{P}^N$.

Let $\mathcal{F}_0 = (f \times id)^*\mathcal{U}_0$ and let $k =$ deg (f). Then it follows from (1.5) and the fact that $(\mathcal{F}_0)_p \cong O_C(x_0)$, for any $p \in \mathbb{P}^1$, that we have

$$c_1(\mathcal{F}_0) = \mathbb{P}^1 \times \{x_0\} + k\{pt\} \times C \in \mathcal{A}^1(\mathbb{P}^1 \times C) \tag{1.7}$$

It follows from (1.5) and (1.7) that, for any $x \in C$, the degree of f is:

$$\deg\,(f) = \deg\,(c_1(\mathcal{F}_{0|\mathbb{P}^1\times\{x\}})) = \deg\,(c_2(\mathcal{F}_0)) \tag{1.8}$$

It is easy to see now that if $\mathcal{F}$ is an arbitrary bundle that gives f we have the following formula for the degree of f:

$$\deg(f) = \deg(2c_2(\mathcal{F}) - c_1(\mathcal{F}_{|\mathbb{P}^1\times\{x\}})) \tag{1.9}$$

This is because $\mathcal{F}$ has to be of the form $\mathcal{F}_0 \otimes p_1^*O(m)$, where m is some integer.

Note that it follows from the observations in 1.1.1 and from (1.3) that it makes sense to talk about the Kontsevich moduli space $\overline{M}_0(M,k)$ as a parameter space for rational curves of degree k.

The pull-backs of the classes α and β via a morphism $S \to M$

In this section we'll give a way of computing the pull backs of the topological classes α and β via a morphism $S \to M$, in terms of the topological Chern classes of any rank 2 bundle $\mathcal{F}$ on $S \times C$ that gives the morphism $S \to M$.

Let $\kappa : S \to M$ be a morphism. If $\mathcal{U}$ is a Poincaré vector bundle, not necessarily the rigidified one, we let $\mathcal{F} := (\kappa \times id)^*\mathcal{U}$. From (1.5) we get:

$$c_1^{\text{top}}(\mathcal{F}) = \kappa^*(\phi) + f \qquad c_2^{\text{top}}(\mathcal{F}) = \kappa^*(\chi) + \kappa^*(\psi) + \kappa^*(\omega) \otimes f$$

where

$$\kappa^*(\phi), \kappa^*(\omega) \in \mathrm{H}^2(S;\mathbb{Z}), \quad \kappa^*(\chi) \in \mathrm{H}^4(S;\mathbb{Z}), \quad \kappa^*(\psi) \in \mathrm{H}^3(S;\mathbb{Z}) \otimes \mathrm{H}^1(C;\mathbb{Z}).$$

Then $\kappa^*(\alpha) = 2\kappa^*(\omega) - \kappa^*(\phi)$ and $\kappa^*(\beta) = \kappa^*(\phi^2) - \kappa^*(\chi)$.

Denote u and v the following maps, coming from the Kunneth decomposition:

$$\begin{aligned} u &: \mathrm{H}^2(S \times C;\mathbb{Z}) \to \mathrm{H}^2(S) \otimes \mathrm{H}^0(C) \cong H^2(S) \\ v &: \mathrm{H}^4(S \times C;\mathbb{Z})) \to \mathrm{H}^2(S) \otimes H^2(C) \cong H^2(S) \\ w &: \mathrm{H}^4(S \times C;\mathbb{Z}) \to \mathrm{H}^4(S) \otimes \mathrm{H}^0(C) \cong \mathrm{H}^4(S) \end{aligned}$$

Then we have:

$$\kappa^*(\alpha) = 2v(c_2(\mathcal{F})) - u(c_1(\mathcal{F})) \quad \text{and} \quad \kappa^*(\beta) = w(c_1(\mathcal{F})^2 - 4c_2(\mathcal{F})) \qquad (1.10)$$

Note that these expressions are invariant when twisting with a line bundle from S. Hence, if $\mathcal{F}'$ is another vector bundle on $S \times C$ that gives the morphism κ, then the same relations hold for the pull-backs of the classes α and β via κ.

Note that we get as a special case the formula (1.9).

1.3.3 A useful result about rational curves on moduli of vector bundles

Consider the scheme $\mathrm{Mor}_k(\mathbb{P}^1, M)$ introduced in Section 1.1.2. Since the canonical bundle K_M is -2Θ and $\dim M = 3g-3$, it follows that if $f : \mathbb{P}^1 \to M$ is a morphism of degree k, then

$$\chi(\mathbb{P}^1, f^*T_M) = \deg f^*T_M + \dim M = -\deg f^*K_M + \dim M = 2k + 3g - 3$$

The expected dimension is:

$$\text{exp. dim. } \mathrm{Mor}_k(\mathbb{P}^1, M) = 2k + 3g - 3 \tag{1.11}$$

Consider the Kontsevich space $\overline{M}_0(M,k)$. Note that by (1.3) and (1.2), M satisfies 1.1, hence, it makes sense to consider the class $k \in \mathrm{H}_2(X, ; \mathbb{Z})$.

The expected dimension of $\overline{M}_0(M,k)$ is:

$$\text{exp. dim. } \overline{M}_0(M,k) = 2k + 3g - 6 \tag{1.12}$$

We say that a rank 2 vector bundle on $\mathbb{P}^1$ is *balanced* if it splits as $O(a) \oplus O(b)$, for some integers a and b with $|b-a| \geq 1$.

Lemma 1.11. *If $f : \mathbb{P}^1 \to M$ is a morphism of degree k given by the vector bundle $\mathcal{F}$ on $\mathbb{P}^1 \times C$ and for general $x \in C$, the bundle $\mathcal{F}_x$ is balanced, then*

$$H^1(\mathbb{P}^1, f^*T_M) = 0.$$

The point $[f]$ is an unobstructed point of the Kontsevich space $\overline{M}_0(M,k)$, as well as of the space of morphisms $Mor_k(\mathbb{P}^1, M)$.

Proof. From (1.4), the tangent bundle of M is given by:

$$T_M \cong \mathrm{R}^1 p_{1*}(\mathcal{E}nd^0(\mathcal{U}))$$

where $\mathcal{E}nd^0(\mathcal{U})$ is the sheaf of traceless endomorphisms of some Poincaré bundle $\mathcal{U}$ on $M \times C$. Moreover, we have that for any $i \neq 1$:

$$\mathrm{R}^i p_{1*}(\mathcal{E}nd^0(\mathcal{U})) = 0.$$

If we let $\mathcal{F}' = (f \times \mathrm{id}_C)^*\mathcal{U}$ we have

$$f^*T_M \cong \mathrm{R}^1 p_{1*}(\mathcal{E}nd^0(\mathcal{F}')).$$

Since both $\mathcal{F}$ and $\mathcal{F}'$ give the same morphism f, it follows that $\mathcal{F}$ and $\mathcal{F}'$ differ by a twist with a line bundle from $\mathbb{P}^1$. Therefore, we can assume that $\mathcal{F}' \cong \mathcal{F}$.

We have for any $i \neq 1$:

$$\mathrm{R}^i p_{1*}(\mathcal{E}nd^0(\mathcal{F})) = 0.$$

There are two Leray spectral sequences:

$$\mathrm{H}^i(\mathbb{P}^1, \mathrm{R}^j p_{1*}(\mathcal{E}nd^0(\mathcal{F})) \Longrightarrow \mathrm{H}^{i+j}(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{F}))$$
$$\mathrm{H}^i(C, \mathrm{R}^j p_{2*}(\mathcal{E}nd^0(\mathcal{F})) \Longrightarrow \mathrm{H}^{i+j}(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{F}))$$

Then, as for $i \geq 2$ we have $\mathrm{R}^i p_{2*}(\mathcal{E}nd^0(\mathcal{F})) = 0$, it follows that

$$\mathrm{H}^1(\mathbb{P}^1, \mathrm{R}^1 p_{1*}(\mathcal{E}nd^0(\mathcal{F})) \cong \mathrm{H}^2(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{F})) \cong \mathrm{H}^1(\mathrm{R}^1 p_{2*}(\mathcal{E}nd^0(\mathcal{F})).$$

The sheaf $\mathrm{R}^1 p_{2*}(\mathcal{E}nd^0(\mathcal{F}))$ is supported at a finite number of points of C, as for general $x \in C$ we have that the rank 2 vector bundle $\mathcal{F}_x$ on $\mathbb{P}^1$ is balanced. This is because at the generic point ξ of C we have:

$$\mathrm{R}^1 p_{2*}(\mathcal{E}nd^0(\mathcal{F}))_\xi \cong \mathrm{H}^1(\mathbb{P}^1, \mathcal{E}nd^0(\mathcal{F}_\xi)) = 0.$$

This is because if $\mathcal{E} = O(a) \oplus O(b)$ is a vector bundle on $\mathbb{P}^1$, then

$$\mathcal{E}nd(\mathcal{E}) \cong O \oplus O \oplus O(a-b) \oplus O(b-a)$$
$$\mathcal{E}nd^0(\mathcal{E}) \cong O \oplus O(a-b) \oplus O(b-a)$$

□

An observation about the Kodaira-Spencer map

Consider the scheme $\mathrm{Mor}_k(\mathbb{P}^1, M)$ and the evaluation morphism

$$\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, M) \to M.$$

Define on $\mathbb{P}^1 \times C \times \mathrm{Mor}_k(\mathbb{P}^1, M)$ the bundle:

$$\mathcal{H} = (ev \times id_C)^* \mathcal{U}_0.$$

If $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, M)$ is a closed point, then if we let $\mathcal{H}_f = \mathcal{H}_{|\mathbb{P}^1 \times C \times \{f\}}$, we have:

$$\mathcal{H}_f \cong (f \times id)^* \mathcal{U}_0.$$

The bundle $\mathcal{H}$ gives a family of vector bundles on $\mathbb{P}^1 \times C$ and induces a Kodaira-Spencer map at the point $[f]$:

$$\omega : T_{[f]}\mathrm{Mor}_k(\mathbb{P}^1, M) \to \mathrm{Def}\ (\mathcal{H}_f)$$

where Def $(\mathcal{H}_f)$ is the space of infinitesimal deformations of the bundle $\mathcal{H}_f$:

$$\mathrm{Def}\ (\mathcal{H}_f) \cong \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{H}_f, \mathcal{H}_f) \cong \mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd(\mathcal{H}_f)) \cong \mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{H}_f))$$

Since M is smooth, there is a canonical isomophism:

$$T_{[f]}\mathrm{Mor}_k(\mathbb{P}^1, M) \to \mathrm{H}^0(\mathbb{P}^1, f^*T_M)$$

By (1.4) we have:

$$\mathrm{H}^0(\mathbb{P}^1, f^*T_M) \cong \mathrm{H}^0(\mathbb{P}^1, \mathrm{R}^1 p_{1*}\mathcal{E}nd^0(\mathcal{H}_f))$$

From the Leray spectral sequence associated to p_1, we have an exact sequence:

$$0 \to \mathrm{H}^1(\mathbb{P}^1, p_{1*}\mathcal{E}nd^0(\mathcal{H}_f)) \to \mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{H}_f)) \to \mathrm{H}^0(\mathbb{P}^1, \mathrm{R}^1 p_{1*}\mathcal{E}nd^0(\mathcal{H}_f)) \to 0$$

Since the bundle $\mathcal{H}_{f,p}$ on C is stable for any $p \in \mathbb{P}^1$, it follows that

$$p_{1*}\mathcal{E}nd^0(\mathcal{H}_f)) = 0.$$

So there is a canonical isomorphism:

$$\mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{H}_f)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathrm{R}^1 p_{1*}\mathcal{E}nd^0(\mathcal{H}_f))$$

There is a commutative diagram:

$$\begin{array}{ccc} T_{[f]}\mathrm{Mor}_k(\mathbb{P}^1, M) & \xrightarrow{\omega} & \mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd^0(\mathcal{H}_f)) \\ \| & & \downarrow \cong \\ T_{[f]}\mathrm{Mor}_k(\mathbb{P}^1, M) & \xrightarrow{\cong} & \mathrm{H}^0(\mathbb{P}^1, \mathrm{R}^1 p_{1*}\mathcal{E}nd^0(\mathcal{H}_f)) \end{array} \tag{1.13}$$

It follows that the Kodaira-Spencer map ω is an isomorphism.

Chapter 2

Constructing Rational Curves on moduli of vector bundles

Let C be a genus $g \geq 2$ smooth projective curve. Fix a point $x_0 \in C$ and let $M = M(2, O_C(x_0))$ be the moduli space of rank 2 stable vector bundles on C with determinant $O_C(x_0)$.

We effectively construct rational curves of odd and even degree on M and find that there is a *nice* component of the space of rational curves of degree k: it has the expected dimension and the general point is unobstructed. The idea for constructing the odd degree curves is a classical one: look at spaces of extensions of line bundles on C. As these spaces map to M, we obtain rational curves on M by constructing rational curves in the spaces of extensions. Similarly, for the even degree curves, we look at extensions of skyscraper sheaves by rank 2 sheaves on C and construct rational curves in them. We also find the MRC fibration for the nice component to be the Jacobian of the curve C.

2.1 Spaces of extensions of line bundles

2.1.1 Local construction

Let $e \geq 0$ be an integer and let $\mathcal{L}$ to be a line bundle over C of degree $-e$. Consider extensions:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}(x_0) \to 0. \tag{*}$$

Such extensions are classified by the vector space

$$V_{\mathcal{L}} = \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0), \mathcal{L}) \cong \mathrm{H}^1(C, \mathcal{L}^2(-x_0)) \tag{2.1}$$

By Riemann-Roch, $V_{\mathcal{L}}$ is a vector space of dimension

$$\dim\,(V_{\mathcal{L}}) = (2e+1) - 1 + g = 2e + g \tag{2.2}$$

Clearly, any two nonzero elements v, v' of $V_{\mathcal{L}}$ which differ by a scalar define isomorphic vector bundles $\mathcal{E}$. Therefore the isomorphism classes of non-trivial extensions as above are parametrized by the projective space $\mathbb{P}(V_{\mathcal{L}})$.

The locus of unstable extensions

We have the following result about the extensions (*) for which $\mathcal{E}$ is unstable. Such an extension will be called an *unstable extension.*

Proposition 2.1. *For each $\mathcal{L} \in Pic^{-e}(C)$, there is a closed integral subscheme $Z_{\mathcal{L}} \subset \mathbb{P}(V_{\mathcal{L}})$ corresponding to the unstable extensions. The codimension of $Z_{\mathcal{L}}$ is at least g. In the particular case when $e = 0$, we have $Z_{\mathcal{L}} = \emptyset$.*

Proof. The idea is as follows.

The bundle $\mathcal{E}$ in the extension (*) is unstable if and only if there exists a line bundle $\mathcal{L}'$ on C of degree 1 and a non-zero morphism

$$\mathcal{L}' \to \mathcal{E}.$$

Then $\mathcal{L}' \to \mathcal{L}^{-1}(x_0)$ is non-zero as well, since there are no non-zero morphisms $\mathcal{L}' \to \mathcal{L}$, as deg $(\mathcal{L}') >$ deg $(\mathcal{L})$. It follows that there is an effective divisor D on C of degree e such that:

$$\mathcal{L}' \cong \mathcal{L}^{-1}(x_0 - D).$$

Let $\mathcal{E}'$ be the kernel of the composition $\mathcal{E} \to \mathcal{L}^{-1}(x_0) \to \mathcal{L}^{-1}(x_0)_{|D}$. Then there is a commutative diagram with the two horizontal sequences exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L}^{-1}(x_0)_{|D} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & \mathcal{L}^{-1}(x_0 - D) & \longrightarrow & \mathcal{L}^{-1}(x_0) & \longrightarrow & \mathcal{L}^{-1}(x_0)_{|D} & \longrightarrow & 0
\end{array}$$

Using the snake lemma, we get an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E}' \to \mathcal{L}^{-1}(x_0 - D) \to 0. \tag{2.3}$$

Consider the composition morphism

$$\mathcal{L}' \cong \mathcal{L}^{-1}(x_0 - D) \to \mathcal{E} \to \mathcal{L}^{-1}(x_0)_{|D}$$

By the commutativity of the previous diagram, this is zero. Hence, the map from $\mathcal{L}' \cong \mathcal{L}^{-1}(x_0 - D)$ to $\mathcal{E}$ factors through $\mathcal{E}'$. By chasing diagrams, it follows that the exact sequence (2.3) is split.

Consider the vector space

$$V = \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0), \mathcal{L}) \cong \mathrm{H}^1(C, \mathcal{L}^2(-x_0)).$$

By the previous discussion, the vector $v \in V$ corresponding to an unstable vector bundle $\mathcal{E}$ is in the kernel of the surjective map

$$\mathrm{H}^1(C, \mathcal{L}^2(-x_0)) \to \mathrm{H}^1(C, \mathcal{L}^2(-x_0 + D)).$$

From the long exact sequence coming from

$$0 \to \mathcal{L}^{-1}(x_0) \to \mathcal{L}^{-1}(x_0 + D) \to \mathcal{L}^{-1}(x_0 + D)_{|D} \to 0,$$

by applying $\mathrm{Ext}^1_C(-, \mathcal{L})$, we get that

$$0 \to \mathrm{H}^0(C, \mathcal{L}^2(-x_0 + D)_{|D}) \to \mathrm{H}^1(C, \mathcal{L}^2(-x_0)) \to \mathrm{H}^1(C, \mathcal{L}^2(-x_0 + D)) \to 0.$$

Therefore the unstable extensions in V form an e-dimensional linear subspace for each D effective divisor of degree e. If we let D vary in $\mathrm{Sym}^e(C)$, we get that the unstable extensions in V form a family of dimension at most $2e$. Hence, the codimension of this locus is at $(2e + g) - 2e = g \geq 2$.

Note that for $e = 0$, there are no unstable extensions.

To make this precise, on $C^e \times C$ consider the divisors $\Delta_{1,e+1}, \ldots, \Delta_{e,e+1}$, where $\Delta_{i,e+1}$ is the diagonal given by the i-th and $(e+1)$-th components in C^{e+1}. Let

$$\Delta = \Delta_{1,e+1} + \ldots + \Delta_{e,e+1}.$$

If $u = \{(p_1, \ldots, p_e)\} \in C^e$ we let $D = p_1 + \ldots + p_e$.

We have an exact sequence on $C^e \times C$:

$$0 \to O \to O(\Delta) \to O(\Delta)_\Delta \to 0 \tag{2.4}$$

Let π_1, π_2 be the two projections from $C^e \times C$. After tensoring the exact sequence (2.4) with $\pi_2^*(\mathcal{L}^2(-x_0))$ and applying the $\pi_{1*}(-)$ functor, we get an exact sequence:

$$0 \to \pi_{1*}(O(\Delta)_{|\Delta} \otimes \pi_2^*\mathcal{L}^2(-x_0)) \to \mathrm{R}^1\pi_{1*}\pi_2^*\mathcal{L}^2(-x_0) \to$$
$$\to \mathrm{R}^1\pi_{1*}(O(\Delta) \otimes \pi_2^*\mathcal{L}^2(-x_0)) \to 0$$

This is because $\mathrm{H}^1(C, \mathcal{L}^2(-x_0 + D)_{|D}) = 0$ and $\mathrm{H}^0(C, \mathcal{L}^2(-x_0 + D)) = 0$, as $\deg \mathcal{L}^2(-x_0 + D) = -(e + 1)$ and therefore:

$$\pi_{1*}O(\Delta) \otimes \pi_2^*(\mathcal{L}^2(-x_0)) = 0 \quad \mathrm{R}^1\pi_{1*}(O(\Delta)_{|\Delta} \otimes \pi_2^*(\mathcal{L}^2(-x_0))) = 0$$

We define the following sheaves on C^e:

$$\mathcal{E} = \mathrm{R}^1\pi_{1*}\pi_2^*(\mathcal{L}^2(-x_0))$$
$$\mathcal{E}' = \pi_{1*}(\mathcal{D}_{|\Delta} \otimes \pi_2^*(\mathcal{L}^2(-x_0)), \quad \mathcal{E}'' = \mathrm{R}^1\pi_{1*}(O(\Delta) \otimes \pi_2^*\mathcal{L}^2(-x_0))$$

We have an exact sequence:

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

Note that $\mathcal{E}$ is a trivial vector bundle with fiber at $u \in C^e$:

$$\mathcal{E}_u \cong \mathrm{H}^1(C, \mathcal{L}^2(-x_0)) \cong V.$$

In a similar way, $\mathcal{E}'$ and $\mathcal{E}''$ are locally free and the fibers at u are:

$$\mathcal{E}'_u \cong \mathrm{H}^0(C, \mathcal{L}^2(-x_0 + D)_{|D}), \quad \mathcal{E}''_u \cong \mathrm{H}^1(C, \mathcal{L}^2(-x_0 + D))$$

We have that rk $(\mathcal{E}) = 2e + g$, rk $(\mathcal{E}') = e$ and rk $(\mathcal{E}'') = e + g$.

The injective morphism $\mathcal{E}' \to \mathcal{E}$ induces a closed immersion

$$\mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E}).$$

Since $\mathcal{E}$ is a trivial vector bundle on C^e with fiber V, it follows that we have a canonical isomorphism:

$$\mathbb{P}(\mathcal{E}) \cong C^e \times \mathbb{P}(V) \tag{2.5}$$

Let Z be the image of $\mathbb{P}(\mathcal{E}')$ in $\mathbb{P}(V)$ via the isomorphism 2.5, followed by the projection of $C^e \times \mathbb{P}(V)$ onto $\mathbb{P}(V)$. This is the locus of unstable extensions in $\mathbb{P}(V)$. As dim $\mathbb{P}(\mathcal{E}') = 2e - 1$ and dim $\mathbb{P}(V) = 2e + g - 1$, it follows that the codimension of Z is at most $(2e + g - 1) - (2e - 1) = g$. This proves proposition. □

The morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$

We would like to define a morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$ on the locus in $\mathbb{P}(V_{\mathcal{L}})$ which corresponds to associating to every extension (*) the isomorphism class of the vector bundle $\mathcal{E}$.

We find a bundle $\mathcal{G}$ on $(\mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}) \times C$, such that for any $p \in \mathbb{P}(V_{\mathcal{L}}) \setminus Z$ the bundle $\mathcal{G}_p$ is stable. This will detremine the morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$.

Lemma 2.2. *For each $\mathcal{L} \in Pic^{-e}(C)$, there is a vector bundle $\mathcal{G}$ on $\mathbb{P}(V_{\mathcal{L}}) \times C$ and a universal exact sequence*

$$0 \to q_1^* O(1) \otimes q_2^* \mathcal{L} \to \mathcal{G} \to q_2^*(\mathcal{L}^{-1}(x_0)) \to 0 \tag{2.6}$$

where q_1, q_2 are the projections onto $\mathbb{P}(V_{\mathcal{L}})$ and C respectively. It has the property that its restriction to $\{p\} \times C$ is an extension

$$0 \to \mathcal{L} \to \mathcal{G}_p \to \mathcal{L}^{-1}(x_0) \to 0$$

which gives an element in $V_{\mathcal{L}}$ whose class in $\mathbb{P}(V_{\mathcal{L}})$ is p.

Proof. This is a particular case of the Lemma A.1, in which we take $S = \mathrm{Spec}(\mathbb{C})$, $\mathcal{T} = \mathcal{L}$, $\mathcal{V} = \mathcal{L}^{-1}(x_0)$. We have that $\mathrm{Hom}(\mathcal{L}^{-1}(x_0), \mathcal{L}) = 0$ so conditions in Lemma A.1 are satisfied. It follows that there is an extension:

$$0 \to q_1^* O(1) \otimes q_2^* \mathcal{L} \to \mathcal{G} \to q_2^*(\mathcal{L}^{-1}(x_0)) \to 0$$

□

Corollary 2.3. *For each $\mathcal{L} \in Pic^{-e}(C)$ there is a morphism*

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M \tag{2.7}$$

such that for any $p \in \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$, we have that $\kappa_{\mathcal{L}}(p) \in M$ is the isomorphism class of the stable bundle on C which is the middle term of an extension in $V_{\mathcal{L}}$ corresponding to $p \in \mathbb{P}(V_{\mathcal{L}})$.

Proof. Consider the vector bundle $\mathcal{G}$ of the universal extension (2.6). From Lemma 2.1, if $p \in \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$, the vector bundle $\mathcal{G}_p$ is stable. By the definition of the moduli scheme M, there is an associated morphism $\kappa_{\mathcal{L}}$ corresponding to the restriction of $\mathcal{G}$ to $\mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$ □

When is the morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$ dominant?

Note that we have dim $M = 3g - 3$ and if $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ then by (2.2) we have

$$\dim \mathbb{P}(V_{\mathcal{L}}) = 2e + g - 1$$

For $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ we have:

i. If $e < (g-1)$, the morphism κ is not dominant

ii. If $e \geq (g-1)$ then κ is dominant

Note that part i. is clear by dimension considerations. Part ii. follows from the fact that if $e \geq (g-1)$ then for any $\mathcal{E}$ vector bundle of rank 2 and degree 1, the bundle $\mathcal{E} \otimes \mathcal{L}^{-1}$ has sections, hence, there is a non-zero morphism $\mathcal{L} \to \mathcal{E}$. If $\mathcal{E}$ is general, then the line subbundle $\mathcal{L}$ of $\mathcal{E}$ is saturated in $\mathcal{E}$, i.e. there is no non-zero effective divisor D such that $\mathcal{L} \to \mathcal{E}$ is obtained as a composition $\mathcal{L} \to \mathcal{L}(D) \to \mathcal{E}$. Hence, if $\mathcal{E} \in M$ then $\mathcal{E}$ comes from an exact sequence (*).

In particular, this proves that M is unirational. If $e = g-1$ the two dimensions are the same and the morphism $\kappa_{\mathcal{L}}$ is birational.

Computation of $\kappa^*\Theta$

Let $\alpha \in \mathrm{H}^2(M;\mathbb{Z})$ and $\beta \in \mathrm{H}^4(M;\mathbb{Z})$ be the classes defined in (1.6). We have the following lemma.

Lemma 2.4. *Let $\mathcal{L} \in Pic^{-e}(C)$. The morphism*

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$$

has the property that $\kappa_{\mathcal{L}}^*(\alpha) = (2e+1)h$ and $\kappa_{\mathcal{L}}^*(\beta) = h^2$, where h is the Poincaré dual to the fundamental class of a hyperplane in $\mathbb{P}(V_{\mathcal{L}})$.

Proof. Let's denote $V = V_{\mathcal{L}}$ and $\kappa = \kappa_{\mathcal{L}}$. Let $\{H\} \in \mathrm{A}^1(\mathbb{P}(V))$ be the class of a hyperplane in $\mathbb{P}(V)$. The Chern classes $c_1(\mathcal{G}), c_2(\mathcal{G}) \in \mathrm{A}^*(\mathbb{P}(V) \times C)$ of $\mathcal{G}$ can be computed from the exact sequence (2.6) as:

$$c_1(\mathcal{G}) = \{H\} \times C + \mathbb{P}(V) \times \{x_0\}, \quad c_2(\mathcal{G}) = \{H\} \times (\{x_0\} - c_1(\mathcal{L})). \tag{2.8}$$

It follows that if $f \in \mathrm{H}^2(C)$ is the positive generator, then

$$c_1^{\mathrm{top}}(\mathcal{G}) = h + f \in \mathrm{H}^2(\mathbb{P}(V) \times C), \quad c_2^{\mathrm{top}}(\mathcal{G}) = (1+e)h \otimes f \in \mathrm{H}^4(\mathbb{P}(V) \times C).$$

Consider the morphisms coming from the Kunneth decomposition of the cohomology of $\mathbb{P}(V) \times C$:

$$u : \mathrm{H}^2(\mathbb{P}(V) \times C) \to \mathrm{H}^2(\mathbb{P}(V)), \quad v : \mathrm{H}^4(\mathbb{P}(V) \times C) \to \mathrm{H}^2(\mathbb{P}(V))$$
$$w : \mathrm{H}^4(\mathbb{P}(V) \times C) \to \mathrm{H}^4(\mathbb{P}(V))$$

It follows that

$$u(c_1^{\mathrm{top}}(\mathcal{G})) = h, \quad v(c_2^{\mathrm{top}}(\mathcal{G})) = (1+e)h, \quad w(c_1^{\mathrm{top}}(\mathcal{G})^2) = h^2, \quad w(c_2^{\mathrm{top}}(\mathcal{G})) = 0$$

Then from the formulas in (1.10), we have:

$$\kappa^*(\alpha) = 2v(c_2^{\mathrm{top}}(\mathcal{G})) - u(c_1^{\mathrm{top}}(\mathcal{G})) = (2e+1)h$$
$$\kappa^*(\beta) = w(c_1^{\mathrm{top}}(\mathcal{G})^2 - 4c_2^{\mathrm{top}}(\mathcal{G})) = h^2$$

This proves the lemma. □

Corollary 2.5. *The morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$ has the property that*

$$\kappa_{\mathcal{L}}^*(\Theta) = O(2e+1)$$

Note 2.6. *If we let $\mathcal{G}_0 = (\kappa_{\mathcal{L}} \times id)^* \mathcal{U}_0$, where $\mathcal{U}_0$ is the rigidified Poincaré bundle and $\mathcal{G}$ is the universal bundle on $\mathbb{P}(V_{\mathcal{L}}) \times C$ of (2.6), then*

$$\mathcal{G}_0 = \mathcal{G} \otimes q_1^* O(e).$$

Proof. Let $V = V_{\mathcal{L}}$ and $\kappa = \kappa_{\mathcal{L}}$. Since both $\mathcal{G}$ and $\mathcal{G}_0$ correspond to the same morphism κ, it follows that there is an integer m such that

$$\mathcal{G} \cong \mathcal{G}_0 \otimes {q_1}^* O(m)$$

Since $c_1(\mathcal{U}_{0M\times\{x\}}) = O(\Theta)$, by Corollary 2.5, we have

$$c_1(\mathcal{G}_{0|\mathbb{P}(V)\times\{x\}}) = (2e+1)\{H\}$$

where $\{H\} \in \mathrm{A}^1(\mathbb{P}(V))$ is the class of a hyperplane. Since by (2.8) we have

$$c_1(\mathcal{G}_{\mathbb{P}(V)\times\{x\}}) = \{H\}$$

it follows that $m = -e$. □

The bundle $\mathcal{G}_0$ sits in an exact sequence:

$$0 \to q_1^* O(1+e) \otimes q_2^* \mathcal{L} \to \mathcal{G}_0 \to q_1^* O(e) \otimes q_2^*(\mathcal{L}^{-1}(x_0)) \to 0. \tag{2.9}$$

2.1.2 Global construction

The space of extensions of line bundles

We would like to let $\mathcal{L}$ vary in $\mathrm{Pic}^{-e}(C)$ and consider the spaces of extensions $V_{\mathcal{L}} = \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0), \mathcal{L})$ as fibers of some vector bundle on $\mathrm{Pic}^{-e}(C)$.

Lemma 2.7. *There is a projective bundle $p : X \to Pic^{-e}(C)$ such that for any $\mathcal{L} \in Pic^{-e}(C)$ the fiber $p^{-1}(\{\mathcal{L}\})$ is canonically isomorphic to $\mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{2e+g-1}$.*

Proof. Let $\mathcal{A}$ be a Poincaré bundle on $\mathrm{Pic}^{-e}(C) \times C$:

$$\mathcal{A}_{|\{\mathcal{L}\}\times C} \cong \mathcal{L}.$$

Let π_1, π_2 the two projections from $\mathrm{Pic}^{-e}(C) \times C$. Define on $\mathrm{Pic}^{-e}(C)$ the relative extension sheaf

$$\mathcal{S} := \mathcal{E}xt^1_{\mathrm{Pic}^{-e}(C)\times C|\mathrm{Pic}^{-e}(C)}(\pi_1^*\mathcal{A}^{-1} \otimes \pi_2^* O_C(x_0), \pi_1^*\mathcal{A})$$

Note that $\mathcal{S}$ is locally free and

$$\mathcal{S} \cong \mathrm{R}^1\pi_{1*}(\mathcal{A}^2 \otimes \pi_2^* O_C(-x_0)).$$

Consider the projective bundle

$$p : \mathbb{P}(\mathcal{S}) \to \mathrm{Pic}^{-e}(C).$$

Then we have that for any $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$

$$\mathcal{S}_{|\{\mathcal{L}\}} \cong V_{\mathcal{L}} \cong \mathrm{H}^1(\{\mathcal{L}\} \times C, \mathcal{L}^2(-x_0)) \quad \text{and} \quad p^{-1}(\{\mathcal{L}\}) \cong \mathbb{P}(V_{\mathcal{L}})$$

We let $X = \mathbb{P}(\mathcal{S})$. We have

$$\dim X = 2e + 2g - 1 \tag{2.10}$$

□

The projective bundle X from depends on the Poincaré bundle $\mathcal{A}$ on $\mathrm{Pic}^{-e}(C) \times C$. If $\mathcal{A}' = \mathcal{A} \otimes \mathcal{M}$, where $\mathcal{M}$ is a line bundle on $\mathrm{Pic}^{-e}(C)$, and we consider the projective bundle X' constructed as in the proof of Lemma 2.7, using the Poincaré bundle $\mathcal{A}'$, it follows that we have an isomorphism of projective bundles over $\mathrm{Pic}^{-e}(C)$:

$$\phi : X \to X', \quad \text{such that} \quad \phi^* O_{X'}(-1) \cong O_X(-1) \otimes \mathcal{M}^2 \tag{2.11}$$

This is clear, since we have $\mathcal{S}' \cong \mathcal{S} \otimes \mathcal{M}^2$, where

$$\mathcal{S}' = \mathcal{E}xt^1_{\mathrm{Pic}^{-e}(C) \times C | \mathrm{Pic}^{-e}(C)}(\mathcal{A'}^{-1} \otimes \pi_2^* O_C(x_0), \mathcal{A}').$$

Let ν_1, ν_2 be the two projections from $X \times C$. In the following Lemma, $\mathcal{A}$ is the Poincaré bundle on $\mathrm{Pic}^{-e}(C) \times C$ from the proof of Lemma 2.7, used to construct X.

Lemma 2.8. *There is a universal extension on $X \times C$:*

$$0 \to \nu_1^* O_X(1) \otimes p^* \mathcal{A} \to \tilde{\mathcal{G}} \to p^*(\mathcal{A})^{-1} \otimes \nu_2^* O(x_0) \to 0. \tag{2.12}$$

It has the property that, when we restrict to $\{x\} \times C$, where $x \in X$ is a point and we let $\mathcal{L} = p(x) \in Pic^{-e}(C)$, we get an exact sequence:

$$0 \to \mathcal{L} \to \tilde{\mathcal{G}}_x \to \mathcal{L}^{-1}(x_0) \to 0$$

which corresponds to an element in $V_{\mathcal{L}}$, whose class in $\mathbb{P}(V_{\mathcal{L}}) \cong p^{-1}(\{\mathcal{L}\})$ is x.

Proof. This is another application of Lemma A.1 in Appendix, in which we take $S = \mathrm{Pic}^{-e}(C)$, $\mathcal{T} = \mathcal{A}$ and $\mathcal{V} = p^*(\mathcal{A})^{-1} \otimes \nu_2^* O(x_0)$. □

The locus of unstable extensions

Let $\mathcal{A}$ be a Poincaré bundle on $\mathrm{Pic}^{-e}(C) \times C$ and let X be the projective bundle over $\mathrm{Pic}^{-e}(C)$ from Lemma 2.7.

Lemma 2.9. *The locus of unstable extensions in X is a closed integral subscheme $Z \subset X$ of codimension at least g. If $e = 0$, then $Z = \emptyset$.*

Proof. Consider the universal extension on $X \times C$:

$$0 \to \nu_1^* O_X(1) \otimes p^*\mathcal{A} \to \tilde{\mathcal{G}} \to p^*(\mathcal{A})^{-1} \otimes \nu_2^* O(x_0) \to 0.$$

Let $Z \subset X$ be the locus of unstable extensions in X. This is precisely the locus of $x \in X$ for which $\tilde{\mathcal{G}}_x$ is unstable. By Fact 1.9, this is a closed subset of X.

Let $p : X \to \mathrm{Pic}^{-e}(C)$ be the projection. If $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ then $Z \cap p^{-1}(\{\mathcal{L}\})$ is precisely the locus of points in $\mathbb{P}(V_{\mathcal{L}})$ corresponding to unstable extensions, $Z_{\mathcal{L}} \subset \mathbb{P}(V_{\mathcal{L}})$. By Proposition 2.1, $Z_{\mathcal{L}}$ has codimension in $\mathbb{P}(V_{\mathcal{L}})$ at least g and if $e = 0$, then $Z_{\mathcal{L}} = \emptyset$. Since this is true for any $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$, it follows that $Z \subset X$ has codimension at least g and if $e = 0$, then $Z = \emptyset$. □

Note that one can make a construction of the locus of unstable extensions $Z \subset X$ by globalizing the constructions in the proof of Proposition 2.1.

Corollary 2.10. *There is a morphism $\kappa : X \setminus Z \to M$ which restricted to the fiber of $p : X \to Pic^{-e}(C)$ at $\mathcal{L}$ gives exactly the morphism (2.7):*

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M.$$

If $\mathcal{A}' = \mathcal{A} \otimes \mathcal{M}$ is another Poincaré bundle, consider the isomorphism (2.11) of projective bundles on $\mathrm{Pic}^{-e}(C)$:

$$\phi : X \to X'.$$

Let $Z \subset X$ and $Z' \subset X'$ be the loci in Lemma 2.9 and consider the morphisms defined by Corollary 2.10:

$$\kappa : X \setminus Z \to M, \qquad \kappa' : X \setminus Z' \to M$$

Then there is a commutative diagram:

$$\begin{array}{ccc} X \setminus Z & \xrightarrow{\phi} & X' \setminus Z' \\ \kappa \downarrow & & \downarrow \kappa' \\ M & = & M \end{array} \tag{2.13}$$

2.2 Rational curves coming from extensions of line bundles

Let $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$. Consider the morphism (2.7) $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$. The main observation is that for any integer $n \geq 0$ and any morphism:

$$g : \mathbb{P}^1 \to \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \quad \text{such that} \quad g^*O(1) \cong O(n)$$

we get a rational curve $f = \kappa \circ g : \mathbb{P}^1 \to M$ of degree $n(2e+1)$.

2.2.1 The loci $M(e,n)$ in $\overline{M}_0(M, n(2e+1))$

For $k \geq 1$ let $\overline{M}_0(M,k)$ be the Kontsevich space 1.1.1. Note that we could use the space of morphisms $\mathrm{Mor}_k(\mathbb{P}^1, M)$ as well.

Proposition 2.11. *Let e and n be integers such that $e \geq 0$ and $n \geq 1$. There exist integral closed subschemes $M(e,n) \subset \overline{M}_0(M, n(2e+1))$ such that a general element of $M(e,n)$ is a morphism $f : \mathbb{P}^1 \to M$ obtained by a composition:*

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M \tag{2.14}$$

where $\mathcal{L} \in Pic^{-e}(C)$ and $g^*O(1) \cong O(n)$.

Proof. Let $\mathcal{A}$ be a fixed Poincaré bundle on $\mathrm{Pic}^{-e}(C) \times C$ and consider the projective bundle of Lemma 2.7:

$$p : X \to \mathrm{Pic}^{-e}(C).$$

Let $[l] \in \mathrm{H}_2(X;\mathbb{Z})$ be the class of a line contained in a fiber. Note that a morphism $f : \mathbb{P}^1 \to X$ representing the class $n[l]$ lies in a fiber of p. Consider the Kontsevich space $\overline{M}_0(X, n[l])$. By Fact 2.13, there is a morphism:

$$\pi : \overline{M}_0(X, n[l]) \to \mathrm{Pic}^{-e}(C)$$

such that for $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ the fiber $\pi^{-1}(\{\mathcal{L}\})$ is isomorphic to $\overline{M}_0(\mathbb{P}(V_{\mathcal{L}}), n)$. The scheme $\overline{M}_0(X, n[l])$ is integral and it has the expected dimension.

Since the locus of unstable extensions $Z \subset X$ has codimension at least $g \geq 2$, it follows that the morphism

$$\kappa : X \setminus Z \to M$$

induces a rational map between the corresponding Kontsevich spaces:

$$\Psi : \overline{M}_0(X, n[l]) \dashrightarrow \overline{M}_0(M, n(2e+1)) \tag{2.15}$$

Note that, since κ depends on a Poincaré bundle $\mathcal{A}$ on $\mathrm{Pic}^{-e}(C) \times C$, Ψ depends on $\mathcal{A}$.

Define the closure of the image of the morphism Ψ to be:

$$M(e,n) \subset \overline{M}_0(M, n(2e+1)).$$

By taking $M(e,n)$ with the induced reduced structure, we have that $M(e,n)$ is an irreducible closed subscheme of $\overline{M}_0(M, n(2e+1))$. Note that $M(e,n)$ does not depend on the Poincaré bundle $\mathcal{A}$ (see (2.13)). □

Fact 2.12. *[FP] The projective scheme $\overline{M}_0(\mathbb{P}^r, n)$ is irreducible, reduced and it has the expected dimension, which is $(r+1)(n+1)-4$. Moreover, $\overline{M}_0(\mathbb{P}^r, n)$ is a rational variety.*

The following Fact is straightforward to prove using Fact 2.12.

Fact 2.13. *Let $p : X \to S$ be a projective bundle over an integral projective scheme S, given by $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank $(r+1)$ on S. Let $[l]$ be the class of a line in a fiber of p and n a positive integer. The Kontsevich space $\overline{M}_0(X, n[l])$ is an integral projective scheme of the expected dimension. There is a morphism*

$$\pi : \overline{M}_0(X, n[l]) \to S$$

with fiber at $s \in S$ canonically isomorphic to $\overline{M}_0(\mathbb{P}(\mathcal{E}_s), n)$. The dimension of $\overline{M}_0(X, n[l])$ is:

$$dim\ \overline{M}_0(X, n[l]) = dim\ S + (r+1)(n+1) - 4.$$

Theorem 2.14. *The closed subschemes $M(e, n) \subset \overline{M}_0(M, n(2e+1))$ have dimension*

$$dim\ M(e, n) = g + (2e+g)(n+1) - 4.$$

Their MRC fibration is given by a rational map:

$$\rho : M(e, n) \dashrightarrow Pic^{-e}(C)$$

The map ρ associates to a rational curve f which is a composition as in (2.14), the line bundle $\mathcal{L} \in Pic^{-e}(C)$.

Proof. Consider the projective bundle $p : X \to \mathrm{Pic}^{-e}(C)$ of Lemma 2.7 and consider the morphism (2.15) induced from the morphism (2.10) $\kappa : X \setminus Z \to M$

$$\Psi : \overline{M}_0(X, n[l]) \dashrightarrow \overline{M}_0(M, n(2e+1))$$

Since X is a $\mathbb{P}^{2e+g-1}$-bundle over $\mathrm{Pic}^{-e}(C)$, by Fact 2.13, we have that there is a morphism:

$$\pi : \overline{M}_0(X, n[l]) \to \mathrm{Pic}^{-e}(C)$$

and that $\dim \overline{M}_0(X, n[l]) = g + (2e+g)(n+1) - 4$. By Fact 2.12, the fibers of π are rational. Since $\mathrm{Pic}^{-e}(C)$ is an abelian variety, it follows that π gives the MRC fibration of $\dim \overline{M}_0(X, n[l])$.

We will prove that Ψ is birational onto its image, hence, $M(e, n)$ is birational to $\overline{M}_0(X, n[l])$, and then our theorem follows. We will prove that the morphism Ψ is injective on the dense open consisting of morphisms $\mathbb{P}^1 \to X \setminus Z$.

Let g and g' be morphisms $\mathbb{P}^1 \to X \setminus Z$ such that $\kappa \circ g$ and $\kappa \circ g'$ represent the same stable map in $\overline{M}_0(M, n(2e+1))$, i.e., there is an isomorphism $\mu : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\kappa \circ g' \circ \mu = \kappa \circ g.$$

Let $h = g' \circ \mu$. Then we have

$$\kappa \circ h = \kappa \circ g.$$

We prove that $g = h$, hence g and g' represent the same stable map in $\overline{M}_0(X, n[l])$, so Ψ is injective on a dense open.

Assume that g, respectively g', have image in the fiber of p over some $\mathcal{L}$, respectively $\mathcal{L}'$, with $\mathcal{L}, \mathcal{L}' \in \text{Pic}^{-e}(C)$. We have that the following two morphisms are *equal*:

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

$$\mathbb{P}^1 \xrightarrow{h} \mathbb{P}(V_{\mathcal{L}'}) \setminus Z_{\mathcal{L}'} \xrightarrow{\kappa_{\mathcal{L}'}} M$$

By Proposition 2.15 we have $\mathcal{L} \cong \mathcal{L}'$ and $g = h$. ☐

Proposition 2.15. *Let $f : \mathbb{P}^1 \to M$ be a morphism given by a vector bundle $\mathcal{F}$. Assume that $\mathcal{F}$ is an extension of line bundles on $\mathbb{P}^1 \times C$. Then there is a unique line bundle $\mathcal{L}$ of degree $-e$ on C (with $e \leq 0$), an integer $n \geq 0$ and a morphism $g : \mathbb{P}^1 \to \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$ such that $g^*O(1) \cong O(n)$ and f is the composition $\kappa_{\mathcal{L}} \circ g$:*

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

Proof. First, note that if there is an extension

$$0 \to \mathcal{M}_1 \to \mathcal{F} \to \mathcal{M}_2 \to 0$$

with $\mathcal{M}_1, \mathcal{M}_2$ line bundles on $\mathbb{P}^1 \times C$ then there exist $\mathcal{L}, \mathcal{L}' \in \text{Pic}(C)$, $n, m \in \mathbb{Z}$ such that

$$\mathcal{M}_1 = p_1^*O(n) \otimes p_2^*\mathcal{L} \qquad \mathcal{M}_2 = p_1^*O(m) \otimes p_2^*(\mathcal{L}')$$

Since f does not change if we twist $\mathcal{F}$ by a line bundle from $\mathbb{P}^1$, we can assume that $\mathcal{F}$ sits in an extension:

$$0 \to p_1^*O(n) \otimes p_2^*\mathcal{L} \to \mathcal{F} \to p_2^*\mathcal{L}' \to 0.$$

Restriction to $\{p\} \times C$ gives an exact sequence

$$0 \to \mathcal{L} \to \mathcal{F}_p \to \mathcal{L}' \to 0.$$

Since det $(\mathcal{F}_p) \cong O_C(x_0)$, it follows that $\mathcal{L}' \cong \mathcal{L}^{-1}(x_0)$.

Let $e = -\deg(\mathcal{L})$. Since $\mathcal{F}_p$ is stable, $e \geq 0$. Hence, we can assume that $\mathcal{F}$ sits in an extension:

$$0 \to p_1^*O(n) \otimes p_2^*\mathcal{L} \to \mathcal{F} \to p_2^*(\mathcal{L}^{-1}(x_0)) \to 0 \tag{2.16}$$

Note that there is a unique twist of our initial $\mathcal{F}$ that sits in such an extension.

Note that n has to be a non-negative integer. This is because we can compute the degree of f to be $n(2e+1)$, as we have from the exact sequence (2.16)

$$c_1(\mathcal{F}_{|\mathbb{P}^1 \times \{x\}}) = n \qquad \deg c_2(\mathcal{F}) = n(1+e)$$

By formula (1.9) it follows that deg $(f) = n(1+2e)$.

We claim that $\mathcal{L}$ and n are uniquely determined by $\mathcal{F}$. Assume that there is another extension:

$$0 \to p_1^* O(m) \otimes p_2^* \mathcal{L}' \to \mathcal{F} \to p_2^* \mathcal{L}'^{-1}(x_0) \to 0$$

where $\mathcal{L}'$ is a line bundle on C. Note that since there are no non-zero morphisms $O(n) \to O$, it follows that there are no non-zero morphisms

$$p_1^* O(n) \otimes p_2^* \mathcal{L} \to p_2^* \mathcal{L}'^{-1}(x_0)$$

and therefore, there is an induced diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^* O(n) \otimes p_2^* \mathcal{L} & \longrightarrow & \mathcal{F} & \longrightarrow & p_2^* \mathcal{L}^{-1}(x_0) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p_1^* O(m) \otimes p_2^* \mathcal{L}_1 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & p_2^* \mathcal{L}_1^{-1}(x_0) & \longrightarrow & 0 \end{array}$$

It follows from Lemma A.3 in the Appendix that the vertical morphisms are isomorphisms. Hence $\mathcal{L} \cong \mathcal{L}'$ and $m = n$.

Note also that $\mathcal{F}$ determines uniquely, up to a scalar multiple, the class of the extension (2.16) in the space of extensions

$$\mathrm{Ext}^1_{\mathbb{P}^1 \times C}(p_2^*(\mathcal{L}^{-1}(x_0)), p_1^* O(n) \otimes p_2^* \mathcal{L}).$$

Consider the universal extension (2.6) and denote it with (υ):

$$0 \to q_1^* O(1) \otimes q_2^* \mathcal{L} \to \mathcal{G} \to q_2^*(\mathcal{L}^{-1}(x_0)) \to 0 \qquad (\upsilon)$$

By Lemma A.2 in Appendix (for $\mathcal{V} = \mathcal{L}^{-1}(x_0)$ and $\mathcal{T} = \mathcal{L}$) we get that there there is a morphism

$$g : \mathbb{P}^1 \to \mathbb{P}(V_{\mathcal{L}}) \quad \text{such that} \quad g^* O(1) \cong O(n)$$

and the extension (2.16) is a multiple scalar of the extension $g^*\upsilon$:

$$0 \to p_1^* O(n) \otimes p_2^* \mathcal{L} \to \mathcal{F} \to p_2^*(\mathcal{L}^{-1}(x_0)) \to 0 \qquad (g^*\upsilon)$$

The morphism g sends a point $p \in \mathbb{P}^1$ to the class in $\mathbb{P}(V_{\mathcal{L}})$ of the element in $V_{\mathcal{L}}$ given by the extension obtained by restricting (2.16) to $\{p\} \times C$:

$$0 \to \mathcal{L} \to \mathcal{F}_p \to \mathcal{L}^{-1}(x_0) \to 0.$$

Since $\mathcal{F}_p$ is stable, it follows that g has image in $\mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$.

Note that we have $\mathcal{F} \cong (g \times \mathrm{id})^* \mathcal{G}$. Since $\mathcal{F}$ determines the morphism f and

$(g \times \mathrm{id})^*\mathcal{G}$ determines the morphism $\kappa \circ g$, it follows that $f = \kappa \circ g$.

Since (2.16) is equal, modulo multiplication by a scalar, to $g^*(v)$ and $\mathcal{F}$ determines uniquely, up to a scalar multiple, the class of the extension (2.16) in the space of extensions

$$\mathrm{Ext}^1_{\mathbb{P}^1 \times C}(p_2^*(\mathcal{L}^{-1}(x_0)), p_1^*O(n) \otimes p_2^*\mathcal{L}),$$

it follows from Lemma A.2 in Appendix, that g satisfying the properties in our Theorem is unique. □

Note 2.16. *The expected dimension of* $\overline{M}_0(M, n(2e+1))$ *is*

$$2n(2e+1) + 3g - 6.$$

The dimensions of the subschemes $M(e,n)$ *are as follows:*

i. *dim* $M(e,n)$ = *expected dimension if and only if* $n = 1$ *or* g *is even and* $e = \frac{g}{2} - 1$

ii. *dim* $M(e,n) >$ *expected dimension if and only if* $n > 1$ *and* $e < \frac{g}{2} - 1$

2.2.2 The nice component of $\overline{M}_0(M,k)$ for k odd

In this section k will be an odd integer. We let $k = 2e+1$, with $e \geq 0$. For $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$, we have the morphism (2.7):

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M \quad \text{and} \quad \kappa_{\mathcal{L}}^*\Theta = O(k)$$

Recall from 1.12 that the expected dimension of $\overline{M}_0(M,k)$ is $2k + 3g - 6$.

Theorem 2.17. *There is a* nice *irreducible component* $\mathfrak{M}$ *of the moduli space* $\overline{M}_0(M,k)$. *By* nice *component, we mean:*

i. $\mathfrak{M}$ *has the expected dimension*

ii. *A general point* $[f] \in \mathfrak{M}$ *is obtained as a composition:*

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

where $\mathcal{L} \in Pic^{-e}(C)$ *and* $g^*O(1) \cong O(1)$

iii. *A general point* $[f] \in \mathfrak{M}$ *is an unobstructed point of* $\overline{M}_0(M,k)$

iv. *The MRC fibration of* $\mathfrak{M}$ *is given by a map*

$$\mathfrak{M} \dashrightarrow Pic^{-e}(C)$$

which sends the point $[f] \in \mathfrak{M}$ *to* $\mathcal{L} \in Pic^{-e}(C)$

Proof. Consider the closed subscheme $M(1,e) \subset \overline{M}_0(M,k)$. We claim this is an irreducible component that satisfies all the conditions in the Theorem. By Proposition 2.11, a general point $[f] \in M(e,1)$ is obtained as a composition:

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

where $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ and $g^*O(1) \cong O(1)$.

By the proof of Proposition 2.15 the morphism f is given by a vector bundle $\mathcal{F}$ on $\mathbb{P}^1 \times C$, which sits in an exact sequence:

$$0 \to p_1^*O(1) \otimes p_2^*\mathcal{L} \to \mathcal{F} \to p_2^*\mathcal{L}^{-1}(x_0) \to 0.$$

It follows that for any $x \in C$, we have that the bundle $\mathcal{F}_x$ has balanced splitting

$$\mathcal{F}_x \cong O(1) \oplus O.$$

It follows from Lemma 1.11, that f is an unobstructed point in $\overline{M}_0(M,k)$. Hence, $[f]$ is contained in a unique irreducible component of $\overline{M}_0(M,k)$, which also has the expected dimension. Since $M(e,1)$ is an irreducible scheme of the expected dimension (see Note 2.16), it must be the unique irreducible component containing $[f]$. Part iv. follows from Theorem 2.14. □

2.3 Spaces of extensions of skyscraper sheaves by rank 2 vector bundles

2.3.1 Local construction

Let $e \geq 1$ be an integer. Let $y_1, \ldots, y_e$ be points on C (not necessarily distinct) and let $D = y_1 + \ldots + y_e$. Fix $\mathcal{E}$ a rank 2 vector bundle on C with $\det(\mathcal{E}) \cong O_C(x_0 - D)$. Consider extensions:

$$0 \to \mathcal{E} \to \mathcal{E}' \to O_D \to 0. \tag{*}$$

Then

$$c_1(\mathcal{E}') = c_1(\mathcal{E}) + D, \qquad \det(\mathcal{E}') \cong O_C(x_0)$$

Such extensions are classified by the $2e$-dimensional vector space:

$$V_{D,\mathcal{E}} := \mathrm{Ext}^1_C(O_D, \mathcal{E}) \cong \mathrm{H}^0(C, \mathcal{E}(D)_{|D}) \tag{2.17}$$

Clearly, any two nonzero elements v, v' of $V_{D,\mathcal{E}}$ which differ by a scalar define isomorphic vector bundles $\mathcal{E}$. Therefore the isomorphism classes of non-trivial extensions as above are parametrized by the projective space $\mathbb{P}(V_{D,\mathcal{E}})$.

The locus of not locally free extensions

We have the following result about the locus of extensions (*), where $\mathcal{E}'$ is not locally free. We call these *not locally free extensions.*

Lemma 2.18. *For $D \in Sym^e C$ and $\mathcal{E}$ a rank 2 vector bundle on C with*

$$det\,(\mathcal{E}) \cong O_C(x_0 - D)$$

there exist a closed subscheme $\Gamma_{D,\mathcal{E}}$ in $\mathbb{P}(V_{D,\mathcal{E}})$ corresponding to not locally free extensions (*). This locus is a union of e codimension 2 linear subspaces if $e \geq 2$ and it is empty if $e = 1$.

Proof. Let $D = y_1 + \ldots + y_e$, with $y_1, \ldots, y_e$ points on C. For simplicity, denote $V = V_{D,\mathcal{E}}$. Consider the 2-dimensional vector spaces

$$V_i = \mathrm{Ext}^1_C(O_{y_i}, \mathcal{E}).$$

Then there is an isomorphism $V \cong \oplus_{i=1}^e V_i$. A vector $v \in V$ corresponds $(v_1, \ldots, v_e)$, with $v_i \in V_i$, in the following way: if v corresponds to the extension

$$0 \to \mathcal{E} \to \mathcal{E}' \to O_D \to 0 \qquad (*)$$

then v_i will correspond to the exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to O_{y_i} \to 0 \qquad (2.18)$$

obtained by applying the snake lemma to the short exact sequences in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' & \longrightarrow & O_D & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E}' & \longrightarrow & O_{D_i} & \longrightarrow & 0 \end{array} \qquad (2.19)$$

where $D_i := D - y_i$ and the morphism $\mathcal{E}' \to O_{D_i}$ is the morphism obtained by composing $\mathcal{E}' \to O_D$ with the projection $O_D \to O_{D_i}$.

We claim that the locus of the extensions (*) inside the space $V \cong \oplus V_i$, for which $\mathcal{E}'$ is not locally free, is given by $v = (v_1, \ldots, v_e)$ such that $v_i = 0$ for some $i \in \{1, \ldots, e\}$.

By Lemma A.11 in the Appendix, we have that $\mathcal{E}'$ is not locally free if and only if there is $y \in \{y_1, \ldots, y_e\}$ such that the sequence (*) on stalks at $\{y\}$ splits. If in the commutative diagram (2.19) we look at the stalks at the point $y = y_i$, we get that the exact sequence (*) on stalks at y splits if and only if the short exact sequence (2.18) on stalks at y splits. Since at all the other points of C the short exact sequence (2.18) on stalks splits, it follows, by Lemma A.12 in Appendix, that the sequence (2.18) is split. Hence, $v_i = 0$.

Denote by Γ_i the codimension 2 linear subspace

$$\Gamma_i = \mathbb{P}(\oplus_{j \neq i} V_j) \subset \mathbb{P}(V).$$

Note that if $D_i = D - y_i$ then $\Gamma_i \cong \mathbb{P}(\mathrm{Ext}^1_C(O_{D_i}, \mathcal{E}))$.

Then the union

$$\Gamma := \cup_{i=1}^{e} \Gamma_i$$

is precisely the locus of extensions (*) in $\mathbb{P}(V)$ for which $\mathcal{E}'$ is not locally free. Cleraly, if $e = 1$, then $\Gamma = \emptyset$. □

The locus of unstable extensions

We have the following result about the locus of extensions (*), where $\mathcal{E}'$ is not a stable vector bundle. Such an extension will be called an *unstable extension.*

Proposition 2.19. *For $D \in Sym^e C$ and $\mathcal{E}$ a rank 2 general* stable *vector bundle on C with*

$$det\,(\mathcal{E}) \cong O_C(x_0 - D),$$

there exists a closed integral subscheme $Y_{D,\mathcal{E}}$ in $\mathbb{P}(V_{D,\mathcal{E}}) \setminus \Gamma_{D,\mathcal{E}}$ corresponding to extensions (*), with $\mathcal{E}'$ is an unstable vector bundle. The codimension of $Y_{D,\mathcal{E}}$ is at least 2. If $e = 1$ then $Y_{D,\mathcal{E}} = \emptyset$.

Proof. The idea is as follows.

The bundle $\mathcal{E}'$ in the extension (*) is not stable if and only if there is a line bundle $\mathcal{L}$ on C of degree at least 1 and a non-zero morphism

$$h : \mathcal{L} \to \mathcal{E}'.$$

Since we can take the saturation of $\mathcal{L}$ in $\mathcal{E}'$, we can assume that the quotient is torsion-free. Hence,we can assume that there is an extension:

$$0 \to \mathcal{L} \to \mathcal{E}' \to \mathcal{L}^{-1}(x_0) \to 0.$$

Consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{L} & = & \mathcal{L} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' & \longrightarrow & O_D & \longrightarrow & 0 \end{array}$$

where $f : \mathcal{L} \to O_D$ is the morphism induced by composing h with $\mathcal{E}' \to O_D$.

Let $\mathcal{K} = \ker\ (f)$. By the snake lemma, it follows that there is an exact sequence:

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{L}^{-1}(x_0) \to \text{coker } (f) \to 0.$$

If $f = 0$ then $\mathcal{K} \cong \mathcal{L}$ is a subbundle of $\mathcal{E}$. Since $\mathcal{E}$ is a stable bundle of degree $1 - e \leq 0$, we get a contradiction. Hence, $f \neq 0$.

If $i \in \{1, \ldots, e\}$, we denote

$$D_i = y_1 + \ldots + y_i, \quad D_i' = y_{i+1} + \ldots + y_{e+1}.$$

Assume, without loss of generality, that there is $i \in \{1, \ldots, e\}$ such that im $(f) = O_{D_i}$. Then coker $(f) \cong O_{D_i'}$. From the exact sequence

$$0 \to \mathcal{K} \to \mathcal{L} \to O_{D_i} \to 0, \qquad (**)$$

it follows that $\mathcal{K} \cong \mathcal{L}(-D_i)$.

The injective morphism $\mathcal{K} \to \mathcal{E}$ gives rise to an exact sequence:

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{K}^{-1}(x_0 - D) \to 0.$$

We denote with u the induced morphism $\mathcal{K} \to \mathcal{E}$. There is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{L} & \xrightarrow{f} & O_{D_i} & \longrightarrow & 0 \\ \downarrow & & \downarrow u & & \downarrow h & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' & \longrightarrow & O_D & \longrightarrow & 0 \end{array}$$

The first exact sequence corresponds to the element (**) in $\text{Ext}^1(O_{D_i}, \mathcal{K})$.

From $0 \to O_{D_i} \to O_D \to O_{D_i'} \to 0$ we get:

$$0 \to \text{Ext}^1(O_{D_i}, \mathcal{K}) \to \text{Ext}^1(O_D, \mathcal{K}) \to \text{Ext}^1(O_{D_i'}, \mathcal{K}) \to 0.$$

The extension (**) can be regarded therefore as an element in $\text{Ext}^1(O_D, \mathcal{K})$.

On the other hand, there is a canonical map coming from the injective map $\mathcal{K} \to \mathcal{E}$:

$$\text{Ext}^1(O_D, \mathcal{K}) \to \text{Ext}^1(O_D, \mathcal{E}).$$

The key observation is that, by chasing diagrams, the extension (**) maps to the extension (*) by the composition:

$$\text{Ext}^1(O_{D_i}, \mathcal{K}) \to \text{Ext}^1(O_D, \mathcal{K}) \to \text{Ext}^1(O_D, \mathcal{E})$$

Let f be some integer such that deg $(\mathcal{K}) = -f$. We have

$$-f = \deg\ (\mathcal{K}) = \deg\ (\mathcal{L}) - i \geq (1 - e)$$

Since $\mathcal{E}$ is stable, we have

$$-f = \deg\,(\mathcal{K}) < \frac{\deg\,(\mathcal{E})}{2} = \frac{1-e}{2} \leq 0.$$

It follows that the line subbundle $\mathcal{K}$ of $\mathcal{E}$ has degree $-f$, with f an integer satisfying:

$$e - 1 \leq f > \frac{e-1}{2} \geq 0$$

We proved that $\mathcal{E}'$ is unstable if and only if there are

i. an integer f such that $e - 1 \geq f > \frac{e-1}{2}$

ii. an integer i such that $e \geq i \geq 1$

iii. a cycle $D_i \subset D$ of length i

iv. a saturated line subundle $\mathcal{K}$ of $\mathcal{E}$ with $\deg\,(\mathcal{K}) = -f$

v. an element (**) in $\mathrm{Ext}^1(O_{D_i}, \mathcal{K})$

such that (**) is sent to (*) via the morphisms:

$$\mathrm{Ext}^1(O_{D_i}, \mathcal{K}) \to \mathrm{Ext}^1(O_D, \mathcal{K}) \to \mathrm{Ext}^1(O_D, \mathcal{E}.) \tag{2.20}$$

Note that to give an injective morphism $\mathcal{K} \to \mathcal{E}$ with torsion free cokernel is the same as giving an extension:

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{K}^{-1}(x_0 - D) \to 0 \tag{2.21}$$

The extension is unique up to multiplication with a scalar. Note that the middle term of the extension in (2.21) is our fixed vector bundle $\mathcal{E}$.

The idea is to use Lemma 2.39 from Section 2.5 to parametrize the data $(\mathcal{K}, (**))$, when we fix f, i in the given range and we also fix $D_i \subset \mathcal{D}$. We will use Corollary 2.42 from the Section 2.5 to estimate that the dimension of this family is at most

$$g + (2f - e + g - 1) - (3g - 3) + i = 2f - e + i - g + 2.$$

Hence, the codimension in the $2e$-dimensional space of extensions $V_{D,\mathcal{E}}$ is at least

$$\begin{gathered}(2e - 1) - (2f - e - g + i + 1) = 3e - 2f - i + g - 2 = \\ 2(e - f) + (e - i) + (g - 2) \geq 2 + (g - 2) \geq g \geq 2\end{gathered}$$

To make this precise, we fix an f in the range $e - 1 \geq f > \frac{e-1}{2}$ and i in the range $e \geq i \geq 1$. We fix one of the finitely many choices $D_i \subset D$ of a cycle of length i in our fixed cycle D. We effectively construct a parameter space for the space of pairs $(\mathcal{K}, (**))$ and count dimensions.

We fix an f in the range $e-1 \geq f > \frac{e-1}{2}$. We let $\xi = O_C(x_0 - D)$ and consider the projective bundle X constructed in Lemma 2.39 from the Section 2.5 and the map κ constructed in Corollary 2.42 from the same Section:

$$p : X \to \mathrm{Pic}^{-f}(C) \quad \text{and} \quad \kappa : X \setminus Z \to M_\xi.$$

The space X parametrizes extensions (2.21), up to scalar multiplication.

Moreover, if $\mathcal{A}$ is a Poincaré bundle on $\mathrm{Pic}^{-f}(C) \times C$, then there is a universal sequence on $X \times C$:

$$0 \to \pi_1^* O_X(1) \otimes \mathcal{A} \to \tilde{\mathcal{G}} \to \mathcal{A}^{-1} \otimes \pi_2^* \xi \to 0$$

where π_1 and π_2 are the projections from $X \times C$ onto X and C respectively.

The morphism κ sends the extension (2.21) to the isomorphism class of $\mathcal{E}$ in $\mathcal{M}_\xi$. Let $X_\mathcal{E} \subset X$ be the fiber of X at $\mathcal{E} \in \mathcal{M}_\xi$. If for the given integer f the morphism κ is not dominant (it can happen! see comments after Corollary 2.42 from the Section 2.5), then $X_\mathcal{E} = \emptyset$ and we are done. If κ is dominant, then since $\mathcal{E}$ is general in $\mathcal{M}_\xi$, it follows that

$$\dim X_\mathcal{E} = \dim X - \dim M_\xi =$$
$$g + (2f - e + g - 1) - (3g - 3) = 2f - e - g + 2$$

Let $D_i' = D - D_i$. Consider the following relative extensions sheaves on $X_\mathcal{E}$:

$$\mathcal{F}' = \mathcal{E}xt^1_{X_\mathcal{E} \times C | X_\mathcal{E}}(\pi_2^* O_{D_i}, \mathcal{A}), \quad \mathcal{F} = \mathcal{E}xt^1_{X_\mathcal{E} \times C | X_\mathcal{E}}(\pi_2^* O_D, \tilde{\mathcal{G}})$$
$$\mathcal{F}'' = \mathcal{E}xt^1_{X_\mathcal{E} \times C | X_\mathcal{E}}(\pi_2^* O_{D_i}, \mathcal{A}^{-1} \otimes \pi_2^* \xi) \oplus \mathcal{E}xt^1_{X_\mathcal{E} \times C | X_\mathcal{E}}(\pi_2^* O_{D_i'}, \tilde{\mathcal{G}})$$

Here $\mathcal{A}$ and $\tilde{\mathcal{G}}$ are restricted to $X_\mathcal{E} \times C$. We have an exact sequence:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

The sheaves $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{F}''$ are locally free and if we restrict to $\{u\} \times C$, where u is a point in X given by an extension (2.21), we have:

$$\mathcal{F}'_u \cong \mathrm{Ext}^1_C(O_{D_i}, \mathcal{K}), \quad \mathcal{F}_u \cong \mathrm{Ext}^1_C(O_D, \mathcal{E})$$
$$\mathcal{F}''_u \cong \mathrm{Ext}^1_C(O_{D_i}, \mathcal{K}^{-1}(x_0)) \oplus \mathrm{Ext}^1_C(O_{D_i'}, \mathcal{E})$$

The injective morphism $\mathcal{F}' \to \mathcal{F}$ induces a closed immersion of projective bundles $\mathbb{P}(\mathcal{F}') \hookrightarrow \mathbb{P}(\mathcal{F})$ over $X_\mathcal{E}$. Since $\tilde{\mathcal{G}}_u \cong \mathcal{E}$ for any $u \in X_\mathcal{E}$, we have that there is some line bundle $\mathcal{M}$ on $X_{\mathcal{E}}$ such that

$$\tilde{\mathcal{G}} \cong \pi_2^*\mathcal{E} \otimes \mathcal{M}.$$

Then, if we let $V = V_{D,\mathcal{E}}$, we have:

$$\mathcal{F} = \mathcal{E}xt^1_{X_{\mathcal{E}}\times C|X_{\mathcal{E}}}(\pi_2^* O_D, \tilde{\mathcal{G}}) \cong \mathcal{M} \otimes V$$

There is a canonical isomorphism:

$$\mathbb{P}(\mathcal{F}) \cong X_{\mathcal{E}} \times \mathbb{P}(V).$$

Let $\Gamma \subset \mathbb{P}(V)$ be the closed subscheme from Lemma 2.18, of extensions (*) for which $\mathcal{E}'$ is not locally free.

The extensions in $\mathbb{P}(V) \setminus \Gamma$ that have $\mathcal{E}'$ unstable are given by intersecting $\mathbb{P}(V) \setminus \Gamma$ with the image of $\mathbb{P}(\mathcal{F}')$ in $\mathbb{P}(V)$ via

$$\mathbb{P}(\mathcal{F}') \subset \mathbb{P}(\mathcal{F}) \cong X_{\mathcal{E}} \times \mathbb{P}(V) \to \mathbb{P}(V).$$

Hence, the dimension of the locus of unstable extensions in $\mathbb{P}(V) \setminus \Gamma$ is at most:

$$\begin{aligned} &\dim\ \mathbb{P}(\mathcal{F}') = \dim\ (X_{\mathcal{E}}) + \dim\ \mathrm{Ext}^1(O_{D_i}, \mathcal{K}) - 1 = \\ &\quad = (2f - e - g + 2) + i - 1 = 2f - e - g + i + 1 \end{aligned}$$

As $f \leq e$ and $i \leq e$ we have that the unstable locus has codimension at least:

$$\begin{aligned} &(2e - 1) - (2f - e - g + i + 1) = 3e - 2f - i + g - 2 = \\ &\quad = 2(e - f) + (e - i) + (g - 2) \geq 2 + (g - 2) \geq g \geq 2 \end{aligned}$$

Note that if $e < (g - 1)$ then dim $X <$ dim M_ξ, hence $\kappa : X \setminus Z \to \mathcal{M}_\xi$ cannot be dominant. If $\mathcal{E}$ is general in $\mathcal{M}_\xi$ it follows that if in the extension (*) $\mathcal{E}'$ is locally free, then $\mathcal{E}'$ is stable. We have $Y_{D,\mathcal{E}} = \emptyset$. In particular, if $e = 1$ (D is equal to a point), then we have $\Gamma_{D,\mathcal{E}} = \emptyset$ and $Y_{D,\mathcal{E}} = \emptyset$. □

Note about what $\mathcal{E}$ "general" means in Proposition 2.19

Note 2.20. *For each integer f in the range $e - 1 \geq f \geq \frac{e-1}{2}$, we denote by $X(f)$ the projective bundle X defined in Lemma 2.39 in Section 2.5. Hence, we have a projective bundle:*

$$X(f) \to Pic^{-f}(C)$$

For each f, let $\kappa(f)$ be the morphism κ from Corollary 2.42:

$$\kappa(f) : X(f) \setminus Z(f) \to M_\xi$$

In Proposition 2.19), if we let $\xi = O(x_0 - D)$, then by $\mathcal{E}$ general in M_ξ, we *mean*

i. *$\mathcal{E}$ is not in the image of the map $\kappa(f)$ for those f for which $\kappa(f)$ is not dominant*

i. *the fiber of the map $\kappa(f)$ at the point $\mathcal{E} \in M_\xi$ has dimension dim $X(f)$ − dim M_ξ, for those f for which $\kappa(f)$ is dominant*

The morphism $\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$

Let D and $\mathcal{E}$ be as in Proposition 2.19. We denote by $Z_{D,\mathcal{E}}$ the union $\Gamma_{D,\mathcal{E}} \cup \overline{Y_{D,\mathcal{E}}}$, where $\overline{Y_{D,\mathcal{E}}}$ is the closure of $Y_{D,\mathcal{E}}$ in $\mathbb{P}(V_{D,\mathcal{E}})$.

We would like to define a morphism $\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$ on the locus in $\mathbb{P}(V_{D,\mathcal{E}})$ which corresponds to associating to every extension (*) the isomorphism class of the vector bundle $\mathcal{E}$.

In order to define a morphism $\mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$ we need to give a bundle $\mathcal{J}$ on $(\mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}}) \times C$, such that for any $p \in \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}}$ the bundle $\mathcal{J}_p$ is stable.

Lemma 2.21. *For D and $\mathcal{E}$ as in Proposition 2.19, there is a vector bundle $\mathcal{J}$ on $\mathbb{P}(V_{D,\mathcal{E}}) \times C$ and a universal exact sequence*

$$0 \to q_1^*O(1) \otimes q_2^*\mathcal{E} \to \mathcal{J} \to q_2^*O_D \to 0 \tag{2.22}$$

where q_1, q_2 are the projections onto $\mathbb{P}(V_{D,\mathcal{E}})$ and C respectively. It has the property that its restriction to $\{p\} \times C$ is an extension

$$0 \to \mathcal{E} \to \mathcal{J}_p \to O_D \to 0$$

which gives an element in $V_{D,\mathcal{E}}$ whose class in $(\mathbb{P}(V_{D,\mathcal{E}})$ is p.

Proof. This is a particular case of the Lemma A.1 in the Appendix. We take $S = \mathrm{Spec}(\mathbb{C})$, $\mathcal{T} = \mathcal{E}$, $\mathcal{V} = O_D$. We have that $\mathrm{Hom}(O_D, \mathcal{E}) = 0$, so all the conditions in Lemma A.1 are satisfied. It follows that there is an extension:

$$0 \to q_1^*O(1) \otimes q_2^*\mathcal{E} \to \mathcal{J} \to q_2^*O_D \to 0$$

□

Corollary 2.22. *For D and $\mathcal{E}$ as in Proposition 2.19, there is a morphism:*

$$\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M \tag{2.23}$$

such that for any $p \in (\mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}}$, we have that $\eta_{D,\mathcal{E}}(p) \in M$ is the isomorphism class of the stable bundle on C which is the middle term of an extension in $V_{D,\mathcal{E}}$ corresponding to $p \in \mathbb{P}(V_{D,\mathcal{E}})$.

Proof. From Proposition 2.19, if $p \in \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}}$, the vector bundle $\mathcal{J}_p$ is stable. By the definition of the moduli scheme M, there is an associated morphism $\eta_{D,\mathcal{E}}$ corresponding to the restriction of $\mathcal{J}$ to $\mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}}$. □

Let $\alpha \in \mathrm{H}^2(M;\mathbb{Z})$ and $\beta \in \mathrm{H}^4(M;\mathbb{Z})$ be the classes defined in 1.6. We have the following lemma.

Lemma 2.23. *For D and $\mathcal{E}$ as in Proposition 2.19, the morphism*

$$\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$$

has the property that $\eta^*_{D,\mathcal{E}}(\alpha) = (2e)h$ and $\eta^*_{D,\mathcal{E}}(\beta) = 4h^2$, where h is the Poincaré dual to the fundamental class of a hyperplane in $\mathbb{P}(V_{D,\mathcal{E}})$.

Proof. Let's denote $V = V_{D,\mathcal{E}}$ and $\eta = \eta_{D,\mathcal{E}}$. Let $\{H\} \in A^1(\mathbb{P}(V))$ be the class of a hyperplane in $\mathbb{P}(V)$. The Chern classes $c_1(\mathcal{J})$ and $c_2(\mathcal{J})$ in $A^*(\mathbb{P}(V) \times C)$ can be computed from the exact sequence (2.21) as:

$$c_1(\mathcal{J}) = 2\{H\} \times C + \mathbb{P}(V) \times \{x_0\}, \quad c_2(\mathcal{J}) = \{H\} \times (\{x_0\} + D). \tag{2.24}$$

It follows that if $f \in \mathrm{H}^2(C)$ is the positive generator, then

$$c_1^{\text{top}}(\mathcal{J}) = 2h + f \in \mathrm{H}^2(\mathbb{P}(V) \times C), \quad c_2^{\text{top}}(\mathcal{J}) = (1+e)h \otimes f \in \mathrm{H}^4(\mathbb{P}(V) \times C).$$

Consider the morphisms coming from the Kunneth decomposition of the cohomology of $\mathbb{P}(V) \times C$:

$$u : \mathrm{H}^2(\mathbb{P}(V) \times C) \to \mathrm{H}^2(\mathbb{P}(V)), \quad v : \mathrm{H}^4(\mathbb{P}(V) \times C) \to \mathrm{H}^2(\mathbb{P}(V))$$
$$w : \mathrm{H}^4(\mathbb{P}(V) \times C) \to \mathrm{H}^4(\mathbb{P}(V))$$

It follows that

$$u(c_1^{\text{top}}(\mathcal{J})) = 2h, \quad v(c_2^{\text{top}}(\mathcal{J})) = (1+e)h, \quad w(c_1^{\text{top}}(\mathcal{J})^2) = 4h^2, \quad w(c_2^{\text{top}}(\mathcal{J})) = 0$$

Then from the formulas 1.10, we have:

$$\kappa^*(\alpha) = 2v(c_2^{\text{top}}(\mathcal{J})) - u(c_1^{\text{top}}(\mathcal{J}) = (2e)h$$
$$\kappa^*(\beta) = w(c_1^{\text{top}}(\mathcal{J})^2 - 4c_2^{\text{top}}(\mathcal{J}) = 4h^2$$

This proves the lemma. □

Corollary 2.24. *The morphism $\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$ has the property that*

$$\eta^*_{D,\mathcal{E}}(\Theta) = O(2e)$$

Note 2.25. *If we let $\mathcal{J}_0 = (\eta_{D,\mathcal{E}} \times id)^*\mathcal{U}_0$, where $\mathcal{U}_0$ is the rigidified Poincaré bundle and $\mathcal{J}$ is the universal bundle on $\mathbb{P}(V_{D,\mathcal{E}}) \times C$ of (2.21), then*

$$\mathcal{J}_0 = \mathcal{J} \otimes q_1^* O(e).$$

Proof. Let $V = V_{D,\mathcal{E}}$ and $\eta = \eta_{D,\mathcal{E}}$. Since both $\mathcal{J}$ and $\mathcal{J}_0$ correspond to the same morphism η, it follows that there is an integer m such that

$$\mathcal{J} \cong \mathcal{J}_0 \otimes q_1{}^* O(m)$$

Since $c_1(\mathcal{U}_{0M\times\{x\}}) = O(\Theta)$, by Corollary 2.24, we have

$$c_1(\mathcal{J}_{0\mathbb{P}(V)\times\{x\}}) = (2e)\{H\}$$

where $\{H\} \in \mathrm{A}^1(\mathbb{P}(V)$ is the class of a hyperplane. Since by (2.24)

$$c_1(\mathcal{J}_{\mathbb{P}(V)\times\{x\}}) = \{H\}$$

it follows that $m = -e$. □

Note that the bundle $\mathcal{J}_0$ sits in an exact sequence:

$$0 \to q_1^* O(1+e) \otimes q_2^* \mathcal{E} \to \mathcal{J}_0 \to q_1^* O(e) \otimes q_2^*(O_D) \to 0. \tag{2.25}$$

A group action on $\mathbb{P}(V_{D,\mathcal{E}})$

For any $D \in \mathrm{Sym}^e C$ we have a torus group T_D, acting on $V_{D,\mathcal{E}}$, given by

$$\mathrm{Aut}\ (O_D) \cong \mathrm{H}^0(D, O_D^*).$$

If $\lambda \in \mathrm{H}^0(D, O_D^*)$, we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' & \longrightarrow & O_D & \longrightarrow & 0 \\ \downarrow & & \| & & \cong\downarrow & & \cong\downarrow\lambda & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' & \longrightarrow & O_D & \longrightarrow & 0 \end{array}$$

If D is made of distinct points, then $T_D \cong (\mathbb{G}_m)^e$.

The automorphism group of $\mathcal{E}$ also acts on $V_{D,\mathcal{E}}$. Let $H_{\mathcal{E}}$ be this group. If $\mathcal{E}$ is stable, then $H_{\mathcal{E}} = \mathbb{G}_m$. Note that since in both actions scalar multiplication gives the same action, we define:

$$G_{D,\mathcal{E}} = T_D \times_{\mathbb{G}_m} H_{\mathcal{E}}$$

The group $G_{D,\mathcal{E}}$ acts on $V_{D,\mathcal{E}}$ and $\mathbb{P}(V_{D,\mathcal{E}})$, without changing the isomorphism class of the middle term $\mathcal{E}'$. Hence, the action preserves the locus $Z_{D,\mathcal{E}} \subset \mathbb{P}(V_{D,\mathcal{E}})$.

2.3.2 Global construction

Let $e \geq 1$. We would like to let D vary in $\mathrm{Sym}^e C$ and $\mathcal{E}$ vary among the rank 2 stable vector bundles on C such that det $(\mathcal{E}) \cong O(x_0 - D)$ and construct a space that parametrizes pairs $(D, \mathcal{E}, (*))$, where (*) is an extension in $V_{D,\mathcal{E}}$.

Consider the morphism

$$u : \mathrm{Sym}^e(C) \to \mathrm{Pic}^{1-e}(C), \quad \text{given by} \quad D \mapsto O_C(x_0 - D)$$

Let $M(2, 1-e)$ be the moduli scheme of rank 2 vector bundles on C with determinant of degree $1-e$. We have the determinant map

$$\det\ : M(2, 1-e) \to \mathrm{Pic}^{1-e}(C).$$

The moduli scheme $M(2, 1-e)$ is the geometric quotient of a projective integral scheme $\overline{M}(2, 1-e)$ by the action of an algebraic group PGL(r), for some integer r. Let the quotient map be:

$$\tau : \overline{M}(2, 1-e) \to M(2, 1-e) \tag{2.26}$$

Denote by $\overline{\det}$ the composition $(\det\ ) \circ \tau$:

$$\overline{\det}\ : M(2, 1-e) \to \mathrm{Pic}^{1-e}(C).$$

Define P to be the fibered product in the following diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Sym}^e(C) \\ \downarrow & & \downarrow u \\ \overline{M}(2, -e) & \xrightarrow[\overline{\det}]{} & \mathrm{Pic}^{1-e}(C) \end{array} \tag{2.27}$$

Then P parametrizes pairs $\zeta = (D, \mathcal{E}, t)$, where $D \in \mathrm{Sym}^e C$, $\mathcal{E}$ is a rank 2 stable vector bundles on C such that det $(\mathcal{E}) \cong O(x_0 - D)$ and g is an element in $\tau^{-1}(\{\mathcal{E}\})$.

Lemma 2.26. *There is a projective bundle $p : X \to P$ such that for any $\zeta = (D, \mathcal{E}, t) \in P$, the fiber $p^{-1}(\{\zeta\})$ is canonically isomorphic to $\mathbb{P}(V_{D,\mathcal{E}}) \cong \mathbb{P}^{2e-1}$.*

Proof. Let $\mathcal{W}$ be a Poincaré bundle on $\overline{M}(2, 1-e) \times C$. Note that such a Poincaré bundle exists, regardless of the parity of e. Define the divisor $\Delta \subset \mathrm{Sym}^e(C) \times C$ to be the universal scheme coming from $D \in \mathrm{Sym}^e(C)$.

Let $\mathcal{W}_P$ be the vector bundle on $P \times C$ coming from $\mathcal{W}$ and let Δ_P be the universal subscheme $\Delta_P \subset P \times C$ coming from Δ.

Define on P the relative extension sheaf

$$\mathcal{S} = \mathcal{E}xt^1_{P \times C|P}(O_{\Delta_P}, \mathcal{W}_P)$$

Note that if we let $\pi_1 : P \times C \to P$ to be the first projection, we have:

$$\mathcal{S} \cong \pi_{1*}(\mathcal{W}_P(\Delta_P)_{\Delta_P}).$$

Consider the projective bundle $p : \mathbb{P}(\mathcal{S}) \to P$. Let $X = \mathbb{P}(\mathcal{S})$. Then for any $\zeta = (D, \mathcal{E}, g) \in P$, we have:

$$\mathcal{S}_{|\{\zeta\}} \cong V_{D,\mathcal{E}} \cong \mathrm{H}^0(\{\zeta\} \times C, \mathcal{E}(D)_D) \quad \text{and} \quad p^{-1}(\{\zeta\}) \cong \mathbb{P}(V_{D,\mathcal{E}})$$

□

In a similar way as in 2.1.2, we have that the projective bundle X from Lemma 2.26 depends on the Poincaré bundle $\mathcal{W}$ on $\overline{M}(2, 1-e) \times C$. If $\mathcal{W}' = \mathcal{W} \otimes \mathcal{M}$, where $\mathcal{M}$ is a line bundle on $\overline{M}(2, 1-e)$, and we consider a projective bundle X', constructed as in the proof of Lemma 2.26, using the Poincaré bundle $\mathcal{W}'$, it follows that we have an isomorphism of projective bundles over P:

$$\phi : X \to X', \quad \text{such that} \quad \phi^* O(-1) \cong O(-1) \otimes p^*(\mathcal{M})^2. \tag{2.28}$$

Let ν_1, ν_2 be the two projections from $X \times C$. In the following Lemma, $\mathcal{W}$ is the Poincaré bundle on $\overline{M}(2, 1-e) \times C$ from the proof of Lemma 2.26, used to construct the scheme X.

Lemma 2.27. *There is a universal extension on $X \times C$:*

$$0 \to \nu_1^* O_X(1) \otimes p^* \mathcal{W}_P \to \tilde{\mathcal{J}} \to p^* O_{\Delta_P} \to 0. \tag{2.29}$$

It has the property that, when we restrict to $\{x\} \times C$, where $x \in X$ is a point and we let $\zeta = p(x) \in P$, with $\zeta = (D, \mathcal{E}, g)$, we get an exact sequence:

$$0 \to \mathcal{E} \to \tilde{\mathcal{J}}_x \to O_D \to 0$$

which corresponds to an element in $V_{D,\mathcal{E}} = Ext^1_C(O_D, \mathcal{E})$, whose class is x in

$$\mathbb{P}(V_{D,\mathcal{E}}) \cong p^{-1}(\{\zeta\}).$$

Proof. This is another application of Lemma A.1 in Appendix. We take $S = P$, $\mathcal{T} = \mathcal{W}_P$ and $\mathcal{V} = O_{\Delta_P}$. □

The locus of not locally free/unstable extensions

Let $\mathcal{W}$ be a Poincaré bundle on $\mathrm{Sym}^e(C) \times C$ and let $p : X \to P$ be the projective bundle over P constructed in Lemma 2.26.

Lemma 2.28. *The locus of not locally free extensions in X is a closed subscheme $\Gamma \subset X$ of codimension 2 if $e \geq 2$ and empty if $e = 1$. The locus of unstable extensions in $X \setminus \Gamma$ is a closed integral subscheme $Y \subset X \setminus \Gamma$ of codimension at least g. If $e = 1$, then $Y = \emptyset$.*

Proof. Consider the universal extension 2.29 on $X^0 \times C$:

$$0 \to \nu_1^* O_X(1) \otimes p^* \mathcal{W}_P \to \tilde{\mathcal{J}} \to p^* O_{\Delta_P} \to 0. \tag{2.30}$$

Let $\Gamma \subset X$ be the locus of not locally free extensions in X. This is precisely the locus of $x \in X$ for which $\tilde{\mathcal{J}}_x$ is not locally free. This is a closed subset of X.

If $\zeta = (D, \mathcal{E}, g) \in P$ then $\Gamma \cap p^{-1}(\{\zeta\})$ is the locus in Lemma 2.18 corresponding to not locally free extensions:

$$\Gamma_{D,\mathcal{E}} \subset \mathbb{P}(V_{D,\mathcal{E}}) \cong p^{-1}(\{\zeta\}.$$

By Lemma 2.18, $\Gamma_{D,\mathcal{E}}$ has codimension 2 in $\mathbb{P}(V_{D,\mathcal{E}})$ if $e \geq 2$ and $\Gamma_{D,\mathcal{E}} = \emptyset$ if $e = 1$. Since this is true for any $\zeta = (D, \mathcal{E}, g) \in P^0$, it follows that $\Gamma \subset X$ has codimension at least g and if $e = 0$, then $Z = \emptyset$. □

Note that one can make a construction of the locus of not-locally free extensions by globalizing to X the constructions in the proof of Lemma 2.18.

Proposition 2.29. *The locus of unstable extensions in $X \setminus \Gamma$ is a closed integral subscheme $Y \subset X \setminus \Gamma$ of codimension at least g. If $e = 1$, then $Y = \emptyset$.*

Proof. In a similar way as in Lemma 2.28, we let $Y \subset X \setminus \Gamma$ be the locus of unstable extensions in $X \setminus \Gamma$. This is precisely the locus of $x \in X \setminus \Gamma$ for which $\tilde{\mathcal{J}}_x$ is unstable, where J is as in (2.30). By Fact 1.9, this is a closed subset of $X \setminus \Gamma$.

If $\zeta = (D, \mathcal{E}, g) \in P$ then $Y \cap p^{-1}(\{\zeta\})$ is the locus corresponding to unstable extensions:

$$Y_{D,\mathcal{E}} \subset \mathbb{P}(V_{D,\mathcal{E}}) \setminus \Gamma_{D,\mathcal{E}} \subset \mathbb{P}(V_{D,\mathcal{E}}) \cong p^{-1}(\{\zeta\}.$$

By Lemma 2.19, for points $\zeta = (D, \mathcal{E}, t)$ in a dense open $P^0 \subset P$ (for each D, $\mathcal{E}$ is general in the sense of Note 2.20), $Y_{D,\mathcal{E}}$ has codimension at least g in $\mathbb{P}(V_{D,\mathcal{E}}) \setminus \Gamma_{D,\mathcal{E}}$ if $e \geq 2$ and $\Gamma_{D,\mathcal{E}} = \emptyset$ if $e = 1$. However, this is not enough to prove that Y has codimension at least g. We have to explicitly identify Y. We will use the proof of Proposition 2.29.

Let f be in the range $e - 1 \geq f > \frac{e-1}{2}$ and i in the range $e \geq i \geq 1$. Note that it is enough to make the construction for $i = e$ – when we obtain the maximal possible dimension for the unstable locus, see (2.20). So we assume that $i = e$ and $D_i = D$ in (2.20). Consider the fiber product (see (2.27)):

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Sym}^e C \\ \downarrow & & \downarrow u \\ \overline{M}(2, 1-e) & \xrightarrow{\det} & \mathrm{Pic}^{1-e}(C) \end{array}$$

Consider the morphism (2.45):

$$\tilde{\kappa} : \mathcal{X} \setminus \mathcal{Z} \to M(2, 1-e),$$

where $\mathcal{X}$ is a projective bundle over $\mathrm{Pic}^{-f}(C) \times \mathrm{Pic}^{1-e}(C)$. Consider the fiber product:

$$\begin{array}{ccc} S & \longrightarrow & P \\ \downarrow & & \downarrow \\ \mathcal{X} \setminus \mathcal{Z} & \xrightarrow{\tilde{\kappa}} & \overline{M}(2, 1-e) \end{array}$$

Then S parametrizes pairs $(D, \mathcal{E}, t, \mathcal{K})$ such that

i. $D \in \mathrm{Sym}^e C$

ii. $\mathcal{E}$ is an element in $\mathcal{M}(2, O_C(x_0 - D))$

iii. t is in the fiber of τ at $\mathcal{E}$ (see (2.26))

iv. $\mathcal{K}$ is a saturated line subbundle of $\mathcal{E}$, of degree $-f$

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Poincaré bundles on $S \times C$ coming from $\mathrm{Pic}^{-f}(C) \times C$ and from $\mathrm{Pic}^{1-e}(C) \times C$. Consider the universal extension (2.43) pulled back to $S \times C$:

$$0 \to \mathcal{A}_1 \otimes O_{\mathcal{X}}(1) \to \mathcal{H} \to {\mathcal{A}_1}^{-1} \otimes \mathcal{A}_2 \to 0$$

Let $\Delta \subset \mathcal{S} \times C$ the universal subscheme coming from $D \in \mathrm{Sym}^e(C)$. Consider the following relative extension sheaves on S:

$$\mathcal{F}' = \mathcal{E}xt^1_{S\times C|S}(O_\Delta, \mathcal{A}_1), \quad \mathcal{F} = \mathcal{E}xt^1_{S\times C|S}(O_\Delta, \mathcal{H})$$
$$\mathcal{F}'' = \mathcal{E}xt^1_{S\times C|S}(O_\Delta, \mathcal{A}_1^{-1} \otimes \mathcal{A}_2)$$

If $s = (D, \mathcal{E}, t, \mathcal{K}) \in S$ then we have:

$$\mathcal{F}'_s = \mathrm{Ext}^1_C(O_D, \mathcal{K}), \quad \mathcal{F}_s = \mathrm{Ext}^1_C(O_D, \mathcal{E})$$
$$\mathcal{F}''_s = \mathrm{Ext}^1_C(O_D, \mathcal{K}^{-1}(x_0))$$

The data (2.20) (which makes $\mathcal{E}$ unstable) is parametrized by $\mathbb{P}(\mathcal{S}')$. Note that $\mathcal{F}', \mathcal{F}$ and $\mathcal{F}''$ are locally free, sitting in an exact sequence:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

The injective morphism $\mathcal{F}' \to \mathcal{F}$ induces a closed immersion $\mathbb{P}(\mathcal{F}') \hookrightarrow \mathbb{P}(\mathcal{F})$ of projective bundles over S. Since if $s = (D, \mathcal{E}, t, \mathcal{K}) \in S$, then $\mathcal{H}_s \cong \mathcal{E}$, it follows that

$$\mathcal{H} \cong \mathcal{W} \otimes \mathcal{M}$$

where $\mathcal{W}$ is a Poincaré bundle on $S \times C$ coming from $\overline{M}(2, 1-e) \times C$, and $\mathcal{M}$ is a line bundle on S. It follows that

$$\mathbb{P}(\mathcal{F}) \cong S \times_P X$$

The unstable locus is given by taking the image (for every integer f) of $\mathbb{P}(\mathcal{F}')$ in X via

$$\mathbb{P}(\mathcal{F}') \hookrightarrow \mathbb{P}(\mathcal{F}) \cong S \times_P X \to X.$$

Computing dimensions as in the proof of Proposition 2.19, it follows that the unstable locus has codimension at least g. □

We let $\overline{Y}$ to be the closure of Y in X and define:

$$Z = \Gamma \cup \overline{Y} \subset X \tag{2.31}$$

Corollary 2.30. *There is a morphism $\eta : X \setminus Z \to M$ which restricted to the fiber of $p : X \to P$ gives exactly the morphism $\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$ defined in Corollary 2.23.*

If $\mathcal{W}' = \mathcal{W} \otimes \mathcal{M}$ is another Poincaré bundle on $\overline{M}(2, 1-e) \times C$, where $\mathcal{M}$ is a line bundle on $\overline{M}(2, 1-e)$, consider the isomorphism (2.28) of projective bundles over P :

$$\phi : X \to X'.$$

Let $Z \subset X$ and $Z' \subset X'$ the loci (2.31), coming from Lemma 2.28 and Proposition 2.29. Consider the morphisms defined by Corollary 2.30:

$$\eta : X \setminus Z \to M \qquad \eta' : X' \setminus Z' \to M$$

Then there is a commutative diagram:

$$\begin{array}{ccc} X \setminus Z & \xrightarrow{\phi} & X' \setminus Z' \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta'} \\ M & = & M \end{array} \tag{2.32}$$

Group actions on X

Let P be the scheme defined in (2.27). There is a group scheme $T \to P$ such that at $\zeta = (D, \mathcal{E}, t) \in P$ the group T_ζ is canonically isomorphic to T_D and it acts as in (2.3.1). We have that T acts on X.

Let G' be the group acting on $\overline{M}(2, 1-e)$ giving $M(2, 1-e)$ as a geometric quotient. Then G' acts on P, and therefore on X.

Since scalar multiplication of extensions appears in both the action of T and G', we define:

$$G = T \times_{\mathbb{G}_m} G' \tag{2.33}$$

We have that G acts on X and, as we remarked in (2.3.1), it preserves the not locally free/unstable locus Z.

2.4 Rational curves coming from extensions of rank 2 bundles by skyscraper sheaves

Let $D \in \mathrm{Sym}^e(C)$ and $\mathcal{E}$ a general rank 2 stable vector bundle on C such that

$$\deg\,(\mathcal{E}) \cong O_C(x_0 - D) \tag{2.34}$$

Consider the morphism (2.23):

$$\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M$$

The main observation is that for any integer $n \geq 0$ and any morphism:

$$g : \mathbb{P}^1 \to \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \qquad \text{such that} \qquad g^*O(1) \cong O(n)$$

we get a rational curve $f = \eta \circ g : \mathbb{P}^1 \to M$ of degree $2en$.

2.4.1 The loci $N(e,n)$ in $\overline{M}_0(M, 2en)$

For $k \geq 1$ let $\overline{M}_0(M,k)$ be the Kontsevich space 1.1.1.

Proposition 2.31. *Let e and n be integers such that $e \geq 1$ and $n \geq 1$. There exist integral closed subschemes $N(e,n) \subset \overline{M}_0(M, 2en)$ such that a general element of $N(e,n)$ is a morphism $f : \mathbb{P}^1 \to M$ obtained by a composition:*

$$\mathbb{P}^1 \xrightarrow{\ g\ } \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \xrightarrow{\ \eta_{D,\mathcal{E}}\ } M \tag{2.35}$$

where $(D,\mathcal{E})$ are as in (2.34) and $g^*O(1) \cong O(n)$.

Proof. Let $\mathcal{W}$ be a fixed Poincaré bundle on $\overline{M}(2, 1-e) \times C$ and consider the projective bundle X of Lemma 2.26:

$$p : X \to P.$$

Let $[l] \in \mathrm{H}_2(X;\mathbb{Z})$ be the class of a line contained in a fiber. Note that a morphism $f : \mathbb{P}^1 \to X$, representing the class $n[l]$, lies in a fiber of p. Consider the Kontsevich space $\overline{M}_0(X, n[l])$. By Fact 2.13, there is a morphism:

$$\pi : \overline{M}_0(X, n[l]) \to P$$

such that for $\zeta = (D, \mathcal{E}, t) \in P$ the fiber $\pi^{-1}(\{\zeta\})$ is isomorphic to $\overline{M}_0(\mathbb{P}(V_{D,\mathcal{E}}), n)$ The scheme $\overline{M}_0(X, n[l])$ is integral and it has the expected dimension by Fact 2.13.

Since the locus of not-locally free or unstable extensions $Z \subset X$ has codimension at least $g \geq 2$, it follows that we have that the morphism

$$\eta : X \setminus Z \to M$$

induces a rational map between the corresponding Kontsevich spaces:

$$\Psi : \overline{M}_0(X, n[l]) \dashrightarrow \overline{M}_0(M, 2en) \tag{2.36}$$

Note that, since η depends on a Poincaré bundle $\mathcal{W}$ on $\overline{M}(2, 1-e) \times C$, it follows that Ψ depends on $\mathcal{W}$.

Define the closure of the image of the morphism Ψ to be:

$$N(e, n) \subset \overline{M}_0(M, 2en).$$

By taking $N(e, n)$ with the induced reduced structure, we have that $N(e, n)$ is an irreducible closed subscheme of $\overline{M}_0(M, 2en)$. Note that $N(e, n)$ does not depend on the Poincaré bundle $\mathcal{W}$ (see (2.32)). □

Theorem 2.32. *The closed subschemes $N(e, n) \subset \overline{M}_0(M, n(2e+1))$ have dimension*

$$dim\ N(e, n) = 2en + 2e + 3g - 6.$$

Their MRC fibration is given by a rational map:

$$\rho : N(e, n) \dashrightarrow Pic^{1-e}(C)$$

The map ρ associates to a rational curve f which is a composition as in 2.35, the line bundle $O(x_0 - D) \in Pic^{1-e}(C)$. The map ρ is surjective if and only if $e \geq g$.

Proof. Consider the projective bundle $p : X \to P$ of Lemma 2.26 and consider the morphism 2.36, induced from the morphism (2.30) $\eta : X \setminus Z \to M$:

$$\Psi : \overline{M}_0(X, n[l]) \dashrightarrow \overline{M}_0(M, 2en)$$

We find first the dimension and the MRC fibration of $\overline{M}_0(X, n[l])$. Recall that P is defined as the fiber product:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Sym}^e(C) \\ \downarrow & & \downarrow u \\ \overline{M}(2, 1-e) & \xrightarrow{\det} & \mathrm{Pic}^{1-e}(C) \end{array} \tag{2.37}$$

where $\tau : \overline{M}(2, 1-e) \to M(2, 1-e)$ is a geometric quotient by the action of some group $G' = \mathrm{PGL}(r)$. Let $N = \dim G'$. Then the fiber of τ at a point in $M(2, 1-e)$ is the orbit of that point by the action of G'. If the point is corresponding to stable bundles $\mathcal{E}$, then the dimension of the orbit is the dimension of G'. We have:

$$\begin{gathered} \dim P = \dim \overline{M}(2, 1-e) + \dim \mathrm{Sym}^e C - \dim \mathrm{Pic}^{1-e}(C) = \\ = \dim M(2, 1-e) + \dim G' = (4g - 3) + N + e - g = 3g - 3 + N + e \end{gathered}$$

Since X is a $\mathbb{P}^{2e-1}$-bundle over P, by Fact 2.13, we have

$$\dim \overline{M}_0(X, n[l]) = \dim P + (2e)(n+1) - 4 = 2en + 3e + 3g + N - 7$$

By the same Fact 2.13, there is a morphism:

$$\pi : \overline{M}_0(X, n[l]) \to P$$

with fibers isomorphic to $\overline{M}_0(\mathbb{P}^{2e-1}, n)$. By Fact 2.12, the fibers are rational projective integral schemes. Let $\theta : P \to \mathrm{Pic}^{1-e}(C)$ be the morphism from (2.37) and let σ be the composition:

$$\sigma : \overline{M}_0(X, n[l]) \xrightarrow{\pi} P \xrightarrow{\theta} \mathrm{Pic}^{1-e}(C)$$

Let $\xi \in \mathrm{Pic}^{1-e}(C)$ be a point which is in the image of the morphism

$$u : \mathrm{Sym}^e C \to \mathrm{Pic}^{1-e}(C).$$

The fiber of θ at the point ξ is isomorphic to:

$$\overline{M}(2, \xi) \times \mathbb{P}(H^0(C, \mathcal{L}))$$

where $\overline{M}(2, \xi)$ is the preimage $M(2, \xi)$ via τ (see (2.26); in particular, $M(2, \xi)$ is a geometric quotient of $\overline{M}(2, \xi)$ by the action of G'. The scheme $M(2, \xi)$ is unirational, for any ξ. Therefore, $\overline{M}(2, \xi)$ is unirational. Since all the fibers of π are rational, it follows by Fact 1.6 that the fibers of σ are rationally connected and, therefore, σ gives the MRC fibration of $\overline{M}_0(X, n[l])$.

We will analyze now the general fiber of Ψ. Let f be a general element in $N(e, n)$, given by a composition as in (2.35):

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \xrightarrow{\eta_{D,\mathcal{E}}} M$$

where $D \in Sym^e C$, $\mathcal{E}$ is a general element in $M(2, \xi)$ and g is has the property that $g^* O(1) \cong O(n)$.

Recall that we defined a group scheme over P:

$$G = T_D \times_{\mathbb{G}_m} G'.$$

Note that the action of G on X (see (2.33)) induces an action on the automorphism free locus in $\overline{M}_0(X, n[l])$. We prove that the fiber of Ψ at f is given by the orbit containing g. Note that for f general (hence, for general D and $\mathcal{E}$) the orbit is

$$(\mathbb{G}_m)^e \times_{\mathbb{G}_m} G'.$$

It follows that the general fiber of Ψ is $(N + e - 1)$-dimensional and we have:

$$\dim N(e, n) = \dim \overline{M}_0(X, n[l]) - (N + e - 1) = 2en + 2e + 3g - 6$$

Moreover, since $G' = PGL(r)$, it follows that the general fiber of Ψ is rational. By the universal property of MRC fibrations, it follows that the morphism σ factors through $N(e,n)$, i.e., there is a morphism

$$\rho : N(e,n) \dashrightarrow \mathrm{Pic}^{1-e}(C) \quad \text{such that} \quad \rho \circ \Psi = \sigma$$

Since the general fiber of ρ is dominated by a fiber of σ (rationally connected), it follows that ρ gives the MRC fibration of $N(e,n)$.

Let's prove that if g' is another element in the fiber Ψ at f, then g' is in the same orbit as g for the action of G on $\overline{M}_0(X, n[l])$. We have $g' : \mathbb{P}^1 \to X \setminus Z$. If $\eta \circ g'$ and f represent the same stable map in $\overline{M}_0(M, 2en)$, there is an isomorphism $\mu : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\eta \circ g' \circ \mu = \eta \circ g.$$

Let $h = g' \circ \mu$. Then we have

$$\eta \circ h = \eta \circ g.$$

We prove that h is in the orbit of g by the action of G. Assume g' lies in the fiber of p at the point $\zeta' = (D', \mathcal{E}', t') \in P$. By Proposition 2.33, it is enough to prove that $D = D'$ and $\mathcal{E} = \mathcal{E}'$. Then t' is in the orbit of t by the action of G', and if $\mathcal{E}$ is stable, then this orbit has dimension dim $G' = N$. Then the orbit of g in $\overline{M}_0(X, n[l])$ is given by the union (over the elements in the orbit of t by the action of G' on P) of the orbits of g by the action of G on $\overline{M}_0(\mathbb{P}(V_{D,\mathcal{E}}), n[l])$ and the theorem follows.

We prove now that $D = D'$ and $\mathcal{E} = \mathcal{E}'$. We have that the following two morphisms are *equal*:

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \xrightarrow{\eta_{D,\mathcal{E}}} M$$
$$\mathbb{P}^1 \xrightarrow{h} \mathbb{P}(V_{D',\mathcal{E}'}) \setminus Z_{D',\mathcal{E}'} \xrightarrow{\eta_{D',\mathcal{E}'}} M$$

Let deg $(g') = m$ and deg $D' = d$. Since $\eta \circ g = \eta' \circ g'$, we have that

$$\deg\ (f) = 2ne = 2md.$$

If $\mathcal{U}_0$ is the rigidified Poincaré bundle on $M \times C$, we let $\mathcal{F}_0 = (f \times id)^* \mathcal{U}_0$. Since $\eta \circ g = \eta' \circ g'$, we have:

$$\mathcal{F}_0 = (g \times id)^* (\eta \times id)^* \mathcal{U}_0 = (g' \times id)^* (\eta' \times id)^* \mathcal{U}_0.$$

From (2.25) we get that there are exact sequences:

$$0 \to p_1^* O(n + ne) \otimes p_2^* \mathcal{E} \to \mathcal{F}_0 \to p_1^* O(ne) \otimes p_2^* O_D \to 0$$
$$0 \to p_1^* O(m + md) \otimes p_2^* \mathcal{E}' \to \mathcal{F}_0 \to p_1^* O(md) \otimes p_2^* O_{D'} \to 0.$$

Note that since $ne = md$, there are no non-zero maps $O(n+ne) \to O(md)$; hence, by Lemma A.5 in the Appendix there is a commutative diagram with all the vertical morphisms isomorphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*O(n+ne)\otimes p_2^*\mathcal{E} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & p_1^*O(ne)\otimes p_2^*O_D & \longrightarrow & 0 \\ \downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \downarrow \\ 0 & \longrightarrow & p_1^*O(m+md)\otimes p_2^*\mathcal{E}' & \longrightarrow & \mathcal{F}_0 & \longrightarrow & p_1^*O(md)\otimes p_2^*O_{D'} & \longrightarrow & 0 \end{array} \tag{2.38}$$

It follows that $n = m, e = d$, $\mathcal{E} \cong \mathcal{E}'$ and $D = D'$. □

Proposition 2.33. *Let $f : \mathbb{P}^1 \to M$ be a morphism given by a vector bundle $\mathcal{F}$ which sits in an exact sequence*

$$0 \to p_1^*O(n)\otimes p_2^*\mathcal{E} \to \mathcal{F} \to p_2^*O_D \to 0. \tag{2.39}$$

with $D \in Sym^e(C)$ and $\mathcal{E}$ a rank 2 vector bundle with

$$det\,(\mathcal{E}) \cong O_C(x_0 - D).$$

Then there is an integer $n \geq 0$ and a morphism

$$g : \mathbb{P}^1 \to \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}}, \quad \textit{such that} \quad g^*O(1) \cong O(n)$$

and $f = \eta_{D,\mathcal{E}} \circ g$:

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \xrightarrow{\eta_{D,\mathcal{E}}} M$$

The morphisms g with this property form an orbit for the action of G on

$$\overline{M}_0(\mathbb{P}(V_{D,\mathcal{E}}), n).$$

Proof. Let (υ) be the universal extension (2.21):

$$0 \to q_1^*O(1)\otimes q_2^*\mathcal{E} \to \mathcal{J} \to q_2^*O_D \to 0. \tag{υ}$$

Let $V = V_{D,\mathcal{E}}$, $Z = Z_{D,\mathcal{E}}$. We apply Lemma A.2 in Appendix for $\mathcal{V} = O_D$ and $\mathcal{T} = \mathcal{E}$ to get that there there is a unique morphism

$$g : \mathbb{P}^1 \to \mathbb{P}(V) \quad \text{such that} \quad g^*O(1) \cong O(n)$$

such that the extension (2.39) is a multiple scalar of the extension $(g^*\upsilon)$, where $(g^*\upsilon)$ is the pull back of (υ) by g:

$$0 \to p_1^*O(n)\otimes p_2^*\mathcal{E} \to \mathcal{F} \to p_2^*O_D \to 0. \tag{$g^*\upsilon$}$$

We note that g has image in $\mathbb{P}(V) \setminus Z$ (see Lemma A.2).

It follows from the fact that (2.39) and $(g^*\upsilon)$ are scalar multiples of each other that there is an isomorphism

$$\mathcal{F} \cong (g \times id)^* \mathcal{J}.$$

Since $\mathcal{F}$ determines the morphism f and $(g \times id)^* \mathcal{J}$ determines the morphism $\eta \circ g$, it follows that $f = \eta \circ g$.

The action of the group scheme G on X restricts to the action of the group T on $\mathbb{P}(V)$. For example, if D is made of distinct points, then $T = (\mathbb{G}_m)^e$ acts on $\mathbb{P}(V)$ as follows:

$$(\lambda_1, \ldots, \lambda_e).(x_1, \ldots, x_{2e}) = (\lambda_1.x_1, \lambda_1.x_2, \lambda_2.x_3, \lambda_2.x_4, \ldots, \lambda_e.x_{2e-1}, \lambda_e.x_{2e}).$$

This action induces an action on the auomorphism free locus $\overline{M}_0^*(\mathbb{P}(V), n)$ of $\overline{M}_0(\mathbb{P}(V), n)$ by :

$$\begin{gathered} T \times \overline{M}_0^*(\mathbb{P}(V), n) \to \overline{M}_0^*(\mathbb{P}(V), n) \\ (\lambda, g) \mapsto g_\lambda, \quad \text{where} \quad g_\lambda(p) = \lambda.g(p) \end{gathered}$$

We prove that g_λ has the properties in the proposition. It is clear that $g_\lambda^* O(1) \cong O(n)$, since the action of $\lambda \in T$ on $\mathbb{P}(V)$ gives an isomorphism of $\mathbb{P}(V)$. We prove that g_λ has the property that $f = \eta \circ g_\lambda$. Consider the pull-back of the universal sequence (v) by g_λ:

$$0 \to p_1^* O(n) \otimes p_2^* \mathcal{E} \to (g_\lambda \times \mathrm{id})^* \mathcal{J} \to p_2^* O_D \to 0. \qquad (g_\lambda^* v)$$

Note that T acts as well on the space of extensions

$$W = \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(p_2^* O_D, p_1^* O(n) \otimes p_2^* \mathcal{E}) \cong \oplus_{i=1}^e \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(p_2^* O_{y_i}, p_1^* O(n) \otimes p_2^* \mathcal{E})$$

where $D = y_1 + \ldots + y_e$. If the points are distinct, then the action is by componentwise multiplication.

By the definition of g_λ, it follows that the extensions $g_\lambda^* v$ and $\lambda.(g^* v)$ are scalar multiples, when restricted to $\{p\} \times C$. It follows that there is a line bundle $\mathcal{M}$ on $\mathbb{P}^1$ such that

$$(g_\lambda \times \mathrm{id})^* \mathcal{J} \cong \mathcal{F} \otimes \mathcal{M}.$$

Since $(g_\lambda \times \mathrm{id})^* \mathcal{J}$ induces $\eta \circ g_\lambda$ and $\mathcal{F}$ induces f, we have $f = \eta \circ g_\lambda$. In fact, one could prove that $\mathcal{M} \cong O$ just by analizing the two extensions $g_\lambda^* v$ and $\lambda.(g^* v)$ and using arguments as in (2.38). It follows that $g_\lambda^* v$ and $\lambda.(g^* v)$ are scalar multiples of each other.

We prove now that any morphism $g' : \mathbb{P}^1 \to \mathbb{P}(V)$, such that $g'^* O(1) \cong O(n)$ and $\eta \circ g' = \eta \circ g = f$ is a morphism g_λ for some $\lambda \in T$. Consider the extension $(g'^* v)$ obtained by pulling back by g' the universal extension (v):

$$0 \to p_1^*O(n) \otimes p_2^*\mathcal{E} \to \mathcal{F}' \to p_2^*O_D \to 0. \qquad (g'^*v)$$

where $\mathcal{F}' = (g' \times id)^*\mathcal{J}$. As $\eta \circ g' = \eta \circ g$, we have

$$(g \times id)^*(\eta \times id)^*\mathcal{U}_0 \cong (g' \times id)^*(\eta \times id)^*\mathcal{U}_0.$$

Recall from (2.25) that if $\mathcal{J}_0 = (\eta \times id)^*\mathcal{U}_0$ then $\mathcal{J}_0 = \mathcal{J} \otimes q_1^*O(e)$. Hence,

$$\mathcal{F}' = (g' \times id)^*\mathcal{J} \cong (g \times id)^*\mathcal{J} = \mathcal{F}.$$

From Lemma A.5 in Appendix it follows that there is $\lambda \in T$ such that

$$(g'^*v) = \lambda.(g^*v)$$

Since $(g_\lambda^*v) = \lambda.(g^*v)$, it follows by Lemma A.2 that $g' = g_\lambda$. □

Note 2.34. *The expected dimension of* $\overline{M}_0(M, 2en)$ *is:*

$$2n(2e+1) + 3g - 6.$$

The dimensions of the subschemes $M(e,n)$ *are as follows:*

i. *dim* $N(e,n) =$ *expected dimension if and only if* $n = 1$

ii. *dim* $N(e,n) <$ *expected dimension if* $n > 1$

2.4.2 The nice component of $\overline{M}_0(M,k)$ for k even

In this subection k will be an even integer, $k = 2e$, with $e \geq 1$. For each $D \in \mathrm{Sym}^e(C)$ and $\mathcal{E}$ stable rank 2 vector bundle with det $(\mathcal{E}) \cong O(x_0 - D)$, we have the morphism (2.23)

$$\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \quad \text{such that} \quad \eta_{D,\mathcal{E}}^*\Theta = O(k)$$

Theorem 2.35. *There is a* nice *irreducible component* $\mathfrak{M}$ *of the moduli space* $\overline{M}_0(M,k)$. *By* nice *component, we mean:*

i. $\mathfrak{M}$ *has the expected dimension*

ii. *A general point* $[f] \in \mathfrak{M}$ *is obtained as a composition:*

$$\mathbb{P}^1 \xrightarrow{\ g\ } \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \xrightarrow{\eta_{D,\mathcal{E}}} M$$

where $D \in Sym^{-e}(C)$ *and* $\mathcal{E}$ *is a rank* 2 *stable vector bundle on* C *with det* $(\mathcal{E}) \cong O_C(x_0 - D)$ *and* $g^*O(1) \cong O(1)$

iii. *A general point* $[f] \in \mathfrak{M}$ *is an unobstructed point of* $\overline{M}_0(M,k)$

iv. *The MRC fibration of $\mathfrak{M}$ is given by a map*

$$\mathfrak{M} \dashrightarrow Pic^{1-e}(C)$$

which sends the point $[f] \in \mathfrak{M}$ *to* $O_C(x_0 - D) \in Pic^{-e}(C)$

Proof. Consider the closed subscheme $N(1,e) \subset \overline{M}_0(M,k)$. We claim $N(1,e)$ is an irreducible component that satisfies all the conditions in the Theorem. By Proposition 2.31, a general point $[f] \in M(e,1)$ is obtained as a composition:

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

where $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$ and $g^*O(1) \cong O(1)$.

By pulling back the universal sequence (2.21) by f, it follows that the morphism f is given by a vector bundle $\mathcal{F}$ on $\mathbb{P}^1 \times C$, which sits in an exact sequence:

$$0 \to p_1^*O(1) \otimes p_2^*\mathcal{E} \to \mathcal{F} \to p_2^*O_D \to 0.$$

It follows that for any $x \in C$, we have that the bundle $\mathcal{F}_x$ has balanced splitting

$$\mathcal{F}_x \cong O(1) \oplus O.$$

It follows from Lemma 1.11, that f is an unobstructed point in $\overline{M}_0(M,k)$. Hence, $[f]$ is contained in a unique irreducible component of $\overline{M}_0(M,k)$ which also has the expected dimension. Since $N(e,1)$ is an irreducible scheme of the expected dimension (see Note 2.34), it must be the unique irreducible component containing $[f]$. Part iv. follows from Theorem 2.32. □

2.5 Some generalizations for the case of M_ξ

In this section we make constructions similar to the ones in Section 2.1 for the case of moduli spaces of semistable, rank 2 vector bundles with fixed determinant of arbitrary degree d.

2.5.1 Extensions of line bundles

We are using the same ideas as in Section 2.1. There are minor modifications due to the fact that the moduli space is not fine in the case when d is even.

Let ξ be a line bundle on C of degree d, where d is a fixed integer. Denote by M_ξ the coarse moduli scheme of isomorphism classes of semistable vector bundles of rank 2 and determinant isomorphic to ξ. Recall that M_ξ is a projective integral scheme of dimension $3g - 3$.

Let e be an integer such that $e \geq -\frac{d}{2}$ and let $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$. Consider extensions:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \otimes \xi \to 0. \qquad (*)$$

Then $\mathcal{E}$ is a vector bundle of rank 2 and such extensions are classified by the vector space

$$V_{\mathcal{L}} = \mathrm{Ext}^1_C(\mathcal{L}^{-1} \otimes \xi, \mathcal{L}) \cong \mathrm{H}^1(C, \mathcal{L}^2 \otimes \xi^{-1}).$$

By Riemann-Roch, $V_{\mathcal{L}}$ is a vector space of dimension

i. $(2e+d)-1+g = 2e+d+g-1$, if $e > -\frac{d}{2}$ or if $e = -\frac{d}{2}$ and $\mathcal{L}^2 \otimes \xi^{-1} \neq O_C$

ii. $2e+d+g$, if $e = -\frac{d}{2}$ and $\mathcal{L}^2 \otimes \xi^{-1} \cong O_C$

Clearly, any two nonzero elements v, v' of $V_{\mathcal{L}}$ which differ by a scalar define isomorphic vector bundles $\mathcal{E}$. Therefore the isomorphism classes of non-trivial extensions as above are parametrized by the projective space $\mathbb{P}(V_{\mathcal{L}})$.

The locus of unstable extensions

Let $e \geq -\frac{d}{2}$. An unstable extension is an extension (*) for which $\mathcal{E}$ is unstable (neither stable, nor semi-stable). A *non-stable extension* is an extensions (*) for which $\mathcal{E}$ is not stable.

Proposition 2.36. *For each $\mathcal{L} \in Pic^{-e}(C)$ there are closed integral suschemes $Z_{\mathcal{L}}$ and $Z'_{\mathcal{L}}$ such that*

$$Z_{\mathcal{L}} \subset Z'_{\mathcal{L}} \subset \mathbb{P}(V_{\mathcal{L}})$$

corresponding to the unstable, respectively non-stable extensions.

i. If d is odd then $Z_{\mathcal{L}} = Z'_{\mathcal{L}}$ has codimension at least g.

ii. If d is even, then $Z_{\mathcal{L}}$ has codimension at least $(g+1)$ and $Z'_{\mathcal{L}}$ has codimension at least $(g-1)$.

In both cases, when $e = \lceil \frac{d+1}{2} \rceil - d$, we have $Z_{\mathcal{L}} = \emptyset$. If d is even and $e = -\frac{d}{2}$, then the non-stable locus $Z'_{\mathcal{L}}$ is the whole $\mathbb{P}(V_{\mathcal{L}})$.

Proof. The idea is the same as in the proof of Proposition 2.1. We analyze first the non-stable locus $Z'_{\mathcal{L}}$. If $e = -\frac{d}{2}$, it follows immediately that $Z'_{\mathcal{L}}$ is the whole $\mathbb{P}(V_{\mathcal{L}})$.

Assume $e > -\frac{d}{2}$. The bundle $\mathcal{E}$ is not stable if and only if there exists a line bundle $\mathcal{L}'$ on C of degree $\lceil \frac{d}{2} \rceil$ and a non-zero morphism

$$\mathcal{L}' \to \mathcal{E}.$$

Then the morphism $\mathcal{L}' \to \mathcal{L}^{-1} \otimes \xi$ is non-zero as well. This is because there is no non-zero morphism $\mathcal{L}' \to \mathcal{L}$ as

$$\deg\,(\mathcal{L}') \geq \frac{d}{2} > -e = \deg\,(\mathcal{L}).$$

Therefore, it follows that there is some effective divisor D on X of degree $e+d-\lceil \frac{d}{2} \rceil$ such that

$$\mathcal{L}' \cong \mathcal{L}^{-1} \otimes \xi(-D).$$

Let $\mathcal{E}'$ be the kernel of the composition

$$\mathcal{E} \to \mathcal{L}^{-1} \otimes \xi \to \mathcal{L}^{-1} \otimes \xi_{|D}.$$

Then there is a commutative diagram with the two horizontal sequences exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L}^{-1} \otimes \xi_{|D} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & \mathcal{L}^{-1} \otimes \xi(-D) & \longrightarrow & \mathcal{L}^{-1} \otimes \xi & \longrightarrow & \mathcal{L}^{-1} \otimes \xi_{|D} & \longrightarrow & 0 \end{array}$$

Using the snake lemma, we get an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E}' \to \mathcal{L}^{-1} \otimes \xi(-D) \to 0. \tag{2.40}$$

From the commutativity of the previous diagram the following composition is zero:

$$\mathcal{L}^{-1} \otimes \xi(-D) \to \mathcal{E} \to \mathcal{L}^{-1} \otimes \xi_{|D}.$$

We get that $\mathcal{L}^{-1} \otimes \xi(-D)$ maps to the subbundle $\mathcal{E}'$ of $\mathcal{E}$. By chasing diagrams, it follows that the exact sequence (2.40) is split.

Denote $V = V_{\mathcal{L}}$. We conclude that the vector $v \in \mathcal{V}$ corresponding to an unstable vector bundle $\mathcal{E}$ is in the kernel of the surjective map

$$V \cong \mathrm{H}^1(C, \mathcal{L}^2 \otimes \xi^{-1}) \to \mathrm{H}^1(C, \mathcal{L}^2 \otimes \xi^{-1}(D)).$$

From the long exact sequence coming from

$$0 \to \mathcal{L}^{-1} \otimes \xi \to \mathcal{L}^{-1} \otimes \xi(D) \to \mathcal{L}^{-1} \otimes \xi(D)_{|D} \to 0,$$

by applying $\mathrm{Ext}^1_C(-, \mathcal{L})$, we get that

$$0 \to \mathrm{H}^0(C, \mathcal{L}^2 \otimes \xi^{-1}(D)_{|D}) \to \mathrm{H}^1(C, \mathcal{L}^2 \otimes \xi^{-1}) \to \mathrm{H}^1(C, \mathcal{L}^2 \otimes \xi^{-1}(D)) \to 0.$$

Therefore the non-stable extensions in V form a

$$l := e + d - \lceil \frac{d}{2} \rceil$$

dimensional linear subspace for each D effective divisor of degree l. If we let D vary in $\mathrm{Sym}^l(C)$ we get that the locus of non-stable extensions in V is at most $2l$-dimensional. Hence, the codimension of the non-stable locus is at least:

$$(2e + d + g - 1) - 2l = 2\lceil \frac{d}{2} \rceil - d + g - 1.$$

If d is odd, then the codimension of the nonstable locus (which is the same as the unstable locus) is at least $g \geq 2$. Note that in this case, when $l = 0$, so

$$e = \lceil \frac{d}{2} \rceil - d = \frac{1-d}{2}$$

the locus $Z_{\mathcal{L}}$ is empty. If d is even and $e > -\frac{d}{2}$ then the codimension of the non-stable extensions $Z'_{\mathcal{L}}$ is at least $g - 1 \geq 1$.

Analyze now the unstable locus $Z_{\mathcal{L}}$ for d even. Let $e \geq -\frac{d}{2}$. By a similar argument, the bundle $\mathcal{E}$ is not semistable if and only if there exists a line bundle $\mathcal{L}'$ on C of degree $\frac{d}{2} + 1$ and a non-zero morphism $\mathcal{L}' \to \mathcal{E}$. Then the morphism $\mathcal{L}' \to \mathcal{L}^{-1} \otimes \xi$ is non-zero as well. This is because

$$\deg\ (\mathcal{L}') = \frac{d}{2} + 1 > \deg\ (\mathcal{L}) = -e.$$

We see that $\mathcal{L}' \cong \mathcal{L}^{-1} \otimes \xi(-D)$ for some effective divisor D on X of degree

$$l := e + \frac{d}{2} - 1.$$

Following the same arguments, we have that the locus of unstable extensions is at most $2l$-dimensional. Hence, the codimension is at least:

$$(2e + d + g - 1) - 2l = g + 1 \geq 3.$$

Note that when $l = 0$, so $e = 1 - \frac{d}{2}$, $Z_{\mathcal{L}}$ is empty. $\square$

The morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$

We would like to define a morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$ on the locus in $\mathbb{P}(V_{\mathcal{L}})$ which corresponds to associating to every extension (*) the isomorphism class of the vector bundle $\mathcal{E}$. We use the same ideas as in Section 2.1.

We have the following lemma.

Lemma 2.37. *For each $\mathcal{L} \in Pic^{-e}(C)$, with $e > -\frac{d}{2}$, there is a vector bundle $\mathcal{G}$ on $\mathbb{P}(V_{\mathcal{L}}) \times C$ and a universal exact sequence*

$$0 \to q_1^* O(1) \otimes q_2^* \mathcal{L} \to \mathcal{G} \to q_2^*(\mathcal{L}^{-1} \otimes \xi) \to 0 \qquad (2.41)$$

where q_1, q_2 are the projections onto $\mathbb{P}(V_{\mathcal{L}})$ and C respectively. It has the property that its restriction to $\{p\} \times C$ is an extension

$$0 \to \mathcal{L} \to \mathcal{G}_p \to \mathcal{L}^{-1} \otimes \xi) \to 0$$

which gives an element in $V_{\mathcal{L}}$ whose class in $\mathbb{P}(V_{\mathcal{L}})$ is p.

Proof. This is a particular case of the Lemma A.1 in the Appendix. We take $S = \mathrm{Spec}(\mathbb{C})$, $\mathcal{T} = \mathcal{L}$, $\mathcal{V} = \mathcal{L}^{-1} \otimes \xi)$. We have that $\mathrm{Hom}(\mathcal{L}^{-1} \otimes \xi), \mathcal{L}) = 0$ as

$$\deg\ (\mathcal{L}^{-1} \otimes \xi)) = e + d > -e = \deg\ (\mathcal{L}).$$

So all the conditions in Lemma A.1 are satisfied. It follows that there is an extension with the required properties:

$$0 \to q_1^* O(1) \otimes q_2^* \mathcal{L} \to \mathcal{G} \to q_2^*(\mathcal{L}^{-1} \otimes \xi)) \to 0$$

□

Corollary 2.38. *For each $\mathcal{L} \in Pic^{-e}(C)$, with $e > -\frac{d}{2}$, there is a morphism*

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M_\xi$$

such that for any $p \in \mathbb{P}(V_{\mathcal{L}})$, we have that $\kappa_{\mathcal{L}}(p) \in M_\xi$ is the isomorphism class of the stable bundle on C which is the middle term of an extension in V corresponding to $p \in \mathbb{P}(V_{\mathcal{L}})$.

Proof. From Proposition 2.36, if $p \in \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$, the vector bundle $\mathcal{G}_p$ is stable. By the definition of the moduli scheme M_ξ, there is an associated morphism $\kappa_{\mathcal{L}}$ corresponding to the restriction of $\mathcal{G}$ to $\mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}}$ □

When is the morphism $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M_\xi$ dominant?

We have $\dim M_\xi = 3g - 3$. If $e > -\frac{d}{2}$ then $\dim \mathbb{P}(V_{\mathcal{L}}) = 2e + d + g - 1$.

i. If $e < (g - (d+1)/2)$, then κ is not dominant by dimension considerations.

ii. If $e \geq (g - (d+1)/2)$, then κ is dominant, by the same ideas as in Section 2.1.

Note that this proves that $\mathcal{M}_\xi$ is unirational.

2.5.2 Global Construction

This will be identical to Section 2.1.2. We will just give the statements and point out some of the differences that appear.

The space of extensions of line bundles

For a fixed integer d, let $\xi \in \mathrm{Pic}^d(C)$ be a fixed line bundle.

Lemma 2.39. *For $e > -\frac{d}{2}$ there is a projective bundle $p : X \to Pic^{-e}(C)$ such that for any $\mathcal{L} \in Pic^{-e}(C)$, the fiber $p^{-1}(\{\mathcal{L}\})$ is canonically isomorphic to $\mathbb{P}(V_{\mathcal{L}})$. The space X is a $\mathbb{P}^{2e+d+g-2}$-bundle.*

Proof. Let $\mathcal{A}$ be a Poincaré bundle on $\mathrm{Pic}^{-e}(C) \times C$. Let π_1, π_2 the two projections from $\mathrm{Pic}^{-e}(C) \times C$. Define on $\mathrm{Pic}^{-e}(C)$ the relative extension sheaf

$$\mathcal{S} := \mathcal{E}xt^1_{\mathrm{Pic}^{-e}(C) \times C|\mathrm{Pic}^{-e}(C)}(\mathcal{A}^{-1} \otimes \pi_2^* O_C \otimes \xi, \mathcal{A})$$

Then $\mathcal{S}$ is locally free (we use here that $e > -\frac{d}{2}$) and we let $X = \mathbb{P}(\mathcal{S})$. □

We have

$$\dim X = 2e + d + 2g - 2$$

Note that if $e = -\frac{d}{2}$, by defining $\mathcal{S}$ as in the previous proof, we get a sheaf whose fiber jumps at special points in $\mathrm{Pic}^{-e}(C) \times C$.

The projective bundle X depeds on he Poincaré bundle $\mathcal{A}$. Everything in Note 2.11 applies in this case.

Let ν_1, ν_2 be the two projections from $X \times C$. We have the following lemma.

Lemma 2.40. *For $e > -\frac{d}{2}$, there is a universal extension on $X \times C$:*

$$0 \to \nu_1^* O_X(1) \otimes p^* \mathcal{A} \to \tilde{\mathcal{G}} \to p^*(\mathcal{A})^{-1} \otimes \nu_2^* \xi \to 0.$$

It has the property that, when we restrict to $\{x\} \times C$, where $x \in X$ and we let $\mathcal{L} = p(x) \in Pic^{-e}(C)$, we get an exact sequence:

$$0 \to \mathcal{L} \to \tilde{\mathcal{G}}_x \to \mathcal{L}^{-1} \otimes \xi \to 0$$

which corresponds to an element in $V_{\mathcal{L}}$, whose class in $\mathbb{P}(V_{\mathcal{L}}) \cong p^{-1}(\{\mathcal{L}\})$ is x.

Proof. This is another application of Lemma A.1 in Appendix. We take $S = \mathrm{Pic}^{-e}(C)$, $\mathcal{T} = \mathcal{A}$ and $\mathcal{V} = p^*(\mathcal{A})^{-1} \otimes \nu_2^* \xi$. □

The locus of unstable extensions

Let X be the projective bundle over $Pic^{-e}(C)$ constructed in Lemma 2.39. We use the same arguments as in 2.9 to prove the following lemma.

Lemma 2.41. *For $e > -\frac{d}{2}$, the locus of unstable extensions in X is a closed integral subscheme $Z \subset X$ of codimension at least g. If d is even, the locus of non-stable extensions has codimension at least $g - 1$. If $e = \lceil \frac{d}{2} \rceil - d$, then $Z = \emptyset$.*

Corollary 2.42. *For $e > -\frac{d}{2}$, there is a morphism $\kappa : X \backslash Z \to M_\xi$ which restricted to the fiber of $p : X \to Pic^{-e}(C)$ gives exactly the morphism 2.38*

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M_\xi.$$

When is the morphism $\kappa : X \setminus Z \to \mathcal{M}_\xi$ dominant?

We have dim $X = 2e + 2g + d - 2$ and dim $M = 3g - 3$.

i. If $e < \frac{g}{2} - \frac{d+1}{2}$ then κ is not dominant, by dimension considerations.

ii. If $e \geq \frac{g}{2} - \frac{d+1}{2}$ then κ is dominant (see arguments in Section 2.1).

2.5.3 When the determinant ξ varies

Putting together the spaces X when ξ varies

Let e and d be integers such that $e > -\frac{d}{2}$. We would like to construct a space $\mathcal{X}$ for extensions:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \otimes \xi \to 0, \tag{*}$$

where $\mathcal{L}$ varies in $\mathrm{Pic}^{-e}(C)$ and ξ varies in $\mathrm{Pic}^{d}(C)$. The space $\mathcal{X}$ will be the union of the spaces X constructed in Lemma 2.39, while we let ξ vary in $\mathrm{Pic}^{d}(C)$.

We denote $V_{\mathcal{L},\xi} = \mathrm{Ext}^1_C(\mathcal{L}^{-1} \otimes \xi, \mathcal{L})$.

Proposition 2.43. *If e an d are integers such that $e > -\frac{d}{2}$, there is a projective bundle*

$$p : \mathcal{X} \to Pic^{-e}(C) \times Pic^{d}(C)$$

such that for every point $(\mathcal{L}, \xi)$ in the base, there is a canonical isomorphism

$$p{-}1(\{\mathcal{L}, \xi\}) \cong \mathbb{P}(V_{\mathcal{L},\xi}).$$

Proof. Denote $S = \mathrm{Pic}^{-e}(C) \times \mathrm{Pic}^{d}(C)$. Let $\mathcal{A}_1$ be the pull-back to $S \times C$ of a Poincaré line bundle on $\mathrm{Pic}^{-e}(C) \times C$ and let $\mathcal{A}_2$ be the the pull-back to $S \times C$ of a Poincaré line bundle on $\mathrm{Pic}^{d}(C) \times C$.

Define $\mathcal{T}$ to be the relative extension sheaf:

$$\mathcal{T} = \mathcal{E}xt^1_{S \times C|S}(\mathcal{A}_1^{-1} \otimes \mathcal{A}_2, \mathcal{A}_1) \tag{2.42}$$

Let $\mathcal{X} = \mathbb{P}(\mathcal{T})$ and let $q : \mathcal{X} \to S$ be the corresponding projection. If $\zeta = (\mathcal{L}, \xi) \in S$, we have:

$$\mathcal{T}_{|\xi} \cong \mathrm{Ext}^1_C(\mathcal{L}^{-1} \otimes \xi, \mathcal{L}) \cong \mathrm{H}^1(C, \mathcal{L}^2 \otimes \xi^{-1}).$$

Note that the space $\mathcal{X}$ depends on $\mathcal{A}_1$ and $\mathcal{A}_2$.

□

Putting together the morphisms κ when ξ varies

Let ρ_1, ρ_2 the two projections from $\mathcal{X} \times C$.

We have the following lemma about the locus of unstable (and non-stable) extensions. The method of proof is as in Proposition 2.36.

Lemma 2.44. *For $e > -\frac{d}{2}$, the locus of unstable extensions in X is a closed integral subscheme $\mathcal{Z} \subset \mathcal{X}$ of codimension at least g. If d is even, the locus of non-stable extensions has codimension at least $g-1$. If $e = \lceil \frac{d}{2} \rceil - d$, then $\mathcal{Z} = \emptyset$.*

Let $M(2,d)$ be the coarse moduli scheme of rank 2 semistable vector bundles of degree d.

Corollary 2.45. *For $e > -\frac{d}{2}$, there is a morphism $\tilde{\kappa} : \mathcal{X} \setminus \mathcal{Z} \to M(2,d)$ which restricted to the fiber of $p : \mathcal{X} \to Pic^{-e}(C) \times Pic^{d}(C)$ at the point $(\mathcal{L}, \xi)$ gives the morphism defined in Corollary 2.38:*

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M_{\xi}$$

Proof. Using Lemma A.1 In Appendix, we have that there is a universal extension on $\mathcal{X} \times C$:

$$0 \to q^*\mathcal{A}_1 \otimes \rho_1^* O_{\mathcal{X}}(1) \to \mathcal{H} \to q^*({\mathcal{A}_1}^{-1} \otimes \mathcal{A}_2) \to 0 \tag{2.43}$$

The bundle $\mathcal{H}$ gives the morphism:

$$\tilde{\kappa} : \mathcal{X} \setminus \mathcal{Z} \to M(2,d).$$

□

Chapter 3

Moduli Spaces of Rank 2 Vector Bundles on $\mathbb{P}^1 \times C$

Let C be a genus $g \geq 2$ smooth projective curve. Fix a point $x_0 \in C$ and let $M = M(2, O_C(x_0))$ be the moduli space of rank 2 stable vector bundles on C with determinant $O_C(x_0)$. We are interested in morphisms $f : \mathbb{P}^1 \to M$.

To give a morphism f is to give a rank 2 vector bundle on $\mathbb{P}^1 \times C$. Such a bundle is unique up to tensoring with a line bundle from $\mathbb{P}^1$. Since there is a rigidified Poincaré vector bundle $\mathcal{U}_0$ on $M \times C$, it follows that there is a unique way of associating a rank 2 vector bundle on $\mathbb{P}^1 \times C$ to f, by taking $(f \times \mathrm{id}_C)^*\mathcal{U}_0$. This gives a one-to one correspondence between rational curves of degree k on M and rank 2 vector bundles on $\mathbb{P}^1 \times C$ with fixed Chern classes and satisfying a stability condition. We will show that the irreducible components of the space of rational curves $\mathrm{Mor}_k(\mathbb{P}^1, M)$ are birational to moduli spaces of vector bundles on $\mathbb{P}^1 \times C$.

To make precise this correspondence, we have the following lemma:

Lemma 3.1. *Let $k \geq 1$ be an integer. There is a one-to-one correspondence between morphisms $f : \mathbb{P}^1 \to M$ of degree k and rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ with the following properties:*

i. The Chern classes $c_1(\mathcal{F}) \in A^1(\mathbb{P}^1 \times C)$ and $c_2(\mathcal{F}) \in A^2(\mathbb{P}^1 \times C)$ satisfy:

$$c_1(\mathcal{F}) = k\{pt\} \times C + \mathbb{P}^1 \times \{x_0\} \quad \text{and} \quad deg\,(c_2(\mathcal{F})) = k \in \mathbb{Z} \tag{3.1}$$

ii. The bundle $\mathcal{F}_p = \mathcal{F}_{|\{p\}\times C}$ on C is stable for any $p \in \mathbb{P}^1$.

Proof. Given $f : \mathbb{P}^1 \to M$ of degree k we associate to it the vector bundle

$$\mathcal{F} = (f \times \mathrm{id}_C)^*\mathcal{U}_0.$$

This bundle determines f (from the definition of the moduli scheme M). The Chern classes of $\mathcal{F}$ satisfy the relations (3.1), as they were computed in (1.7) and (1.8). Conversely, given a bundle $\mathcal{F}$ with such properties, it induces a morphism $f : \mathbb{P}^1 \to M$. Formula (1.9) proves that the degree of f is k. Moreover, since $(f \times \mathrm{id}_C)^*\mathcal{U}_0$ and $\mathcal{F}$ both determine the same morphism f and they have the same first Chern class, it follows that $\mathcal{F} \cong (f \times \mathrm{id}_C)^*\mathcal{U}_0$. □

We construct moduli spaces for rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ satisfying the relations (3.1). This was done by Brosius in [BR1] and [BR2] in the more general case of rank 2 vector bundles on a ruled surface. We reproduce some of his constructions in our particular case of vector bundles on $\mathbb{P}^1 \times C$ and in addition we identify the locus of those rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ satisfying the additional stability property in Lemma 3.1.

3.1 The canonical extension of a rank 2 vector bundle

This section contains definitions and results stated in [BR1]. We give proofs for some of the facts in [BR1]. We will define the canonical extension of a rank 2 vector bundle on $\mathbb{P}^1 \times T$, where T is an arbitrary integral Noetherian scheme. We need this level of generality, since we want a canonical way to obtain the canonical extension when we have a family of vector bundles on $\mathbb{P}^1 \times C$, in order to construct a moduli scheme of such bundles.

Let T be an integral Noetherian scheme. Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times T$ and let k be the fiber degree of $\mathcal{F}$ with respect to the second projection $p_2 : \mathbb{P}^1 \times T \to T$:

$$k = c_1(\mathcal{F}).(\{\mathbb{P}^1\} \times \{pt\}).$$

If F is a fiber of p_2, then the restriction of $\mathcal{F}$ to F has the form $O(a) \oplus O(k-a)$, for some integer $a \geq k/2$. The pair $(a, k-a)$ is called the *fiber type* of $\mathcal{F}$ on F.

The function $h : T \to \mathbb{Z}$, which associates to a point $t \in T$ the integer a from the fiber type of $\mathcal{F}$ on the fiber F_t above t, is upper semicontinuos. Let a be the value of h on the generic point of T. We say that $\mathcal{F}$ has *generic fiber type* $(a, k-a)$. We say $\mathcal{F}$ is of *type* U (unequal) if $a > k/2$ and of *type* E (equal) if $a = k/2$.

The statement of the following lemma is mentioned in [BR1].

Lemma 3.2. *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times T$ of generic fiber type $(a, k-a)$ and let $\mathcal{F}' = (p_2{}^*p_{2*}(\mathcal{F}(-a)))(a)$. Then $\mathcal{F}'$ is a torsion free sheaf of rank 1 in case U and rank 2 in case E. Moreover, the map*

$$g : \mathcal{F}' \to \mathcal{F}$$

coming from the canonical map $p_2^*p_{2*}(\mathcal{F}(-a)) \to \mathcal{F}(-a)$, is injective.

Proof. Note that the sheaf $p_{2*}\mathcal{F}(-a)$ is torsion free because $\mathcal{F}$ is torsion free. By restricting to the generic point ξ of T, one can see easily that the sheaf $p_{2*}\mathcal{F}(-a)$ has rank 1 in case U $(a > k/2)$ and rank 2 in case E $(a = k/2)$ as we have:

$$(p_{2*}\mathcal{F}(-a))_\xi \cong \mathrm{H}^0(\mathbb{P}^1_\xi, \mathcal{F}_\xi(-a))$$

and the restriction of morphism g to $\mathbb{P}^1 \times \{\xi\}$ is:

$$\mathrm{H}^0(\mathbb{P}^1_\xi, \mathcal{F}_\xi(-a)) \otimes O \to \mathcal{F}_\xi(-a), \quad \text{given by} \quad s \otimes t \mapsto s.t$$

The morphism g is injective when restricted to $\mathbb{P}^1 \times \{\xi\}$; hence, it is injective at the generic point of $\mathbb{P}^1 \times T$. It follows that g is injective, since $\mathcal{F}'$ is torsion free. □

Let $\mathcal{J} = \text{coker }(g)$. The *canonical extension of* $\mathcal{F}$ is the exact sequence:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0.$$

The bundle $\mathcal{F}'$ is called the *canonical subbundle* of $\mathcal{F}$.

3.1.1 The canonical extension in Case U

The following lemma is mentioned without proof in [BR1].

Lemma 3.3. *In case U, the sheaf $\mathcal{J}$ is isomorphic to $\mathcal{I}_Z \otimes \mathcal{M}$, for some line bundle $\mathcal{M}$ on $\mathbb{P}^1 \times T$ and $\mathcal{I}_Z$ is the ideal sheaf of a local complete intersection (lci) subscheme Z of $\mathbb{P}^1 \times T$, given by the zeros of the morphism g, which has codimension at least* 2.

The lemma is a consequence of the following Lemma. The method of proof for the Lemma will appear again in the proof of Proposition 3.19 and other future results.

Lemma 3.4. *In case U, the sheaf $\mathcal{F}'$ is a line bundle and the sheaf $\mathcal{J}$ is a torsion-free sheaf.*

Proof. We are in Case U, so $\mathcal{F}'$ and $\mathcal{J}$ have both rank 1. If $\mathcal{J}$ is torsion free, then since $\mathcal{F}'$ is torsion free, it follows by Lemma A.10 in the Appendix that $\mathcal{F}'$ is a reflexive sheaf. But a reflexive sheaf of rank 1 is a line bundle, and so it follows that $\mathcal{F}'$ is a line bundle.

We prove now that $\mathcal{J}$ is torsion-free. Let $t(\mathcal{J})$ be the torsion subsheaf of $\mathcal{J}$ and let $\mathcal{J}_1 = \mathcal{J}/t(\mathcal{J})$. Then $\mathcal{J}_1$ is a torsion free sheaf of rank 1. Let $\mathcal{F}_1$ be the kernel of the composition morphism $\mathcal{F} \to \mathcal{J} \to \mathcal{J}_1$. There is a comutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J}_1 & \longrightarrow & 0
\end{array}$$

By Lemma A.10 in the Appendix applied to the lower exact sequence, it follows that $\mathcal{F}_1$ is a reflexive sheaf. But $\mathcal{F}_1$ has rank 1, and any reflexive sheaf of rank 1 is a line bundle; hence, $\mathcal{F}_1$ is a line-bundle. Then $\mathcal{F}_1 \cong O(b) \boxtimes \mathcal{N}$ for some integer b and some line bundle $\mathcal{N}$ on T. By the snake lemma, we have an exact sequence:

$$0 \to \mathcal{F}' \to \mathcal{F}_1 \to t(\mathcal{J}) \to 0.$$

Let's look at the injective morphism $\mathcal{F}' \to \mathcal{F}_1$. We would like to prove that it is in fact an isomorphism. It is enough to prove that $b = a$ and $\mathcal{N} \cong p_{2*}\mathcal{F}(-a)$. Since $\mathcal{F}' \to \mathcal{F}_1$ is injective, it follows that $b \geq a$.

Assume $b > a$. From the lower exact sequence of the diagram, we have:

$$0 \to p_{2*}\mathcal{F}_1(-b) \to p_{2*}\mathcal{F}(-b) \to p_{2*}\mathcal{J}_1(-b).$$

As $p_{2*}\mathcal{F}_1(-b) \cong \mathcal{N}$ and $p_{2*}\mathcal{F}(-b) = 0$, we arrived at a contradiction. So $b = a$ and $\mathcal{F}_1 \cong O(a) \boxtimes \mathcal{N}$. We look again at the sequence:

$$0 \to p_{2*}(\mathcal{F}_1(-a)) \to p_{2*}\mathcal{F}(-a) \to p_{2*}\mathcal{J}_1(-a).$$

Note that $p_{2*}\mathcal{F}_1(-a) \cong \mathcal{N}$. The sequence becomes:

$$0 \to \mathcal{N} \to p_{2*}\mathcal{F}(-a) \to p_{2*}\mathcal{J}_1(-a)$$

Since $\mathcal{N}$ and $p_{2*}\mathcal{F}(-a)$ have both rank 1, the cokernel is a torsion subsheaf of $p_{2*}\mathcal{J}_1(-a)$, so it must be zero, as $\mathcal{J}_1$ torsion-free implies $p_{2*}\mathcal{J}_1(-a)$ is torsion-free. Therefore, $\mathcal{N} \cong p_{2*}\mathcal{F}(-a)$ and $\mathcal{F}' \cong \mathcal{F}_1$. Hence, $t(\mathcal{J}) = 0$. □

Consider now the case $T = C$ in the previous construction. Let us summarize the results in this particular case:

Lemma 3.5. *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C$ with generic fiber type $(a, k-a)$, where $a > \frac{k}{2}$. Then $\mathcal{F}$ belongs to an exact sequence:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0$$

where $\mathcal{F}'$ and $\mathcal{J}$ are as follows:

$$\mathcal{F}' \cong O(a) \boxtimes \mathcal{L}, \quad \mathcal{M} \cong O(k-a) \boxtimes \mathcal{L}', \quad \mathcal{J} \cong \mathcal{I}_Z \otimes \mathcal{M}$$

whith $\mathcal{L}$ and $\mathcal{L}'$ line bundles on C and $Z \subset \mathbb{P}^1 \times C$ an lci 0-cycle.

Note that $\mathcal{L} = p_{2*}\mathcal{F}(-a)$.

3.1.2 The canonical extension in Case E

The following Fact is proved in [BR1].

Fact 3.6. *[BR1] Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times T$ with generic fiber type (a, a). Let $\mathcal{J}$ be the sheaf from the canonical sequence of $\mathcal{F}$. Then there is a unique* *line bundle $\mathcal{M}$ on $\mathbb{P}^1 \times T$, and a closed subscheme $Z \subset \mathbb{P}^1 \times T$ such that, if D is the scheme theoretic image $p_2(Z) \subset T$, then:*

$$\mathcal{J} \cong \mathcal{I} \otimes \mathcal{M}$$

where $\mathcal{I}$ is the ideal sheaf of Z in $\mathbb{P}^1 \times D$.

The subscheme Z is clearly not unique.

Consider the case $T = C$ in the previous construction. Let us summarize the results:

Fact 3.7. *[BR1] Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C$ with generic fiber type (a, a). Then $\mathcal{F}$ belongs to an exact sequence:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0$$

where $\mathcal{F}'$ and $\mathcal{J}$ are as follows:

$$\mathcal{F}' \cong O(a) \boxtimes \mathcal{E} \quad \text{and} \quad \mathcal{J} \cong \mathcal{I}(a)$$

where $\mathcal{E}$ is a rank 2 vector bundle on C and $\mathcal{I}$ is the ideal sheaf of the 0-cycle Z in $\mathbb{P}^1 \times D$, where Z is a 0-cycle in $\mathbb{P}^1 \times C$ and $D = p_2(Z) \subset T$ (scheme theoretic image). Note that $\mathcal{E} = p_{2*}\mathcal{F}(-a)$.

If $D = n_1p_1 + \ldots + n_rp_r$ for some distinct points $p_1, \ldots, p_r$ on C and some positive integers $n_1, \ldots, n_r$, then we have that the ideal sheaf $\mathcal{I}$ is of the form

$$\mathcal{I} \cong \oplus_i (O(-m_i) \boxtimes O_{n_ip_i})$$

where $m_1, \ldots, m_r$ are positive integers.

Let $c \in C$ and consider the restriction $\mathcal{F}_c$ to $\mathbb{P}^1 \times \{c\}$. We have:

i. If c is not in D, then $\mathcal{F}_c \cong O(a) \oplus O(a)$

ii. If $c = p_i$, then $\mathcal{F}_c \cong O(a + m_i) \oplus O(a - m_i)$

We will work mainly with the case when the morphism $p_2 : Z \to D$ is an isomorphism. In this case we have:

$$\mathcal{I} \cong O(-1) \boxtimes O_D, \quad \text{and} \quad \mathcal{J} \cong O(a-1) \boxtimes O_D$$

We say that $\mathcal{F}$ has type (†) if it has canonical sequence:

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{F} \to O(a-1) \boxtimes O_D \to 0 \qquad (\dagger)$$

where $\mathcal{E}$ is a *stable* vector bundle on C.

Note 3.8. *A vector bundle $\mathcal{F}$ with equal generic fiber type (a, a) has the bundle $\mathcal{I}$ in the canonical quotient of the form $O(-1) \boxtimes O_D$ for some divisor D on C if and only if for any $c \in C$, the bundle $\mathcal{F}_c$ splits either as $O(a) \oplus O(a)$ or $O(a+1) \oplus O(a-1)$.*

3.1.3 Classification of canonical extensions in Case U

This section continues to follow the results stated in [BR1]. All of them are easy consequences of the results of the previous section.

Invertible extensions

Remark 3.9. *Let $\mathcal{F}$ be a rank 2 indecomposable vector bundle of type U on $\mathbb{P}^1 \times C$ with generic fiber type $(a, k-a)$ $(a > \frac{k}{2})$. The bundle $\mathcal{F}$ determines uniquely the sheaves $\mathcal{F}'$, $\mathcal{M}$ and the lci 0-cycle $Z \subset \mathbb{P}^1 \times C$ and a point*

$$\xi \in \mathbb{P}(Ext^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}')).$$

Clearly, the line bundles $\mathcal{L}$ and $\mathcal{L}'$ from Lemma 3.5 are uniquely determined.

The point ξ can also be thought of as an orbit for the action of $\tilde{\mathcal{G}}_m$ on

$$\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}').$$

It will be convenient to refer to it in this way.

We would like to know when we can recover from such data the vector bundle $\mathcal{F}$. For given $\mathcal{F}'$, $\mathcal{M}$ and Z it turns out that only *invertible* orbits ξ will correspond to an extension whose middle term is a vector bundle. We will define this notion a few paragraphs later. The result we are aiming for is the following:

Lemma 3.10. *In Case U, an indecomposable vector bundle $\mathcal{F}$ determines line bundles $\mathcal{F}'$ and $\mathcal{M}$ on C together with an lci zero-cycle Z and an* invertible *orbit ξ in $\mathbb{P}(Ext^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}')$. These data are uniquely determined and they in turn determine $\mathcal{F}$ uniquely (up to isomorphism) as the middle term of an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{M} \to 0$$

corresponding to any extension Ξ in the orbit ξ.

In order to define invertibe orbits, we need the following lemma.

Lemma 3.11. *There is a short exact sequence:*

$$0 \to Ext^1_{O_{\mathbb{P}^1\times C}}(\mathcal{M}, \mathcal{F}') \to Ext^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}') \to Ext^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M}_{|Z}, \mathcal{F}') \to 0.$$

Proof. Consider the exact sequence:

$$0 \to \mathcal{I}_Z \otimes \mathcal{M} \to \mathcal{M} \to \mathcal{M}_{|Z} \to 0.$$

By applying $\mathrm{Hom}(-, \mathcal{F}')$, we get an exact sequence of vector spaces:

$$\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{M}_{|Z},\mathcal{F}') \to \mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{M},\mathcal{F}') \to \mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z\otimes\mathcal{M},\mathcal{F}') \to$$
$$\to \mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M}_{|Z},\mathcal{F}') \to \mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M},\mathcal{F}')$$

Note that

$$\mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M},\mathcal{F}') \cong \mathrm{H}^2(\mathbb{P}^1\times C,\mathcal{M}^*\otimes\mathcal{F}') \cong \mathrm{H}^0(\mathbb{P}^1\times C,\mathcal{M}\otimes\mathcal{F}'^*\otimes K_{\mathbb{P}^1\times C}).$$

Let $\mathcal{L}$ and $\mathcal{L}'$ be the line bundles from Lemma 3.5:

$$\mathcal{M}\cong O(k-a)\boxtimes\mathcal{L}',\quad \text{and}\quad \mathcal{F}'\cong O(a)\boxtimes\mathcal{L}$$

We have

$$K_{\mathbb{P}^1\times C}\cong O(-2)\boxtimes K_C$$
$$\mathcal{M}\otimes\mathcal{F}'^*\otimes K_{\mathbb{P}^1\times C}\cong O(k-2a-2)\boxtimes(\mathcal{L}^{-1}\otimes\mathcal{L}'\otimes K_C)$$

Since $k-2a-2<0$, we have $\mathrm{H}^0(\mathbb{P}^1\times C,\mathcal{M}\otimes\mathcal{F}'^*\otimes K_{\mathbb{P}^1\times C})=0$ and therefore

$$\mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M},\mathcal{F}')=0.$$

Let's look at the space

$$\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{M}_{|Z},\mathcal{F}')\cong\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(O_Z,\mathcal{F}'\otimes\mathcal{M}^*).$$

Using duality, we get:

$$\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(O_{|Z},\mathcal{F}'\otimes\mathcal{M}^*)\cong\mathrm{H}^1(\mathbb{P}^1\times C,\mathcal{F}'^*\otimes\mathcal{M}\otimes K\otimes O_Z)=0.$$

□

We define now the notion of an invertible orbit. Let

$$\Xi\in\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z\otimes\mathcal{M},\mathcal{F}')$$

and let $\overline{\Xi}$ be its image in $\mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M}_{|Z},\mathcal{F}')$. Note that the vector space

$$\mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(\mathcal{M}_{|Z},\mathcal{F}')\cong\mathrm{Ext}^2_{O_{\mathbb{P}^1\times C}}(O_Z,\mathcal{F}'\otimes\mathcal{M}^*)$$

is isomorphic, using duality, to

$$\mathrm{H}^0(\mathbb{P}^1\times C,\mathcal{F}'^*\otimes\mathcal{M}\otimes K\otimes O_Z)\cong\mathrm{H}^0(Z,O_Z).$$

The extension Ξ is called *invertible* if $\overline{\Xi}$ as an element of $\mathrm{H}^0(Z,O_Z)$ does not have zeros over Z. The property of being invertible is invariant under the action of

$\mathbb{G}_m$ and so the notion of an invertible orbit ξ is well-defined. It is straightforward to see that if Z is an lci 0-cycle, then an orbit ξ gives rise to a vector bundle $\mathcal{F}$ if and only if ξ is invertible. Moreover, note that invertible orbits form a dense open set in $\mathbb{P}(\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}'))$.

Correspondence between $\mathcal{F}$ and orbits in the space of extensions

The natural question is when an invertible extension

$$\Xi \in \mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}')$$

determines $\mathcal{F}$ such that the canonical extension of $\mathcal{F}$ determines the same element in $\mathbb{P}(\mathrm{Ext}^1_{O_{\mathbb{P}^1\times C}}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}'))$. The answer is given by the following lemma. We will consider the more general case of vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times T$ (this generality will be useful later).

Lemma 3.12. *Let T be an integral Noetherian scheme and $\mathcal{F}$ is a rank 2 vector bundle on $\mathbb{P}^1 \times T$ of fiber degree k. Assume there is a line bundle $\mathcal{L}$ on T, a torsion free sheaf $\mathcal{J}$ on $\mathbb{P}^1 \times T$, an integer $a > \frac{k}{2}$ and a short exact sequence:*

$$0 \to O(a) \boxtimes \mathcal{L} \to \mathcal{F} \to \mathcal{J} \to 0. \tag{3.2}$$

Then $\mathcal{F}$ has generic fiber type $(a, k-a)$ and $p_{2*}\mathcal{F}(-a) \cong \mathcal{L}$. Moreover, this extension is, up to scalar multiplication, the canonical extension of $\mathcal{F}$.

Proof. Assume $\mathcal{F}$ has generic fiber type $(b, k-b)$. Let η be the generic point of T. If we restrict (3.2) to $\mathbb{P}^1 \times \{\eta\}$ we get an exact sequence:

$$0 \to O(a) \to O(b) \oplus O(k-b) \to \mathcal{J}_\eta \to 0.$$

Since $a > \frac{k}{2}$, it follows that $a \leq b$. Therefore, $\mathcal{J}_\eta \cong O(k-a)$. Tensor (3.2) by $O(-a)$ and take push-forward to T:

$$0 \to \mathcal{L} \to p_{2*}\mathcal{F}(-a) \to p_{2*}\mathcal{J}(-a) \to 0.$$

We have:

$$p_{2*}\mathcal{F}(-a)_\eta \cong \mathrm{H}^0(\mathbb{P}^1 \times \{\eta\}, O(b-a) \oplus O(k-a-b))$$
$$p_{2*}\mathcal{J}(-a)_\eta \cong \mathrm{H}^0(\mathbb{P}^1 \times \{\eta\}, O(k-2a) = 0$$

Hence, $p_{2*}\mathcal{J}(-a)$ is a torsion sheaf. But $\mathcal{J}$ was torsion-free, hence $p_{2*}\mathcal{J}(-a)$ is torsion-free. It follows that $p_{2*}\mathcal{J}(-a)$ is zero and $\mathcal{L} \cong p_{2*}\mathcal{F}(-a)$.

The sheaf $p_{2*}\mathcal{F}(-a)$ has rank $b-a+1$, so we must have $b = a$. Hence, $\mathcal{L} \cong p_{2*}\mathcal{F}(-a)$.

Consider the canonical extension of $\mathcal{F}$:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0$$

where $\mathcal{F}' \cong O(a) \boxtimes \mathcal{L}$.

First note that there are no non-zero morphisms $\mathcal{F}' \to \mathcal{J}$, as there are no non-zero morphisms $O(a) \to O(k-a)$ on $\mathbb{P}^1$. There is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \psi} & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \end{array}$$

where one row is the canonical extension and the other is the exact sequence in our lemma. As ψ is a non-zero morphism and $\mathcal{F}'$ is a line bundle, it follows that it is an isomorphism given by multiplication with a non-zero scalar. Then ϕ is also an isomorphism. Using the fact that an automorphism of a rank 1 torsion free sheaf is given by multiplication by a non-zero scalar, we get that ϕ is given by multiplication with a non-zero scalar. It follows that the two extensions are scalar multiples of each other in $\mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{J}, \mathcal{F}')$. □

Corollary 3.13. *Any isomorphism $\mathcal{F}_1 \cong \mathcal{F}_2$ of vector bundles of type U on $\mathbb{P}^1 \times C$ induces an isomorphism on their canonical sequences. The two exact sequences determine the same element in $\mathbb{P}(Ext^1_{\mathbb{P}^1 \times C}(\mathcal{J}, \mathcal{F}'))$.*

3.1.4 Classification of canonical extensions in Case E

Remark 3.14. *Let $\mathcal{F}$ be a rank 2 indecomposable vector bundle of type E on $\mathbb{P}^1 \times C$ with generic fiber type (a, a). Consider the canonical sequence of $\mathcal{F}$:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0.$$

Assume that we are in the case when $\mathcal{F}$ has type (†):

$$\mathcal{F}' \cong O(a) \boxtimes \mathcal{E} \quad \text{and} \quad \mathcal{J} \cong O(a-1) \boxtimes O_D$$

where the vector bundle $\mathcal{E}$ on C is stable and D is a 0-cycle on C.

Then the bundle $\mathcal{F}$ determines uniquely the sheaves $\mathcal{F}'$, $\mathcal{J}$ and an orbit ξ for the action of $Aut\ (O_D) \cong H^0(D, O_D^*)$ on

$$Ext^1_{O_{\mathbb{P}^1 \times C}}(\mathcal{J}, \mathcal{F}').$$

Clearly, $\mathcal{E}$ and D are uniquely determined.

We would like to know when we can recover from such data the vector bundle $\mathcal{F}$. For given $\mathcal{F}'$, D and ξ, it turns out again that only *invertible* orbits ξ will correspond to an extension whose middle term is a vector bundle. The invertible extensions form a dense open in the space of extensions.

Lemma 3.15. *In Case E, an indecomposable vector bundle $\mathcal{F}$ of type (†), determines the sheaves $\mathcal{F}'$ and $\mathcal{J}$, together with an invertible orbit ξ for the action of Aut (O_D) on $Ext^1_{O_{\mathbb{P}^1\times C}}(\mathcal{J},\mathcal{F}')$. These data are uniquely determined and they in turn determine $\mathcal{F}$ uniquely (up to isomorphism) as the middle term of an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0$$

corresponding to any extension Ξ in the orbit ξ.

Correspondence between $\mathcal{F}$ and orbits in the space of extensions

We ask if an invertible extension

$$\Xi \in \mathrm{Ext}^1(\mathcal{J},\mathcal{F}')$$

determines $\mathcal{F}$ such that the canonical extension of $\mathcal{F}$ determines the same element orbit in $\mathrm{Ext}^1(\mathcal{J},\mathcal{F}')$. The answer is given by the following lemma.

Lemma 3.16. *Let $\mathcal{F}$ on $\mathbb{P}^1 \times C$ a rank 2 vector bundle of fiber degree $k = 2a$. Assume there is a* stable *vector bundle $\mathcal{E}$ of rank 2 on C, an effective divisor D on C and a short exact sequence:*

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{F} \to O(a-1) \boxtimes O_D \to 0. \tag{3.3}$$

Then $\mathcal{F}$ has generic fiber type (a,a) and $\mathcal{E} \cong p_{2*}(\mathcal{F}(-a))$. Moreover, this extension is in the same orbit of the action of Aut (O_D) on the space of extensions

$$Ext^1(O(a-1) \boxtimes O_D, O(a) \boxtimes \mathcal{E})$$

as the canonical extension of $\mathcal{F}$.

Proof. Let η be the generic point of C. If we restrict (3.3) to $\mathbb{P}^1 \times \{\eta\}$ we get an exact sequence:

$$0 \to O(a) \oplus O(a) \to \mathcal{F}_\eta \to 0.$$

Hence, $\mathcal{F}$ has generic fiber type (a,a). Tensor (3.3) by $O(-a)$ and take push-forward to C:

$$0 \to \mathcal{E} \to p_{2*}\mathcal{F}(-a) \to p_{2*}\mathcal{J}(-a) \to 0.$$

We have:

$$p_{2*}\mathcal{J}(-a) \cong O_D \otimes p_{2*}p_1^*O(-1) = 0.$$

Hence, $\mathcal{E} \cong p_{2*}\mathcal{F}(-a)$.

Consider the canonical extension of $\mathcal{F}$:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0$$

where $\mathcal{F}' \cong O(a) \boxtimes \mathcal{E}$.

The fact that $\mathcal{F}$ sits in an exact sequence as (3.3) implies that for any $c \in C$ the bundle $\mathcal{F}_c$ splits either as $O \oplus O$ or $O(1) \oplus O(-1)$. Hence, the sheaf $\mathcal{J}$ has the form $O(a-1) \boxtimes O_{D'}$ for some effective divisor D' on C. Note that since the splitting of $\mathcal{F}_c$ jumps exactly at the points $c \in D'$, it follows that D and D' are supported at the same points. Since by restriction to $\{p\} \times C$, where $p \in \mathbb{P}^1$ is any point, we get an exact sequence:

$$0 \to \mathcal{E} \to \mathcal{F}_p \to O_D \to 0$$

It follows that $D = D'$, hence, $\mathcal{J} \cong O(a-1) \boxtimes O_D$.

First note that there are no non-zero morphisms $\mathcal{F}' \to \mathcal{J}$, as there are no non-zero morphisms $O(a) \to O(a-1)$ on $\mathbb{P}^1$. There is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \\ \downarrow & & \downarrow \psi & & \| & & \downarrow \phi & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \end{array}$$

where one row is the canonical extension and the other is the exact sequence in our lemma. As ψ induces a non-zero morphism $\mathcal{E} \to \mathcal{E}$, it follows that it is an isomorphism. Since $\mathcal{E}$ is stable, ψ is given by multiplication with a non-zero scalar. Then ϕ is also an isomorphism. Since an automorphism of $\mathcal{J}$ is given by an element in $\mathrm{H}^0(D, O_D^*)$, it follows that the two extensions are in the same orbit for the action of Aut (O_D) on $\mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{J}, \mathcal{F}')$. □

Corollary 3.17. *Any isomorphism $\mathcal{F}_1 \cong \mathcal{F}_2$ of vector bundles of type E on $\mathbb{P}^1 \times C$ induces an isomorphism on their canonical sequences. The two exact sequences determine the same orbit for the action of Aut (O_D) on $Ext^1_{\mathbb{P}^1 \times C}(\mathcal{J}, \mathcal{F}')$.*

3.2 Families of rank 2 vector bundles on $\mathbb{P}^1 \times C$

We analyze the behaviour of the canonical sequence in families of vector bundles on $\mathbb{P}^1 \times C$. In this section S will be an integral scheme and we will consider vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C \times S$. As usual, if $s \in S$ and $c \in C$, we denote $\mathcal{F}_s = \mathcal{F}_{|\mathbb{P}^1 \times C \times \{s\}}$ and $\mathcal{F}_{c,s} = \mathcal{F}_{|\{c\} \times \{s\}}$. Moreover, we denote by p_2^s the projection $\mathbb{P}^1 \times C \times \{s\} \to C \times \{s\}$.

3.2.1 A general fact

Lemma 3.18. *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$ with fiber degree k and generic fiber type $(a, k-a)$, with $a \geq k/2$. Then there is a dense open $S^0 \subseteq S$ such that, for any $s \in S^0$, the bundle $\mathcal{F}_s$ on $\mathbb{P}^1 \times C \times \{s\}$ has generic fiber type $(a, k-a)$ and the degree of the vector bundle $p^s_{2*}\mathcal{F}_s(-a)$ on $C \times \{s\}$ is constant.*

Proof. Since $\mathcal{F}$ has generic fiber type $(a, k-a)$, there is a dense open $S^0 \subset S$ on which $\mathcal{F}_s$ on $\mathbb{P}^1 \times C \times \{s\}$ has generic fiber type $(a, k-a)$.

Let's first note that the numerical Chern classes $c_1(\mathcal{F}_s)$ and $c_2(\mathcal{F}_s)$ do not depend on S. This fact has as a consequence, ussing the Grothendieck Riemann-Roch formula for p_2^s and the bundle $\mathcal{F}_s(-a)$, that the Chern character of the following element in the Grothedieck group of $\mathbb{P}^1 \times C \times \{s\}$ does not depend on s:

$$(p_2^s)_! \mathcal{F}_s(-a) = p_{2*}^s \mathcal{F}_s(-a) - \mathrm{R}^1 p_{2*}^s \mathcal{F}_s(-a)$$

In particular, the first Chern class does not depend on s:

$$c_1(p_{2*}^s \mathcal{F}_s(-a)) - c_1(\mathrm{R}^1 p_{2*}^s \mathcal{F}_s(-a)).$$

Note that since $\mathcal{F}$ is a vector bundle, in particular $\mathcal{F}_s(-a)$ is a torsion-free sheaf and therefore $p_{2*}^s \mathcal{F}_s(-a)$ is a torsion free sheaf on $C \times \{s\}$. Since we are on a curve, it follows that $p_{2*}^s \mathcal{F}_s(-a)$ is a vector bundle.

We prove that for any $s \in S$, the degree of the first Chern class of the sheaf $\mathrm{R}^1 p_{2*}^s \mathcal{F}_s(-a)$ does not depend on $s \in S$. It follows that the vector bundle $p_{2*}^s \mathcal{F}_s(-a)$ has constant degree on S.

Let $p_{2,3}$ be the projection onto the the second and third factor. For any $s \in S$ consider the canonical morphism of sheaves on $C \times \{s\}$:

$$\phi : (\mathrm{R}^1 p_{2,3_*} \mathcal{F}(-a))_s \to \mathrm{R}^1 p_{2*}^s \mathcal{F}_s(-a).$$

For any $c \in C$ and $s \in S$ there are canonical isomorphisms $\psi_{s,c}$ and $\chi_{s,c}$ such that:

$$\psi_{s,c} : (\mathrm{R}^1 p_{2,3_*} \mathcal{F}(-a))_{c,s} \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{F}_{c,s}(-a))$$
$$\chi_{s,c} : (\mathrm{R}^1 p_{2*}^s \mathcal{F}_s(-a))_c \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{F}_{c,s}(-a))$$

with $\chi_{c,s} \circ \phi_c = \psi_{c,s}$ ([M]):

$$\begin{array}{ccc} (\mathrm{R}^1 p_{2,3_*} \mathcal{F}(-a))_{c,s} & \xrightarrow{\phi_c} & (\mathrm{R}^1 p_{2*}^s \mathcal{F}_s(-a))_c \\ & \searrow \quad \swarrow & \\ & \mathrm{H}^1(\mathbb{P}^1, \mathcal{F}_{c,s}(-a)) & \end{array}$$

It follows that ϕ_c is an isomorphism for any point $c \in C$, hence, ϕ is an isomorphism.

On $C \times S$ consider the sheaf:

$$\mathcal{H} = \mathrm{R}^1 p_{2,3_*} \mathcal{G}(-a).$$

By generic flatness, the degree of the first Chern class of the sheaf $\mathcal{H}_s$ on $C \times \{s\}$ is constant on a dense open $S^0 \subset S$; hence, our assertion follows, since for any $s \in S$, we have:

$$\mathcal{H}_s \cong \mathrm{R}^1 p^s_{2*}\mathcal{F}_s(-a).$$

□

3.2.2 Families of vector bundles in Case U

Proposition 3.19. *Let $\mathcal{F}_S$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$. Assume that the bundle $\mathcal{F}_S$ has generic fiber type $(a, k-a)$ with $a > \frac{k}{2}$. Then there is a dense open $S^0 \subseteq S$ such that:*

i. For any $s \in S^0$, the restriction of the canonical sequence of $\mathcal{F}_S$ to $\mathbb{P}^1 \times C \times \{s\}$ is the canonical sequence of $\mathcal{F}_s$. In particular, there is an integer d such that for $s \in S^0$ the canonical line subbundle of $\mathcal{F}_s$ has type (a, d).

ii. If $s_0 \in S \setminus S^0$, then $\mathcal{F}_{s_0}$ has canonical line subbundle of type (a_0, d_0) with $a_0 \geq a$ and $d_0 \geq d$.

(We denote by $\mathcal{F}_s$ be the restriction of $\mathcal{F}_S$ to $\mathbb{P}^1 \times C \times \{s\}$.)

Proof. Let η be the generic point of S. The generic fiber type of $\mathcal{F}_\eta$ is $(a, k-a)$. Consider the canonical sequence of $\mathcal{F}_\eta$:

$$0 \to \mathcal{F}'_\eta \to \mathcal{F}_\eta \to \mathcal{J}_\eta \to 0 \tag{3.4}$$

where $\mathcal{F}'_\eta$ and $\mathcal{J}_\eta$ are sheaves on $\mathbb{P}^1 \times C \times \{\eta\}$. Since we are in Case U, it follows that $\mathcal{F}'_\eta$ is a line bundle and $\mathcal{J}_\eta$ is torsion-free. Moreover, $\mathcal{F}'_\eta \cong O(a) \boxtimes \mathcal{N}_\eta$, where $\mathcal{N}_\eta$ line bundle on $C \times \{\eta\}$.

We prove that when we restrict the canonical sequence of $\mathcal{F}_S$ to $\mathbb{P}^1 \times C \times \{\eta\}$ we get exactly the exact sequence (3.4).

Let $\mathcal{F}_1 \to \mathcal{F}_S$ be an injective morphism such that when restricted to $\mathbb{P}^1 \times C \times \{\eta\}$ we get the injective morphism $\mathcal{F}'_\eta \to \mathcal{F}_\eta$. Let $\mathcal{J}_1$ be the cokernel:

$$0 \to \mathcal{F}_1 \to \mathcal{F}_S \to \mathcal{J}_1 \to 0.$$

As restriction to $\mathbb{P}^1 \times C \times \{\eta\}$ is exact, we get an exact sequence:

$$0 \to (\mathcal{F}_1)_\eta \to \mathcal{F}_\eta \to \mathcal{J}_{1\eta} \to 0$$

which is isomorphic to (3.4).

Note that since $\mathcal{F}'_\eta \cong (\mathcal{F}_1)_\eta$ and $\mathcal{F}'_\eta$ is a line bundle, it follows that $\mathrm{rk}\ \mathcal{F}_1 = 1$.

Let $\mathcal{F}'_S$ be the saturation of $\mathcal{F}_1$ in $\mathcal{F}_S$. The cokernel $\mathcal{J}_S$ is isomorphic to the quotient sheaf $\mathcal{J}_1/t(\mathcal{J}_1)$, where $t(\mathcal{J}_1)$ is the torsion subsheaf of $\mathcal{J}_1$. There is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_S & \longrightarrow & \mathcal{J}_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}'_S & \longrightarrow & \mathcal{F}_S & \longrightarrow & \mathcal{J}_S & \longrightarrow & 0
\end{array}$$

We prove that the sequence

$$0 \to \mathcal{F}'_S \to \mathcal{F}_S \to \mathcal{J}_S \to 0 \tag{3.5}$$

is the canonical sequence of $\mathcal{F}_S$.

We notice first that the restriction to $\mathbb{P}^1 \times C \times \{\eta\}$ gives the canonical extension of $\mathcal{F}_\eta$. This follows from the fact that $\mathcal{J}_{1\eta}$ is torsion free. We have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{F}_1)_\eta & \longrightarrow & \mathcal{F}_\eta & \longrightarrow & (\mathcal{J}_1)_\eta & \longrightarrow & 0 \\
\downarrow & & \cong\downarrow & & \| & & \downarrow\cong & & \downarrow \\
0 & \longrightarrow & (\mathcal{F}'_S)_\eta & \longrightarrow & \mathcal{F}_\eta & \longrightarrow & (\mathcal{J}_S)_\eta & \longrightarrow & 0
\end{array}$$

Note rk $\mathcal{F}'_S =$ rk $\mathcal{F}_1 = 1$. Moreover, since $\mathcal{J}_S$ is torsion-free and $\mathcal{F}_S$ is a vector bundle, it follows by Lemma A.10 in the Appendix that $\mathcal{F}'_S$ is a reflexive sheaf. Since a reflexive sheaf of rank 1 is a line bundle, it follows that

$$\mathcal{F}'_S \cong O(\alpha) \boxtimes \mathcal{N}_S$$

for some line bundle $\mathcal{N}_S$ on $C \times S$ and some integer α. From

$$(\mathcal{F}'_S)_\eta \cong (\mathcal{F}_1)_\eta \cong \mathcal{F}'_\eta \cong O(a) \boxtimes \mathcal{N}_\eta$$

we have $\alpha = a$ and $(\mathcal{N}_S)_\eta \cong \mathcal{N}_\eta$. As $\mathcal{J}_S$ is a torsion free sheaf, by Lemma 3.12 , the sequence (3.5) is the canonical sequence of $\mathcal{F}_S$, up to multiplication by a scalar. Since restriction to $\mathbb{P}^1 \times C \times \{\eta\}$ gives the canonical sequence of $\mathcal{F}_\eta$, it follows that (3.5) is the canonical sequence of $\mathcal{F}_S$.

Let $s \in S$. Consider the canonical sequence of the bundle $\mathcal{F}_s$:

$$0 \to \mathcal{F}'_s \to \mathcal{F}_s \to \mathcal{J}_s \to 0$$

where $\mathcal{F}'_s \cong O(a_s) \boxtimes \mathcal{L}_s$ for some integer a_s and $\mathcal{L}_s$ is the vector bundle $p^s_{2*}\mathcal{F}_s(-a)$.

There is a canonical morphism of line bundles:

$$(\mathcal{F}'_S)_s \to \mathcal{F}'_s$$

This is a non-zero map of line bundles, hence, it is injective. It follows that by restriction to $\mathbb{P}^1 \times C \times \{s\}$ we get an exact sequence:

$$0 \to (\mathcal{F}'_S)_s \to \mathcal{F}_s \to (\mathcal{J}_S)_s \to 0 \tag{3.6}$$

There is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{F}'_S)_s & \longrightarrow & \mathcal{F}_s & \longrightarrow & (\mathcal{J}_S)_s & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}'_s & \longrightarrow & \mathcal{F}_s & \longrightarrow & \mathcal{J}_s & \longrightarrow & 0
\end{array} \tag{3.7}$$

As $\mathcal{F}'_s$ is the canonical subbundle of $\mathcal{F}_s$, we have that there is $\mathcal{L}_s$, a bundle on $\mathbb{P}^1 \times C \times \{s\}$, such that:

$$\mathcal{F}'_s \cong O(a_s) \boxtimes \mathcal{L}_s$$

We have:

$$\mathcal{F}'_S \cong O(a) \boxtimes \mathcal{N}_S$$

It follows that $a_s \geq a > \frac{k}{2}$ (which also follows from upper-semicontinuity – note that it follows that $\mathcal{L}_s$ is a line bundle) and that there is an injective morphism

$$(\mathcal{N}_S)_s \to \mathcal{L}_s \tag{3.8}$$

This is an isomorphism if $s = \eta \in S$.

Note that since $\mathcal{N}_S$ is a line bundle on $C \times S$, there is an integer d such that deg $(\mathcal{N}_S)_s = d$ for any $s \in S$. In particular, $d =$ deg $(\mathcal{N}_S)_\eta =$ deg $\mathcal{L}_\eta$.

By previous Lemma, there is a dense open $S^0 \subset S$ on which the degree of the line bundle $\mathcal{L}_s = p^s_{2*}\mathcal{F}_s(-a)$ is constant. In particular, since the degree of the line bundle $p^\eta_{2*}\mathcal{F}_\eta(-a) = \mathcal{L}_\eta$ is d, it follows that deg $\mathcal{L}_s = d$ for $s \in S^0$. Since deg $(\mathcal{N}_S)_s = d$, it follows that (3.8) is an isomorphism.

The commutative diagram (3.7) is an isomorphism of exact sequences and the exact sequence (3.6) is, up to multiplication by a scalar, the canonical sequence of $\mathcal{F}_s$. By the construction of $\mathcal{F}'$, we see that, in fact, this is the canonical sequence of $\mathcal{F}_s$. This proves i.

Part ii. is imediate: if $s \in S \setminus S^0$, then it follows from the injective morphism (3.8) that

$$d_0 = \text{deg } (\mathcal{L}_0) \geq \text{deg } (\mathcal{N}_S)_s = d.$$

□

Side Note

Since $\mathcal{J}_S$ is torsion free of rank 1, there is a lci closed subscheme $\mathcal{D}$ of $\mathbb{P}^1 \times C \times S$, of codimension at least 2 and a line bundle $\mathcal{M}_S$, such that if $\mathcal{I}_\mathcal{D}$ is the ideal sheaf of $\mathcal{D}$, then:

$$\mathcal{J} \cong \mathcal{I}_\mathcal{D} \otimes \mathcal{M}_S.$$

Let $\mathcal{D}_s$ be the scheme theoretic intersection $\mathcal{D} \cap \mathbb{P}^1 \times C \times \{s\}$. There are exact sequences:

$$0 \to \mathcal{T}or_1^{O_{\mathbb{P}^1\times C\times S}}(O_\mathcal{D}, O_{\mathbb{P}^1\times C\times\{s\}}) \to (\mathcal{I}_\mathcal{D})_s \to O \to O_{\mathcal{D}_s} \to 0$$

$$0 \to \mathcal{T}or_1^{O_{\mathbb{P}^1\times C\times S}}(O_\mathcal{D}, O_{\mathbb{P}^1\times C\times\{s\}}) \to (\mathcal{I}_\mathcal{D})_s \to \mathcal{I}_{\mathcal{D}_s} \to 0$$

The sheaf
$$\mathcal{T}_s = \mathcal{T}or_1^{O_{\mathbb{P}^1\times C\times S}}(O_{\mathcal{D}}, O_{\mathbb{P}^1\times C\times\{s\}})$$
is supported on the set $\mathcal{D}_s$.

Note from (3.6) that for any $s \in S$ the sheaf $(\mathcal{J}_S)_s$ has rank 1. Hence, $(\mathcal{I}_{\mathcal{D}})_s$ has rank 1. Since for any $s \in S$ the sheaf $\mathcal{I}_{\mathcal{D}_s}$ is a torsion-free sheaf of rank 1, it follows that the sheaf $\mathcal{T}_s$ is a torsion sheaf; in fact, it is the torsion subsheaf of $(\mathcal{I}_{\mathcal{D}})_s$.

Moreover, it is straightforward to prove that for any $s \in S$ the sheaf $\mathcal{T}_s$ is 0 if and only if $\mathcal{D}$ is flat at the points in $\mathcal{D}_s$.

For $s \in S^0$ we have $(\mathcal{J}_S)_s \cong \mathcal{J}_s$; hence, $(\mathcal{I}_{\mathcal{D}})_s \cong \mathcal{I}_{Z_s}$ where $Z_s \subset \mathbb{P}^1 \times C \times \{s\}$ is such that:
$$\mathcal{J}_s \cong \mathcal{I}_{Z_s} \otimes \mathcal{M}_s.$$
It follows that the sheaf $(\mathcal{I}_{\mathcal{D}})_s$ is torsion free of rank 1 for any $s \in S^0$. By previous comments, it follows that $\mathcal{T}_s = 0$ and $(\mathcal{I}_{\mathcal{D}})_s \cong \mathcal{I}_{\mathcal{D}_s}$. In particular, $\mathcal{D}_s = Z_s$.

It also follows that $\mathcal{D}$ is flat over S^0. It might be that $\mathcal{D}$ does not dominate S and in that case $\mathcal{D}_s = \emptyset$ for $s \in S^0$. Otherwise, over S^0, $\mathcal{D}$ is a codminension 2 lci cycle and it maps finitely onto S^0.

Let $s_0 \in S\setminus S^0$. The restriction of the canonical sequence of $\mathcal{F}_S$ to $\mathbb{P}^1 \times C \times \{s_0\}$ does not give the canonical sequence of $\mathcal{F}_{s_0}$. This is equivalent to any of the following:

i. The injective morphism $\mathcal{F}'_{S s_0} \to \mathcal{F}_{s_0}$ is not an isomorphism.

ii. The sheaf$(\mathcal{J}_S)_{s_0}$ is not torsion-free (see (3.7)

iii. The sheaf $(\mathcal{I}_{\mathcal{D}})_{s_0}$ is not torsion-free

iv. The torsion sheaf $\mathcal{T}_{s_0}$ is not zero

v. $\mathcal{D}$ is not flat over the point s_0.

We can assume that S is a curve, as for $s_0 \in S \setminus S^0$, we can replace S with an irreducible curve in S which passes through s_0 while intersecting S^0. Then $\mathcal{D} \subseteq \mathbb{P}^1 \times C \times S$ has dimension ≤ 1.

Consider the commutative diagram:
$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{F}'_S)_{s_0} & \longrightarrow & \mathcal{F}_{s_0} & \longrightarrow & (\mathcal{J}_S)_{s_0} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}'_{s_0} & \longrightarrow & \mathcal{F}_{s_0} & \longrightarrow & \mathcal{J}_{s_0} & \longrightarrow & 0
\end{array}$$
It follows that the lower sequence in the diagram (the canonical sequence of $\mathcal{F}_{s_0}$) is the saturation of the upper sequence. Hence,
$$\mathcal{F}'_{s_0} \cong (\mathcal{F}'_S)_{s_0}(E)$$

where E is an effective divisor on $\mathbb{P}^1 \times C \times \{s_0\}$ and we have the exact sequence:

$$0 \to (\mathcal{F}'_S)_{s_0} \to \mathcal{F}'_{s_0} \to O_E \to 0.$$

By the snake lemma, it follows that

$$0 \to O_E \to (\mathcal{J}_S)_{s_0} \to \mathcal{J}_{s_0} \to 0.$$

Hence, the torsion part of $(\mathcal{I}_\mathcal{D})_{s_0}$ is isomorphic to O_E. But the torsion part of $(\mathcal{I}_\mathcal{D})_{s_0}$ is $\mathcal{T}_{s_0}$; hence $T_{s_0} = O_E$. In particular, E has support in $\mathcal{D}_{s_0}$.

We already know that since E is an effective divisor of numerical class

$$n(\{pt\} \times C) + m(\mathbb{P}^1 \times \{pt\}) \in \mathrm{NS}(\mathbb{P}^1 \times C)$$

with $n, m \geq 0$, it follows that $\mathcal{F}_{s_0}$ has canonical line subbundle of type (a_0, d_0), where $a_0 = a + n \geq a$ and $d_0 = d + m \geq d$.

3.2.3 Families of vector bundles in Case E

In this section we will work with families of vector bundles on $\mathbb{P}^1 \times C$ with equal generic splitting (a, a). Since we can tensor $\mathcal{F}$ with $O(-a)$, we can assume $a = 0$.

As before, S will be an integral scheme. We can adapt the proof of Proposition 3.19 to prove the following lemma.

Lemma 3.20. *Let $\mathcal{F}_S$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$. Assume that the bundle $\mathcal{F}_S$ has generic fiber type $(0,0)$. Then there is a dense open $S^0 \subseteq S$ such that:*

i. For any $s \in S^0$, the restriction of the canonical sequence of $\mathcal{F}_S$ to $\mathbb{P}^1 \times C \times \{s\}$ is the canonical sequence of $\mathcal{F}_s$. In particular, there is an integer d such that for $s \in S^0$ the canonical subbundle of $\mathcal{F}_s$ has type (a, d).

ii. If $s_0 \in S \setminus S^0$, then $\mathcal{F}_{s_0}$ has canonical line *subbundle of fiber type $a > 0$.*

Stability

Lemma 3.21. *Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$. Assume that for any s in S and for any $c \in C$ the bundle $\mathcal{F}_{s,c}$ splits either as $O \oplus O$ or $O(1) \oplus O(-1)$. Then there is an open $S^0 \subseteq S$ (possibly empty) such that for any $s \in S^0$, $p_{2*}\mathcal{F}_s$ is a rank* 2 stable *bundle.*

Proof. We claim that for any $s \in S$ the canonical morphism

$$\phi : p^s_{2*}\mathcal{F}_s \cong (p_{2,3_*}\mathcal{F})_s$$

is in fact an isomorphism.

This is because for any $s \in S$ and $c \in C$ we have

$$\mathrm{H}^1(\mathcal{F}_{s,c}) = 0.$$

By an argument similar to the proof of Lemma 3.18, it follows that the morphism ϕ is an isomorphism on fibers over C, hence, an isomorphism.

By Fact 1.9, the locus of those $s \in S$ such that $(p_{2,3_*}\mathcal{F})_s$ is stable is open (possibly empty) and our lemma follows. □

Complete families of vector bundles

We will adopt some terminology from [P]. If T and X are integral *smooth* schemes and $\mathcal{F}$ is a vector bundle on $X \times T$, then we say that $\mathcal{F}$ gives a *complete* family of vector bundles on X if the Kodaira-Spencer map

$$\omega : T_tT \to \mathrm{Ext}^1_X(\mathcal{F}_t, \mathcal{F}_t)$$

is surjective for any $t \in T$.

We will use the following fact from [P], p.206.

Fact 3.22. *Let T be an integral smooth scheme and let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times T$. Assume $\mathcal{F}$ is a* complete *family of vector bundles on $\mathbb{P}^1 \times T$. Then the points $t \in T$ such that $\mathcal{F}_t$ splits as $O(\alpha) \oplus O(\beta)$ with $|\alpha - \beta| \geq 3$ form a closed set of codimension at least 2.*

Lemma 3.23. *Assume that S is smooth and let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$ with generic fiber type $(0,0)$. Assume that $\mathcal{F}$ is a* complete *family of vector bundles on $\mathbb{P}^1$. Then there is a dense open $S^0 \subset S$ such that for $s \in S^0$ and for any $c \in C$ we have that $\mathcal{F}_{s,c}$ splits either as $O \oplus O$ or $O(1) \oplus O(-1)$.*

Proof. Let $Z \subset S \times C$ the locus of those $(s,c) \in S \times C$ such that

$$\mathrm{H}^0(\mathcal{F}_{s,c}(-2)) \neq 0.$$

By upper semicontinuity, the locus Z is closed in $S \times C$. By Fact 3.22, the codimension of Z in $S \times C$ is at least 2. Let S' be the image of Z in S. The codimension of S' in S is at least 1. Take $S^0 = S \setminus S'$. This is our required dense open set. □

Note that if S is a smooth integral scheme and $\mathcal{F}$ is a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$, then if we fix any point $(c,s) \in C \times S$, we have a commutative diagram:

$$\begin{array}{ccc} T_sS & \longrightarrow & \mathrm{Ext}^1_{\mathbb{P}^1\times C}(\mathcal{F}_s,\mathcal{F}_s) \\ & \searrow \quad \swarrow & \\ & \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{F}_{s,c},\mathcal{F}_{s,c}) & \end{array}$$

whith two Kodaira-Spencer maps appearing, one given by the family $\mathcal{F}_c$ of vector bundles on $\mathbb{P}^1$, and another given by the family $\mathcal{F}$ of vector bundles on $\mathbb{P}^1 \times C$:

$$T_sS \to \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{F}_{s,c},\mathcal{F}_{s,c}) \tag{3.9}$$

$$T_sS \to \mathrm{Ext}^1_{\mathbb{P}^1\times C}(\mathcal{F}_s,\mathcal{F}_s) \tag{3.10}$$

The following map is given by restriction to $\mathbb{P}^1 \times \{c\}$:

$$\mathrm{Ext}^1_{\mathbb{P}^1\times C}(\mathcal{F}_s,\mathcal{F}_s) \to \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{F}_{s,c},\mathcal{F}_{s,c}) \tag{3.11}$$

Lemma 3.24. *Let $\mathcal{G}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C$ with generic fiber type $(0,0)$. Then the following restriction map is surjective for any closed point $c \in C$:*

$$Ext^1_{\mathbb{P}^1\times C}(\mathcal{G},\mathcal{G}) \to Ext^1_{\mathbb{P}^1}(\mathcal{G}_c,\mathcal{G}_c)$$

Proof. It is enough to prove that the following map given by restriction is surjective:

$$\mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd(\mathcal{G})) \to \mathrm{H}^1(\mathbb{P}^1 \times \{c\}, \mathcal{E}nd(\mathcal{G}_c))$$

Equivalently, if we let $\mathcal{I}_c$ to be the ideal sheaf of $\mathbb{P}^1 \times \{c\}$, it is enough to prove that

$$\mathrm{H}^2(\mathbb{P}^1 \times C, \mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c) = 0$$

We have

$$\mathrm{H}^2(\mathbb{P}^1 \times C, \mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c) \cong \mathrm{H}^1(C, R^1p_{2*}(\mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c)).$$

Let $\xi \in C$ be the generic point. We have:

$$R^1p_{2*}(\mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c)_\xi \cong \mathrm{H}^1(\mathbb{P}^1_\xi, \mathcal{E}nd(\mathcal{G}_\xi).$$

Since the generic splitting of $\mathcal{G}$ is $(0,0)$, we have $\mathcal{G}_\xi \cong \mathcal{O} \oplus \mathcal{O}$ and $\mathcal{E}nd(\mathcal{G}_\xi) \cong \mathcal{O}^4$. It follows that $R^1p_{2*}(\mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c)_\xi = 0$, hence, the sheaf $R^1p_{2*}(\mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c)$ is a torsion sheaf on C. It is supported on a finite set of points of C; it follows that

$$\mathrm{H}^1(C, R^1p_{2*}(\mathcal{E}nd(\mathcal{G}) \otimes \mathcal{I}_c)) = 0.$$

□

We apply Lemma 3.24 to $\mathcal{G} = \mathcal{F}_s$.

Corollary 3.25. *Assume that S is smooth and let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^1 \times C \times S$ with generic fiber type $(0,0)$. Then $\mathcal{F}_c$ is a complete family of vector bundles on $\mathbb{P}^1$ for any $c \in C$ if and only if the family $\mathcal{F}$ of vector bundles on $\mathbb{P}^1 \times C$ is complete.*

3.3 The moduli scheme in Case U

We construct moduli spaces for rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ with the properties in Lemma 3.1 and with $\mathcal{F}$ of type U.

A bit of terminology: we say that a line bundle $\mathcal{M}$ on $\mathbb{P}^1 \times C$ has *type* (a, d) if

$$\mathcal{M} \cong O(a) \boxtimes \mathcal{L},$$

where $\mathcal{L}$ is a line bundle on C of degree d.

Let

$$c_1 = k\{pt\} \times C + \mathbb{P}^1 \times \{x_0\} \in \mathrm{A}^1(\mathbb{P}^1 \times C) \quad \text{and} \quad c_2 = k \in \mathbb{Z}$$

We will consider vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ with $c_1(\mathcal{F}) = c_1$ and deg $(c_2(\mathcal{F})) = c_2$.

Let a and e be integers such that $a > \frac{k}{2}, e \geq 0$. Our goal is to construct moduli schemes for the following contravariant functors:

$$F : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets} \tag{3.12}$$

$F(S) = \{$Isomorphism classes $[\mathcal{F}]$ of vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C \times S$, such that $\forall s \in S$, $\mathcal{F}_s$ has Chern classes c_1 and c_2 and canonical subbundle of type $(a, -e)\}$

$$F^0 : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets} \tag{3.13}$$

$F^0(S) = \{$Isomorphism classes $[\mathcal{F}]$ of vector bundles $\mathcal{F}$, such that $[\mathcal{F}] \in F(S)$ and $\forall (p, s) \in \mathbb{P}^1 \times S$ the vector bundle $\mathcal{F}_{p,s}$ is stable$\}$

Note that the functor F^0 is an open subfunctor of F. The functor F has a fine moduli scheme $\mathfrak{B}(a, e)$, which is constructed in [BR2]. We prove that the functor F^0 has a fine moduli scheme a *dense* open subscheme $\mathfrak{B}^0(a, e)$ of $\mathfrak{B}(a, e)$.

3.3.1 Some computations with Chern classes

Lemma 3.26. *Consider the exact sequence of sheaves on $\mathbb{P}^1 \times C$:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{M} \to 0.$$

where $\mathcal{F}'$ and $\mathcal{M}$ are line bundles and Z a 0-cycle. Then we have:

$$c_1(\mathcal{F}) = c_1(\mathcal{F}') + c_1(\mathcal{M}), \quad \text{and} \quad c_2(\mathcal{F}) = Z + c_1(\mathcal{F}')c_1(\mathcal{M})$$

Lemma 3.27. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}^1 \times C$ with generic fiber type $(a, k-a)$ (where $a \geq \frac{k}{2}$) and Chern classes c_1 and c_2. Assume there is an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{M} \to 0$$

with $\mathcal{F}'$ a line subbundle of type $(a, -e)$ (where $e \in \mathbb{Z}$), $\mathcal{M}$ a line bundle and $\mathcal{I}_Z$ the ideal sheaf of a 0-cycle Z. We have:

$$\mathcal{F}' \cong O(a) \boxtimes \mathcal{L}, \quad \text{deg } (\mathcal{L}) = -e \quad \text{and} \quad \mathcal{M} \cong O(k-a) \boxtimes \mathcal{L}^{-1}(x_0)$$

Let $\delta =$ length (Z). Then

$$\delta = (k-a) - e(2a-k)$$

Lemma 3.28. *Assume we are under the same assumption as in Lemma 3.27. In addition, assume that the bundle $\mathcal{F}$ is such that $\mathcal{F}_p$ is stable for any $p \in \mathbb{P}^1$. Then $e \geq 0$. Moreover, if $\delta > 0$ then $e > 0$.*

Proof. We have that $\mathcal{L}$ is a line subbundle of $\mathcal{F}_p$ for any closed $p \in \mathbb{P}^1$. As $\mathcal{F}_p$ is stable with determinant $O_C(x_0)$ it follows that $-e = \deg(\mathcal{L}) \leq 0$.

If $\delta > 0$ then let $p \in \mathbb{P}^1$ be a point such that $(\{p\} \times C) \cap Z \neq \emptyset$. Let D be the 0-cycle on C such that $(\{p\} \times C) \cap Z = \{p\} \times D$. By restriction to $\{p\} \times C$ we get an exact sequence:

$$0 \to \mathcal{L} \to \mathcal{F}_p \to \mathcal{L}^{-1}(x_0 - D) \oplus \mathcal{L}^{-1}(x_0)_{|D} \to 0$$

If we take the saturation of the line subbundle $\mathcal{L}$ of $\mathcal{F}_p$, we get an exact sequence:

$$0 \to \mathcal{L}(D) \to \mathcal{F}_p \to \mathcal{L}^{-1}(x_0 - D) \to 0$$

It follows that $\mathcal{F}_p$ has a line subbundle of degree $-e + \deg(D)$. If $e = 0$ then since $\deg D > 0$ we have a contradiction of the stability of $\mathcal{F}_p$. Hence, $e > 0$. □

Remark 3.29. *If for integers a and e such that $a > \frac{k}{2}$ and $e \geq 0$ there exists a vector bundle $\mathcal{F}$ on $\mathbb{P}^1 \times C$ with Chern classes c_1 and c_2 and canonical line subbundle of type $(a, -e)$, then we must have $\delta \geq 0$. Equivalently:*

$$\frac{k-a}{2a-k} \geq e \geq 0.$$

In particular, we must have $a \leq k$.

3.3.2 The moduli schemes $\mathfrak{B}(a,e)$

Fact 3.30. *[BR2] For each pair of integers a and e such that $k \geq a > \frac{k}{2}, e \geq 0$, there exists a quasi-projective integral scheme $\mathfrak{B}$ and a universal rank 2 bundle $\mathcal{F}_{\mathfrak{B}}$ on $\mathbb{P}^1 \times C \times \mathfrak{B}$ which makes $\mathfrak{B}$ into a fine moduli scheme for the functor F of (3.12). More precisely, if S is a scheme and $\mathcal{F}_S$ is a family of rank 2 bundles on $\mathbb{P}^1 \times C \times S$ with Chern classes c_1, c_2 and with canonical line subbundle of type $(a,-e)$, then there is a unique morphism*

$$\nu : S \to \mathfrak{B}$$

such that $\mathcal{F}_S \cong \nu^* \mathcal{F}_{\mathfrak{B}} \otimes \mathcal{N}$, for some line bundle $\mathcal{N}$ on S.

We denote by $\mathfrak{B}(a,e)$ the moduli scheme $\mathfrak{B}$ for the given integers a and e.

The previous fact is a consequence of the construction in [BR2]. We fixed the Chern class c_1 as an element of $\mathrm{A}^1(\mathbb{P}^1 \times C)$. If we fix c_1 only up to numerical equivalence, hence, as an element of $\mathrm{NS}(\mathbb{P}^1 \times C)$, then we get the moduli scheme $\mathfrak{B}$ of [BR2]. More precisely, the scheme $\mathfrak{B}$ of [BR2] has a canonical map:

$$\pi : \mathfrak{B} \to \mathfrak{S}, \quad \text{where} \quad \mathfrak{S} = \mathrm{Pic}^{-e}(C) \times \mathrm{Pic}^{e+1}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C).$$

and $\mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$ is the Hilbert scheme parametrizing lci 0-cycles on $\mathbb{P}^1 \times C$ of degree δ. The map π sends the point $\mathcal{F}$ to the point $(\mathcal{L}, \mathcal{M}, Z)$.

The scheme $\mathfrak{B}$ that we construct is the preimage under π of the subscheme of $\mathfrak{S}$ given by the closed immersion:

$$\begin{gathered}\mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C) \hookrightarrow \mathrm{Pic}^{-e}(C) \times \mathrm{Pic}^{e+1}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C) \\ (\mathcal{L}, Z) \mapsto (\mathcal{L}, \mathcal{L}^{-1}(x_0), Z)\end{gathered}$$

Outline of the construction of $\mathfrak{B}$

This follows closely [BR2]. Let (a,e) be a pair of integers such that $a \geq \frac{k}{2}$ and $e \geq 0$.

We want to construct a scheme $\mathfrak{B}(a,e)$ whose points correspond to vector bundles $\mathcal{F}$ sitting in an extension:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{M} \to 0$$

as in Lemma 3.27. By Remark 3.29, we need to restrict our attention to pairs of integers a and e in the range:

$$a \geq \frac{k}{2}, \quad e \geq 0, \quad \delta \geq 0$$

where δ is the length of the 0-cycle Z:

$$\delta = (k-a) - e(2a-k).$$

Note that we allow the case $a = \frac{k}{2}$. Even if not part of Case U, this will be useful when analyzing Case E.

We start by constructing a space $\mathfrak{B}'$ parametrizing extensions:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{I}_Z \otimes \mathcal{M} \to 0 \tag{3.14}$$

where $\mathcal{L} \in \mathrm{Pic}^{-e}(C)$, Z is an element in the Hilbert scheme $\mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$, parametrizing lci 0-cycles on $\mathbb{P}^1 \times C$ of degree δ, and $\mathcal{F}'$ and $\mathcal{M}$ are given by:

$$\mathcal{F}' \cong O(a) \boxtimes \mathcal{L}, \quad \mathcal{M} \cong O(k-a) \boxtimes \mathcal{L}^{-1}(x_0)$$

Let $\mathfrak{S} = \mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$. Let $\mathcal{L}_{\mathfrak{S}}$ be the pull back to $\mathbb{P}^1 \times C \times \mathfrak{S}$ of a Poincaré bundle from $C \times \mathrm{Pic}^{-e}(C)$. On $\mathbb{P}^1 \times C \times \mathfrak{S}$ let:

$$\mathcal{F}'_{\mathfrak{S}} = p_1^* O(a) \otimes \mathcal{L}_{\mathfrak{S}}, \quad \mathcal{M}_{\mathfrak{S}} = p_1^* O(k-a) \otimes \mathcal{L}_{\mathfrak{S}}^{-1} \otimes p_2^* O(x_0)$$

Let $Z_{\mathfrak{S}} \subset \mathbb{P}^1 \times C \times \mathfrak{S}$ be the universal subscheme coming from $\mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$. Define $\mathcal{S}$ to be the relative extension sheaf:

$$\mathcal{S} = \mathcal{E}xt^1_{\mathbb{P}^1 \times C \times \mathfrak{S}/\mathfrak{S}}(\mathcal{I}_{Z_{\mathfrak{S}}} \otimes \mathcal{M}_{\mathfrak{S}}, \mathcal{F}'_{\mathfrak{S}}).$$

The sheaf $\mathcal{S}$ is a locally free sheaf on $\mathbb{P}^1 \times C \times \mathfrak{S}$. If $u = (\mathcal{L}, Z) \in \mathfrak{S}$, then we have that $\mathcal{S}_u \cong \mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}')$.

Define $\mathfrak{B}' = \mathbb{P}(\mathcal{S})$ and let $p' : \mathfrak{B}' \to \mathfrak{S}$ be the canonical map.

We construct now the subscheme $\mathfrak{B} \subset \mathfrak{B}'$ parametrizing extensions (3.14), such that $\mathcal{F}$ is locally free.

Consider the sheaves $\mathcal{F}'_{\mathfrak{B}'}$, $\mathcal{I}_{Z_{\mathfrak{B}'}}$ and $\mathcal{M}_{\mathfrak{B}'}$ on $\mathbb{P}^1 \times C \times \mathfrak{B}'$, which are the pull-backs of the corresponding sheaves on $\mathbb{P}^1 \times C \times \mathfrak{S}$. There is a universal extension:

$$0 \to \mathcal{F}'_{\mathfrak{B}'}(1) \to \mathcal{F}_{\mathfrak{B}'} \to \mathcal{I}_{Z_{\mathfrak{B}'}} \otimes \mathcal{M}_{\mathfrak{B}'} \to 0$$

where the sheaf $O(1)$ stands for the sheaf $O_{\mathbb{P}(\mathcal{S})}(1)$.

There is a dense open subscheme $\mathfrak{B}$ of $\mathfrak{B}'$, such that $\mathcal{F}_{\mathfrak{B}'}$ restricts to a locally free sheaf $\mathcal{F}_{\mathfrak{B}}$ on $\mathfrak{B}$. The scheme $\mathfrak{B} \subset \mathfrak{B}'$ is dense in every fiber of $p' : \mathfrak{B} \to \mathfrak{S}$. This completes the construction of the scheme $\mathfrak{B}$.

In the case $a > \frac{k}{2}$, a vector bundle $\mathcal{F}$, with Chern classes c_1, c_2 and canonical line subbundle of type $(a, -e)$, determines and is uniquely determined by a point in $\mathfrak{B}$. It follows that $\mathfrak{B}$ is a fine moduli scheme for the functor F (see [BR2]).

Note that in the case $a = \frac{k}{2}$, the scheme $\mathfrak{B}$ still exists and it has a universal bundle $\mathcal{F}_{\mathfrak{B}}$. Note that in this case we have $\delta = a$.

Note 3.31. *If $\delta = 0$ then $\mathfrak{B} = \mathfrak{B}'$.*

Proof. If $\delta = 0$ then $Z = \emptyset$. For any extension of line bundles

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{M} \to 0$$

the sheaf $\mathcal{F}$ is locally free, it follows that $\mathfrak{B} = \mathfrak{B}'$. □

Dimension of $\mathfrak{B}(a, e)$

We compute the dimension of the scheme $\mathfrak{B} = \mathfrak{B}(a, e)$. We have:

$$\dim\, \mathfrak{B} = \dim\, \mathfrak{B}' = \dim\, \mathfrak{S} + N - 1 = g + 2\delta + N - 1$$

where $N = \dim\, \mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}')$.

We compute N from the exact sequence in Lemma 3.10:

$$N = \dim\, \mathrm{Ext}^1(\mathcal{M}, \mathcal{F}') + \dim\, \mathrm{H}^0(Z, O_Z) = \dim\, \mathrm{Ext}^1(\mathcal{M}, \mathcal{F}') + \delta.$$

We have:

$$\mathrm{Ext}^1(\mathcal{M}, \mathcal{F}') \cong \mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{F}' \otimes \mathcal{M}^*).$$

This vector space has dimension $-\chi(\mathcal{F}' \otimes \mathcal{M}^*)$, since for $i = 0, 2$ we have

$$\mathrm{H}^i(\mathbb{P}^1 \times C, \mathcal{F}' \otimes \mathcal{M}^*) = 0.$$

This is because

$$\begin{aligned}
\mathcal{F}' \otimes \mathcal{M}^* &\cong O(2a - k) \boxtimes \mathcal{L}^2(-x_0) \quad \text{and} \\
\mathrm{H}^0(O(2a-k) \boxtimes \mathcal{L}^2(-x_0)) &\cong \mathrm{H}^0(\mathbb{P}^1, O(2a-k)) \otimes \mathrm{H}^0(C, \mathcal{L}^2(-x_0)) = 0 \\
\mathrm{H}^2(O(2a-k) \boxtimes \mathcal{L}^2(-x_0)) &\cong \mathrm{H}^0(O(-2-2a+k) \boxtimes (K_C \otimes \mathcal{L}^{-2}(x_0))) \\
&\cong \mathrm{H}^0(\mathbb{P}^1, O(-2-2a+k)) \otimes \mathrm{H}^0(C, K_C \otimes \mathcal{L}^{-2}(x_0)) = 0
\end{aligned}$$

since $\deg\, \mathcal{L}^2(-x_0) = -2e - 1 < 0$ and $2a - k \geq 0$

We compute $\chi(\mathcal{F}' \otimes \mathcal{M}^*)$. Let $\mathrm{NS}(\mathbb{P}^1 \times C) = \mathbb{Z}[F] \oplus \mathbb{Z}[\Sigma]$ be the Neron-Severi group of $\mathbb{P}^1 \times C$, where $[F]$ and $[\Sigma]$ are the classes of a fiber F, respectively a section Σ, of $p_2 : \mathbb{P}^1 \times C \to C$.

Let $D = (2a-k)[\Sigma] - (2e+1)[F]$. By Riemann-Roch, we have

$$\chi(D) = \frac{D^2 - D.K}{2} + 1 - g.$$

Then as $K = -2[\Sigma] + (2g-2)[F]$, we have:

$$\begin{aligned}
D^2 &= -2(2a-k)(2e+1) \\
D.K &= (2a-k)(2g-2) + 2(2e+1) \\
\chi(D) &= -(2a-k+1)(2e+g) N = (2a-k+1)(2e+g) + \delta
\end{aligned}$$

Hence, the dimension N of $\mathrm{Ext}(\mathcal{I}_Z \otimes \mathcal{M}, \mathcal{F}')$ is:

$$N = (2a - k + 1)(2e + g) + \delta = (2a - k + 1)g + (2a - k + 2)e + (k - a) \quad (3.15)$$

Therefore, the dimension of $\mathfrak{B}(a, e)$ is

$$\dim \mathfrak{B}(a, e) = 2\delta + g + N - 1 = (2a - k + 2)g + (3k - 3a - 1) - e(2a - k - 2). \quad (3.16)$$

Note that if we take $k = 2a$ and $e \geq 0$ then $\delta = a$ and we have:

$$\dim \mathfrak{B}(\frac{k}{2}, e) = 3\frac{k}{2} + 2g + 2e - 1. \quad (3.17)$$

Note that since we are working in the range $k \geq a \geq \frac{k}{2}$, $e \geq 0$ and $\delta \geq 0$, we have $N \geq g \geq 2$ and $\dim \mathfrak{B}(a, e) \geq (2g - 1)$. In particular, the schemes $\mathfrak{B}(a, e)$ that we constructed are non-empty.

3.3.3 The good locus $\mathfrak{B}^0(a, e)$ in $\mathfrak{B}(a, e)$

We constructed a non-empty integral quasiprojective scheme $\mathfrak{B} = \mathfrak{B}(a, e)$ and a universal bundle $\tilde{\mathcal{F}} = \mathcal{F}_{\mathfrak{B}}$ for pairs of integers a and e in the following range:

$$a \geq \frac{k}{2}, e \geq 0, \delta \geq 0.$$

We say that a closed point $b \in \mathfrak{B}$ is *good* if the vector bundle $\tilde{\mathcal{F}}_b$ on $\mathbb{P}^1 \times C$ has the property that for any $p \in \mathbb{P}^1$ the bundle $\tilde{\mathcal{F}}_{p,b}$ is stable (equivalently, $\tilde{\mathcal{F}}_b$ induces a morphism $f : \mathbb{P}^1 \to M$). We let $\mathfrak{B}^0 \subset \mathfrak{B}$ be the set of good points.

Consider the locus:

$$\mathcal{Y} = \{(p, b) \in \mathbb{P}^1 \times \mathfrak{B} | \text{ the bundle } \tilde{\mathcal{F}}_{p,b} \text{ is not stable } \}$$

By Fact 1.9, $\mathcal{Y}$ is closed in $\mathbb{P}^1 \times \mathfrak{B}$. Note that $\mathfrak{B}^0$ is the complement of the image of $\mathcal{Y}$ via the projection $\pi : \mathbb{P}^1 \times \mathfrak{B} \to \mathfrak{B}$. Therefore, the set of good points $\mathfrak{B}^0$ is open in $\mathfrak{B}$, but possibly empty.

Note that if $a > \frac{k}{2}$ then the open subscheme $\mathfrak{B}^0$ of $\mathfrak{B}$ is a fine moduli scheme for the open subfunctor (3.13) F^0 of F. (This is not the case anymore when $a = \frac{k}{2}$.)

Theorem 3.32. *Let $a \geq \frac{k}{2}$. The open $\mathfrak{B}^0(a, e)$ is a non-empty, hence, dense, in $\mathfrak{B}(a, e)$ in either of the following cases:*

i. $e > 0$ *and* $\delta \geq 0$

ii. $e = 0$ *and* $\delta = 0$*; equivalently,* $(a, e) = (k, 0)$

Moreover, $\mathfrak{B}^0(a, 0) = \emptyset$ *if* $e = 0$ *and* $\delta > 0$.

Note that if $k = 1$ *and* $a = 1$, *we have* $\mathfrak{B}^0(1, 0) = \mathfrak{B}(1, 0) = \mathfrak{B}'(1, 0)$.

Proof. Recall that the open $\mathfrak{B} \subset \mathfrak{B}'$ is dense in every fiber of the projective bundle $p' : \mathfrak{B}' \to \mathfrak{S}$, where

$$\mathfrak{S} = \mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C).$$

Let $p : \mathfrak{B} \to \mathfrak{S}$ be the restriction of p' to $\mathfrak{B}$.

Let $u = (\mathcal{L}, Z) \in \mathfrak{S}$ be a closed point and let $\mathcal{I}_Z$ be the ideal sheaf of the scheme Z in $\mathbb{P}^1 \times C$. Consider the following line bundles on $\mathbb{P}^1 \times C$:

$$\mathcal{F}' = O(a) \boxtimes \mathcal{L} \quad \text{and} \quad \mathcal{M} = O(k-a) \boxtimes \mathcal{L}^{-1}(x_0)$$

Then the fiber ${p'}^{-1}(u)$ is isomorphic to $\mathbb{P}(W)$, where W denotes the vector space:

$$W = \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}').$$

Let $U = p^{-1}(u) \subseteq \mathbb{P}(W)$ be the open in $\mathbb{P}(W)$ corresponding to extensions of $\mathcal{M} \otimes \mathcal{I}_Z$ by $\mathcal{F}'$, which have the middle term locally free. By Note 3.31, if $\delta = 0$ then $U = \mathbb{P}(W)$.

Let $\tilde{\mathcal{F}}$ be the universal bundle on $\mathbb{P}^1 \times C \times U$. It has the property that if $\xi \in \mathbb{P}(W)$ corresponds to some extension $\Xi \in W$:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0 \qquad (\Xi)$$

then we have that $\tilde{\mathcal{F}}_\xi \cong \mathcal{F}$.

Let $U^0 \subset U$ be the good locus in U. This is the locus of points $\xi \in U$ such that for any $p \in \mathbb{P}^1$, the bundle $\mathcal{F}_{p,\xi}$ is stable. Then $U = \mathfrak{B}^0 \cap p^{-1}(u)$.

We prove the following:

i. for any $u \in \mathfrak{S}$, the open U^0 is not empty if $\delta = 0$

ii. for any $u \in \mathfrak{S}$, the open U^0 is empty if $\delta > 0$ and $e = 0$

iii. for $u \in \mathfrak{S}$ general, the open U^0 is not empty if $\delta > 0$ and $e > 0$

In Case iii by $u = (\mathcal{L}, Z) \in \mathfrak{S}$ general, we mean that the 0-cycle Z is reduced and if $(p,q) \in Z \subset \mathbb{P}^1 \times C$ then $\{p\} \times C \cap Z = \{(p,q)\}$.

Plan of Proof

Consider the following closed set of $\mathbb{P}^1 \times U$:

$$Y = \{(p,\xi) \in \mathbb{P}^1 \times U | \text{ the bundle } \tilde{\mathcal{F}}_{p,\xi} \text{ on C is not stable } \}.$$

Let $\pi : \mathbb{P}^1 \times U \to U$ be the projection. We have:

$$U^0 = U \setminus \pi(Y)$$

Let $\Gamma \subset \mathbb{P}^1$ be the set of points $p \in \mathbb{P}^1$ with the property that there is a point $q \in C$ such that (p,q) is in Z. If $Z = \emptyset$, we let $\Gamma = \emptyset$. Consider the following subschemes of $\mathbb{P}^1 \times U$:

$$Y' = Y \cap ((\mathbb{P}^1 \setminus \Gamma) \times U), \quad \text{and} \quad Y'' = Y \cap (\Gamma \times U)$$

We prove:

1. Y' has codimension at least 2 in $\mathbb{P}^1 \times U$
2. $Y'' = \Gamma \times U$ if $\delta > 0$ and $e = 0$
3. Y'' has codimension at least 2 in $\mathbb{P}^1 \times U$ if $\delta > 0$, $e > 0$ and $u \in \mathfrak{S}$ general

These assertions prove our theorem, as follows:

i. If $\delta = 0$ then $Z = \emptyset$ and $Y' = Y$. By 1, the closed set Y in $\mathbb{P}^1 \times U$ has codimension 2, therefore $\pi(Y)$ is a proper closed subset of U; hence, $U^0 \neq \emptyset$.

ii. If $\delta > 0$ and $e = 0$, by 2. it follows that $\pi(Y) = U$, $U^0 = \emptyset$.

iii. If $\delta > 0$, $e > 0$ and $u \in \mathfrak{S}$ general, by 1. and 3., since $Y = Y' \cup Y'' = \overline{Y'} \cup Y''$ ($\overline{Y'}$ is the closure of Y' in $\mathbb{P}^1 \times U$), we have that

$$\pi(Y) = \pi(\overline{Y'}) \cup \pi(Y'')$$

is a proper closed subset of U; hence, $U^0 \neq \emptyset$.

For a point $p \in \mathbb{P}^1$, denote $Y_p = Y \cap (\{p\} \times U)$. Then Y_p can be identified with the closed subscheme of U corresponding to classes of extensions ξ in $U \subset \mathbb{P}(W)$, such that $\tilde{\mathcal{F}}_{p,\xi}$ is not stable.

Equivalently to 1, 2 and 3, we prove:

1'. Y_p has codimension at least 2 in U if $p \in \mathbb{P}^1 \setminus \Gamma$

2'. $Y_p = U$ if $p \in \Gamma$, when $\delta = 0$ and $e = 0$

3'. Y_p has codimension at least 1 in U if $p \in \Gamma$, when $\delta > 0$, $e > 0$ and $u \in \mathfrak{S}$ general

Proof of 1'.

Consider the vector spaces:

$$W = \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}') \quad \text{and} \quad V = \mathrm{Ext}^1_{\{p\} \times C}((\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\} \times C}, \mathcal{F}'_{|\{p\} \times C}).$$

Let $r : W \to V$ be the restriction morphism of Lemma 3.33. An extension given by an element $\Xi \in W$:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0 \qquad (\Xi)$$

is sent by r to the element in V given by restriction of (Ξ) to $\{p\} \times C$ (restriction is exact in this case):

$$0 \to \mathcal{F}'_{|\{p\}\times C} \to \mathcal{F}_{|\{p\}\times C} \to (\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\}\times C} \to 0.$$

By Lemma 3.33, we have that r is surjective.

Since $p \in \mathbb{P}^1 \setminus \Gamma$, we have:

$$\mathcal{I}_{Z|\{p\}\times C} \cong \mathcal{O}, \quad (\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\}\times C} \cong \mathcal{L}^{-1}(x_0), \quad \mathcal{F}'_{|\{p\}\times C} \cong \mathcal{L}.$$

We have an isomorphism:

$$V \cong \mathrm{Ext}^1(\mathcal{L}^{-1}(x_0), \mathcal{L}).$$

Recall from 2.2, that V is a vector space of dimension $2e + g$.

Consider the locus of unstable extensions in $\mathbb{P}(V)$ (see Proposition 2.1):

$$Z \subset \mathbb{P}(V).$$

Since $e \geq 0$, we have seen that Z is a closed subscheme of $\mathbb{P}(V)$ of codimension at least $g \geq 2$. If $e = 0$ then $Z = \emptyset$.

Let $C(Z) \subseteq V$ be the affine cone over $Z \subset \mathbb{P}(V)$. If $Z = \emptyset$ then let $C(Z) = 0 \in V$. Let $\tilde{Z} \subset \mathbb{P}(W)$ be the projectivisation of the preimage via r of $C(Z)$ in W. Note that

$$Y_p = \tilde{Z} \cap U.$$

Since $\mathrm{codim}\,_W(r^{-1}(C(Z))) = \mathrm{codim}\,_V C(Z) = \mathrm{codim}\,_{\mathbb{P}(V)} Z$, it follows that

$$\mathrm{codim}\,_{\mathbb{P}(W)} \tilde{Z} \geq g \geq 2 \quad \text{and} \quad \mathrm{codim}\,_U Y_p \geq 2.$$

Side Note

We see that $\mathbb{P}(\ker\,(r)) \cap U \subset Y_p$. This is the locus of extensions Ξ with the middle term $\mathcal{F}$ a locally free sheaf and for which the restriction to $\{p\} \times C$ gives a split exact sequence. One should expect that we always have $\ker\,(r) \cap U \neq \emptyset$, i.e. there is a vector bundle $\mathcal{F}$ sitting in an exact sequence (Ξ) with

$$\mathcal{F}_p \cong \mathcal{L} \oplus \mathcal{L}^{-1}(x_0).$$

This would show that $Y_p \neq \emptyset$. It follows that $U^0 \neq U$; hence, $\mathfrak{B}^0 \subset \mathfrak{B}$ is a strict inclusion. Moreover, we have a lower bound for the codimension:

$$\mathrm{codim}\,_U Y_p \leq \dim\, V = 2e + g.$$

We have a better understanding of Y_p in the following particular cases:

i. If $\delta = 0$ then $U = \mathbb{P}(W)$ and $Y_p = \tilde{Z} \neq \emptyset$.

ii. If $e = 0$ then $Z = \emptyset$, $C(Z) = 0$ and $Y_p = \mathbb{P}(\ker(r)) \cap U$.

iii. If $e = 0$ and $\delta = 0$ then $U = \mathbb{P}(W)$, $Z = \emptyset$, $C(Z) = 0$ and $Y_p = \ker(r) \neq \emptyset$.

Proof of 2'

Assume $\delta > 0$ and $e = 0$. Let p be a point in Γ. We prove that $Y_p = U$, i.e., the bundle $\mathcal{F}_p$ is unstable for all extensions

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0 \tag{3.18}$$

Let D be the 0-cycle on C such that $(\{p\} \times C) \cap Z = \{p\} \times D$. We have:

$$\mathcal{F}'_{|\{p\}\times C} \cong \mathcal{L} \quad \text{and} \quad (\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\}\times C} \cong \mathcal{L}^{-1}(x_0 - D) \oplus \mathcal{L}^{-1}(x_0)_{|D}$$

If we restrict (3.18) to $\{p\} \times C$, we get an exact sequence:

$$0 \to \mathcal{L} \to \mathcal{F}_p \to \mathcal{L}^{-1}(x_0 - D) \oplus \mathcal{L}^{-1}(x_0)_{|D} \to 0.$$

If we take the saturation of $\mathcal{L}$ in $\mathcal{F}_p$, we get an exact sequence:

$$0 \to \mathcal{L}(D) \to \mathcal{F}_p \to \mathcal{L}^{-1}(x_0 - D) \to 0.$$

Since $e = 0$, the degree of the line bundle $\mathcal{L}(D)$ is $\deg(D) \geq 1$. Hence, $\mathcal{F}_p$ is not a stable bundle.

Proof of 3'

Assume $\delta > 0$ and $e > 0$. We assume that $u \in \mathfrak{S}$ is general, in the sense that Z is reduced and if $(p, q) \in Z$ then $(\{p\} \times C) \cap Z = \{p\} \times \{q\}$.

Consider the vector spaces:

$$W = \operatorname{Ext}^1_{\mathbb{P}^1\times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}') \quad \text{and} \quad V = \operatorname{Ext}^1_{\{p\}\times C}((\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\}\times C}, \mathcal{F}'_{|\{p\}\times C})$$

We have:

$$\mathcal{F}'_{|\{p\}\times C} \cong \mathcal{L} \quad \text{and} \quad (\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\}\times C} \cong \mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}$$

Then

$$V \cong \operatorname{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L})$$

We prove that for a general extension in W:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0$$

the bundle $\mathcal{F}$ has the property that $\mathcal{F}_p$ is stable.

By Lemma 3.33, we have that the restriction morphism $r : W \to V$ is surjective. Therefore, it is enough to prove that in the vector space

$$V \cong \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L})$$

there is an open set of extensions:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q} \to 0 \qquad (*)$$

with the bundle $\mathcal{E}$ a stable vector bundle.

From Lemma 3.34, we have that there is a surjective morphism $s : V \to V'$, where

$$V' = \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}(q))$$

sending the extension (*) to the extension (**):

$$0 \to \mathcal{L}(q) \to \mathcal{E} \to \mathcal{L}^{-1}(x_0 - q) \to 0 \qquad (**)$$

given by taking the saturation of $\mathcal{L}$ in $\mathcal{E}$.

Since $e > 0$, we have that $\deg \mathcal{L}(q) \leq 0$ and we can apply again Proposition 2.1, to get that a general extension (**) in V' has the middle term $\mathcal{E}$ a stable vector bundle. Since s is surjective, the result follows. This completes the proof of Theorem. □

Lemma 3.33. *For any $p \in \mathbb{P}^1$ the following morphism of vector spaces, given by restriction, is surjective:*

$$r : Ext^1_{\mathbb{P}^1 \times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}') \to Ext^1_{\{p\} \times C}((\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\} \times C}, \mathcal{F}'_{|\{p\} \times C})$$

Proof. Let's fix $p \in \mathbb{P}^1$ and let $\mathcal{I}_{\{p\} \times C}$ be the ideal sheaf of $\{p\} \times C$ in $\mathbb{P}^1 \times C$.

There is a short exact sequence:

$$0 \to \mathcal{F}' \otimes \mathcal{I}_{\{p\} \times C} \to \mathcal{F}' \to \mathcal{F}'_{|\{p\} \times C} \to 0 \qquad (3.19)$$

This induces a morphism:

$$w : \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}') \to \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}'_{|\{p\} \times C})$$

From the restriction

$$\mathcal{M} \otimes \mathcal{I}_Z \to (\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\} \times C}$$

we get a morphism:

$$v : \mathrm{Ext}^1_{\{p\} \times C}((\mathcal{M} \otimes \mathcal{I}_Z)_{|\{p\} \times C}, \mathcal{F}'_{|\{p\} \times C}) \to \mathrm{Ext}^1_{\{p\} \times C}(\mathcal{M} \otimes \mathcal{I}_Z, \mathcal{F}'_{|\{p\} \times C}).$$

It is straightforward to see that actually $v \circ r = w$.

We claim that w is actually surjective. From the long exact sequence coming from (3.19) we have:

$$\text{coker }(w) = \text{Ext}^2_{\mathbb{P}^1\times C}(\mathcal{M}\otimes\mathcal{I}_Z, \mathcal{F}'\otimes\mathcal{I}_{\{p\}\times C})$$

Recall that

$$\mathcal{M}\cong O(k-a)\boxtimes\mathcal{L}^{-1}(x_0), \quad \mathcal{F}'\cong O(a)\boxtimes\mathcal{L}$$
$$\mathcal{I}_{\{p\}\times C}\cong O(-1)\boxtimes O, \quad K_{\mathbb{P}^1\times C}\cong O(-2)\boxtimes K_C$$

Using duality, we have:

$$\text{Ext}^2_{\mathbb{P}^1\times C}(\mathcal{M}\otimes\mathcal{I}_Z, \mathcal{F}'\otimes\mathcal{I}_{\{p\}\times C}) \cong \text{H}^0(\mathbb{P}^1\times C, \mathcal{M}\otimes\mathcal{I}_Z\otimes\mathcal{F}'^{-1}\otimes\mathcal{I}^{-1}_{\{p\}\times C}\otimes K_{\mathbb{P}^1\times C})$$
$$\cong \text{H}^0(\mathbb{P}^1, O(k-2a-1))\otimes\text{H}^0(C,\mathcal{L}^{-2}(x_0)\otimes K_C) = 0$$

This is because $k-2a-1<0$. It follows that u is surjective; hence, r is surjective. □

Lemma 3.34. *The morphism of vector spaces*

$$s: Ext^1_C(\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) \to Ext^1_C(\mathcal{L}^{-1}(x_0-q), \mathcal{L}(q))$$

is surjective and it sends an extension

$$0\to\mathcal{L}\to\mathcal{E}\to\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}\to 0$$

to the extension given by its saturation:

$$0\to\mathcal{L}(q)\to\mathcal{E}\to\mathcal{L}^{-1}(x_0-q)\to 0$$

It is obtained from the projection

$$\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}\to\mathcal{L}^{-1}(x_0-q).$$

Proof. Consider the exact sequence

$$0\to\mathcal{L}\to\mathcal{L}(q)\to\mathcal{L}(q)_{|q}\to 0.$$

Applying $\text{Hom}(\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}, -)$, one gets a morphism

$$u: \text{Ext}^1_C(\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) \to \text{Ext}^1_C(\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}(q)).$$

Consider the split exact sequence:

$$0\to\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q}\to\mathcal{L}^{-1}(x_0)_{|q}\to 0.$$

Applying $\mathrm{Hom}(-, \mathcal{L}(q))$, one gets a morphism

$$t : \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}(q)) \to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}(q)).$$

Let $s = t \circ u$. We prove s is surjective. Note that there is a split exact sequence:

$$0 \to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}(q)) \to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) \to$$
$$\to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}(q)) \to 0.$$

Then t is a retract for this sequence.

In a similar way, there is a split exact sequence:

$$0 \to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}) \to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) \to$$
$$\to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) \to 0.$$

There is a canonical retract:

$$v' : \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) \to \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}).$$

There is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}) & \xrightarrow{t} & \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}, \mathcal{L}(q)) \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}) & \xrightarrow{t'} & \mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}(q)) \end{array}$$

The morphism t' is surjective, because

$$\mathrm{Ext}^1_C(\mathcal{L}^{-1}(x_0 - q), \mathcal{L}(q)_{|q}) \cong \mathrm{H}^1(\mathcal{L}^2(2q - x_0)_{|q}) = 0.$$

As v' is also surjective, it follows that $s = v' \circ t' = v \circ t$ is surjective.

We would like to describe the morphism s. Start with an extension

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q} \to 0 \qquad (*)$$

Consider the morphism $g : \mathcal{E} \to \mathcal{L}^{-1}(x_0 - q)$ obtained by composing the morphism $\mathcal{E} \to \mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q}$ with the projection

$$\mathcal{L}^{-1}(x_0 - q) \oplus \mathcal{L}^{-1}(x_0)_{|q} \to \mathcal{L}^{-1}(x_0 - q).$$

Then $\ker(g) \cong \mathcal{L}(q)$ and let the induced exact sequence be:

$$0 \to \mathcal{L}(q) \to \mathcal{E} \to \mathcal{L}^{-1}(x_0 - q) \to 0 \qquad (**)$$

We claim that s sends $(*)$ into $(**)$. To prove this, note first that there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{L}(q) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L}^{-1}(x_0-q) & \longrightarrow & 0
\end{array}$$

Apply $\mathrm{Hom}(\mathcal{L}^{-1}(x_0-q)\oplus\mathcal{L}^{-1}(x_0)_{|q},-)$ to both exact sequences to get the following commutative diagram. (Since $\mathcal{L}^{-1}(x_0)_{|q}\cong O_q$, for convenience, we replace $\mathcal{L}^{-1}(x_0)_{|q}$ with O_q everywhere in the diagram).

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{L}^{-1}(x_0-q)\oplus O_q,\mathcal{L}^{-1}(x_0-q)\oplus O_q) & \xrightarrow{\delta_1} & \mathrm{Ext}(\mathcal{L}^{-1}(x_0-q)\oplus O_q,\mathcal{L}) \\
\downarrow & & \downarrow t \\
\mathrm{Hom}(\mathcal{L}^{-1}(x_0-q)\oplus O_q,\mathcal{L}^{-1}(x_0-q)) & \longrightarrow & \mathrm{Ext}(\mathcal{L}^{-1}(x_0-q)\oplus O_q,\mathcal{L}(q)) \\
\downarrow\cong & & \downarrow v \\
\mathrm{Hom}(\mathcal{L}^{-1}(x_0-q),\mathcal{L}^{-1}(x_0-q)) & \xrightarrow{\delta_2} & \mathrm{Ext}(\mathcal{L}^{-1}(x_0-q),\mathcal{L}(q))
\end{array}$$

We denoted by δ_1 and δ_2 the connecting morphisms; hence,

$$\delta_1(Id)=(*),\ \delta_2(Id)=(**).$$

Note that the vertical map on the left of the diagram maps the identity element

$$Id\in\mathrm{Hom}(\mathcal{L}^{-1}(x_0-q)\oplus O_q,\mathcal{L}^{-1}(x_0-q)\oplus O_q)$$

to the identity element

$$Id\in\mathrm{Hom}(\mathcal{L}^{-1}(x_0-q),\mathcal{L}^{-1}(x_0-q)).$$

So $s=v\circ t$ maps $(*)$ to $(**)$.

Note that there is a clear retract for s given by the injection $\mathcal{L}\to\mathcal{L}(q)$. □

3.4 The moduli scheme in Case E

In this section we let $k=2a$. We construct moduli spaces for rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1\times C$ with the properties in Lemma 3.1 and such that $\mathcal{F}$ has equal generic fiber type (a,a).

As in Case U, let c_1 and c_2 be:

$$c_1=k\{pt\}\times C+\mathbb{P}^1\times\{x_0\}\quad\text{and}\quad c_2=k$$

We say that $\mathcal{F}$ has type (†) if it has canonical sequence:

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{F} \to O(a-1) \boxtimes O_D \to 0 \qquad (\dagger)$$

where $\mathcal{E}$ is a *stable* vector bundle on C and D is a 0-cycle on C.

Our goal is to construct coarse moduli schemes for the following contravariant functors:

$$F : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets}$$

$$F(S) = \{\text{Isomorphism classes } [\mathcal{F}] \text{ of vector bundles } \mathcal{F} \text{ on } \mathbb{P}^1 \times C \times S, \text{ such that } \forall s \in S,\ \mathcal{F}_s \text{ has Chern classes } c_1 \text{ and } c_2 \text{ and it has type } (\dagger)$$

$$F^0 : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets}$$

$$F^0(S) = \{\text{Isomorphism classes } [\mathcal{F}] \text{ of vector bundles } \mathcal{F}, \text{ such that } [\mathcal{F}] \in F(S) \text{ and } \forall (p,s) \in \mathbb{P}^1 \times S \text{ the vector bundle } \mathcal{F}_{p,s} \text{ is stable}\}$$

As in Case U, the functor F^0 is an open subfunctor of F. The functor F has a coarse moduli scheme $\mathfrak{B}_{even}$, which is constructed in [BR2]. We prove that the functor F^0 has as coarse moduli scheme a *dense* open subscheme $\mathfrak{B}^0_{even}$ of $\mathfrak{B}_{even}$.

3.4.1 Some computations with Chern classes

Lemma 3.35. *Consider the exact sequence of sheaves on $\mathbb{P}^1 \times C$:*

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{F} \to O(a-1) \boxtimes O_D \to 0 \qquad (\dagger)$$

where $\mathcal{E}$ is a rank 2 vector bundle, and D a 0-cycle on C. Then we have:

$$c_1(\mathcal{F}) = \mathbb{P}^1 \times (c_1(\mathcal{E}) + D) + k\{pt\} \times C$$
$$deg\ c_2(\mathcal{F}) = (a+1) deg\ (D) + a.deg\ (\mathcal{E})$$

Corollary 3.36. *If $\mathcal{F}$ as in the Lemma 3.35 has Chern classes c_1 and c_2 then:*

$$c_1(\mathcal{E}) = \{x_0\} - D$$
$$deg\ (D) = a = \frac{k}{2}$$

3.4.2 The moduli scheme $\mathfrak{B}_{even}$

Fact 3.37. *[BR2] There exists a quasi-projective integral scheme $\mathfrak{B}_{even}$ which is a coarse moduli scheme for the functor F, i.e., for rank 2 vector bundles on $\mathbb{P}^1 \times C \times S$ with equal generic fibre type (a, a), with Chern classes (c_1, c_2) and for which the canonical quotient sheaf $\mathcal{J}$ is of the form $O_Y(-1)$, where Y is a union of fibres.*

The previous fact is a consequence of the construction in [BR2]. As in the Case U, we fixed the Chern class c_1 as an element of $\mathrm{A}^1(\mathbb{P}^1 \times C)$. If we fix c_1 only up to numerical equivalence, hence, as an element of $\mathrm{NS}(\mathbb{P}^1 \times C)$, then we get the moduli scheme $\mathfrak{B}$ of [BR2]. More precisely, the scheme $\mathfrak{B}$ of [BR2] has a canonical map:

$$\pi : \mathfrak{B} \to \mathfrak{S}, \quad \text{where} \quad \mathfrak{S} = \mathrm{Sym}^a(C) \times M(2, 1-a)$$

and $M(2, 1-a)$ is the coarse moduli scheme of rank 2 semistable vector bundles on C of degree $(1-a)$. The map π sends the point $\mathcal{F}$ to the point $(D, \mathcal{E})$.

The scheme $\mathfrak{B} = \mathfrak{B}_{even}$ that we construct is the preimage under π of the subscheme of $\mathfrak{S}$ given by the closed immersion:

$$\mathrm{Sym}^a(C) \times_{\mathrm{Pic}^a(C)} M(2, 1-a) \hookrightarrow \mathrm{Sym}^a(C) \times M(2, 1-a)$$

where $M(2, 1-a) \to \mathrm{Pic}^{1-a}(C)$ is the determinant map and $\mathrm{Sym}^a(C) \to \mathrm{Pic}^{1-a}(C)$ is given by

$$D \mapsto O(x_0 - D).$$

Outline of the construction of $\mathfrak{B}$

This follows closely [BR2]. We want to construct a scheme $\mathfrak{B}$ for vector bundles $\mathcal{F}$ that sit in an extension as in Lemma 3.35:

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{F} \to O(a-1) \boxtimes O_D \to 0 \qquad (\dagger)$$

We start by constructing a space $\overline{\mathfrak{B}}'$ parametrizing extensions $\dagger$.

Let $M(2, 1-a)$ be the coarse moduli scheme of semi-stable rank 2 and degree $(1-a)$ vector bundles on C and let $\overline{M}(2, 1-a)$ be the scheme whose geometric quotient by the action of a general linear group G' is the scheme $M(2, 1-a)$.

On $C \times \overline{M}(2, 1-a)$ there is a rank 2 vector bundle $\mathcal{W}$, which is a locally universal family of stable rank 2 bundles of degree $1-a$. Moreover, $\overline{M}(2, 1-a)$ is smooth. If a is even, then $\mathcal{W}$ is in fact a universal bundle. In this case, $\mathcal{W}$ descends to a universal bundle on $C \times M(2, 1-a)$ and the scheme $M(2, 1-a)$ is smooth.

Let $\overline{\mathfrak{S}}$ be the product

$$\begin{array}{ccc} \overline{\mathfrak{S}} & \longrightarrow & \overline{M}(2, 1-a) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^a(C) & \xrightarrow{u} & \mathrm{Pic}^{1-a}(C). \end{array}$$

where the morphism u is given by $D \mapsto O(x_0 - D)$.

Let $\mathcal{E}_{\overline{\mathfrak{S}}}$ be the vector bundle on $\mathbb{P}^1 \times C \times \overline{\mathfrak{S}}$ induced from the bundle $\mathcal{W}$ on $C \times \overline{M}(2, 1-a)$. Let $\mathcal{D}_{\overline{\mathfrak{S}}} \subset C \times \overline{\mathfrak{S}}$ be the subscheme induced from the universal divisor in $C \times \mathrm{Sym}^a(C)$. Define $\mathcal{S}$ to be the relative extension sheaf

$$\mathcal{S} = \mathcal{E}xt^1_{\mathbb{P}^1 \times C \times \overline{\mathfrak{S}} / \overline{\mathfrak{S}}}(O(a-1) \boxtimes O_{\mathcal{D}_{\overline{\mathfrak{S}}}}, O(a) \boxtimes \mathcal{E}_{\overline{\mathfrak{S}}}).$$

The sheaf $\mathcal{S}$ is a locally free sheaf on $\mathbb{P}^1 \times C \times \overline{\mathfrak{S}}$. If $u = (D, \mathcal{E}) \in \overline{\mathfrak{S}}$, then we have that $\overline{\mathfrak{S}}_{|u} \cong \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(O(a-1) \boxtimes O_D, O(a) \boxtimes \mathcal{E})$.

Define $\overline{\mathfrak{B}}' = \mathbb{V}(\mathcal{S})$ and let $\overline{p}' : \overline{\mathfrak{B}}' \to \overline{\mathfrak{S}}$ be the canonical map.

In a similar way to the U case, we construct now the subscheme $\overline{\mathfrak{B}} \subset \overline{\mathfrak{B}}'$ parametrizing extensions (†), such that $\mathcal{F}$ is locally free.

Consider the sheaves $\mathcal{E}_{\overline{\mathfrak{B}}'}$ and $O_{\mathcal{D}_{\overline{\mathfrak{B}}'}}$ on $\mathbb{P}^1 \times C \times \overline{\mathfrak{B}}'$, which are the pull-backs of the corresponding sheaves on $\mathbb{P}^1 \times C \times \overline{\mathfrak{S}}$. There is a universal extension:

$$0 \to (O(a) \boxtimes \mathcal{E}_{\overline{\mathfrak{B}}'}) \otimes \mathcal{L} \to \mathcal{F}_{\overline{\mathfrak{B}}'} \to O(a-1) \boxtimes O_{\mathcal{D}_{\overline{\mathfrak{S}}}} \to 0$$

where $\mathcal{L}$ is some line bundle on $\overline{\mathfrak{B}}$.

As in Case U, there is a dense open subscheme $\overline{\mathfrak{B}}$ of $\overline{\mathfrak{B}}'$, such that $\mathcal{F}_{\overline{\mathfrak{B}}'}$ restricts to a locally free sheaf $\mathcal{F}_{\overline{\mathfrak{B}}}$ on $\overline{\mathfrak{B}}$. The scheme $\overline{\mathfrak{B}} \subset \overline{\mathfrak{B}}'$ is dense in every fiber of $\overline{p}' : \overline{\mathfrak{B}}' \to \overline{\mathfrak{S}}$.

Let now T be the group scheme on $\overline{\mathfrak{S}}$ which has as fiber over the point $u = (D, \mathcal{E}) \in \overline{\mathfrak{S}}$ the group given by Aut (O_D). As Aut (O_D) acts on the fiber of $\overline{p} : \overline{\mathfrak{B}}' \to \overline{\mathfrak{S}}$ at u:

$$\mathrm{Ext}^1_{\mathbb{P}^1 \times C}(O(a-1) \boxtimes O_D, O(a) \boxtimes \mathcal{E})$$

it follows that the group scheme T acts on $\overline{\mathfrak{B}}$.

Let G' be the group which acts on $\overline{M}(2, 1-a)$ giving as geometric quotient the coarse moduli scheme $M(2, 1-a)$. The group G' acts on the locally universal bundle $\mathcal{W}$, and therefore on $\overline{\mathfrak{B}}$.

Since scalar multiplication of extensions appears in both the action of T and G', we define the group scheme:

$$G = T \times_{\mathbb{G}_m} G'$$

The following fact is stated in [BR2]; it follows easily from Lemma A.5

Fact 3.38. *Let v and w be points of $\overline{\mathfrak{B}}$. Then the bundles $\mathcal{F}_v$ and $\mathcal{F}_w$ induced from the locally universal bundle $\mathcal{F}_{\overline{\mathfrak{B}}}$ are isomorphic if and only if v and w lie in the same orbit for the action of G on $\overline{\mathfrak{B}}$.*

The following fact is a result from [BR2].

Fact 3.39. *There is a universal geometric quotient $\mathfrak{B}$ for the action of G on $\overline{\mathfrak{B}}$. The scheme $\mathfrak{B}$ is integral quasiprojective and it is a coarse moduli scheme for the functor F of (3.4), i.e., for rank 2 vector bundles on $\mathbb{P}^1 \times C$ with Chern classes c_1 and c_2 and with $\mathcal{F}$ of type (†).*

We call this scheme with $\overline{\mathfrak{B}}$. The following fact is a result from [BR2].

Fact 3.40. *There is a geometric quotient $\mathfrak{S}$ of $\overline{\mathfrak{S}}$ by G, which is isomorphic to $Sym^a(C) \times_{Pic^{1-a}(C)} M(2, 1-a)$. The projection map $\overline{p} : \overline{\mathfrak{B}} \to \overline{\mathfrak{S}}$ descends to give a map*

$$p : \mathfrak{B}_{even} \to \mathfrak{S}.$$

Moreover, away from the ramification locus of the map $C^a \to Sym^a(C)$, p is a $(\mathbb{P}GL(1))^a$-bundle in the fpqf-topology ("finite flat surjective" topology).

Dimension of $\mathfrak{B}_{even}$

Since dim $M(2, 1-a) = 4g - 3$ and dim $\mathrm{Sym}^a(C) = a$, we have:

$$\dim\ (\mathfrak{S}) = a + 3g - 3.$$

It follows by Fact 3.40 that

$$\dim\ (\mathfrak{B}_{even}) = 4a + 3g - 3 = 2k + 3g - 3. \tag{3.20}$$

3.4.3 The good locus $\mathfrak{B}^0_{even}$ in $\mathfrak{B}_{even}$

We constructed a scheme $\mathfrak{B} = \mathfrak{B}_{even}$ which is the geometric quotient of an integral quasiprojective scheme $\overline{\mathfrak{B}}$ by the action of the group G. The scheme $\overline{\mathfrak{B}}$ has a locally universal bundle $\tilde{\mathcal{F}} = \mathcal{F}_{\overline{\mathfrak{B}}}$.

We say that a closed point $b \in \overline{\mathfrak{B}}$ is *good* if the vector bundle $\tilde{\mathcal{F}}_b$ on $\mathbb{P}^1 \times C$ has the property that for any $p \in \mathbb{P}^1$ the bundle $\tilde{\mathcal{F}}_{p,b}$ is stable (equivalently, $\tilde{\mathcal{F}}_b$ induces a morphism $f : \mathbb{P}^1 \to M$). We let $\overline{\mathfrak{B}}^0 \subset \overline{\mathfrak{B}}$ be the set of good points.

Consider the locus:

$$\mathcal{Y} = \{(p, b) \in \mathbb{P}^1 \times \overline{\mathfrak{B}} |\text{ the bundle } \tilde{\mathcal{F}}_{p,b} \text{ is not stable }\}$$

By Fact 1.9, $\mathcal{Y}$ is closed in $\mathbb{P}^1 \times \overline{\mathfrak{B}}$. Note that $\overline{\mathfrak{B}}^0$ is the complement of the image of $\mathcal{Y}$ via the projection $\pi : \mathbb{P}^1 \times \overline{\mathfrak{B}} \to \overline{\mathfrak{B}}$. Therefore, the set of good points $\overline{\mathfrak{B}}^0$ is open in $\overline{\mathfrak{B}}$, but possibly empty.

Lemma 3.41. *The action of the group scheme G on $\overline{\mathfrak{B}}$ induces an action on $\overline{\mathfrak{B}}^0$. There is a geometric quotient $\mathfrak{B}^0$ for the action of G on $\overline{\mathfrak{B}}^0$, which is also a coarse moduli scheme for the functor F^0 of (3.4). If the geomtric quotient of $\overline{\mathfrak{B}}$ is given by:*

$$q : \overline{\mathfrak{B}} \to \mathfrak{B}$$

then the scheme $\mathfrak{B}^0$ is the open subscheme of $\mathfrak{B}$ given by $q(\overline{\mathfrak{B}}^0)$.

Proof. Note that by Lemma 3.38, the orbit of a point in $\overline{\mathfrak{B}}^0$ by the action of G on $\overline{\mathfrak{B}}$ is contained in $\overline{\mathfrak{B}}^0$. It follows that if we let $\mathfrak{B}^0 = q(\overline{\mathfrak{B}}^0)$ then $\mathfrak{B}^0 \subset \mathfrak{B}$ is open and $\overline{\mathfrak{B}}^0 = q^{-1}(\mathfrak{B}^0)$ has $\mathfrak{B}^0$ as a geometric quotient for the action of G. □

Theorem 3.42. *The open $\overline{\mathfrak{B}}^0$ is non-empty, hence, dense, in $\overline{\mathfrak{B}}$.*

Proof. Recall that the open $\overline{\mathfrak{B}} \subset \overline{\mathfrak{B}}'$ is dense in every fiber of the vector bundle $\overline{p}' : \overline{\mathfrak{B}}' \to \overline{\mathfrak{S}}$. Let $\overline{p} : \overline{\mathfrak{B}} \to \overline{\mathfrak{S}}$ be the restriction of $\overline{p}'$ to $\overline{\mathfrak{B}}$.

We prove that for a general closed point u in $\mathfrak{S}$ we have:

$$\overline{\mathfrak{B}}^0 \cap p^{-1}(u) \neq \emptyset$$

Let $u = (\mathcal{E}, D) \in \mathfrak{S}$ be a closed point. Consider the following sheaves on $\mathbb{P}^1 \times C$:

$$\mathcal{F}' = O(a) \boxtimes \mathcal{E}, \quad \text{and} \quad \mathcal{J} = O(a) \boxtimes O_D$$

Then the fiber $\overline{p}'^{-1}(u)$ is isomorphic to the affine space W given by

$$W = \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{J}, \mathcal{F}').$$

Let $U = \overline{p}^{-1}(u) \subseteq W$ be the open in W corresponding to extensions of $\mathcal{J}$ by F', which have the middle term locally free.

Let $\tilde{\mathcal{F}}$ be the universal bundle on $\mathbb{P}^1 \times C \times U$. It has the property that if $\xi \in W$ corresponds to some extension

$$0 \to \mathcal{F}' \to \mathcal{F} \to O(a) \boxtimes O_D \to 0 \qquad (\xi)$$

then we have that $\tilde{\mathcal{F}}_\xi \cong \mathcal{F}$.

Let $U^0 \subset U$ be the good locus in U. This is the locus of points $\xi \in U$ such that for any $p \in \mathbb{P}^1$, the bundle $\mathcal{F}_{p,\xi}$ is stable. Then $U = \overline{\mathfrak{B}}^0 \cap \overline{p}^{-1}(u)$.

Consider the following closed set of $\mathbb{P}^1 \times U$:

$$Y = \{(p, \xi) \in \mathbb{P}^1 \times U | \text{ the bundle } \tilde{\mathcal{F}}_{p,\xi} \text{ on C is not stable } \}.$$

Let $\pi : \mathbb{P}^1 \times U \to U$ be the projection. We have:

$$U^0 = U \setminus \pi(Y)$$

For a point $p \in \mathbb{P}^1$, denote $Y_p = Y \cap (\{p\} \times U)$. Then Y_p can be identified with the closed subscheme of U corresponding to classes of extensions ξ in $U \subset W$, such that $\tilde{\mathcal{F}}_{p,\xi}$ is not stable.

If $u = (D, \mathcal{E}) \in \overline{\mathfrak{S}}$ is such that D consists of distinct points then we prove that $\pi(Y)$ is a proper closed subset of U by proving that for any $p \in \mathbb{P}^1$ the closed subscheme Y_p of U has codimension at least 2.

Consider the vector spaces:

$$W = \mathrm{Ext}^1_{\mathbb{P}^1 \times C}(\mathcal{J}, \mathcal{F}') \quad \text{and} \quad V = \mathrm{Ext}^1_{\{p\} \times C}(\mathcal{J}_{|\{p\} \times C}, \mathcal{F}'_{|\{p\} \times C}).$$

Let $r : W \to V$ be the restriction morphism of Lemma 3.43. An extension given by an element $\xi \in W$:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{J} \to 0 \qquad (\xi)$$

is sent by r to the element in V given by the restriction of (ξ) to $\{p\} \times C$ (restriction is exact in this case):

$$0 \to \mathcal{F}'_{|\{p\}\times C} \to \mathcal{F}_{|\{p\}\times C} \to \mathcal{J}_{|\{p\}\times C} \to 0.$$

By Lemma 3.43, we have that r is surjective.

We have:

$$\mathcal{F}'_{|\{p\}\times C} \cong \mathcal{E} \quad \text{and} \quad \mathcal{J}_{|\{p\}\times C} \cong O_D$$

Then

$$V \cong \mathrm{Ext}^1(O_D, \mathcal{E}).$$

So V is the vector space (2.17). It has dimension $k = 2a$.

Consider the locus of unstable extensions in $\mathbb{P}(V)$ (Proposition 2.19):

$$Z \subset \mathbb{P}(V).$$

We have seen that Z is a closed subscheme of $\mathbb{P}(V)$ of codimension at least 2. If $a = 1$ then $Z = \emptyset$. Let $C(Z) \subseteq V$ be the affine cone over $Z \subset \mathbb{P}(V)$. If $Z = \emptyset$ then we let $C(Z) = 0 \in V$. Let $\tilde{Z} \subset W$ be the preimage via r of $C(Z)$ in W. Note that

$$Y_p = \tilde{Z} \cap U.$$

Since

$$\text{codim}\ _W(r^{-1}(C(Z))) = \text{codim}\ _V C(Z) = \text{codim}\ _{\mathbb{P}(V)} Z \geq 2$$

it follows that codim $_U Y_p \geq 2$. □

Lemma 3.43. *For any $p \in \mathbb{P}^1$ there is a morphism of vector spaces:*

$$r : Ext^1_{\mathbb{P}^1\times C}(\mathcal{J}, \mathcal{F}') \to Ext^1_{\{p\}\times C}(\mathcal{J}_{|\{p\}\times C}, \mathcal{F}'_{|\{p\}\times C})$$

given by restriction and r is surjective.

Proof. Let's fix $p \in \mathbb{P}^1$ and let $\mathcal{I}_{\{p\}\times C}$ be the ideal sheaf of $\{p\} \times C$ in $\mathbb{P}^1 \times C$. There is a short exact sequence:

$$0 \to \mathcal{F}' \otimes \mathcal{I}_{\{p\}\times C} \to \mathcal{F}' \to \mathcal{F}'_{|\{p\}\times C} \to 0 \qquad (3.21)$$

This induces a morphism:

$$w : \mathrm{Ext}^1_{\mathbb{P}^1\times C}(\mathcal{J}, \mathcal{F}') \to \mathrm{Ext}^1_{\mathbb{P}^1\times C}(\mathcal{J}, \mathcal{F}'_{|\{p\}\times C})$$

From

$$\mathcal{J} \to \mathcal{J}_{|\{p\}\times C}$$

we get a morphism:

$$v : \mathrm{Ext}^1_{\{p\}\times C}(\mathcal{J}_{|\{p\}\times C}, \mathcal{F}'_{|\{p\}\times C}) \to \mathrm{Ext}^1_{\{p\}\times C}(\mathcal{J}, \mathcal{F}'_{|\{p\}\times C}).$$

We have $v \circ r = w$.

We claim that w is actually surjective. From the long exact sequence coming from (3.21), we have:

$$\text{coker } (w) = \mathrm{Ext}^2_{\mathbb{P}^1\times C}(\mathcal{J}, \mathcal{F}' \otimes \mathcal{I}_{\{p\}\times C}).$$

Using duality, we have:

$$\begin{aligned}\mathrm{Ext}^2_{\mathbb{P}^1\times C}(\mathcal{J}, \mathcal{F}' \otimes \mathcal{I}_{\{p\}\times C}) &\cong \mathrm{Ext}^2_{\mathbb{P}^1\times C}(O(a-1) \boxtimes O_D, O(a-1) \boxtimes \mathcal{E}) \\ \cong \mathrm{Ext}^2_{\mathbb{P}^1\times C}(p_2^*(O_D \otimes \mathcal{E}^*), O) &\cong \mathrm{H}^0(\mathbb{P}^1 \times C, p_2^*(O_D \otimes \mathcal{E}^*) \otimes K_{\mathbb{P}^1\times C}) \\ &\cong \mathrm{H}^0(\mathbb{P}^1, O(-2)) \otimes \mathrm{H}^0(C, O_D \otimes \mathcal{E}^* \otimes K_C)\end{aligned}$$

□

Chapter 4

Irreducible Components of the Space of Rational Curves on M

We find all the irreducible components of $\mathrm{Mor}_k(\mathbb{P}^1, M)$. They all correspond to moduli of rank 2 vector bundles on $\mathbb{P}^1 \times C$. We also find their maximally rationally connected fibrations.

Consider the evaluation morphism ev : $\mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, M) \to M$ and the universal bundle on $\mathbb{P}^1 \times C \times \mathrm{Mor}_k(\mathbb{P}^1, M)$:

$$\mathcal{H} = (ev \times id_C)^* \mathcal{U}_0.$$

If $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, M)$, then if $\mathcal{H}_f = \mathcal{H}_{|\mathbb{P}^1 \times C \times \{f\}}$, we have:

$$\mathcal{H}_f \cong (f \times id)^* \mathcal{U}_0.$$

Recall that if $[f]$ is a closed point of $\mathrm{Mor}_k(\mathbb{P}^1, M)$, we have from (1.7) and (1.8) that the bundle $\mathcal{H}_f$ has Chern classes c_1 and c_2:

$$c_1 = k\{pt\} \times C + \mathbb{P}^1 \times \{x_0\} \in \mathrm{A}^1(\mathbb{P}^1 \times C), \quad \deg\,(c_2) = k \in \mathbb{Z} \tag{4.1}$$

4.1 The subschemes $\mathfrak{M}(a, e)$ of $\mathrm{Mor}_k(\mathbb{P}^1, M)$

In this section we define the subschemes $\mathfrak{M}(a, e) \subset \mathrm{Mor}_k(\mathbb{P}^1, M)$ such that $[f] \in \mathfrak{M}(a, e)$ corresponds to a bundle $[\mathcal{H}_f] \in \mathfrak{B}^0(a, e)$. We prove that $\mathfrak{M}(a, e) \cong \mathfrak{B}^0(a, e)$ such that $\mathfrak{M}(a, e) \cong \mathfrak{B}^0(a, e)$. We prove that any irreducbile component of $\mathrm{Mor}_k(\mathbb{P}^1, M)$ for which the general point $[f]$ has the property that the bundle $\mathcal{H}_f$ has unequal generic splitting is one of the closure of one of the schemes $\mathfrak{M}(a, e)$.

The moduli schemes $\mathfrak{B}^0(a,e)$ map to $\mathbf{Mor}_k(\mathbb{P}^1, M)$

Let a be an integer such that $a > \frac{k}{2}$. Consider the range for the pairs (a,e) for which the good locus $\mathfrak{B}^0(a,e)$ of Theorem 3.32 is non-empty:

$$\delta \geq 0, \quad e > 0 \quad \text{or} \quad \delta = 0, \quad e = 0$$

This is precisely the range that we will call the range $(\star)$:

$$\{(a,e)| \quad k \geq a > k/2, \quad \frac{k-a}{2a-k} \geq e > 0\} \cup \{(k,0)\} \tag{$\star$}$$

Proposition 4.1. *Let a and e be integers in the range $(\star)$. Let $\mathfrak{B}(a,e)$ be the moduli scheme in 3.30 and let $\mathcal{F}_{\mathfrak{B}(a,e)}$ be the universal bundle on $\mathbb{P}^1 \times C \times \mathfrak{B}(a,e)$. Let $\mathfrak{B}^0(a,e) \subseteq \mathfrak{B}(a,e)$ be the dense open in Theorem 3.32 corresponding to vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ inducing morphisms $f : \mathbb{P}^1 \to M$. Then there is a unique morphism:*

$$\mu : \mathfrak{B}^0(a,e) \to Mor_k(\mathbb{P}^1, M) \tag{4.2}$$

such that for some line bundle $\mathcal{N}$ coming from $\mathfrak{B}^0(a,e)$ we have

$$\mathcal{F}_{\mathfrak{B}^0(a,e)} \cong (id_{\mathbb{P}^1} \times id_C \times \mu)^* \mathcal{H} \otimes \mathcal{N}. \tag{4.3}$$

The map μ sends the point $[\mathcal{F}] \in \mathfrak{B}(a,e)$ to the point $[f] \in Mor_k(\mathbb{P}^1, M)$, corresponding to the morphism f induced by $\mathcal{F}$.

Proof. Let $\mathfrak{B} = \mathfrak{B}(a,e)$. The non-empty open $\mathfrak{B}^0 \subseteq \mathfrak{B}$ has the property that for any closed point $b \in \mathfrak{B}^0$ and any closed point $p \in \mathbb{P}^1$, the bundle $\mathcal{F}_{\mathfrak{B}|\{p\}\times C \times \{b\}}$ is stable. Then, by the definition of the moduli scheme M, the restriction $\mathcal{F}_{\mathfrak{B}^0}$ of $\mathcal{F}_{\mathfrak{B}}$ to $\mathbb{P}^1 \times C \times \mathfrak{B}^0$ induces a morphism:

$$g : \mathbb{P}^1 \times \mathfrak{B}^0 \to M, \quad \text{such that} \quad \mathcal{F}_{\mathfrak{B}^0} \cong (g \times \mathrm{id}_C)^* \mathcal{U}_0 \otimes \mathcal{N} \tag{4.4}$$

for some line bundle $\mathcal{N}$ coming from $\mathbb{P}^1 \times \mathfrak{B}^0$. (It is straightforward to see from a computation with Chern classes, that the line bundle $\mathcal{N}$ is in fact the pull back of a line bundle from $\mathfrak{B}^0$.)

It follows from the universal property of the Hilbert scheme $\mathrm{Mor}_k(\mathbb{P}^1, M)$ that there is a unique morphism $\mu : \mathfrak{B}^0 \to \mathrm{Mor}_k(\mathbb{P}^1, M)$, such that g is the composition $\mathrm{ev} \circ (\mathrm{id}_{\mathbb{P}^1} \times \mu)$:

$$\mathbb{P}^1 \times \mathfrak{B}^0 \xrightarrow{(id_{\mathbb{P}^1} \times \mu)} \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, M) \xrightarrow{\ ev\ } M \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\mathcal{F}_{\mathfrak{B}^0} \cong (g \times \mathrm{id}_C)^* \mathcal{U}_0 \otimes \mathcal{N} \cong (id_{\mathbb{P}^1} \times \mathrm{id}_C \times \mu)^* \mathcal{H} \otimes \mathcal{N}.$$

□

Corollary 4.2. *The morphism μ is injective.*

Proof. The morphism μ is sending the point $[\mathcal{F}] \in \mathfrak{B}^0$ to the point $[f]$ corresponding to the morphism $f : \mathbb{P}^1 \to M$ induced by $\mathcal{F}$. Moreover, from (4.3), we have $\mathcal{F} \cong \mathcal{H}_f$. Hence, $\mathcal{F}$ is uniquely determined by f. □

Remark 4.3. *It will be very useful later to notice that in Theorem 4.2 we can allow* $a = \frac{k}{2}$ *and* $e > 0$. *In this case, there exists a scheme* $\mathfrak{B} = \mathfrak{B}(\frac{k}{2}, e)$ *parametrizing extensions (up to scalar multiplication):*

$$0 \to O(\frac{k}{2}) \boxtimes \mathcal{L} \to \mathcal{F} \to (O(\frac{k}{2}) \boxtimes \mathcal{L}^{-1}(x_0)) \otimes \mathcal{I}_Z \to 0. \tag{4.6}$$

where $\mathcal{L}$ *is a line bundle on* C *of degree* $-e$ *and* $Z \subset \mathbb{P}^1 \times C$ *is a* 0*-cycle.*

There exists a universal vector bundle $\mathcal{F}_{\mathfrak{B}}$ *on* $\mathbb{P}^1 \times C \times \mathfrak{B}$. *It parametrizes the middle term of the extensions (4.6). In this case the vector bundle* $\mathcal{F}$ *does not determine the exact sequence (4.6). Therefore,* $\mathfrak{B}$ *is not a moduli scheme for vector bundles* $\mathcal{F}$ *anymore.*

However, the same proof as for the case $a > \frac{k}{2}$ *shows that there is a morphism:*

$$\mu : \mathfrak{B}^0(\frac{k}{2}, e) \to Mor_k(\mathbb{P}^1, M).$$

This morphism sends the extension (4.6) to the morphism induced by $\mathcal{F}$. *This morphism does not need to be injective.*

Components of $\mathbf{Mor}_k(\mathbb{P}^1, M)$ map to $\mathfrak{B}^0(a, e)$

Lemma 4.4. *For each irreducible component* $\mathfrak{M}$ *of the scheme* $Mor_k(\mathbb{P}^1, M)$ *let* $\mathcal{H}_{\mathfrak{M}}$ *be the restriction of the universal vector bundle* $\mathcal{H}$ *to* $\mathbb{P}^1 \times C \times \mathfrak{M}$. *Then there exist integers* a *and* d *such that with* $a \geq \frac{k}{2}$ *and a dense open*

$$\mathfrak{M}^0 \subseteq \mathfrak{M}$$

such that for any closed point $[f] \in \mathfrak{M}^0$, *the vector bundle* $\mathcal{H}_f$ *has generic fiber type* $(a, k-a)$ *and the vector bundle* $p_{2*}(\mathcal{H}_f(-a))$ *has degree* d.

Proof. This is an application of the Lemma 3.18. □

Lemma 4.5. *If in the Lemma 4.4 we have* $a \geq \frac{k}{2}$, *then* $d \leq 0$. *If we let* $e = -d$ *we have that* a *and* e *are in the range* $(\star)$.

Proof. Let $[f] \in \mathfrak{M}^0$ to be a closed point and let $\mathcal{L} = p_{2*}(\mathcal{H}_f(-a))$. By Lemma 3.28, deg $(\mathcal{L}) \leq 0$.

Since the bundle $\mathcal{H}_f$ has Chern classes c_1 and c_2 as in (4.1) and canonical line subbundle of type $(a, -e)$, by Remark 3.29, it follows that a and e satisfy the required inequalities. □

Proposition 4.6. *For each irreducible component* $\mathfrak{M}$ *of the scheme* $Mor_k(\mathbb{P}^1, M)$ *let* $a \geq \frac{k}{2}$ *be the integer such that for a general* $[f] \in \mathfrak{M}$ *the bundle* $\mathcal{H}_f$ *has generic fiber type* $(a, k-a)$. Assume that $a > \frac{k}{2}$. *Then there is an integer* $e \geq 0$ *(*$e > 0$ *if* $a < k$*), a dense open* $\mathfrak{M}^0 \subseteq \mathfrak{M}$ *and a unique morphism*

$$\nu : \mathfrak{M}^0 \to \mathfrak{B}^0(a, e) \tag{4.7}$$

such that there is a line bundle $\mathcal{N}$ *from* $\mathfrak{M}^0$ *with the property that*

$$\mathcal{H} \cong (id_{\mathbb{P}^1} \times id_C \times \nu)^* \mathcal{F}_{\mathfrak{B}^0(a,e)} \otimes \mathcal{N} \tag{4.8}$$

The map ν *sends a point* $[f] \in \mathfrak{M}_0$ *to* $[\mathcal{H}_f] \in \mathfrak{B}^0(a, e)$.

Proof. Consider the open $\mathfrak{M}^0$ from Lemma 4.4 and let $\mathcal{H}_{\mathfrak{M}^0}$ be the restriction of the universal bundle $\mathcal{H}$ to $\mathbb{P}^1 \times C \times \mathfrak{M}^0$. Then for any point $[f] \in \mathfrak{M}^0$ the vector bundle $\mathcal{H}_f$ has Chern classes c_1, c_2 as in (4.1. By Lemma 4.4 and Lemma 4.5, there is an integer $e \geq 0$ such that $\mathcal{H}_f$ has subcanonical line bundle of type $(a, -e)$. Since $\mathfrak{B} = \mathfrak{B}(a, e)$ is a fine moduli space for such vector bundles, there is a unique morphism:

$$\nu : \mathfrak{M}^0 \to \mathfrak{B}, \quad \text{such that} \quad \mathcal{H}_{\mathfrak{M}^0} \cong (\mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_C \times \nu)^* \mathcal{F}_{\mathfrak{B}} \otimes \mathcal{N}$$

for some line bundle $\mathcal{N}$ from $\mathfrak{M}^0$.

Note that since all closed points of $\mathfrak{M}^0$ are mapped to $\mathfrak{B}^0$, the morphism ν factors through $\mathfrak{B}^0$. □

Corollary 4.7. *The morphism* ν *of (4.7) is injective.*

Proof. The morphism ν is induced by $\mathcal{H}_{\mathfrak{M}^0}$. Since $\mathcal{H}_f$ determines the morphism f, ν is injective. □

Irreducible components with unequal generic splitting

Proposition 4.8. *For each irreducible component* $\mathfrak{M}$ *of the scheme* $Mor_k(\mathbb{P}^1, M)$*, for which there is an integer* $a > \frac{k}{2}$ *such that the general* $[f] \in \mathfrak{M}$ *has the property that the bundle* $\mathcal{H}_f$ *has generic fiber type* $(a, k-a)$*, and there is an integer* e *such that* $\mathfrak{M}$ *is the closure of the image of the morphism*

$$\mu : \mathfrak{B}^0(a, e) \to Mor_k(\mathbb{P}^1, M)$$

Proof. Consider the morphisms (4.7) and (4.2):

$$\nu : \mathfrak{M}^0 \to \mathfrak{B}^0(a, e) \quad \text{and} \quad \mu : \mathfrak{B}^0(a, e) \to \mathfrak{M}^0$$

Let $\mathfrak{B} = \mathfrak{B}(a, e)$. Consider the composition:

$$\mathfrak{M}^0 \xrightarrow{\ \nu\ } \mathfrak{B}^0 \xrightarrow{\ \mu\ } \mathrm{Mor}_k(\mathbb{P}^1, M)$$

We prove that this composition is the inclusion morphism

$$i : \mathfrak{M}^0 \hookrightarrow \mathrm{Mor}_k(\mathbb{P}^1, M).$$

Using the universal property of the evaluation morphism, it is enough to prove that the following morphisms are the same:

$$\mathbb{P}^1 \times \mathfrak{M}^0 \xrightarrow{(\mathrm{id}_{\mathbb{P}^1}\times i)} \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, M) \xrightarrow{ev} M$$
$$\mathbb{P}^1 \times \mathfrak{M}^0 \xrightarrow{(\mathrm{id}_{\mathbb{P}^1}\times \mu\nu)} \mathbb{P}^1 \times \mathrm{Mor}_k(\mathbb{P}^1, M) \xrightarrow{ev} M$$

By the definition of the moduli scheme M, the two morphisms are the same if and only if there is a line bundle $\mathcal{L}$ from $\mathbb{P}^1 \times \mathfrak{M}^0$ such that:

$$\mathcal{H}_{\mathfrak{M}^0} = (\mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_C \times i)^*\mathcal{H} \cong (\mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_C \times \mu\nu)^*\mathcal{H} \otimes \mathcal{L}.$$

From formulas (4.3) and (4.8) we have that there are line bundles $\mathcal{N}_{\mathfrak{B}^0}$ from $\mathfrak{B}^0$ and $\mathcal{N}_{\mathfrak{M}^0}$ from $\mathfrak{M}^0$ such that:

$$(\mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_C \times \mu)^*\mathcal{H} \cong \mathcal{F}_{\mathfrak{B}^0} \otimes \mathcal{N}_{\mathfrak{B}^0}^{-1}$$
$$(\mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_C \times \nu)^*\mathcal{F}_{\mathfrak{B}^0} \cong \mathcal{H}_{\mathfrak{M}^0} \otimes \mathcal{N}_{\mathfrak{M}^0}^{-1}$$

It follows that $i = \mu \circ \nu$.

The scheme $\mathfrak{B}$ is reduced and irreducible, therefore the morphism μ factors through an irreducible component $\mathfrak{M}'$ of $\mathrm{Mor}_k(\mathbb{P}^1, M)$. As $i = \mu \circ \nu$, it follows that $\mathfrak{M}'$ contains $\mathfrak{M}^0$. Hence, $\mathfrak{M}' = \mathfrak{M}$ and μ factors through $\mathfrak{M}^0$. The two morphisms

$$\mu : \mathfrak{B}^0 \to \mathfrak{M}^0 \quad \text{and} \quad \nu : \mathfrak{M}^0 \to \mathfrak{B}^0$$

satisfy $\mu \circ \nu = \mathrm{id}_{\mathfrak{M}^0}$. It follows that μ and ν are isomorphisms.

Note that by the same formulas, (4.3) and (4.8), we have that there is a line bundle $\mathcal{L}$ from $\mathfrak{B}^0$ such that:

$$(\mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_C \times \nu\mu)^*\mathcal{F}_{\mathfrak{B}^0} \cong \mathcal{F}_{\mathfrak{B}^0} \otimes \mathcal{L}.$$

This proves that $\nu \circ \mu = id_{\mathfrak{B}^0}$. □

Using the same methods as in the proofs of Proposition 4.6 and Proposition 4.8 one can prove the following Proposition.

Proposition 4.9. *The morphism μ is an isomorphism onto its image.*

Definition 4.10. *We define the subscheme $\mathfrak{M}(a, e) \subset Mor_k(\mathbb{P}^1, M)$ to be the quasi-projective scheme given by the image of the morphism μ (taken with the induced reduced structure).*

Note that $\mathfrak{M}(a, e) \cong \mathfrak{B}^0(a, e)$ is an integral scheme.

4.2 The nice component

We prove that there is a special subscheme $\mathfrak{M} \subset \mathrm{Mor}_k(\mathbb{P}^1, M)$: the general element $[f] \in \mathfrak{M}$ has the property that the bundle $\mathcal{H}_f$ has generic splitting type balanced. We prove that this subscheme is an irreducible component of the expected dimension with the general point unobstructed. We call this component the *nice* component.

4.2.1 The nice component for k odd

We prove that for k odd the nice component is given by one of the subschemes $\mathfrak{M}(a, e)$, namely, $\mathfrak{M}(\frac{k+1}{2}, \frac{k-1}{2})$.

Theorem 4.11. *For any* odd *integer* $k = 2a + 1$, *the scheme* $\mathfrak{M} = \overline{\mathfrak{M}(a + 1, a)}$ *is the unique irreducible component of* $Mor_k(\mathbb{P}^1, M)$ *for which the general* $[f] \in \mathfrak{M}$ *has the property that the bundle* $\mathcal{H}_f$ *has generic fiber type* $(a + 1, a)$. *In addition,* $\mathfrak{M}$ *has the following properties:*

i. *It has the expected dimension* $2k + 3g - 3$

ii. *It is unobstructed at the general point*

Proof. For a point $[f] \in \mathfrak{M}(a + 1, a)$ the bundle $\mathcal{H}_f$ has fiber type $(a + 1, a)$. In fact, for *any* $c \in C$ we have:

$$\mathcal{H}_{f|\mathbb{P}^1 \times \{c\}} \cong O(a + 1) \oplus O(a).$$

By Lemma 1.11, for such an f we have

$$H^1(\mathbb{P}^1, f^*T_M) = 0.$$

Therefore, there is a unique irreducible component $\mathfrak{M}$ containing f and moreover, $\mathfrak{M}$ has the expected dimension $2k + 3g - 3$ and is smooth at the point f. Since $\mathfrak{M}(a + 1, a)$ has the expected dimension, it follows that

$$\mathfrak{M} = \overline{\mathfrak{M}(a + 1, a)}.$$

□

We call this component the *nice* component and we denote it by $\mathfrak{M}_{\mathfrak{nice}}$.

4.2.2 The nice component for k even

In this section we assume k is even. We let $k = 2a$. We define a subscheme $\mathfrak{M}_{even} \subset \mathrm{Mor}_k(\mathbb{P}^1, M)$ such that if $[f] \in \mathfrak{M}_{even}$ corresponds to a bundle $[\mathcal{H}_f]$ in the good locus $\mathfrak{B}^0_{even}$ of Theorem 3.42.

We prove that $\mathfrak{M}_{even} \cong \mathfrak{B}^0_{even}$. We prove that $\mathfrak{M}_{even}$ is the unique irreducbile component of $\mathrm{Mor}_k(\mathbb{P}^1, M)$ such that for a general point $[f]$ the bundle $\mathcal{H}_f$ has generic splitting type (a, a).

The subscheme $\mathfrak{M}_{even}$ of $\mathrm{Mor}_k(\mathbb{P}^1, M)$

Recall that $\mathfrak{B}_{even}$ is the moduli scheme for rank 2 vector bundles $\mathcal{F}$ on $\mathbb{P}^1 \times C$ with canonical sequence of the form:

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{F} \to O(a) \boxtimes O_D \to 0 \qquad (\dagger)$$

where $\mathcal{E}$ is a stable vector bundle on C and D is a zero cycle on C.

Let $\mathfrak{B}^0_{even} \subseteq \mathfrak{B}_{even}$ be the dense open in Theorem 3.42, corresponding to vector bundles $\mathcal{F}$ inducing morphisms $f : \mathbb{P}^1 \to M$. Recall that $\mathfrak{B}^0_{even}$ is the coarse moduli scheme for the functor F^0:

$$F^0 : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Sets}$$

$$F^0(S) = \{\text{Isomorphism classes of rank 2 vector bundles } \mathcal{G} \text{ on } \mathbb{P}^1 \times C \times S$$
$$\text{such that } \forall s \in S, \text{ closed point, the bundle } \mathcal{G}_s \text{ has canonical sequence of type } (\dagger)\}$$

Define a transformation of functors:

$$T : F^0 \longrightarrow \mathrm{Hom}(-, \mathrm{Mor}_k(\mathbb{P}^1, M))$$
$$T(S) : F^0(S) \to \mathrm{Hom}(S, \mathrm{Mor}_k(\mathbb{P}^1, M))$$
$$\mathcal{G} \mapsto [g]$$

where if $\mathcal{G}$ is a vector bundle on $\mathbb{P}^1 \times C \times S$ then $g : S \to \mathrm{Mor}_k(\mathbb{P}^1, M)$ comes from the morphism $\mathbb{P}^1 \times S \to M$ induced $\mathcal{G}$.

Theorem 4.12. *There is a morphism*

$$\rho : \mathfrak{B}^0_{even} \to Mor_k(\mathbb{P}^1, M)$$

such that $\rho([\mathcal{F}]) = [f]$, where f is the morphism induced by $\mathcal{F}$. Moreover, there is a comutative diagram:

$$\begin{array}{ccc} F^0 & \xrightarrow{\;T\;} & Hom(-, Mor_k(\mathbb{P}^1, M)) \\ & \searrow \quad \nearrow & \\ & Mor(-, \mathfrak{B}^0_{even}) & \end{array} \qquad (4.9)$$

where $F^0 \longrightarrow Hom(-, \mathfrak{B}^0_{even})$ *is the morphism coming from the fact that* $\mathfrak{B}^0_{even}$ *is a coarse moduli scheme for the functor* F^0 *of (3.4).*

Irreducible components with equal generic splitting

Theorem 4.13. *Each irreducible component* $\mathfrak{M}$ *of the scheme* $Mor_k(\mathbb{P}^1, M)$*, for which the general* $[f] \in \mathfrak{M}$ *has the property that the bundle* $\mathcal{H}_f$ *has generic fiber type* (a, a) *(where* $k = 2a$*), has the following properties:*

i. *It has the expected dimension* $2k + 3g - 3$

ii. $Mor_k(\mathbb{P}^1, M)$ *is unobstructed at the general point of* $\mathfrak{M}$

iii. *There is a dense open* $\mathfrak{M}^0 \subseteq \mathfrak{M}$ *such that for* $[f] \in \mathfrak{M}^0$*, the vector bundle* $\mathcal{H}_f$ *has canonical sequence of the form:*

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{H}_f \to O(a-1) \boxtimes O_D \to 0$$

with D *some divisor on* C *of degree* a *and* $\mathcal{E}$ *some* stable *rank* 2 *vector bundle with det* $(\mathcal{E}) \cong O(x_0 - D)$.

Proof. Parts i. and ii. are consequences of Lemma 1.11.

For Part iii, we have from Lemma 4.4 that there is an integer d and a dense open $\mathfrak{M}^0 \subseteq \mathfrak{M}$ with the property that for $[f] \in \mathfrak{M}^0$, the vector bundle $\mathcal{H}_f$ has canonical sequence of the form:

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{H}_f \to p_1^* O(a) \otimes \mathcal{I} \to 0 \qquad (4.10)$$

where $\mathcal{E}$ is some rank 2 vector bundle of degree d, $Z \subset \mathbb{P}^1 \times C$ is a 0-cycle, D is the scheme theoretic image $p_2(Z)$ and $\mathcal{I}$ is the ideal sheaf of Z in $\mathbb{P}^1 \times D$.

Note that by ii. we have that $\mathfrak{M}^0$ is a smooth scheme. Consider the Kodaira-Spencer map at a point $f \in \mathfrak{M}^0$, for the family over $\mathfrak{M}^0$ of vector bundles on $\mathbb{P}^1 \times C$ given by the universal bundle $\mathcal{H}$:

$$\omega : T_f \mathfrak{M}^0 \to \mathrm{H}^1(\mathbb{P}^1 \times C, \mathcal{E}nd(\mathcal{H}_f))$$

By Observation (1.13) and since $\mathrm{Mor}_k(\mathbb{P}^1, M)$ is smooth at $[f]$, it follows that ω is an isomorphism for any $f \in \mathfrak{M}^0$. Hence, by Corollary 3.25 and Lemma 3.23 and eventually shrinking $\mathfrak{M}^0$, we get that $\mathcal{I} \cong O_{\mathbb{P}^1 \times D}(-1) \cong O(-1) \boxtimes O_D$. By Corollary 3.35, it follows that

$$\det(\mathcal{E}) \cong O(x_0 - D) \quad \text{and} \quad d = 1 - a$$

From Lemma 1.9, it follows that there is an open in $\mathfrak{M}^0$, possibly empty, such that the bundle $\mathcal{E}$ is stable. We prove that this open is dense.

We claim that if $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, M)$ is an element such that the bundle $\mathcal{H}_f$ has canonical sequence (†) with the bundle $\mathcal{E}$ not stable, then f is in the image of the morphism μ_d from Remark 4.3, for some $0 < d \leq \frac{a-1}{2}$:

$$\mu_d : \mathfrak{B}^0(\frac{k}{2}, d) \to \mathrm{Mor}_k(\mathbb{P}^1, M)$$

Then Part iii. of our Theorem follows, since we have, for any $0 < d \leq \frac{a-1}{2}$:

$$\dim \mathfrak{B}(\frac{k}{2}, d) = 2g + 3a + 2d - 1 \leq 2g + 4a - 2 <$$
$$< 4a + 3g - 3 = 2k + 3g - 3 = \dim \mathfrak{M}.$$

We prove now the claim. Assume that the bundle $\mathcal{H}_f$ has canonical sequence as in (4.10), with the bundle $\mathcal{E}$ not stable. Then there is a line subbundle $\mathcal{L}'$ of $\mathcal{E}$, with

$$\deg\ (\mathcal{L}') \geq \frac{\deg\ (\mathcal{E})}{2} = \frac{1-a}{2}.$$

Consider the injective morphism:

$$0 \to O(a) \boxtimes \mathcal{L}' \to \mathcal{H}_f.$$

Let $\mathcal{F}'$ be the saturation of $O(a) \boxtimes \mathcal{L}'$ in $\mathcal{H}_f$ and let $\mathcal{J}$ be the torsion free quotient:

$$0 \to \mathcal{F}' \to \mathcal{H}_f \to \mathcal{J} \to 0 \tag{4.11}$$

The sheaves $\mathcal{F}'$ and $\mathcal{J}$ have rank 1. Since $\mathcal{H}_f$ is locally free, it is in particular a reflexive sheaf. It follows from Lemma A.10 in Appendix that the sheaf $\mathcal{F}'$ is reflexive. Since a reflexive sheaf of rank 1 is a line bundle, it follows that there is a line bundle $\mathcal{L}$ on C such that

$$\mathcal{F}' \cong O(b) \boxtimes \mathcal{L}.$$

Let $d = -\deg\ (\mathcal{L})$. Since $O(a) \boxtimes \mathcal{L}'$ is a subsheaf of $\mathcal{F}'$, we have:

$$b \geq a \quad \text{and} \quad -d \geq \deg\ (\mathcal{L}') > \frac{1-a}{2}$$

If we push forward to C the short exact sequence (4.11) twisted by $O(-b)$, we get another short exact sequence:

$$0 \to \mathcal{L} \to p_{2*}\mathcal{F}(-b) \to p_{2*}\mathcal{J}(-b) \to 0.$$

If $b > a$ then $p_{2*}\mathcal{F}(-b)$ is a torsion sheaf, since the stalk at the generic point is 0. But then $p_{2*}\mathcal{F}(-b) = 0$, since by pushing forward a torsion free sheaf we still get a torsion free sheaf. Therefore, we deduce that if $b > a$ then $\mathcal{L} = 0$, contradiction. We have $b = a$.

So if we assume that $\mathcal{E}$ is not stable, $\mathcal{H}_f$ sits in an exact sequence:

$$0 \to O(a) \boxtimes \mathcal{L} \to \mathcal{H}_f \to \mathcal{J} \to 0$$

where $\mathcal{J}$ is a torsion free sheaf of rank 1 and $\mathcal{L}$ a line bundle of degree

$$-d > \frac{1-a}{2}.$$

As $\mathcal{J}$ is torsion free, it follows that $\mathcal{J} \cong \mathcal{I}_Z \otimes \mathcal{M}$, for $Z \subset \mathbb{P}^1 \times C$ a zero-cycle and $\mathcal{M}$ is a line bundle on $\mathbb{P}^1 \times C$. From a computation with Chern classes it follows that $\mathcal{M} \cong O(\alpha) \boxtimes \mathcal{L}^{-1}(x_0)$. We have an exact sequence:

$$0 \to O(a) \boxtimes \mathcal{L} \to \mathcal{H}_f \to (O(a) \boxtimes \mathcal{L}^{-1}(x_0)) \otimes \mathcal{I}_Z \to 0.$$

By Lemma 3.28, it follows that $d > 0$.

We conclude that if f corresponds to a bundle $\mathcal{H}_f$ with $\mathcal{E}$ unstable, then f is in the image of the map μ_d, for some $0 \leq d \leq \frac{a-1}{2}$. □

Note that if $k = 2$ then $\mathcal{E}$ is always semistable.

Proposition 4.14. *Let $\mathfrak{M}$ be an irreducible component of $Mor_k(\mathbb{P}^1, M)$ for which the general $[f] \in \mathfrak{M}$ has the property that the bundle $\mathcal{H}_f$ has generic fiber type (a, a). Let $\mathfrak{M}$ be the open of Theorem 4.13. Then there is a morphism*

$$\tau : \mathfrak{M}^0 \to \mathfrak{B}^0_{even}.$$

The morphism τ sends a point $[f]$ to the point $[\mathcal{H}_f] \in \mathfrak{B}^0_{even}$.

Proof. Let $\mathcal{H}_{\mathfrak{M}^0}$ be the restriction of the universal bundle $\mathcal{H}$ to $\mathbb{P}^1 \times C \times \mathfrak{M}^0$. Since $\mathcal{H}_{\mathfrak{M}^0}$ is a family of vector bundles with the properties in Theorem 4.13 and since $\mathfrak{B}^0_{even}$ is a coarse moduli scheme for such vector bundles, it follows that $\mathcal{H}_{\mathfrak{M}^0}$ corresponds to a morphism $\tau : \mathfrak{M}^0 \to \mathfrak{B}^0_{even}$. □

Theorem 4.15. *For any* even *integer $k = 2a$, there is a unique irreducible component $\mathfrak{M}$ of $Mor_k(\mathbb{P}^1, M)$ for which the general $[f] \in \mathfrak{M}$ has the property that the bundle $\mathcal{H}_f$ has generic fiber type (a, a) and there is a dense open $\mathfrak{M}^0 \subset \mathfrak{M}$ such that $\mathfrak{M}^0 \cong \mathfrak{B}^0_{even}$. In addition $\mathfrak{M}$ has the following properties:*

i. It has the expected dimension $2k + 3g - 3$

ii. It is unobstructed at the general point

We call this component the *nice* component and sometimes denote it with $\mathfrak{M}_{\text{nice}}$.

Definition 4.16. *We define the subscheme $\mathfrak{M}_{even} \subset Mor_k(\mathbb{P}^1, M)$ to be given by the image of ρ in $Mor_k(\mathbb{P}^1, M)$ (taken with the induced reduced structure).*

Note that $\mathfrak{M}_{even} \cong \mathfrak{B}^0_{even}$ is an integral scheme.

Proof. Consider the morphism $\tau : \mathfrak{M}^0 \to \mathfrak{B}^0_{even}$. It is the image of the element $\mathcal{H}_{\mathfrak{M}^0}$ under the map:

$$T(\mathfrak{M}^0) : F^0(\mathfrak{M}^0) \longrightarrow \mathrm{Hom}(\mathfrak{M}^0, \mathfrak{B}^0_{even}).$$

The element $\mathcal{H}_{\mathfrak{M}^0} \in F^0(\mathfrak{M}^0)$ is sent by T to the inclusion morphsim

$$i : \mathfrak{M}^0 \hookrightarrow \mathrm{Mor}_k(\mathbb{P}^1, M).$$

From 4.12, there is a commutative diagram:

$$\begin{array}{ccc} F^0(\mathfrak{M}^0) & \xrightarrow{\ T\ } & \mathrm{Hom}(\mathfrak{M}^0, \mathrm{Mor}_k(\mathbb{P}^1, M)) \\ & \searrow \quad \nearrow & \\ & \mathrm{Mor}(\mathfrak{M}^0, \mathfrak{B}^0_{even}) & \end{array}$$

It follows that the following composition is the inclusion morphism i:

$$\mathfrak{M}^0 \xrightarrow{\ \tau\ } \mathfrak{B}^0_{even} \xrightarrow{\ \rho\ } \mathrm{Mor}_k(\mathbb{P}^1, M)$$

Since $\mathfrak{B}_{even}$ is reduced and irreducible, we have

$$\mathfrak{M}^0 \cong \mathfrak{B}^0_{even}.$$

Hence, there is a unique irreducible component with the given properties and it has a dense open isomorphic to $\mathfrak{B}^0_{even}$. □

Note that we could have used the fact that the scheme $\mathfrak{B}$ is a coarse moduli scheme for some functor F to give another proof of Theorem 4.8.

Conclusion about the relations between the schemes $\mathfrak{M}(a, e)$ and $\mathfrak{M}_{even}$

Lemma 4.17. *Let $\mathfrak{M}(a, e)$ and $\mathfrak{M}_{even}$ be the subschemes corresponding to $\mathfrak{B}^0(a, e)$ and $\mathfrak{B}^0_{even}$. The following relations hold:*

i. The schemes $\mathfrak{M}(a, e)$ for all a and e in the range $(\star)$ are mutually disjoint.

ii. In the case when k is even, $\mathfrak{M}_{even}$ is disjoint from any of the $\mathfrak{M}(a, e)$. In particular, the nice component $\overline{\mathfrak{M}_{even}}$ cannot be contained in any of the $\overline{\mathfrak{M}(a, e)}$.

iii. If $[f] \in \overline{\mathfrak{M}(a, e)} \setminus \mathfrak{M}(a, e)$ then $[f] \in \mathfrak{M}(a', e')$, with $a' > a$ and $e' < e$

iv. If $[f] \in \overline{\mathfrak{M}_{even}} \setminus \mathfrak{M}_{even}$ then $[f] \in \mathfrak{M}(a, e)$, with $a > \frac{k}{2}$.

v. The scheme $Mor_k(\mathbb{P}^1, M)$ is a disjoint union of the locally closed sets $\mathfrak{M}(a, e)$ (add $\mathfrak{M}_{even}$, if k even).

Proof. This Lemma is a straightforward consequence of the previous results. The schemes $\mathfrak{M}(a,e)$ and $\mathfrak{M}_{even}$ correspond to the moduli of vector bundles on $\mathbb{P}^1 \times C$. An element $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, M)$ corresponds to the bundle $\mathcal{H}_f$. Since the bundle $\mathcal{H}_f$ is uniquely determined by its canonical sequence, it follows that the schemes $\mathfrak{M}(a,e)$ (for all a and e in the range $(\star)$) and $\mathfrak{M}_{even}$ are mutually disjoint. Since any element $[f] \in \mathrm{Mor}_k(\mathbb{P}^1, M)$ corresponds to a bundle $\mathcal{H}_f$ in some $\mathfrak{B}^0(a,e)$ or in $\mathfrak{B}^0_{even}$, it follows that $\mathrm{Mor}_k(\mathbb{P}^1, M)$ is covered by these sets. By Proposition 3.19, it follows that if we specialize, then a goes up, while e goes down. □

4.3 Irreducible components $\mathfrak{M}(a,e)$

We prove that all the subschemes $\mathfrak{M}(a,e)$ which have at least the expected dimension are dense opens in some irreducible components.

Let $k \geq 1$ a fixed integer and consider the range $(\star)$ for the pairs of integers (a,e):

$$\{(a,e)| \quad k \geq a > k/2, \quad \frac{k-a}{2a-k} \geq e > 0\} \cup \{(k,0)\} \tag{$\star$}$$

Let A be the branch of the hyperbola in the plane with axis a and e given by:

$$e = \frac{k-a}{2a-k} = -\frac{1}{2} + \frac{k}{2(2a-k)}, \quad a > \frac{k}{2} \tag{A}$$

Basic Lemma 4.18. *For each (a,e) in the range $(\star)$, there is a closed integral subscheme $\mathfrak{M}(a,e)$ of $Mor_k(\mathbb{P}^1, M)$ satisfying the following properties:*

i. *The dimension of $\mathfrak{M}(a,e)$ is:*

$$dim\ \mathfrak{M}(a,e) = (2a-k+2)g + (3k-3a-1) - e(2a-k-2) \tag{4.12}$$

ii. *If $\overline{\mathfrak{M}(a,e)}$ is an irreducible component, then it has dimension at least the expected one:*

$$dim\ \mathfrak{M}(a,e) \geq 2k + 3g - 3 = \ exp.\ dim.\ Mor_k(\mathbb{P}^1, M)$$

iii. *If $\mathfrak{M}(a,e) \subset \overline{\mathfrak{M}(a',e')}$, then $a' \geq a$ and $e' \leq e$.*

iv. *If k is* odd, *then $\overline{\mathfrak{M}(\frac{k+1}{2}, \frac{k-1}{2})}$ is the nice component $\mathfrak{M}_{\mathsf{nice}}$*

v. *If $\mathfrak{M}$ is an irreducible component which is different from the nice component, then there are uniquely determined a and e in the range $(\star)$, such that*

$$\mathfrak{M} = \overline{\mathfrak{M}(a,e)}$$

Proof. These are easy consequences or restatements of previous results.

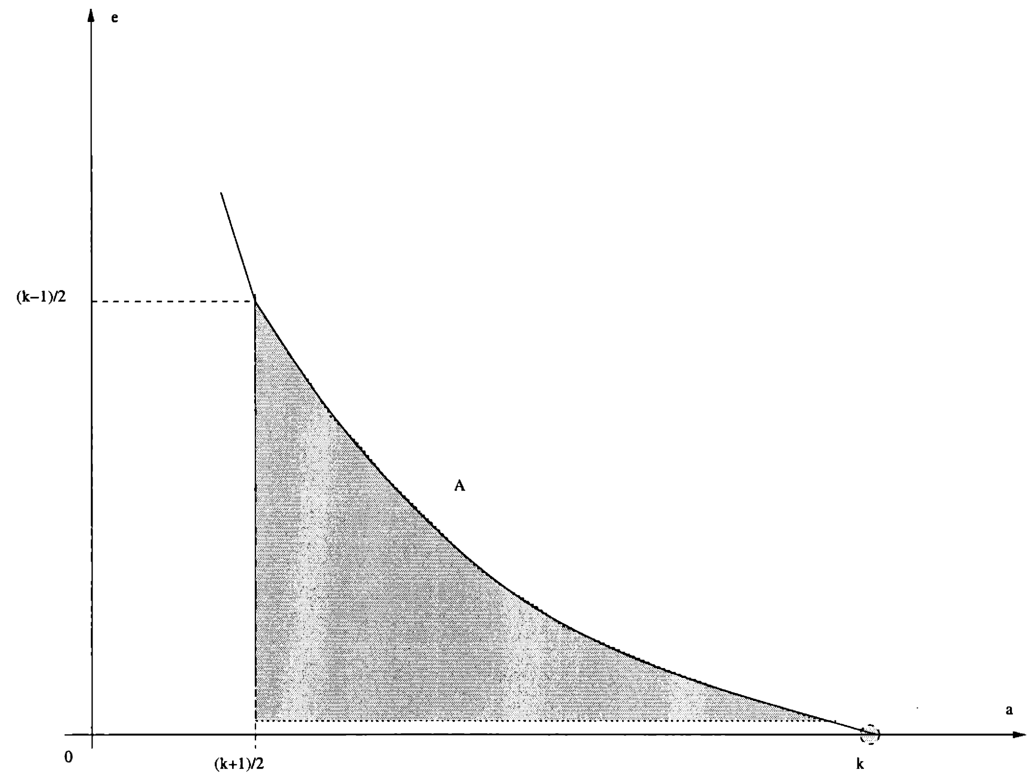

Figure 4-1: This shows the range ($\star$) of the pairs of integers (a, e). The curve A has equation $e = \frac{k-a}{2a-k}$.

i. It follows from Proposition 4.9 and formula (3.16).

ii. This is the statement of Theorem 4.11

iv. It follows from Proposition 3.19.

v. It is a consequence of Theorem 4.8, Theorem 4.15 and Corollary 4.17.

$\square$

Basic Lemma 4.19. *For integers a and e in the range ($\star$), the dimensions of the subschemes $\mathfrak{M}(a, e)$ grow as follows:*

i. *If $a = \frac{k+1}{2}$ and $e' > e$ then*

$$dim\ \mathfrak{M}(a, e') > dim\ \mathfrak{M}(a, e)$$

ii. *If $a = \frac{k}{2} + 1$ then for any e*

$$dim\ \mathfrak{M}(a, e) = 4g + \frac{3k}{2} - 4$$

iii. *If* $k \geq a > \frac{k}{2}+1$ *and* $e' > e$ *then*

$$dim\ \mathfrak{M}(a,e') < dim\ \mathfrak{M}(a,e)$$

iv. *If* $a > a'$ *and* $e \geq (g-1)$, *then*

$$dim\ \mathfrak{M}(a,e) < dim\ \mathfrak{M}(a',e)$$

v. *If* $a > a'$ *and* $e \leq (g-2)$, *then*

$$dim\ \mathfrak{M}(a,e) > dim\ \mathfrak{M}(a',e)$$

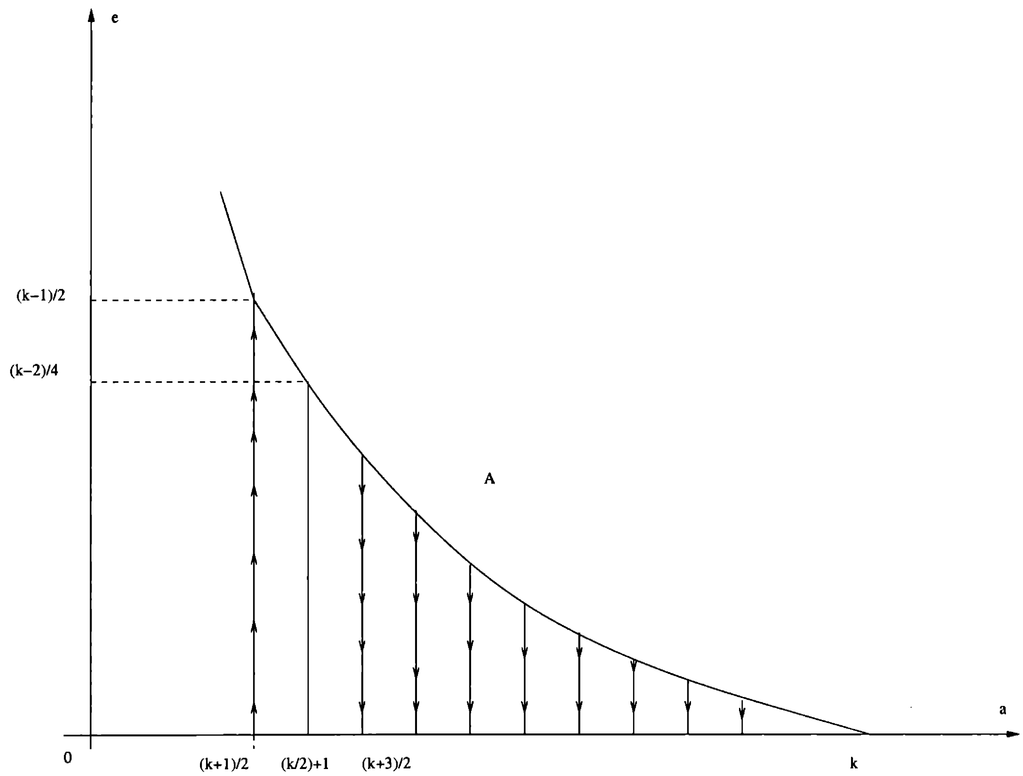

Figure 4-2: The arrows show the direction in which dimensions grow when a is constant. If k is even and $a = \frac{k}{2}+1$, then the dimension is constant for any e in the given range.

Proof. This is clear since

$$\begin{aligned}\dim \mathfrak{M}(a,e') - \dim \mathfrak{M}(a,e) &= (2a-k-2)(e-e') \\ \dim \mathfrak{M}(a',e) - \dim \mathfrak{M}(a,e) &= (2g-2e-3)(a'-a).\end{aligned}$$

□

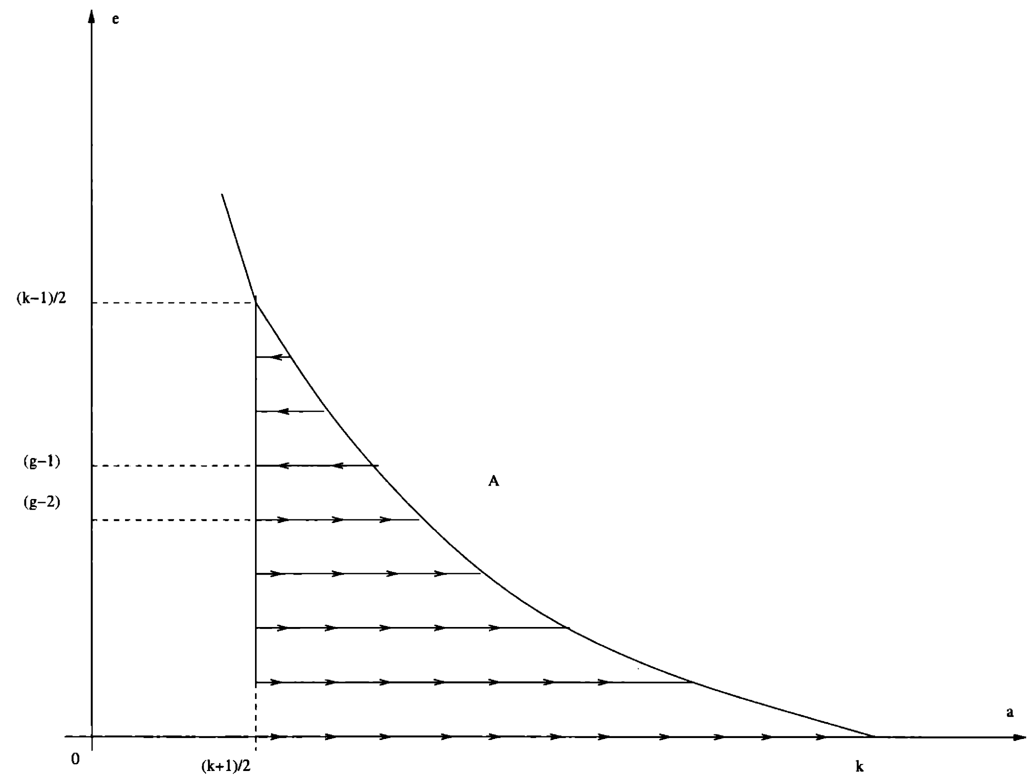

Figure 4-3: The arrows show the direction in which dimensions grow when e is constant. In this picture we assumed $\frac{k-1}{2} \geq (g-1)$.

Basic Lemma 4.20. *The locus of pairs (a, e) in the range $(\star)$ where the dimension of $\mathfrak{M}(a, e)$ is the expected dimension $2k + 3g - 3$ can be described as follows:*

i. *On the line $a = \frac{k+1}{2}$, at the point $(a, e) = (\frac{k+1}{2}, \frac{k-1}{2})$; note that for all the other e ($e < \frac{k-1}{2}$), the dimension is strictly less than expected*

ii. *On the line $a = \frac{k}{2} + 1$ (we assume k even) if and only if $k = 2g - 2$; note that on this line the dimension is constant and we have*

a. *If $k > (2g - 2)$, the dimension is less than the expected dimension*

b. *If $k = 2g - 2$, the dimension is equal to the expected dimension*

c. *If $k < (2g - 2)$, the dimension is bigger than the expected dimension*

ii. *If $a \geq \frac{k+3}{2}$, on the hyperbola B, given by equation:*

$$e = \frac{2g-3}{2} - \frac{2g-2-k}{2(2a-k-2)} \tag{B}$$

inside the range $(\star)$ for $a \geq \frac{k+3}{2}$.

Note that if $k = 2g-2$ then B is the line $e = \frac{2g-3}{2}$ union with the line $a = g = \frac{k}{2}+1$.

Definition 4.21. *Define the region R to be the region inside the region $(\star)$ where the dimension is at least the expected one.*

The region R consists of the following:

i. The point $(\frac{k+1}{2}, \frac{k-1}{2})$, if k odd

ii. The part of the line $a = \frac{k}{2}+1$ inside the region $(\star)$, if k is even and $k \leq (2g-2)$

iii. The region between the graphs of the curves A and B for $a \geq \frac{k+3}{2}$ inside the range $(\star)$, where the dimension is at least the expected one (see Figures 4-4 to 4-12)

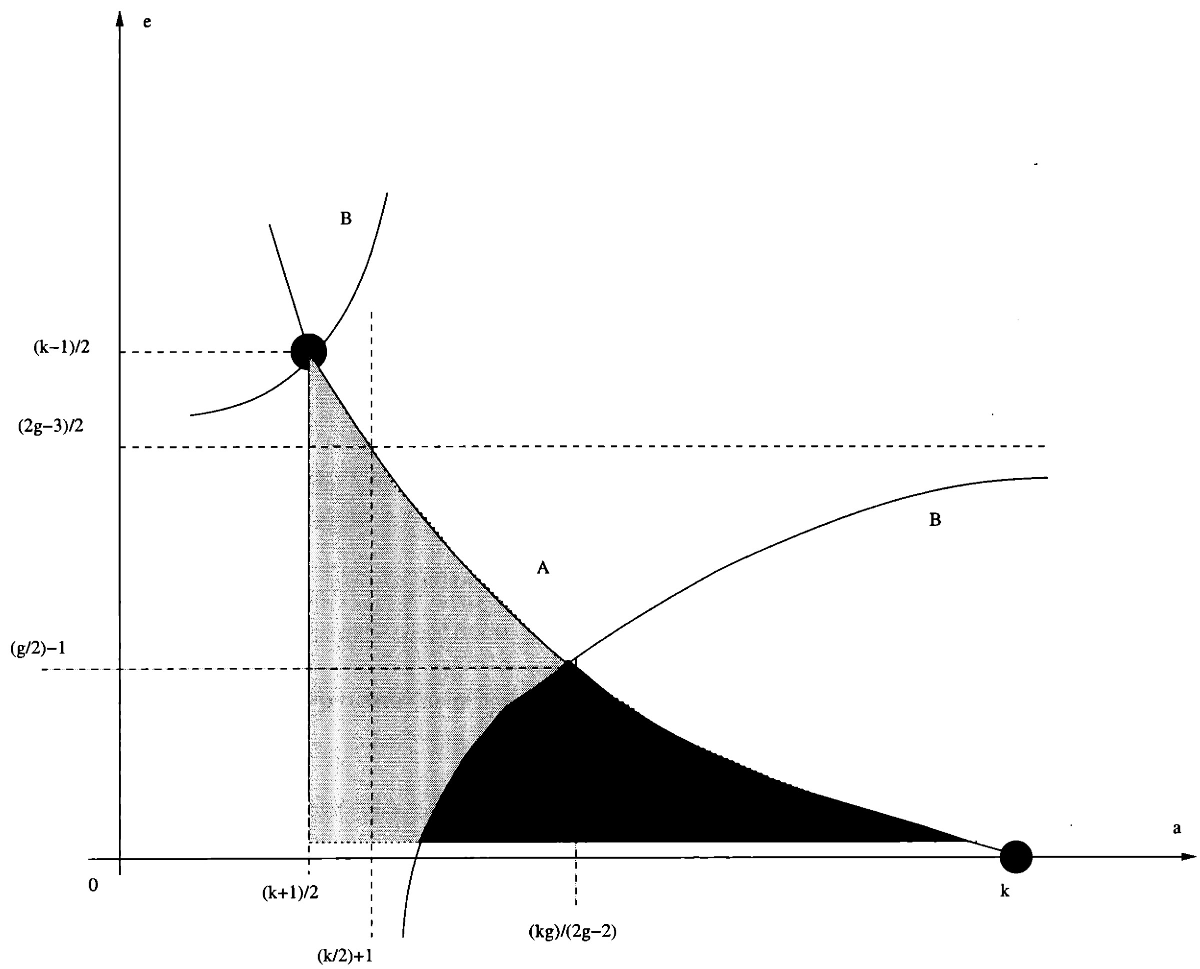

Figure 4-4: The dark shaded part is the region R when $g \geq 3$ and k is an odd integer with $k > 2g-2$

Basic Lemma 4.22. *The hyperbola B intersects the curve A at the points:*

$$a = \frac{kg}{2(g-1)}, \quad e = \frac{g}{2} - 1 \quad \textit{and} \quad a = \frac{k+1}{2}, \quad e = \frac{k-1}{2}$$

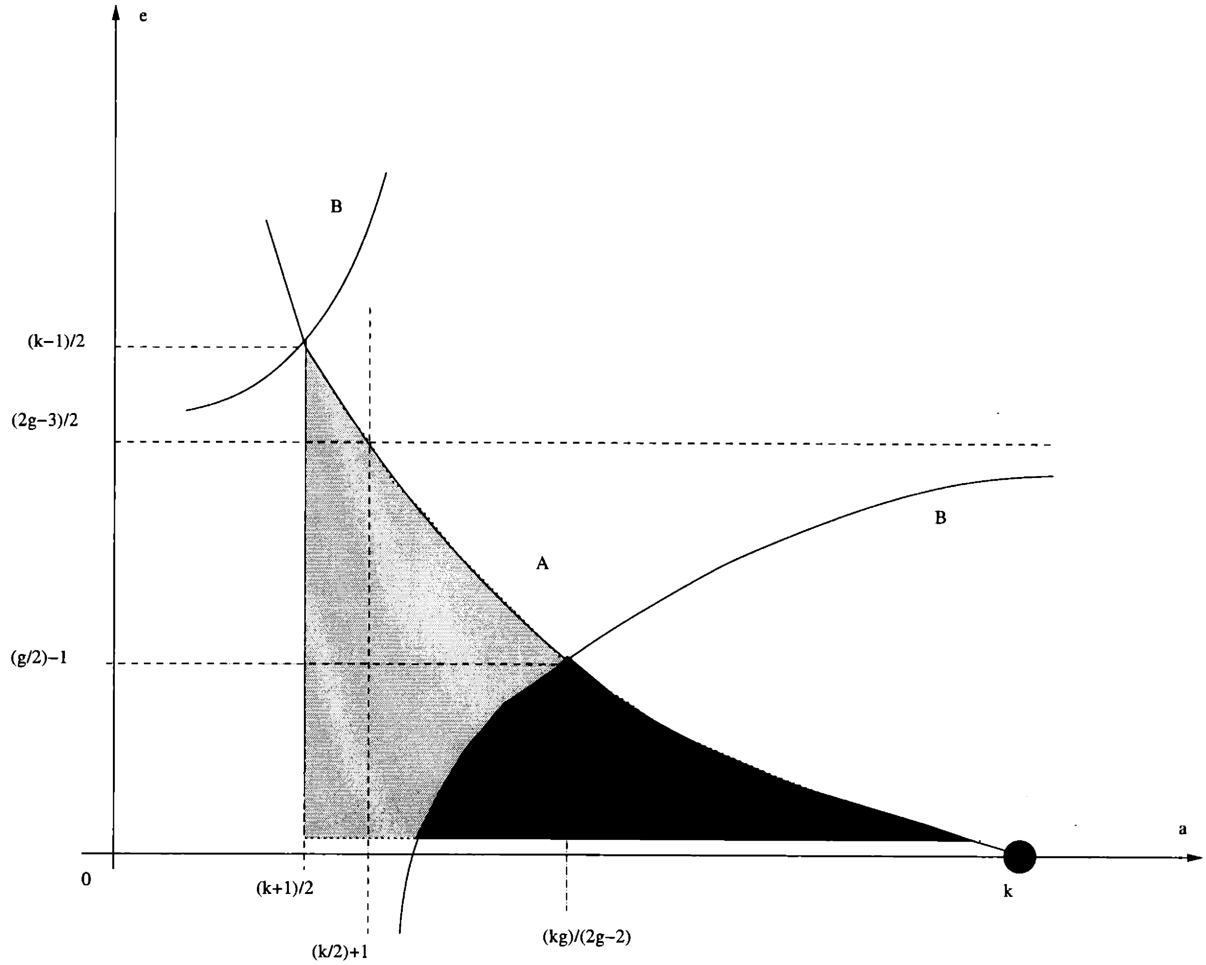

Figure 4-5: The dark shaded part is the region R when $g \geq 3$ and k is an even integer with $k > 2g-2$

The curve B intersects the a-axis at the point

$$a = \frac{(k+1)(g-1)-1}{2g-3}$$

Theorem 4.23. *For a pair of integers (a,e) in the range $(\star)$ the scheme $\overline{\mathfrak{M}(a,e)}$ is an irreducible component of $Mor_k(\mathbb{P}^1, M)$ if and only if (a,e) is in the region R, or, equivalently, if the dimension of $\mathfrak{M}(a,e)$ is at least the exepected dimension. We have:*

i. If k is odd, these are all the irreducible components; note that the nice component is among them: $\mathfrak{M}_{\text{nice}} = \overline{\mathfrak{M}(\frac{k+1}{2}, \frac{k-1}{2})}$

ii. If k is even, then the schemes $\mathfrak{M}(a,e)$ together with the nice component $\mathfrak{M}_{\text{nice}}$, are all the irreducible components.

Moreover, the schemes $\overline{\mathfrak{M}(a,e)}$, for pairs of integers (a,e) in the region $(\star)$, but not in the region R, are contained in the nice component $\mathfrak{M}_{\text{nice}}$.

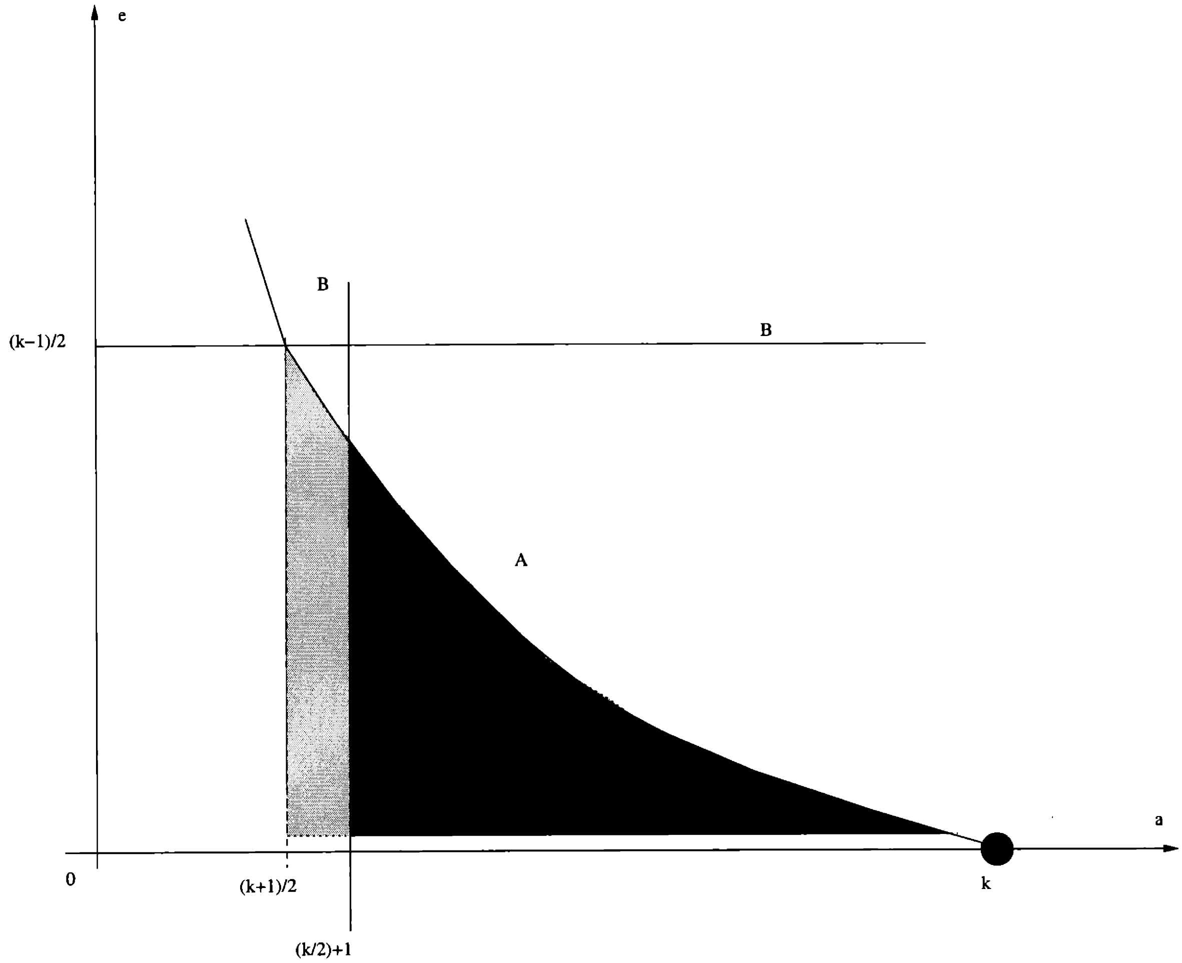

Figure 4-6: The dark shaded part is the region R when $g \geq 3$ and $k = 2g - 2$

Proof. Let $\mathfrak{M}$ be an irreducible component of $\mathrm{Mor}_k(\mathbb{P}^1, M)$, which is not the nice component. Since $\mathfrak{M}$ is not the nice component, by Basic Lemma 4.18, it follows that there are integers a and e such that $\mathfrak{M} = \overline{\mathfrak{M}(a,e)}$. Since $\mathfrak{M}$ has at least the expected dimension, the pair (a, e) is in the range R. By Theorem 4.11 and since $\mathfrak{M}$ is not the nice component, we have

$$(a, e) \neq (\frac{k+1}{2}, \frac{k-1}{2}).$$

We prove now that for any (a, e) in the range R, the subscheme $\overline{\mathfrak{M}(a,e)}$ is an irreducible component. We can assume that

$$(a, e) \neq (\frac{k+1}{2}, \frac{k-1}{2}).$$

If there is an irreducible component $\mathfrak{M} = \overline{\mathfrak{M}(a',e')}$ such that $\mathfrak{M}(a,e) \subset \overline{\mathfrak{M}(a',e')}$,

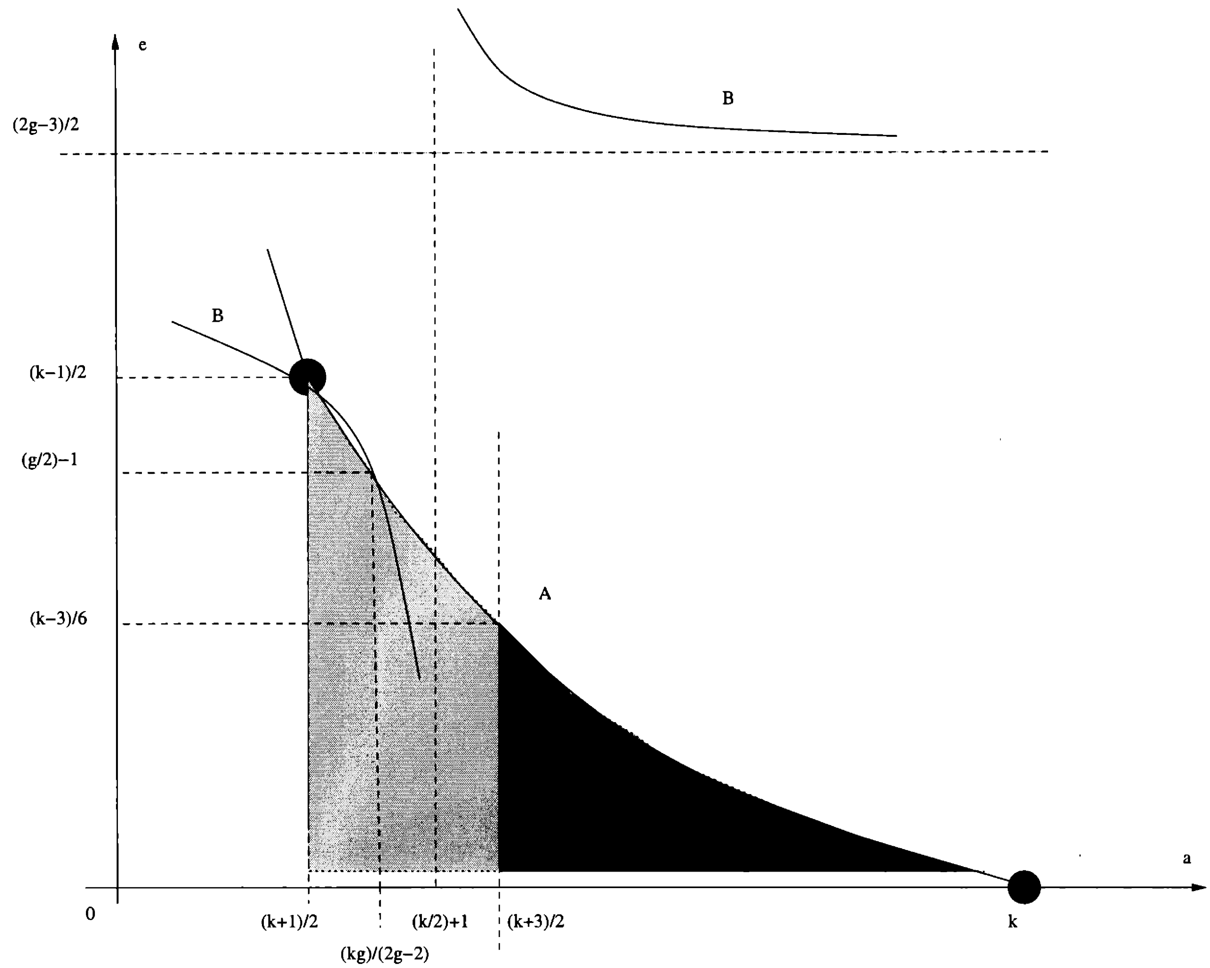

Figure 4-7: The dark shaded part is the region R when $g \geq 3$ and k is an odd integer in the interval $g-1 \leq k < 2g-2$

then, by Basic Lemma 4.18:

$$a \geq a' \text{ and } e \leq e'$$

Note that in the range R we have $e \leq \frac{g}{2} - 1 \leq (g-2)$. By Basic Lemma 4.19, it follows that:

$$\dim \mathfrak{M}(a,e) \geq \dim \mathfrak{M}(a',e) \geq \dim \mathfrak{M}(a',e') \tag{4.13}$$

If one of the inequalities (4.13) is strict, then

$$\dim \mathfrak{M}(a,e) > \dim \mathfrak{M}(a',e').$$

But since $\mathfrak{M}(a,e) \subset \overline{\mathfrak{M}(a',e')}$, we have $\dim \mathfrak{M}(a,e) \leq \dim \mathfrak{M}(a',e')$. It follows that $(a,e) = (a',e')$ and $\overline{\mathfrak{M}(a,e)}$ is an irreducible component.

Let $\mathfrak{M}(a,e)$ be the scheme corresponding to a pair in the range $(\star)$, but not in the region R. Then $\mathfrak{M}(a,e)$ does not have the expected dimension and there is an

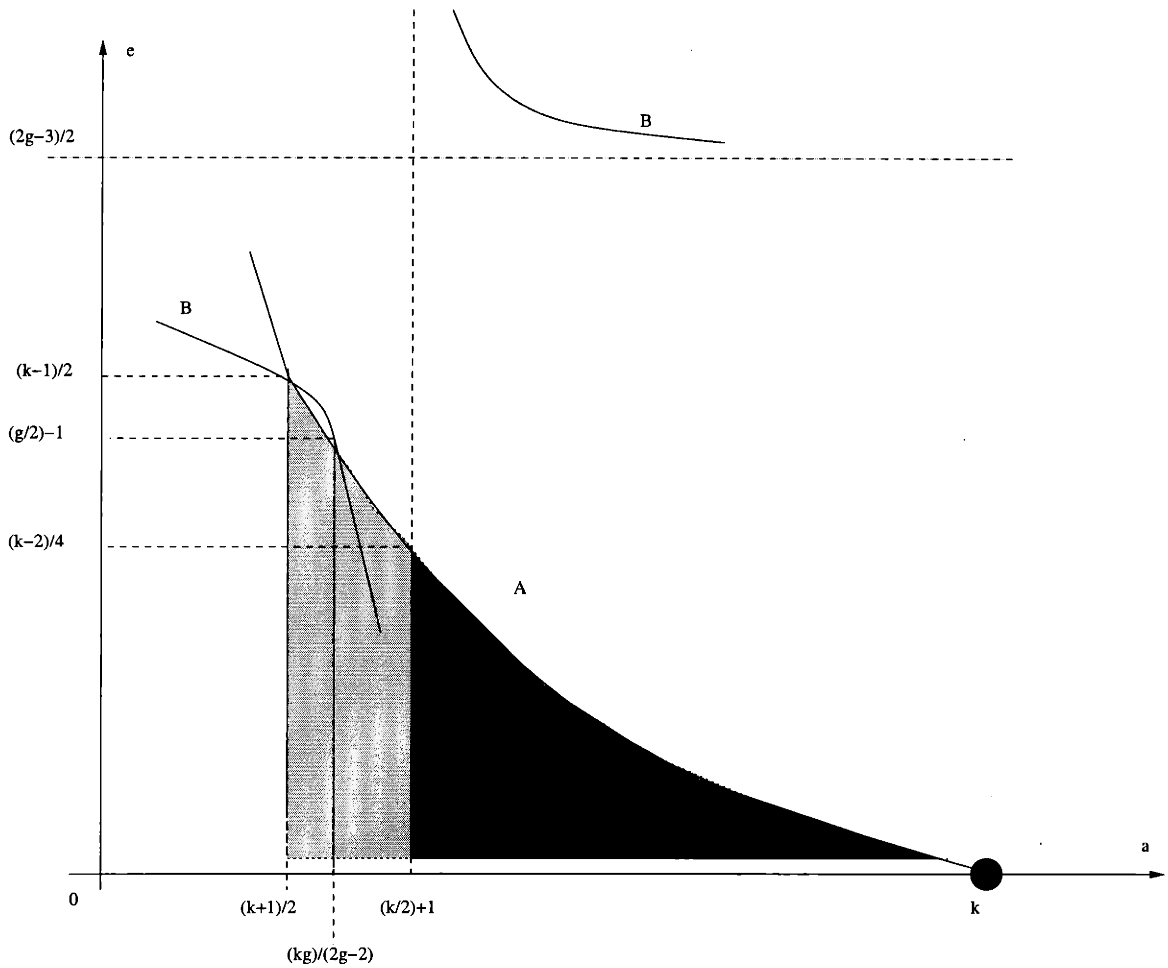

Figure 4-8: The dark shaded part is the region R when $g \geq 3$ and k is an even integer in the interval $g-1 \leq k < 2g-2$

irreducible component $\mathfrak{M}$ of $\underline{\mathrm{Mor}_k(\mathbb{P}^1, M)}$ such that $\mathfrak{M}(a,e) \subset \mathfrak{M}$. If $\mathfrak{M}$ is not the even nice component, then $\mathfrak{M} = \overline{\mathfrak{M}(a',e')}$ for some a' and e' in the region R (the scheme $\mathfrak{M}$ has at least the expected dimension). Then, again by Basic Lemma 4.18:

$$a \geq a' \text{ and } e \leq e'$$

The only pair of integers in the range R with this property is $(\frac{k+1}{2}, \frac{k-1}{2})$ which corresponds to the nice component. It follows that $\mathfrak{M}$ must be the nice component. $\square$

4.4 Description of the irreducible components

4.4.1 Description of the schemes $\mathfrak{M}(a, e)$ and $\mathfrak{M}_{even}$

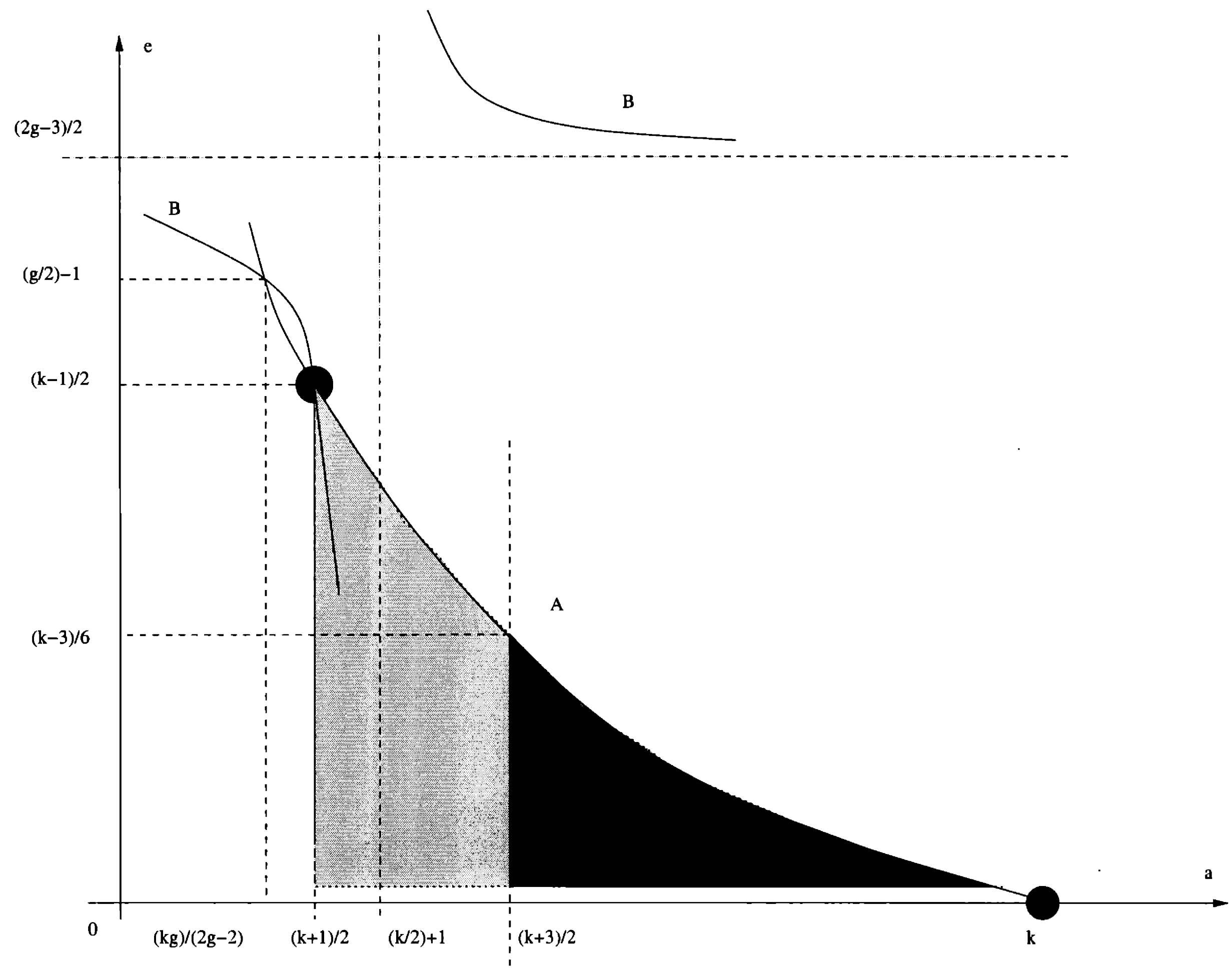

Figure 4-9: The dark shaded part is the region R when $g \geq 3$ and k is an odd integer in the interval $0 < k < g-1$

MRC fibration of $\mathfrak{M}(a,e)$

Consider the schemes $\mathfrak{M}(a,e)$ for pairs of integers in the range $(\star)$. We let

$$\delta = (k-a) - e(2a-k).$$

Since $\mathfrak{M}(a,e)$ is isomorphic to $\mathfrak{B}^0(a,e)$, there is a morphism:

$$\pi : \mathfrak{M}(a,e) \to \mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C) \tag{4.14}$$

An element $[f] \in \mathfrak{M}(a,e)$ is sent by π to the point $\{(\mathcal{L}, Z)\}$, where

$$\mathcal{L} \in \mathrm{Pic}^{-e}(C) \quad \text{and} \quad Z \in \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$$

are from the canonical sequence of the vector bundle $\mathcal{H}_f$:

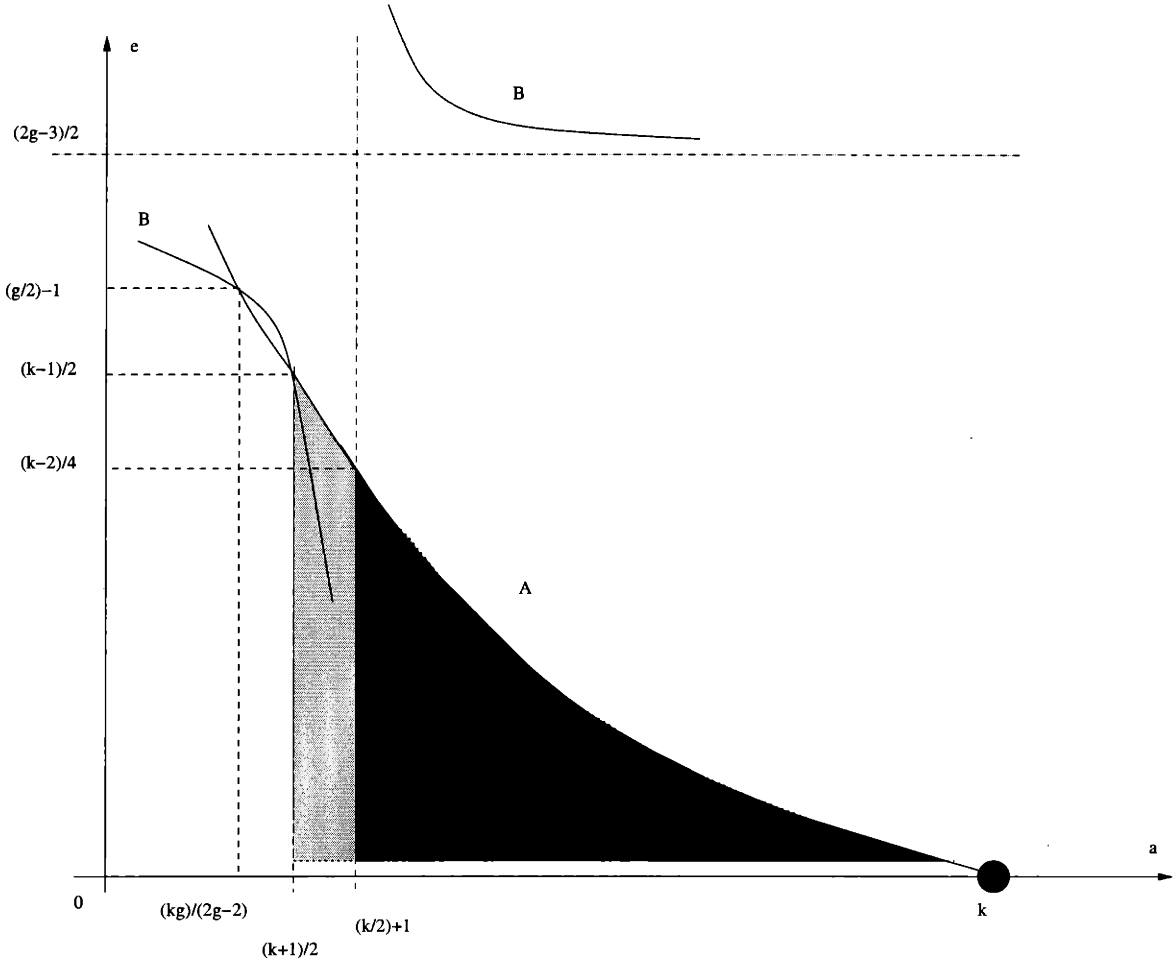

Figure 4-10: The dark shaded part is the region R when $g \geq 3$ and k is an even integer in the interval $0 < k < g-1$

$$0 \to O(a) \boxtimes \mathcal{L} \to \mathcal{H}_f \to (O(k-a) \boxtimes \mathcal{L}^{-1}(x_0)) \otimes \mathcal{I}_Z \to 0. \qquad (4.15)$$

Proposition 4.24. *If $\delta = 0$, the MRC fibration of the scheme $\mathfrak{M}(a,e)$ is given by the morphism π of (4.14):*

$$\pi : \mathfrak{M}(a,e) \to Pic^{-e}(C)$$

If $\delta > 0$, the MRC fibration of the scheme $\mathfrak{M}(a,e)$ is given by the morphism:

$$\rho : \mathfrak{M}(a,e) \to Pic^{-e}(C) \times Pic^{\delta}(C)$$

obtained by composing the morphism π of (4.14) with the canonical morphism coming from:

$$Hilb^{\delta}(\mathbb{P}^1 \times C) \to Sym^{\delta}(\mathbb{P}^1 \times C) \to Sym^{\delta}(C) \to Pic^{\delta}(C).$$

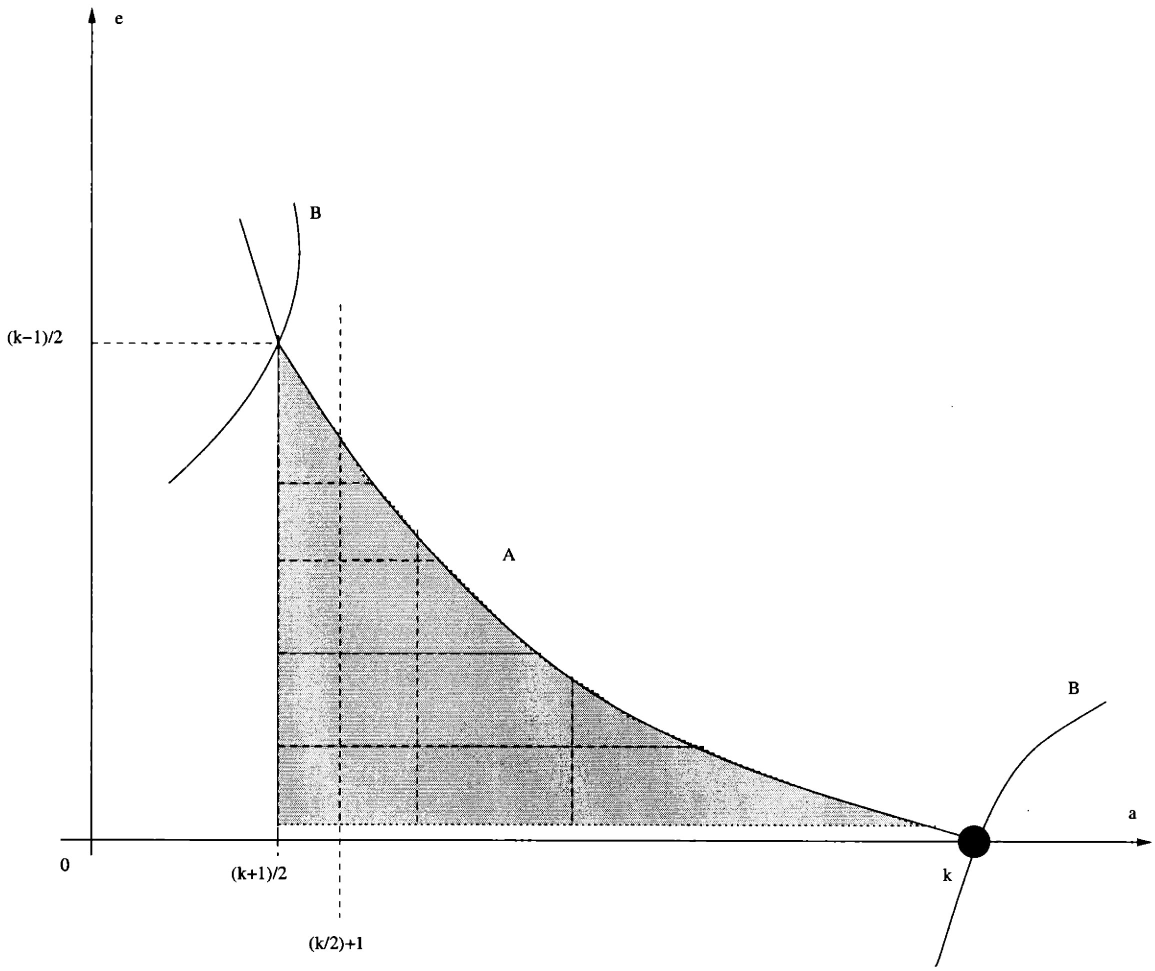

Figure 4-11: The region R when $g = 2$ and k is even is just the point $\{(k, 0)\}$

The morphism ρ is dominant if and only if $\delta \geq g$.

Proof. The scheme $\mathfrak{M}(a, e)$ is isomorphic to a dense open in the scheme $\mathfrak{B}'(a, e)$ which is a projective bundle over $\mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$:

$$p' : \mathfrak{B}'(a, e) \to \mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$$

The morphism p' induces the morphism ρ. Hence, it is enough to find the MRC fibration of $\mathrm{Pic}^{-e}(C) \times \mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$, or equivalently, the MRC fibration of $\mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C)$.

If $\delta = 0$, then we are done. If $\delta > 0$ we consider the birational morphism

$$\mathrm{Hilb}^{\delta}(\mathbb{P}^1 \times C) \to \mathrm{Sym}^{\delta}(\mathbb{P}^1 \times C).$$

We note that

$$\mathrm{Sym}^{\delta}(\mathbb{P}^1 \times C) \cong \mathrm{Sym}^{\delta}(\mathbb{P}^1) \times \mathrm{Sym}^{\delta}(C)$$

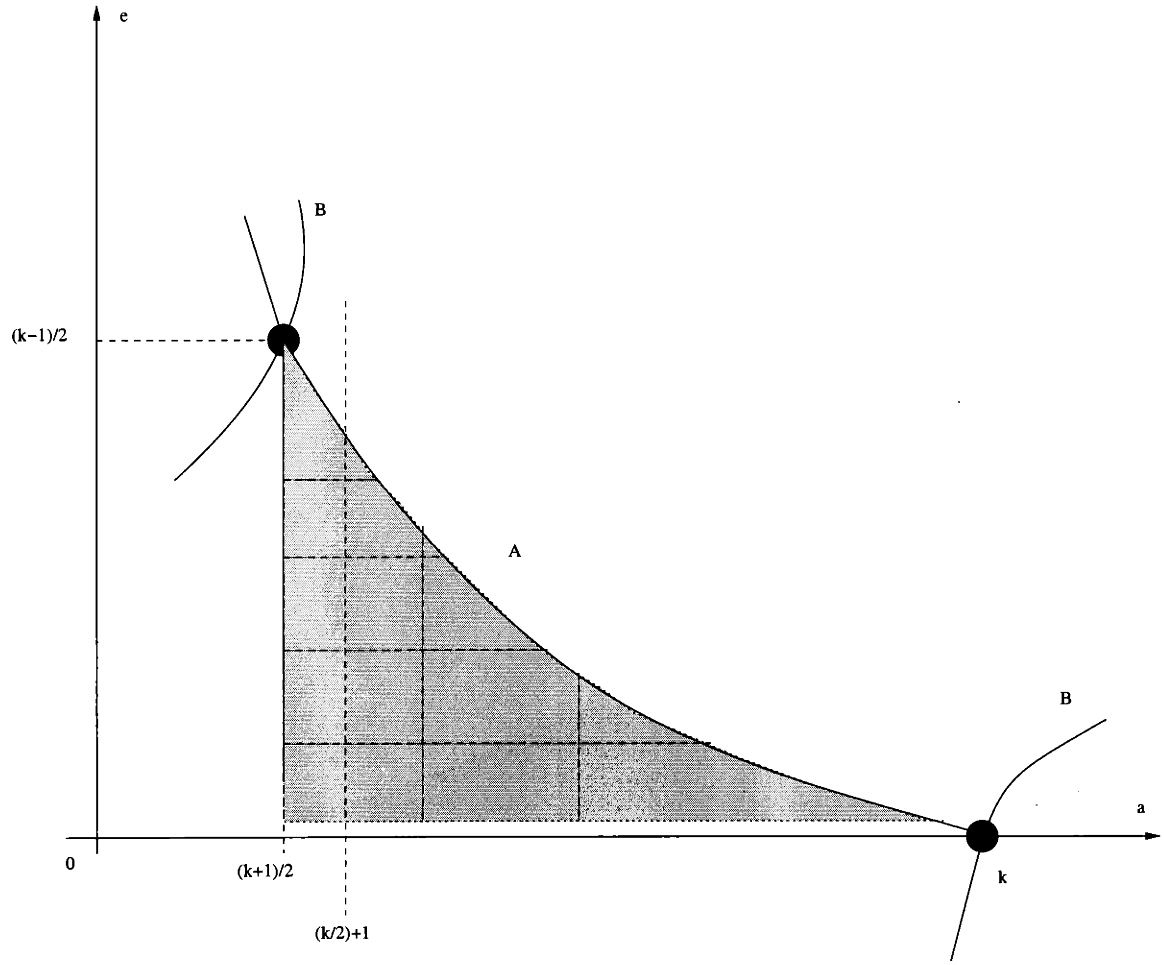

Figure 4-12: The region R when $g = 2$ and k is odd consists of two points: $\{(\frac{k+1}{2}, \frac{k-1}{2}), (k, 0)\}$

Since $\mathrm{Sym}^\delta(\mathbb{P}^1)$ is a rational variety, we are left with finding the MRC fibration of $\mathrm{Sym}^\delta(C)$. This is given by the canonical morphism

$$u : \mathrm{Sym}^\delta(C) \to \mathrm{Pic}^\delta(C).$$

The general fiber of the morphism p' at a point in the image of u is therefore a rational scheme and since $\mathrm{Pic}^\delta(C)$ is an abelian variety, it follows that the map p' gives the MRC fibration of $\mathfrak{B}'(a, e)$. □

General description of $\mathfrak{M}(a, e)$

Lemma 4.25. *An element $[f]$ in $\mathfrak{M}(a, e)$ is the extension of a rational map obtained as a composition:*

$$\mathbb{P}^1 \dashrightarrow \mathbb{P}(V_\mathcal{L}) \setminus Z_\mathcal{L} \to M$$

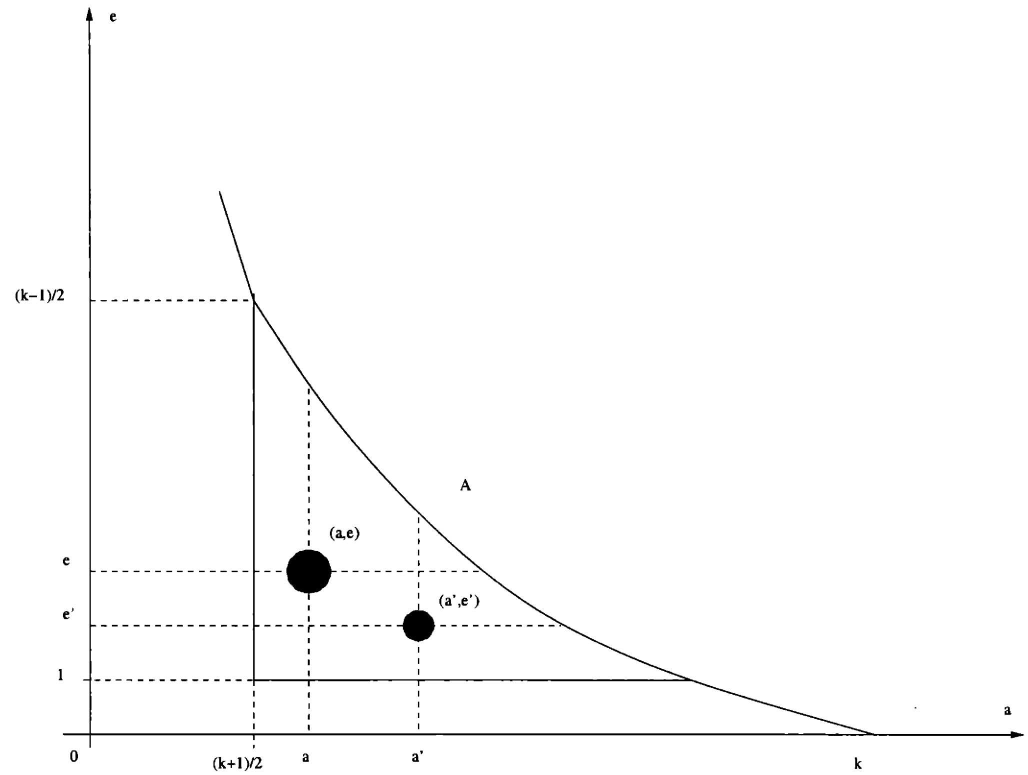

Figure 4-13: The locus $\mathfrak{M}(a', e')$ is contained in $\mathfrak{M}(a, e)$, then $a' \geq a$ and $e' \leq e$

where $\mathcal{L} \in Pic^{-e}(C)$ is the line bundle in Proposition 4.24 and

$$\mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$$

is the morphism (2.7).

If $\delta = 0$ then there is a morphism $\mathbb{P}^1 \to \mathbb{P}(V_{\mathcal{L}})$ (defined everywhere) of degree $2a - k$ such that f is the composition

$$\mathbb{P}^1 \to \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M$$

Proof. Let $\mathcal{L}$ and Z be as in 4.15. Let $\Gamma \subset \mathbb{P}^1$ be the set of points $p \in \mathbb{P}^1$ in the image of Z (as a set). Consider the restriction of the morphism f to $\mathbb{P}^1 \setminus \Gamma$. By the same methods as in Lemma A.2, it follows that f factors as

$$\mathbb{P}^1 \setminus \Gamma \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

where $Z_{\mathcal{L}} \subset \mathbb{P}(V_{\mathcal{L}})$ is the locus of unstable extensions of Proposition 2.1. Recall that $\mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{2e+g-1}$ and we have

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \to M \quad \text{with} \quad \kappa^*\Theta = O(2e+1)$$

If $\delta = 0$ then $Z = \emptyset$ and $\Gamma = \emptyset$. Hence, the morphism g is defined everywhere.

We have:

$$e = \frac{k-a}{2a-k} \in \mathbb{Z}, \quad 2e+1 = \frac{k}{2a-k} \in \mathbb{Z}$$

It follows that $g^*O(1) = O(2a-k)$. □

Note that in the case when $\delta = 0$, the scheme $\mathfrak{M}(a,e)$ corresponds to the subscheme $M(e,n)$ (for $n = 2a-k$) of the Kontsevich space $\overline{M}_0(M,k)$. The canonical sequence of the bundle $\mathcal{H}_f$ has the form:

$$0 \to O(a) \boxtimes \mathcal{L} \to \mathcal{H}_f \to O(k-a) \boxtimes \mathcal{L}^{-1}(x_0) \to 0. \tag{4.16}$$

Note that if we twist by $O(a-k)$, we get that the sequence 4.16 is the pull back by $(g \times \mathrm{id}_C)$ of the universal sequence (2.6).

Description of $\mathfrak{M}(k,0)$

The scheme $\overline{\mathfrak{M}(k,0)}$ is an irreducible component of $\mathrm{Mor}_k(\mathbb{P}^1, M)$ (see Theorem 4.23). It is the irreducible component with the largest dimension (see Figures 4-2 and 4-3):

$$\dim \mathfrak{M}(k,0) = (k+2)g - 1$$

Lemma 4.26. *We have* $\overline{\mathfrak{M}(k,0)} = \mathfrak{M}(k,0)$.

Proof. Let $\mathfrak{M} = \overline{\mathfrak{M}(k,0)}$. We prove that the open $\mathfrak{M}^0 \subseteq \mathfrak{M}$ of Lemma 4.4 is in fact the whole $\mathfrak{M}$. For a general closed point $[f] \in \mathfrak{M}$, the vector bundle $\mathcal{H}_f$ has generic fiber type $(k,0)$. At a special point $[f]$, by upper semicontinuity, $\mathcal{H}_f$ has generic fiber type $(a, k-a)$, with $a \geq k$. But by Remark 3.29, we must have $a \leq k$. It follows that for any $[f] \in \mathfrak{M}$, $\mathcal{H}_f$ has generic fiber type $(k,0)$.

In a similar way, for a general $[f] \in \mathfrak{M}$, the line bundle $p_{2*}(\mathcal{H}_f(-a))$ has degree 0. At a special point $[f]$, the line bundle $p_{2*}(\mathcal{H}_f(-a))$ has degree bigger or equal than 0. By Lemma 3.28, we have that $p_{2*}(\mathcal{H}_f(-a))$ has non-positive degree. It follows that for any $[f] \in \mathfrak{M}$, the line bundle $p_{2*}(\mathcal{H}_f(-a))$ has degree 0. Hence, in the proof of Theorem 4.6 we have $\mathfrak{M}^0 = \mathfrak{M}$. □

Corollary 4.27. *The irreducible component of the largest dimension $\mathfrak{M}(k,0)$ of $Mor_k(\mathbb{P}^1, M)$ is isomorphic to $\mathfrak{B}^0(k,0)$. In particular, it is a dense open in a projective bundle over $Pic^0(C)$:*

$$\pi : \mathfrak{M}(k,0) \to Pic^0(C)$$

Proof. We have that $\mathfrak{M}(k,0)$ maps isomorphically to $\mathfrak{B}^0(k,0)$. Since when $(a,e) = (k,0)$ we have $\delta = 0$, by Note 3.31, we have

$$\mathfrak{B}(k,0) = \mathfrak{B}'(k,0)$$

and $\mathfrak{B}'(k,0)$ is a projective bundle over $\mathrm{Pic}^0(C)$. □

Note 4.28. *Any element in the component $\mathfrak{M}(k,0)$ of the largest dimension is a rational curve obtained as a composition*

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \xrightarrow{\kappa_{\mathcal{L}}} M$$

where $\mathcal{L} \in Pic^0(C)$ is the line bundle $\mathcal{L} = \pi([f])$ and g and $\kappa_{\mathcal{L}}$ have the following properties:

- *The morphism g embeds $\mathbb{P}^1$ as a rational curve of degree k in the projective space*
$$\mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{g-1}$$
- *The morphism $\kappa_{\mathcal{L}}$ embeds $\mathbb{P}(V_{\mathcal{L}})$ as a linear subspace contained in M*

Proof. Recall that dim $(V_{\mathcal{L}}) = g$ and

$$\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \to M, \quad \text{and} \quad \kappa_{\mathcal{L}}^*\Theta = O(1)$$

The morphism $\kappa_{\mathcal{L}}$ is a linear embedding. By Proposition 2.15, it follows that there is a unique morphism g with the required properties. □

Description of the nice component when k odd

Note 4.29. *If $k = 2a+1$ is an odd integer, the general element of the nice component $\mathfrak{M}(a+1,a)$ is given by a rational curve obtained as a composition*

$$\mathbb{P}^1 \xrightarrow{g} \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \xrightarrow{\kappa_{\mathcal{L}}} M$$

where $\mathcal{L} \in Pic^0(C)$ is the line bundle $\mathcal{L} = \pi([f])$ and g and $\kappa_{\mathcal{L}}$ have the following properties:

- *The morphism g embeds $\mathbb{P}^1$ as a line in the projective space*
$$\mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{k+g-2}$$
- *The morphism $\kappa_{\mathcal{L}}$ has the property that $\kappa^*\Theta = O(k)$*

The proof is similar to the proof of Note 4.28.

Description of the nice component when k even

Assume that $k = 2a$ is an even integer and consider the scheme $\mathfrak{M}_{even}$. Since $\mathfrak{M}_{even}$ is isomorphic to $\mathfrak{B}_{even}$, from Fact 3.40 it follows that there is a morphism:

$$\pi : \mathfrak{M}_{even} \to \mathrm{Sym}^a(C) \times_{\mathrm{Pic}^{1-e}(C)} M(2, 1-e)$$

An element $[f] \in \mathfrak{M}_{even}$ is sent by π to the point $(D, \mathcal{E})$, where

$$D \in \mathrm{Sym}^a(C) \quad \text{and} \quad \mathcal{E} \in M(2, 1-e)$$

are from the canonical sequence of the vector bundle $\mathcal{H}_f$:

$$0 \to O(a) \boxtimes \mathcal{E} \to \mathcal{H}_f \to O(a-1) \boxtimes O_D \to 0. \tag{4.17}$$

Proposition 4.30. *The MRC fibration of the nice irreducible component $\mathfrak{M}_{even}$ is given by the morphism*

$$\rho : \mathfrak{M}_{even} \to Pic^{1-a}(C)$$

obtained by composing the morphism π with the canonical morphism:

$$Sym^a(C) \times_{Pic^{1-a}(C)} M(2, 1-a) \to Pic^{1-a}(C)$$

where $Sym^a(C) \to Pic^{1-a}(C)$ is given by $D \mapsto O(x_0 - D)$ and

$$M(2, 1-e) \to Pic^{1-a}(C)$$

is the determinant map. The morphism ρ is dominant if and only if $k \geq 2g$.

Proof. The scheme $\mathfrak{M}_{even}$ is isomorphic to the scheme $\mathfrak{B}_{even}$ and by Fact 3.40 there is a morphism

$$p : \mathfrak{B}_{even} \to \mathrm{Sym}^a(C) \times_{\mathrm{Pic}^{1-a}(C)} M(2, 1-a)$$

whose general fiber is isomorphic to $\mathrm{PGL}(1)^a$. Consider the canonical morphism

$$\mathrm{Sym}^a(C) \times_{\mathrm{Pic}^{1-a}(C)} M(2, 1-a) \to \mathrm{Pic}^{1-a}(C).$$

The fiber at a point $\xi \in \mathrm{Pic}^{1-a}(C)$ in the image of p is isomorphic to

$$\mathbb{P}(\mathrm{H}^0(C, \xi)) \times M(2, \xi)$$

It is unirational, since $M(2, \xi)$ is unirational. It follows by Fact 1.6 that the general fiber of p is rationally connected. It follows that p gives the MRC fibration of $\mathfrak{B}$. $\square$

Note 4.31. *If $k = 2a$ is an even integer, an element $[f]$ of $\mathfrak{M}_{even}$ is given by a rational curve obtained as a composition*

$$\mathbb{P}^1 \xrightarrow{\ g\ } \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \xrightarrow{\eta_{D,\mathcal{E}}} M$$

where $(D, \mathcal{E}) = \pi([f])$ (see Proposition 4.30) and we have the morphism (2.23):

$$\eta_{D,\mathcal{E}} : \mathbb{P}(V_{D,\mathcal{E}}) \setminus Z_{D,\mathcal{E}} \to M.$$

and the morphisms g and $\eta_{D,\mathcal{E}}$ have the following properties:

- *The morphism g embeds $\mathbb{P}^1$ as a line in the projective space*

$$\mathbb{P}(V_{D,\mathcal{E}}) \cong \mathbb{P}^{k-1}$$

- *The morphism $\eta_{D,\mathcal{E}}$ has the property that $\eta^*\Theta = O(k)$*

Note that the scheme $\mathfrak{M}_{even}$ corresponds to the subscheme $N(a,1)$ of the Kontsevich space $\overline{M}_0(M,k)$. The canonical sequence of the bundle $\mathcal{H}_f$ is the pull back by $(g \times \mathrm{id}_C)$ of the universal sequence (2.21).

4.4.2 Low degree rational curves

Lines on M

Proposition 4.32. *The scheme $Mor_1(\mathbb{P}^1, M)$ is isomorphic to a projective bundle over $Pic^0(C)$. It has the expected dimension $3g-1$.*

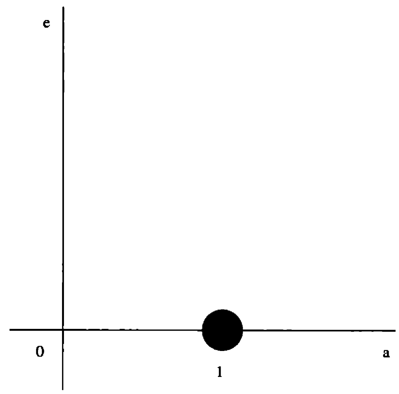

Figure 4-14: The case of lines $k=1$

Proof. Let $k=1$. Then the range $(\star)$ for a and e is just the point $(1,0)$. Let $\mathfrak{M} = \mathfrak{M}(1,0)$; this is the nice component. It is the unique irreducible component of $\mathrm{Mor}_1(\mathbb{P}^1, M)$. It has dimension $3g-1$.

By Lemma 4.26 and as $\mathfrak{B}^0(1,0) = \mathfrak{B}(1,0) = \mathfrak{B}'(1,0)$ (see Theorem 3.32), it follows that $\mathfrak{M}$ is a projective bundle over $Pic^0(C)$.

For any $[f] \in \mathfrak{M}$ the bundle $\mathcal{H}_f$ has generic fiber type $(1,0)$. By Lemma 1.11, $[f]$ is a smooth point of $\mathrm{Mor}_1(\mathbb{P}^1, M)$. $\square$

Conics on M

Note 4.33. *The scheme $Mor_2(\mathbb{P}^1, M)$ has two irreducible components: the nice component $\overline{\mathfrak{M}_{even}}$ of the expected dimension $3g+1$ and the component $\mathfrak{M}(2,0)$ of dimension $4g-1$. An element in the component $\mathfrak{M}(2,0)$ is obtained by taking conics in some $(g-1)$-planes contained in M.*

Proof. There is only one subscheme of the form $\mathfrak{M}(a,e)$, namely $\mathfrak{M}(2,0)$. It follows from Theorem 4.23 that $\mathfrak{M}(2,0)$ and $\mathfrak{M}_{even}$ are the only irreducible components. □

Cubics on M

Note 4.34. *The scheme $Mor_3(\mathbb{P}^1, M)$ has two irreducible components: the nice component $\overline{\mathfrak{M}_{even}}$ of the expected dimension $3g+3$ and the component $\mathfrak{M}(3,0)$ of dimension $5g-1$. An element in the component $\mathfrak{M}(3,0)$ is obtained by taking cubics in some $(g-1)$-planes contained in M.*

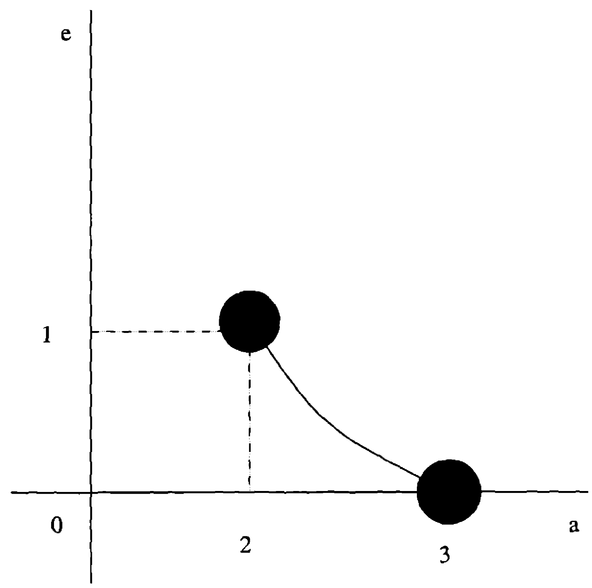

Figure 4-15: The case of cubics $k=3$

Proof. There are only two subschemes of type $\mathfrak{M}(a,e)$: $\mathfrak{M}(2,1)$ and $\mathfrak{M}(3,0)$. It follows from Theorem 4.23 that they are irreducible components. □

4.4.3 Examples for $g = 2, 3$

Proposition 4.35. *If $g=2$ and $k \geq 2$ is any integer, there are two irreducible components in $Mor_k(\mathbb{P}^1, M)$, both of the expected dimension $2k+3$: the nice component and the component $\mathfrak{M}(k,0)$, which has as a general element a morphism $f: \mathbb{P}^1 \to M$ which maps k-to-1 onto a line in M.*

Proof. The region R contains only the point $(k,0)$. By Theorem 4.23, it follows that the only other irreducible component than the nice component is $\mathfrak{M}(k,0)$. □

Proposition 4.36. *If $g = 3$ and $k \geq 2$ is any integer, there are two irreducible components in $Mor_k(\mathbb{P}^1, M)$, the nice component which has the expected dimension $2k + 6$ and the component $\mathfrak{M}(k, 0)$, of dimension $3k + 7$, which has as a general element a morphism $f : \mathbb{P}^1 \to M$ obtained by taking rational curves of degree k in some 2-dimensional planes contained in M.*

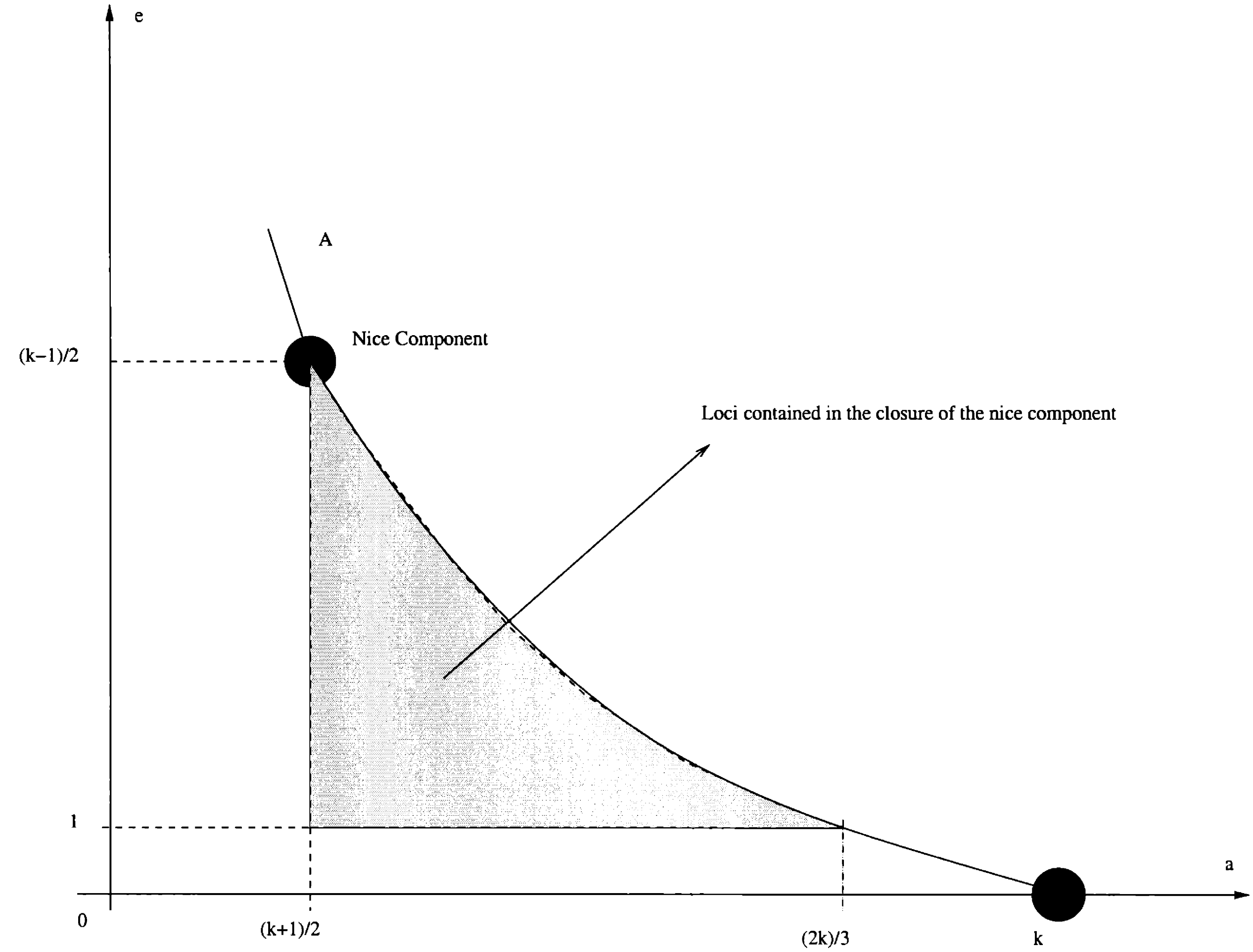

Figure 4-16: Case $g = 2$ or $g = 3$ when k is odd

Note that all the other loci $\mathfrak{M}(a, e)$ with (a, e) in the range $(\star)$, but different than $(k, 0)$, are contained in the nice component. This is because of Basic Lemma 4.18.

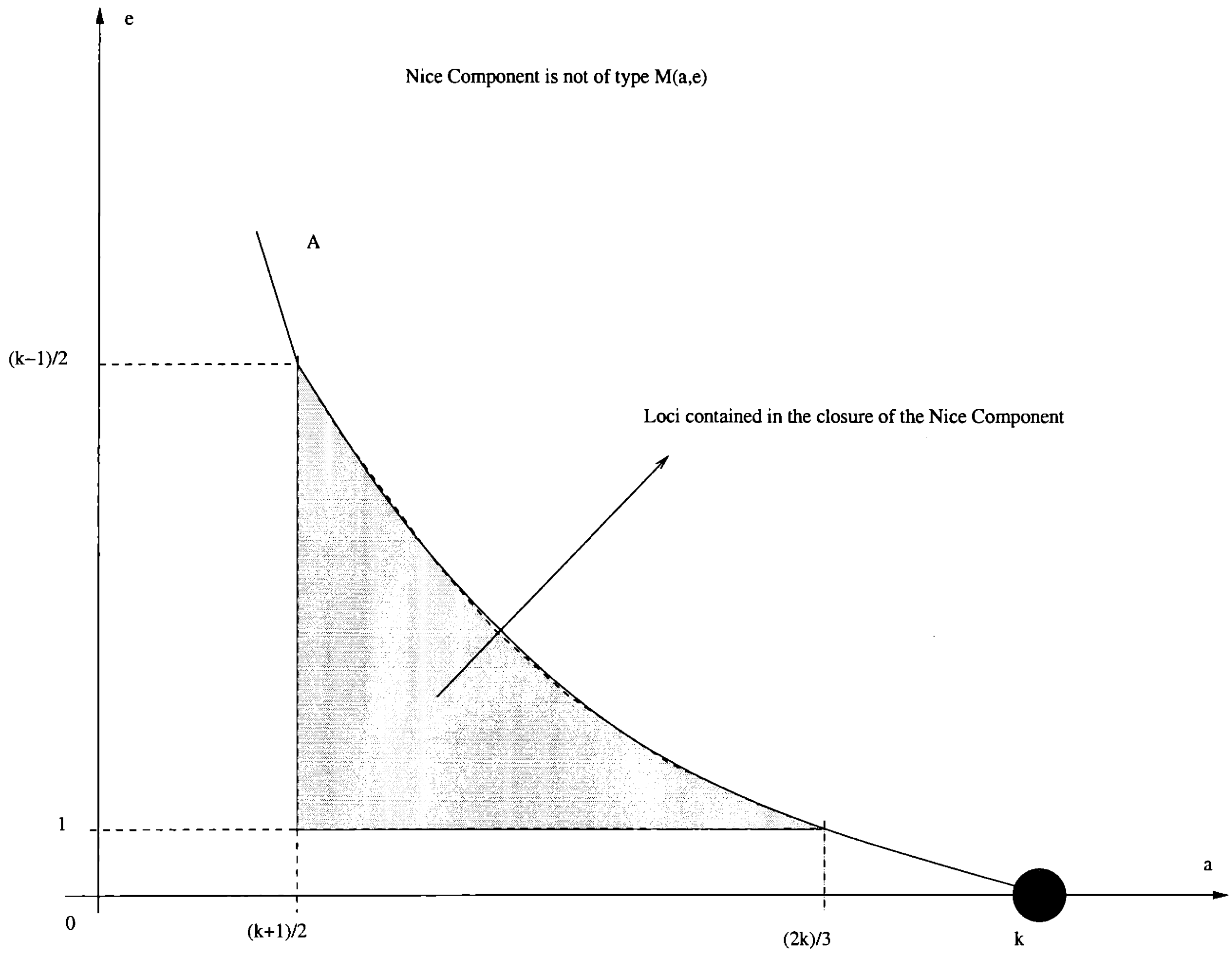

Figure 4-17: Case $g = 2$ or $g = 3$ when k is even

Appendix A

A.1 Extensions

Lemma A.1. *Let S be a projective integral scheme and $\mathcal{V}$ and $\mathcal{T}$ be coherent sheaves on $S \times C$ with the property that $Hom(\mathcal{V}_s, \mathcal{T}_s) = 0$ for any $s \in S$. Let π_1, π_2 be the projections from $S \times C$ onto S and C respectively. Define $\mathcal{S}$ to be the following relative extension sheaf on S:*

$$\mathcal{S} = \mathcal{E}xt^1_{S\times C|S}(\mathcal{V}, \mathcal{T}).$$

Assume that $\mathcal{S}$ is a vector bundle on $S \times C$ and let $\mathcal{Y} = \mathbb{P}(\mathcal{S})$ with $p : \mathcal{Y} \to S$ the canonical projection. Let $\mathcal{V}_\mathcal{Y}$ and $\mathcal{T}_\mathcal{Y}$ be the pull-backs of $\mathcal{V}$ and $\mathcal{T}$ to $\mathcal{Y} \times C$. We have that on $\mathcal{Y} \times C$ there is a universal extension:

$$0 \to O_\mathcal{Y}(1) \otimes \mathcal{T}_\mathcal{Y} \to \mathcal{G} \to \mathcal{V}_\mathcal{Y} \to 0. \qquad (\delta)$$

It is universal in the sense that for any $y \in \mathcal{Y}$, if we let $s = p(y)$, then the extension (δ_y) obtained by restricting (δ) to $\{y\} \times C$

$$0 \to \mathcal{T}_s \to \mathcal{G}_y \to \mathcal{V}_s \to 0 \qquad (\delta_y)$$

gives an element in $Ext^1_C(\mathcal{V}_s, \mathcal{T}_s) \cong \mathcal{S}_s$ which has class in $\mathbb{P}(\mathcal{S}_s)$ the element y.

Proof. The vector space $\mathcal{S}_s \cong \mathrm{Ext}^1_C(\mathcal{V}_s, \mathcal{T}_s)$ parametrizes extensions

$$0 \to \mathcal{T}_s \to \mathcal{E} \to \mathcal{V}_s \to 0.$$

To construct (δ) consider the space:

$$W = \mathrm{Ext}^1_{\mathcal{Y}\times C}(\mathcal{T}_\mathcal{Y}, O_\mathcal{Y}(1) \otimes \mathcal{V}_\mathcal{Y}).$$

There is an exact sequence:

$$0 \to \mathrm{H}^1(\mathcal{Y}, \mathcal{H}om_{\mathcal{Y}\times C|\mathcal{Y}}(\mathcal{V}_\mathcal{Y}, O_\mathcal{Y}(1) \otimes \mathcal{T}_\mathcal{Y})) \to \mathrm{Ext}^1_{\mathcal{Y}\times C}(\mathcal{V}_\mathcal{Y}, O_\mathcal{Y}(1) \otimes \mathcal{T}_\mathcal{Y}) \to$$
$$H^0(\mathcal{Y}, \mathcal{E}xt^1_{\mathcal{Y}\times C|\mathcal{Y}}(\mathcal{V}_\mathcal{Y}, O_\mathcal{Y}(1) \otimes \mathcal{T}_\mathcal{Y})) \to \mathrm{H}^2(\mathcal{Y}, \mathcal{H}om_{\mathcal{Y}\times C|\mathcal{Y}}(\mathcal{V}_\mathcal{Y}, O_\mathcal{Y}(1) \otimes \mathcal{T}_\mathcal{Y}))$$

Since we have $\mathrm{Hom}(\mathcal{V}_s, \mathcal{T}_s) = 0$ for any $s \in S$, it follows that

$$\mathcal{H}om_{S\times C|S}(\mathcal{V}, \mathcal{T}) = 0$$
$$\mathcal{H}om_{\mathcal{Y}\times C|\mathcal{Y}}(\mathcal{V}_{\mathcal{Y}}, O_{\mathcal{Y}}(1) \otimes \mathcal{T}_{\mathcal{Y}}) \cong \mathcal{H}om_{\mathcal{Y}\times C|\mathcal{Y}}(\mathcal{V}_{\mathcal{Y}}, \mathcal{T}_{\mathcal{Y}}) \otimes O_{\mathcal{Y}}(1) \cong$$
$$\cong p^*\mathcal{H}om_{S\times C|S}(\mathcal{V}, \mathcal{T}) \otimes O_{\mathcal{Y}}(1) = 0$$

Moreover we have:

$$\mathcal{E}xt^1_{\mathcal{Y}\times C|\mathcal{Y}}(\mathcal{V}_{\mathcal{Y}}, O_{\mathcal{Y}}(1) \otimes \mathcal{T}_{\mathcal{Y}})) \cong p^*\mathcal{E}xt^1_{S\times C|S}(\mathcal{V}, \mathcal{T}) \otimes O_{\mathcal{Y}}(1) = (p \times \mathrm{id}_C)^*\mathcal{S} \otimes O_{\mathcal{Y}}(1)$$

It follows that we have an isomorphism:

$$W \cong \mathrm{H}^0(\mathcal{Y}, O_{\mathcal{Y}}(1) \otimes p^*\mathcal{S}).$$

We also have canonical isomorphisms:

$$\mathrm{H}^0(\mathcal{Y}, O_{\mathcal{Y}}(1) \otimes p^*\mathcal{S}) \cong \mathrm{H}^0(S, \mathcal{S}^* \otimes \mathcal{S}) \cong \mathrm{Hom}_{\mathcal{Y}}(\mathcal{S}, \mathcal{S}).$$

Take the element $\mathrm{id} \in \mathrm{Hom}_{\mathcal{Y}}(\mathcal{S}, \mathcal{S})$ and take the corresponding extension in W:

$$0 \to O_{\mathcal{Y}}(1) \otimes \mathcal{T}_{\mathcal{Y}} \to \mathcal{G} \to \mathcal{V}_{\mathcal{Y}} \to 0. \qquad (\delta)$$

We claim that this extension has the required properties. To prove this, fix $y \in \mathcal{Y}$ and let $s = p(y)$. We look first at how the extension (δ) restricts to the fiber of p at s. For simplicity, make the notation:

$$V = \mathrm{Ext}^1_{\{s\}\times C}(\mathcal{V}_s, \mathcal{T}_s)$$

Then we have:

$$V \cong \mathcal{S}_s \quad \text{and} \quad \mathbb{P}(V) \cong p^{-1}(\{s\})$$

Let q_1, q_2 be the projections from $\mathbb{P}(V) \times C$ onto $\mathbb{P}(V)$ and C respectively. There are commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathcal{Y}\times C}(\mathcal{T}_{\mathcal{Y}}, O_{\mathcal{Y}}(1) \otimes \mathcal{V}_{\mathcal{Y}}) & \xrightarrow{r_s} & \mathrm{Ext}^1_{\mathbb{P}(V)\times C}(q_2^*\mathcal{V}_s, q_1^*O(1) \otimes q_2^*\mathcal{T}_s) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{H}^0(\mathcal{Y}, O(1) \otimes p^*\mathcal{S}) & \xrightarrow{r'_s} & \mathrm{H}^0(\mathbb{P}(V), O(1) \otimes V) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}_S(\mathcal{S}, \mathcal{S}) & \xrightarrow{r''_s} & \mathrm{Hom}(V, V)
\end{array} \qquad (\mathrm{A}.1)$$

where r_s, r'_s, r''_s are given by restrictions.

Let (δ_s) be the restriction of the extension (δ) to $\mathbb{P}(V) \times C$:

$$0 \to q_1^*O(1) \otimes q_2^*\mathcal{T}_s \to \mathcal{G}_{|\mathbb{P}(V)\times C} \to q_2^*\mathcal{V}_s \to 0 \qquad (\delta_s)$$

Using (A.1), the extension (δ) maps to (δ_s), while $\mathrm{id} \in \mathrm{Hom}_S(\mathcal{S}, \mathcal{S})$ maps to $\mathrm{id} \in \mathrm{Hom}(V, V)$.

Let V_y be the one dimensional linear subspace of V corresponding to the point $y \in \mathbb{P}(V)$. If from $\mathbb{P}(V) \times C$ we restrict (δ) further to $\{y\} \times C$, then we have another sequence of commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathbb{P}(V)\times C}(q_2^*\mathcal{V}_s, q_2^*\mathcal{T}_s \otimes q_1^*O(1)) & \xrightarrow{r_y} & \mathrm{Ext}^1_{\{y\}\times C}(\mathcal{V}_s, V_y^* \otimes \mathcal{T}_s) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{H}^0(\mathbb{P}(V), O(1) \otimes V) & \xrightarrow{r'_y} & \mathrm{H}^0(\{y\}, V_y^* \otimes V) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}(V, V) & \xrightarrow{r''_y} & \mathrm{Hom}(V_y, V)
\end{array}$$

where r_y, r'_y, r''_y are given by restrictions and we used the fact that $O_{\mathbb{P}(V)}(1)_{|\{y\}} \cong V_y^*$.

Note that r''_y sends a morphism $h : V \to V$ to the restriction of h to $V_y \subset V$, while r_y sends an extension in W to its restriction to $\{y\} \times C$. Therefore, the extension (δ_s) goes by r_y to the extension:

$$0 \to \mathcal{T}_s \to \mathcal{G}_y \to \mathcal{V}_s \to 0 \qquad (\delta_y)$$

At the level of r''_y this corresponds to sending $\mathrm{id} \in \mathrm{Hom}(V, V)$ to the inclusion morphism $V_y \hookrightarrow V$ in $\mathrm{Hom}(V_y, V)$.

Let's fix an isomorphism $\mathrm{Hom}(V_y, V) \cong V$ by taking an element $v \in V_y$ and letting $e : \mathrm{Hom}(V_y, V) \to V$ to be the evaluation map $e(f) = f(v)$. This fixes an isomorphism:

$$\mathrm{Ext}^1_{\{y\}\times C}(\mathcal{V}_s, V_y^* \otimes \mathcal{T}_s) \cong V.$$

The extension (δ_y) will correspond to the element $v \in V$ (by evaluating the inclusion morphism at $v \in V_y$). Hence, the extension $(\delta_y) \in V$ gives the element $y \in \mathbb{P}(V)$ (note that it does not depend on which v we choose). $\square$

Lemma A.2. *Let $\mathcal{V}$ and $\mathcal{T}$ be coherent sheaves on the curve C such that*

$$\mathcal{H}om_C(\mathcal{V}, \mathcal{T}) = 0.$$

Let $V = Ext^1_C(\mathcal{V}, \mathcal{T})$. Let $n \geq 0$ and define

$$W = Ext^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n) \otimes p_2^*\mathcal{T}).$$

If (ε) is an element in W given by:

$$0 \to p_1^*O(n) \otimes p_2^*\mathcal{T} \to \mathcal{F} \to p_2^*\mathcal{V} \to 0. \qquad (\varepsilon)$$

such that for any $p \in \mathbb{P}^1$, its restriction to $\{p\} \times C$

$$0 \to \mathcal{T} \to \mathcal{F}_{|\{p\}\times C} \to \mathcal{V} \to 0. \qquad (\varepsilon_p)$$

is a non-split extension, then there is a unique morphism

$$g : \mathbb{P}^1 \to \mathbb{P}(V), \quad \textit{such that} \quad g^*O(1) \cong O(n)$$

such that (ε) *is a scalar multiple in* W *of the extension* $g^*(\delta)$. *The morphism* g *is sending a point* $p \in \mathbb{P}^1$ *to the class of the extension* (ε_p) *in* $\mathbb{P}(V)$.

Note that this gives a one-to one correspondence between rational curves of degree n on the space of extensions $\mathbb{P}(V)$ and certain classes of extensions in $\mathbb{P}(W)$.

Proof. By the arguments in Lemma A.1, there is an isomorphism:

$$W = \mathrm{Ext}^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n) \otimes p_2^*\mathcal{T}) \cong \mathrm{H}^0(\mathbb{P}^1 \times C, \mathcal{E}xt^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n) \otimes p_2^*\mathcal{T}))$$

Note that we have

$$\mathcal{E}xt^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n) \otimes p_2^*\mathcal{T}) \cong \mathcal{E}xt^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_2^*\mathcal{T}) \otimes p_1^*O(n) \cong$$
$$\cong p_2^*\mathcal{E}xt^1_C(\mathcal{V}, \mathcal{T}) \otimes p_1^*O(n)$$

It follows that

$$p_{1*}\mathcal{E}xt^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n) \otimes p_2^*\mathcal{T})) \cong O(n) \otimes \mathrm{H}^0(C, \mathcal{E}xt^1_C(\mathcal{V}, \mathcal{T})) \cong O(n) \otimes V$$

where $V = \mathrm{Ext}^1_C(\mathcal{V}, \mathcal{T})$. It follows that there is a canonical isomorphism

$$W \cong \mathrm{H}^0(\mathbb{P}^1, O(n) \otimes V)).$$

Let $\mathbb{P}^1 = \mathbb{P}(U)$ for U some 2-dimensional vector space over $\mathbb{C}$. Note that if $u : \mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C})$, we have:

$$u_*(O(n) \otimes V) \cong (\mathrm{Sym}^n U)^* \otimes V$$
$$\mathrm{H}^0(\mathbb{P}^1, O(n) \otimes V) \cong (\mathrm{Sym}^n U)^* \otimes V \cong \mathrm{Hom}(\mathrm{Sym}^n U, V).$$

If u, v is a basis for U, then as $u^n, u^{n-1}v, \ldots, v^n$ form a basis for $\mathrm{Sym}^n U$, to give such a morphism of vector spaces $\mathrm{Sym}^n U \to V$ is to give polynomials in u and v, linear in $u^n, u^{n-1}v, \ldots, v^n$, hence, some homogenous polynomials of degree n. Hence, a non-zero linear morphism of vector spaces $\mathrm{Sym}^n U \to V$ induces a morphism $g : \mathbb{P}^1 \to \mathbb{P}(V)$.

We need to prove that for an extension $(\varepsilon) \in W$ with the property that the restriction to $\{p\} \times C$ gives a non-split extension (ε_p), for any $p \in \mathbb{P}^1$, the corresponding the map $\mathrm{Sym}^n U \to V$ is non-zero and it induces a morphism $g : \mathbb{P}^1 \to \mathbb{P}(V)$ of degree n.

Note that $(\varepsilon) \in W$ is not zero. Otherwise, the exact sequence (ε) is split. In particular, the restriction to $\{p\} \times C$ is a split exact sequence. This contradicts our assumption about (ε). It follows that the induced linear morphism $\mathrm{Sym}^n U \to V$ is not zero, hence (ε) induces a morphism $g : \mathbb{P}^1 \to \mathbb{P}(V)$.

We denote by U_p the 1-dimensional linear subspace of U corresponding to the point p. Note that we have

$$O_{\mathbb{P}^1}(n)_{|\{p\}} \cong (\mathrm{Sym}^n U_p)^*.$$

We have commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n)\otimes p_2^*\mathcal{T}) & \xrightarrow{r_p} & \mathrm{Ext}^1_{\{p\}\times C}(\mathcal{V}, O(n)_{|\{p\}}\otimes \mathcal{T}) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{H}^0(\mathbb{P}^1, O(n)\otimes V) & \xrightarrow{r'_p} & \mathrm{H}^0(\{p\}, O(n)_{|\{p\}}\otimes V) \\
\cong\downarrow & & \downarrow\cong \\
(\mathrm{Sym}^n U)^*\otimes V & \xrightarrow{r''_p} & (\mathrm{Sym}^n U_p)^*\otimes V \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}(\mathrm{Sym}^n U, V) & \xrightarrow{e_p} & \mathrm{Hom}(\mathrm{Sym}^n U_p, V)
\end{array}$$

The morphisms r_p, r'_p, r''_p, e_p are given by restriction. The map e_p is the evaluation map, given by restricting a morphism $\mathrm{Sym}^n U \to V$ to $\mathrm{Sym}^n U_p \subset \mathrm{Sym}^n U$.

If (ε) is an extension in W then its image via the restriction morphism r_p is the extension (ε_p). If (ε) corresponds, via the vertical maps on the left, to a morphism $h : \mathrm{Sym}^n U \to V$ which induces $g : \mathbb{P}^1 \to \mathbb{P}(V)$ then $g(p)$ will be the point in $\mathbb{P}(V)$ given by the morphism $\mathrm{Sym}^n U_p \to V$ induced, via the vertical maps on the right, by the extension (ε_p).

We need to prove that g has degree n. Since (ε_p) is non-zero for any $p \in \mathbb{P}^1$, it follows that the corresponding morphism $\mathrm{Sym}^n U_p \to V$ is non-zero. Hence, the homogenous polynomials of degree n giving the morphism $h : \mathrm{Sym}^n U \to V$ do not have a common zero at p. Since this is true for any $p \in \mathbb{P}^1$, the morphism g has degree n.

Clearly, if two extension in W are multiple scalars of each other, then they induce the same morphism g.

On the other hand, to each $g : \mathbb{P}^1 \to \mathbb{P}(V)$ of degree n we can associate in a canonical way an extension in W, by pulling back the universal extension (δ) via $(g \times id)$. If g has degree m, we have an extension:

$$0 \to p_1^*O(n)\otimes p_2^*\mathcal{T} \to \mathcal{F} \to p_2^*(\mathcal{V}) \to 0. \qquad (g^*(\delta))$$

There are commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathbb{P}(V)\times C}(q_2^*\mathcal{V}, q_1^*O(1)\otimes q_2^*\mathcal{T}) & \xrightarrow{(g\times id)^*} & \mathrm{Ext}^1_{\mathbb{P}^1\times C}(p_2^*\mathcal{V}, p_1^*O(n)\otimes p_2^*\mathcal{T}) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{H}^0(\mathbb{P}(V), O(1)\otimes V) & \xrightarrow{g^*} & \mathrm{H}^0(\mathbb{P}^1, O(n)\otimes V) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{Hom}(V,V) & \xrightarrow{oh_g} & \mathrm{Hom}(\mathrm{Sym}^n U, V)
\end{array}$$

Here we denoted by h_g the linear morphism

$$h_g : \mathrm{Sym}^n U \to V$$

which is the image of the extension $(g^*(\delta))$ via the vertical maps on the right – recall that the universal extension (δ) induces via the vertical maps on the right, the identity id $\in \mathrm{Hom}(V,V)$.

Since a morphism $g : \mathbb{P}(U) \to \mathbb{P}(V)$ is induced by a linear morphism $h : \mathrm{Sym}^n U \to V$, unique up to multiplication by a scalar, it follows that since both (ε) and $g^*(\delta)$ induce the same morphism g, they must be scalar multiples of each other. □

Lemma A.3. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ line bundles on X such that $H^0(\mathcal{N}_2\otimes\mathcal{M}_1^{-1}) = 0$. Assume $\mathcal{F}$ is a fixed rank 2 vector bundle on Y such that there are two exact sequences*

$$0 \to \mathcal{M}_1 \to \mathcal{F} \to M_2 \to 0$$

$$0 \to \mathcal{N}_1 \to \mathcal{F} \to N_2 \to 0.$$

Then there are induced isomorphisms *$\mathcal{M}_i \to \mathcal{N}_i$ for $i = 1,2$ such that the following diagram is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow\cong & & \| & & \downarrow\cong & & \downarrow \\
0 & \longrightarrow & \mathcal{N}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{N}_2 & \longrightarrow & 0
\end{array}$$

Proof. There is an induced map $\mathcal{M}_1 \to \mathcal{N}_2$ coming from the two exact sequences. But $\mathrm{H}^0(\mathcal{N}_2 \otimes \mathcal{M}_1^{-1}) = 0$ implies that there is no non-zero morphism $\mathcal{M}_1 \to \mathcal{N}_2$. Hence there is an induced morphism $\mathcal{M}_1 \to \mathcal{N}_1$ and consequently a morphism $\mathcal{M}_2 \to \mathcal{N}_2$ such that the diagram is commutative. Clearly, from the snake lemma $\mathcal{M}_1 \to \mathcal{N}_1$ is injective, $\mathcal{M}_2 \to \mathcal{N}_2$ is surjective and

$$\text{coker } (\mathcal{M}_1 \to \mathcal{N}_1) \cong \ker\ (\mathcal{M}_2 \to \mathcal{N}_2).$$

But coker $(\mathcal{M}_1 \to \mathcal{N}_1)$ is either zero or a torsion sheaf and as we have that ker $(\mathcal{M}_2 \to \mathcal{N}_2)$ is torsion-free, it must be that they are both zero. Hence, they are both isomorphisms. □

Corollary A.4. *If $\mathcal{M}_1, \mathcal{M}_2$ are line bundles on X such that $H^0(\mathcal{M}_2 \otimes \mathcal{M}_1^{-1}) = 0$ and $\mathcal{F}$ is a fixed rank 2 vector bundle on X, then any two extensions*

$$0 \to \mathcal{M}_1 \to \mathcal{F} \to \mathcal{M}_2 \to 0$$

differ by a scalar in the space of extensions $Ext^1(\mathcal{M}_2, \mathcal{M}_1)$.

Proof. From the previous lemma, there are induced isomorphisms $\mathcal{M}_i \to \mathcal{M}_i$ for $i = 1, 2$. As $\mathcal{M}_1, \mathcal{M}_2$ are line bundles, any non-zero morphism $\mathcal{M}_i \to \mathcal{M}_i$ is multiplication by a non-zero scalar, say λ_i. There is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \\ \downarrow & & \cong\downarrow\lambda_1 & & \| & & \cong\downarrow\lambda_2 & & \downarrow \\ 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \end{array}$$

Hence, the two exact sequences differ by a scalar in the space of extensions

$$\mathrm{Ext}^1(\mathcal{M}_2, \mathcal{M}_1).$$

□

Lemma A.5. *If C is a curve and $D \in Sym^e C$ and $\mathcal{E}$ is a rank 2 vector bundle on C, then for any two exact sequences on $\mathbb{P}^1 \times C$ with the same middle term and of the form*

$$0 \to p_1^* O(n) \otimes p_2^* \mathcal{E} \to \mathcal{F} \to p_2^* O_D \to 0$$

with $n > 0$, there is an induced commutative diagram with the vertical arrows isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*O(n) \otimes p_2^*\mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & p_2^*O_D & \longrightarrow & 0 \\ \downarrow & & \cong\downarrow & & \| & & \downarrow\cong & & \downarrow \\ 0 & \longrightarrow & p_1^*O(n) \otimes p_2^*\mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & p_2^*O_D & \longrightarrow & 0 \end{array} \tag{A.2}$$

Proof. Since $n > 0$, there are no non-zero isomorphisms

$$p_1^* O(n) \otimes p_2^* \mathcal{E} \to p_2^* O_D$$

It follows that there is an induced diagram as in (A.2). By the snake lemma, it follows that the following morphism is surjective.

$$p_2^* O_D \to p_2^* O_D$$

Since any such morphism is coming from a morphism $O_D \to O_D$, and such a morphism is surjective if and only if it is an isomorphism, we have that all the vertical arrows are in fact isomorphisms. □

A.2 Torsion-free sheaves

Recall that for any sheaf $\mathcal{F}$ on X there is a canonical morphism

$$\mathcal{F} \to \mathcal{F}^{**}$$

The sheaf $\mathcal{F}$ is *torsion free* if and only if the canonical morphism is injective and is *reflexive* if and only if the canonical morphism is an isomorphism.

Fact A.6. *A torsion free coherent sheaf on X is locally free in codimension 2.*

Fact A.7. *A torsion free coherent sheaf on X is isomorphic to $\mathcal{I}_Z \otimes \mathcal{M}$ for some line bundle $\mathcal{M}$ on X and $\mathcal{I}_Z$ the ideal sheaf of a closed subscheme $Z \subset X$ of codimension at least 2.*

Fact A.8. *A reflexive coherent sheaf on X is locally free in codimension 3.*

Fact A.9. *A reflexive coherent sheaf on X of rank 1 is a line bundle.*

Lemma A.10. *Consider an exact sequence of sheaves on X:*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \longrightarrow 0$$

If $\mathcal{F}$ is reflexive and $\mathcal{F}''$ is torsion free, then $\mathcal{F}'$ is reflexive.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} \mathcal{F}' & \xrightarrow{u} & \mathcal{F} & \xrightarrow{v} & \mathcal{F}'' \\ \beta\downarrow & & \alpha\downarrow & & \gamma\downarrow \\ \mathcal{F}'^{**} & \xrightarrow[u^{**}]{} & \mathcal{F}^{**} & \xrightarrow[v^{**}]{} & \mathcal{F}''^{**} \end{array}$$

The morphism α is an isomorphism because $\mathcal{F}$ is reflexive and the morphism γ is injective because $\mathcal{F}''$ is torsion-free. Note that since $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, which is reflexive, in particular torsion-free, $\mathcal{F}'$ is torsion-free. Hence, α is injective.

Let $\phi : \mathcal{F}'^{**} \to \mathcal{F}$ be the composition $\alpha^{-1} \circ u^{**}$. Note that $v \circ \phi = 0$. Hence ϕ factors as $u \circ \phi'$, where $\phi' : \mathcal{F}'^{**} \to \mathcal{F}'$. It is easy to check that ϕ' is an inverse for β. First, one proves easily that $\phi' \circ \beta = Id$. Then one gets that the short exact sequence:

$$0 \to \mathcal{F}' \to \mathcal{F}'^{**} \to \text{coker } \beta \to 0$$

is split. Since coker β is a torsion sheaf and $\mathcal{F}'^{**}$ is torsion free, it follows that β is an isomorphism, i.e., $\mathcal{F}'$ is reflexive. □

Lemma A.11. *Let $y_1, \ldots, y_e$ be points on a curve C, not necessarily distinct and let $D = y_1 + \ldots + y_e$. Let $\mathcal{E}$ be a rank 2 vector bundle on C and assume there is a short exact sequence:*

$$0 \to \mathcal{E} \to \mathcal{E}' \to O_D \to 0. \tag{A.3}$$

Then $\mathcal{E}'$ is not a locally free O_C-module if and only if there is $y \in \{y_1, \dots, y_e\}$ such that the sequence (*) on stalks at $\{y\}$ splits.

Proof. We denote by $\mathcal{E}_y$ and $\mathcal{E}'_y$ the stalks of $\mathcal{E}$ and $\mathcal{E}'$ at y. Clearly, if $y \in \{y_1, \dots, y_e\}$ and the sequence (A.3) splits at $\{y\}$, then $\mathcal{E}'_y$ is not a free O_y-module.

Let's assume now that that $\mathcal{E}'$ is not locally free. If $y \in C \setminus \{y_1, \dots, y_e\}$ then clearly, $\mathcal{E}'_y \cong \mathcal{E}_y$, so it is a free O_y-module. If $y \in \{y_1, \dots, y_e\}$ appears with multiplicity $r \geq 1$, then consider the exact sequence on stalks

$$0 \to \mathcal{E}_y \to \mathcal{E}'_y \to \tau \to 0.$$

Here, τ is isomorphic to a direct sum of r copies of the residue field $k(y)$ at the point y.

Assume that the multiplicity r is 1. The general case works the same. Then $\mathcal{E}'_y$ is not a free O_y-module if and only if it is not torsion-free. Assume that there is $e \in \mathcal{E}'_y, s \in O_y$ such that $s.e = 0$ and $e \neq 0, s \neq 0$. Denote with u the morphism $\mathcal{E}'_y \to \tau$. If $u(e) = 0$ then $e \in \mathcal{E}_y$, but as $s.e = 0$ this is a contradiction with E_y being a free O_y-module. So $u(e) = \lambda \neq 0$. Let $m_y \subset O_y$ be the maximal ideal and let π be a generator; hence, $m_y = (\pi)$. As $s.e = 0$, $0 = u(s.e) = \overline{s}.\lambda$, where $\overline{s}$ is the class of s modulo m_y. Hence, $s \in m_y$ and we can write $s = \mu.\pi^p$ for some non-zero $\mu \in k(y)$ and some positive integer p. We have $\pi^p.e = 0$. We claim that $\pi.e = 0$. If $p = 1$ we are done. Assume $p \geq 2$. We'll prove that $\pi^{p-1}.e = 0$. But $u(\pi^{p-1}.e) = \overline{\pi^{p-1}}.\lambda = 0.\lambda = 0$; hence, $\pi^{p-1}.e \in \mathcal{E}_y$. But as $\pi.(\pi^{p-1}.e) = 0$ and $\mathcal{E}_y$ is torsion-free, it follows that $\pi^{p-1}.e = 0$. We continue by induction and get that $\pi.e = 0$. Then we can define a morphism $v : k(y) \to \mathcal{E}'_y$ by $1 \mapsto \lambda^{-1}.e$. Because $\pi.e = 0$, this is a well-defined morphism of O_y-modules and $u \circ v = id$. ☐

Lemma A.12. *Let $\mathcal{E}, \mathcal{F}$ be coherent sheaves on X, projective integral scheme, with the property that $\mathcal{H}om_{O_X}(\mathcal{F}, \mathcal{E}) = 0$. Then an element $v \in Ext^1_{O_X}(\mathcal{F}, \mathcal{E})$ is 0 if and only if the exact sequence corresponding to the extension v splits on stalks everywhere on X.*

Proof. Consider the morphism

$$\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{E}) \to \Gamma(C, \mathcal{E}xt^1(\mathcal{F}, \mathcal{E})) \tag{A.4}$$

coming from the spectral sequence relating the local and global Ext. Since

$$\mathcal{H}om_{O_X}(\mathcal{F}, \mathcal{E}) = 0,$$

it follows that (A.4) is an isomorphism.

The lemma follows from the following commutative diagram for any $x \in X$:

$$\begin{array}{ccc} \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{E}) & \longrightarrow & \mathrm{Ext}^1_{O_x}(\mathcal{F}_x, \mathcal{E}_x) \\ \cong\downarrow & & \downarrow\cong \\ \Gamma(X, \mathcal{E}xt^1(\mathcal{F}, \mathcal{E})) & \longrightarrow & \mathcal{E}xt^1(\mathcal{F}, \mathcal{E})_x \end{array}$$

The assertion follows. □

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