Euclidean Network Information Theory

by

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Abstract

Many network information theory problems face the similar difficulty of single letterization. We argue that this is due to the lack of a geometric structure on the space of probability distributions. In this thesis, we develop such a structure by assuming that the distributions of interest are all close to each other. Under this assumption, the Kullback-Leibler (K-L) divergence is reduced to the squared Euclidean metric in an Euclidean space. In addition, we construct the notion of coordinate and inner product, which will facilitate solving communication problems. We will present the application of this approach to the point-to-point channels, general broadcast channels (BC), multiple access channels (MAC) with common sources, interference channels, and multi-hop layered communication networks without or with feedback. It can be shown that with this approach, information theory problems, such as the single-letterization, can be reduced to some linear algebra problems. Solving these linear algebra problems, we will show that for the general broadcast channels, transmitting the common message to receivers can be formulated as the trade-off between linear systems. We also provide an example to visualize this trade-off in a geometric way. For the MAC with common sources, we observe a coherent combining gain due to the cooperation between transmitters, and this gain can be obtained quantitively by applying our technique. In addition, the developments of the broadcast channels and multiple access channels suggest a trade-off relation between generating common messages for multiple users and transmitting them as the common sources to exploit the coherent combining gain, when optimizing the throughputs of communication networks. To study the structure of this trade-off and understand its role in optimizing the network throughput, we construct a deterministic model by our local approach that captures the critical channel parameters and well models the network. With this deterministic model, for multi-hop layered networks, we analyze the optimal network throughputs, and illustrate what kinds of common messages should be generated to achieve the optimal throughputs. Our results provide the insight of how users in a network should cooperate with each other to transmit information efficiently.

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Chapter 1

Introduction

In this thesis, we study a certain class of information theory problems for discrete memoryless communication networks, which we call the *linear information coupling problems.*

1.1 The Fundamental Set-up Of The Linear Information Coupling Problems

For a communication network, the corresponding linear information coupling problem asks the question that how we can efficiently transmit a thin layer of information through this network. More rigorously, we assume that there are sequences of input symbols generated at each transmitter from an i.i.d. distribution P_X. We also assume that the network is composed of some discrete memoryless channels, whose outputs are sequences with an i.i.d. distribution P_Y. We take this setup as an operating point. To encode an information $U = u$, we alter some of these input symbols, such that the empirical distribution changes to $P_{X|U=u}$. We insist that for each u, $P_{X|U=u}$ is close to P_X, which means we can only alter a small fraction of the input symbols. Moreover, when averaging over all different values of u, the marginal distribution of X remains unchanged. The receivers can then decode the information by distinguishing empirical output distributions with respect to different u. The goal of the linear information coupling problem is to design $P_{X|U=u}$ for different u, such that the receivers can distinguish different empirical output distributions the most efficiently. Mathematically, for the point-to-point channel with input X and output Y, the linear information coupling problem of this channel can be formulated as the multi-letter problem

$$\max_{U \to X^n \to Y^n} \frac{1}{n} I(U; Y^n), \tag{1.1}$$

$$\text{subject to: } \frac{1}{n} I(U; X^n) \leq \delta, \tag{1.2}$$

$$\frac{1}{n} \|P_{X^n|U=u} - P_{X^n}\|^2 = O(\delta), \ \forall u, \tag{1.3}$$

where δ is the amount of information modulated in per input symbol X, and assumed to be small. Here, both $P_{X^n|U=u}$ and P_{X^n} in (1.3) are viewed as $|\mathcal{X}|^n$ dimensional vectors, and the norm square is simply the Euclidean metric.

In fact, the problem (1.1) is almost the same as the traditional capacity problem

$$\max_{U \to X^n \to Y^n} \frac{1}{n} I(U; Y^n), \tag{1.4}$$

where U is the message transmitted in the channel. This traditional problem has the solution $\max_{P_X} I(X; Y)$ [1]. The difference between (1.4) and (1.1) lies in the constraint (1.2) and (1.3). Somewhat surprisingly, we will show that with these differences, the linear information coupling problem (1.1) can be solved quite differently from the corresponding capacity problem (1.4).

The linear information coupling problem (1.1) indeed captures some fundamental aspects of the traditional capacity problem. We will demonstrate in chapter 2 that the problem (1.1) is a sub-problem of the capacity problem. In general, the problem (1.1) is a local version of the global optimization problem (1.4), and the solutions of (1.1) are local optimal solutions of the corresponding capacity problem. In addition, we can "integral" the solutions of a set of linear information coupling problems back to a solution of the capacity problem.

One important feature of the linear information coupling problems is that there is a systematic approach for single-letterization for general multi-terminal communication problems. We first show in section 2.2 that, for the point-to-point channel case, the single-letterization of the linear information coupling problem (1.1) to its single-letter version

$$\begin{aligned} &\max_{U \to X \to Y} I(U;Y), \qquad (1.5)\\ &\text{subject to: } I(U;X) \le \delta,\\ &\qquad \|P_{X|U=u} - P_X\|^2 = O(\delta),\ \forall u, \end{aligned}$$

i.e., the process to establish the fact that the optimal solution to (1.1) is the same as that to (1.5), is a linear algebra problem. Then, we illustrate in chapter 3 and 4 that for general multi-terminal communication problems, the single-letterization procedures are conceptually the same as the point-to-point channel case. Note that the single-letterization is precisely the difficulty to generalize the conventional capacity results on the point-to-point channels to general multi-terminal problems, this systematic procedure for the linear information coupling problems thus makes these problems particularly attractive.

The main technique of our approach is based on the local approximation of the Kullback-Leibler (K-L) divergence. Note that in the linear information coupling problem for a communication network, we assume that the conditional distributions $P_{X|U=u}$ are close to the empirical distribution P_X for all u. Therefore, we can approximate the K-L divergence of two distributions around P_X by a quadratic function, which turns out to be related to the Euclidean distance between these two distributions. With this local approximation, the space of the input distributions is locally approximated as an Euclidean space around P_X. Similarly, the space of the output distributions can also be locally approximated as an Euclidean space around P_Y. We can construct geometric structures in these Euclidean spaces, such as orthonormal bases and inner products. Moreover, it can be shown that the channel behaves as a linear map between the input and output Euclidean spaces. Our purpose is to find the directions to perturb from P_X, according to the information $U = u$ to be encoded, in the input distribution space; or equivalently, to design $P_{X|U=u} - P_X$, such that after the channel map, the image of this perturbation at the output distribution space is as large as possible.

In chapter 3 and 4, we apply this technique to solve the linear information coupling problems of the general broadcast channels, and multiple access channels with common sources. From that, we find that, under the local assumptions, the transmissions of different types of messages, such as private and common messages, can be viewed as transmitted through separated deterministic links, and the channel gains of these links can be computed as some linear algebra problems. As a consequence, for a multi-terminal channel, we can quantify both the difficulty of broadcasting common messages than sending private messages, and the gain of transmitting common messages by the cooperation between transmitters. This development is particularly useful when studying the multi-hop networks, because it quantifies the tradeoff between the gain of sending a common message and the cost to create such common message from the previous layer, and hence evaluates whether or not a certain common message should be created.

Motivated by the above idea, in chapter 6, we apply the local geometric approach to the multi-hop layered networks. It turns out that with this approach, we can construct a deterministic network model that captures the channel parameters, which models the channels in the ability of transmitting private and common messages. Then, the linear information coupling problems become linear optimization problems of the network throughputs, and the solutions indicate what kind of common messages should be generated to optimize the throughputs. We also consider the large scale layered networks with identical layers. In these cases, the optimal communication schemes are composed of some fundamental transmission modes, and we specify these transmission modes in section 6.1. Our results in general provide the insights of how users in a communication network should cooperate with each other to increase the network throughputs.

In addition, we also explore the role of feedback using this local geometric approach. For that purpose, we consider the same networks as in chapter 6, but with additionally feedback links from each node to the nodes of the preceding layers. For these networks, we characterize the optimal communication scheme of each node that maximizes the amount of information that flows into destinations. As one consequence, we find that the layer-by-layer feedback strategy, which allows feedback only for the nodes in the immediately-preceding layer, yields the same performance as the most idealistic one where feedback is available to all the nodes of all the preceding layers. Moreover, we find that feedback can provide a multiplicative gain for a class of discrete-memoryless networks than the case without feedback. Specifying the optimal communication schemes in these cases illustrate how users in networks should cooperatively exploit the feedback to communicate efficiently.

1.2 The Relation To Hirschfeld-Gebelein-Rényi Maximal Correlation

Our results are related to [11], which investigates the efficiency of investment in stock markets. In [11], the authors showed that when $\delta \to 0$, the maximum of the ratio between $I(U;Y)$ and $I(U;X)$, subject to $\|P_{X|U} - P_X\|^2 = O(\delta)$[1], approaches the square of the Hirschfeld-Gebelein-Rényi maximal correlation (or simply Rényi maximal correlation) [12]-[15] between random variables X and Y, which is given as:

$$\rho_m(X,Y) = \sup E\left[f(X)g(Y)\right], \tag{1.6}$$

where the supremum is over all Borel-measurable functions f and g such that

$$\begin{aligned} E\left[f(X)\right] &= E\left[g(Y)\right] = 0, \\ E\left[f^2(X)\right] &= E\left[g^2(Y)\right] = 1. \end{aligned} \tag{1.7}$$

In section 2.4, we present an alternative way of showing this fact by using the local approximation approach, which deals precisely with the region of small δ. In fact,

[1] In [11], the authors did not explicitly mention this constraint, but implicitly used it in their proof.

this illustrates the connection between our local approach and the Rényi maximal correlation. In addition, the applications of our local approach to multi-terminal communication systems in chapter 3 and 4 also demonstrate the potential of generalizing the Rényi maximal correlation to more than two random variables, and provide new insights between communication problems and the statistics of random variables.

1.3 Some Remarks On The Local Constraint

Note that in our linear information coupling problem (1.1), we not only assume that the mutual information $\frac{1}{n}I(U;X^n)$ is small, but also restrict the conditional distributions $P_{X^n|U=u}$ satisfy the local constraint $\frac{1}{n}\|P_{X^n|U=u} - P_{X^n}\|^2 = O(\delta)$, for all u. With the local constraint on $P_{X^n|U=u}$, we can then guarantee the validity of the local approximation of KL divergence (see section 2.1 for the detail).

It is important to note that assuming $\frac{1}{n}I(U;X^n)$ to be small does not necessarily imply the local constraint on all the conditional distributions. It is possible that the joint distribution P_{X^nU} satisfies $\frac{1}{n}I(U;X^n) \leq \delta$, but the conditional distributions $P_{X^n|U=u}$ behave as some "tilted distributions" of u, i.e., for some u, the conditional distributions $P_{X^n|U=u}$ are far from P_{X^n}, but with $P_U(u) = O(\delta)$, and for other u, $P_{X^n|U=u}$ are close to P_{X^n} with $P_U(u) = O(1)$. Therefore, optimizing the mutual information $\frac{1}{n}I(U;Y^n)$ with only the constraint $\frac{1}{n}I(U;X^n) \leq \delta$ can be a different problem from our linear information coupling problem.

In fact, Ahlswede and Gács in [24], and a recent paper by Nair *et al.* [25] considered the following quantity

$$s(X^n, Y^n) = \lim_{I(U;X^n)\to 0} \sup_{U\to X^n \to Y^n} \frac{I(U;Y^n)}{I(U;X^n)}, \tag{1.8}$$

where they established two important statements:

(i) For i.i.d. $P_{X^nY^n} = P_{XY}^n$, the $s(X^n, Y^n)$ can be tensorized (single-letterized), i.e., $s(X^n, Y^n) = s(X, Y)$.

(ii) In general, $s(X^n, Y^n)$ can be strictly larger than $\rho_m(X^n, Y^n)$.

The statement (i) is an important property of $s(X^n, Y^n)$, because it addresses the single-letterization of the multi-letter problem in information theory, which reduces a computationally impossible problem to a computable one. On the other hand, the $\rho_m(X^n, Y^n)$ we consider in our local geometry can also be tensorized by a linear algebra approach (see section 2.2 for the detail). So, both $s(X^n, Y^n)$ and $\rho_m(X^n, Y^n)$ have this nice tensorization property in the point-to-point case.

Moreover, the statement (ii) implies that, without the local constraint, the optimal achievable information rate $\frac{1}{n}I(U; Y^n)$, subject to $\frac{1}{n}I(U; X^n) \leq \delta$, is $s(X^n, Y^n) \cdot \delta$. This is strictly better than the case with the local constraint, where the optimal achievable information rate is $\rho_m(X^n, Y^n) \cdot \delta$. Therefore, the $s(X^n, Y^n)$ is indeed a stronger quantity than the one we considered in our linear information coupling problems, i.e., $\rho_m(X^n, Y^n)$.

However, we would like to argue that it is still worth considering the quantity $\rho_m(X^n, Y^n)$, and studying the linear information problem can still provide many insights in network problems. The reason here is that the local geometric approach we developed for solving the linear information coupling problem in the point-to-point case can be easily extended to general networks, and the corresponding tensorization property can still be obtained just by some linear algebra. However, for harder network problems, the tensorization of the rather global problem (1.8) are impossible to be solved. To see this, let us consider a slightly harder problem, where now we want to broadcast messages to two receivers Y_1 and Y_2 through a general broadcast channel. Then, the natural extension of (1.8) becomes

$$s(X^n, Y_1^n, Y_2^n) = \lim_{I(U;X^n)\to 0} \sup_{U\to X^n\to(Y_1^n, Y_2^n)} \min \left\{ \frac{I(U; Y_1^n)}{I(U; X^n)}, \frac{I(U; Y_2^n)}{I(U; X^n)} \right\}.$$

It turns out that the approachs in [24] and [25] for proving the tensorization $s(X^n, Y^n) = s(X, Y)$ can not be directly applied to tensorize $s(X^n, Y_1^n, Y_2^n)$, and the tensorization of $s(X^n, Y_1^n, Y_2^n)$ is overall a non-convex optimization problem over an infinite dimensional space, which is an extremely difficult problem. In fact, one can show that the proving technique in [25] for tensorization is closely related to the single-letterization of the degraded broadcast channel, which can not be generalized to general networks. On the other hand, with the local constraint on the conditional distributions $P_{X^n|U=u}$, we can develop the local geometric structure, which allows us to solve the problem

$$\max_{U \to X^n \to (Y_1^n, Y_2^n):\ \frac{1}{n} I(U;X^n) \leq \delta,\ \frac{1}{n}\|P_{X^n|U} - P_{X^n}\|^2 = O(\delta)} \min \left\{ \frac{I(U;Y_1^n)}{I(U;X^n)}, \frac{I(U;Y_2^n)}{I(U;X^n)} \right\}$$

just as a simple linear algebra problem, and we can show that this problem can be single-letterized, i.e., has the single-letter optimal solution (see chapter 3 for the detail). Moreover, this approach can be easily generalized to general networks, which provides a systematic way to solve the single-letterization for communication networks. Therefore, this local geometric structure gives us a tool to study how information can be efficiently exchanged between different terminals in general networks, and is useful in investigating network communication problems.

Finally, we would like to also point out that the global optimization problem,

$$\max_{U \to X^n \to Y^n : \frac{1}{n} I(U;X^n) \leq \delta} \frac{1}{n} I(U;Y^n),$$

where the δ is not assumed to be small, but can be an arbitrary number, is also known as the information bottleneck [27, 28, 29]. In contrast to the small δ regime, for general δ, the authors in [27] employed the Lagrange multiplier method, and developed an algorithm to compute the optimal conditional distribution $P_{X|U}$. Moreover, when X and Y are distributed as jointly Gaussian random variables, some analytical solutions can be obtained [28].

1.4 Thesis Outline

This thesis is organized as follows. In chapter 2, we study the linear information coupling problems for point-to-point channels. We first introduce the notion of local approximation, and show that the K-L divergence can be approximated as the squared Euclidean metric. Then, the single-letter version of the linear information coupling problems will be solved by exploiting the local geometric structure. Moreover, the single-letterization of the linear information coupling problems will be shown to be equivalent to simple linear algebra problems. We will discuss the relation between our work and the capacity results as well as the code designs in section 2.3, and the relation to the Rényi maximal correlation in section 2.4. Chapter 3 is dedicated to applying the local approach to the general broadcast channels. It will be shown that the linear information coupling problems of general broadcast channels are different from that for the point-to-point channels in general: the single-letter solutions are not optimal, however finite-letter optimal solutions always exist. This provides a new way to visualize the local optimality of the celebrated Marton's coding scheme in general broadcast channels [8]. The application of the local approach to the multiple access channels with common sources is presented in chapter 4. We show that there are coherent combing gains in transmitting the common sources, and also quantify these gains. In chapter 5, we apply the local geometric approach to the interference channels, and we construct a deterministic model that captures the critical channel parameters. We extend this deterministic model to multi-hop layered networks in chapter 6. A Vitrtbi algorithm is proposed for characterizing the optimal network throughputs under our deterministic model. For the multi-hop layered networks with identical layers, in section 6.1, we further simplify the optimal communication schemes to some fundamental communication modes. Moreover, in section 7, we consider the situation where the feedback is allowed in our deterministic model. In this case, we observe certain feedback gain due to the extra available feedback path. Finally, the conclusion of this paper is given in chapter 8.

Chapter 2

The Point-to-Point Channel

We start with formulating and demonstrating the solutions of the linear information coupling problems for point-to-point channels. For a discrete memoryless point-to-point channel, with input $X \in \mathcal{X}$ and output $Y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite sets, let the $|\mathcal{Y}| \times |\mathcal{X}|$ channel matrix W denote the conditional distributions corresponding to the channel. For this channel, it is known that the capacity is given by

$$\max_{P_X} I(X;Y). \tag{2.1}$$

This simple expression is resulted from a multi-letter problem. If we encode a message U in n-dimensional vector X^n, and decode it from the corresponding n-dimensional channel output, we can write the problem as

$$\max_{U \to X^n \to Y^n} \frac{1}{n} I(U;Y^n), \tag{2.2}$$

where $U \to X^n \to Y^n$ denotes a Markov relation. It turns out that for the point-to-point channel, there is a simple procedure to prove that (2.2) and (2.1) have the same maximal value [1]:

$$\begin{aligned}\frac{1}{n}I(U;Y^n) &\leq \frac{1}{n}I(X^n;Y^n)\\ &= \frac{1}{n}\sum_i H(Y_i|Y_1^{i-1}) - H(Y_i|X^n, Y^{i-1})\\ &\leq \frac{1}{n}\sum_i H(Y_i) - H(Y_i|X_i)\\ &= \frac{1}{n}\sum_i I(X_i;Y_i) \leq \max I(X;Y). \end{aligned} \tag{2.3}$$

This procedure is known as the *single-letterization*, that is, to reduce a multi-letter optimization problem to a single-letter one. It is a critical step in general capacity problems, since without such a procedure, the optimization problems can potentially be over infinite dimensional spaces, and even numerical solutions of these problems may not be possible. Unfortunately, for general multi-terminal problems, we do not have a systematic way of single-letterization, which is why many of such problems remain open. The most famous examples of such problems are the general (not degraded) broadcast channels.

In contrast to the capacity problems, we study in this thesis an alternative class of problems, called linear information coupling problems. In this chapter, we consider the linear information coupling problems for point-to-point channels. Assuming as before that X and Y are the input and output of a point-to-point channel, the linear information coupling problem of this channel is the following multi-letter optimization problem[1]

$$\max_{U\to X^n\to Y^n} \frac{1}{n}I(U;Y^n), \tag{2.4}$$

$$\text{subject to: } \frac{1}{n}I(U;X^n) \leq \delta, \tag{2.5}$$

$$\frac{1}{n}\|P_{X^n|U=u} - P_{X^n}\|^2 = O(\delta),\ \forall u, \tag{2.6}$$

[1]In the assumption $\frac{1}{n}I(U;X^n) \leq \delta$, we implicitly assume that δ is a function $\delta(n)$ of n, and $n \cdot \delta(n) \ll 1$, for all n. Thus, the approximation in chapter 2.1 will be valid for any number of letters.

where δ is assumed to be small. The difference between (2.2) and (2.4) lies in the constraints (2.5) and (2.6). In the capacity problem, the entire input sequence is dedicated to encoding U; on the other hand, for the linear information coupling problems, we can only alter the input sequence "slightly" to carry the information from U. Operationally, we assume that sequences of i.i.d. P_X distributed symbols are transmitted, and the corresponding P_Y distributed symbols are received at the channel output. This can also be viewed as having P_{XY} jointly distributed multi-source. Then, we encode the message $U = u$ by altering a small number of symbols in these sequences, such that the empirical distribution changes to $P_{X|U=u}$. As we only alter a small number of symbols, the conditional distribution $P_{X|U=u}$ is close to P_X. For the rest of this thesis, we assume that the marginal distribution P_{X^n} is an i.i.d. distribution over the n letters[2]. Our goal is to find the conditional distributions $P_{X|U=u}$ for different values u, which satisfy the marginal constraint P_X, such that a thin layer of information can be conveyed to the Y end the most efficiently.

Although we assume that the operating point has i.i.d. P_X distribution, it is not priorly clear that $P_{X^n|U=u}$ should be also i.i.d.. Therefore, (2.4) has a multi-letter form. In fact, we will show in section 2.2 that, unlike the capacity problems, the linear information coupling problems allow easy single-letterization, and the optimal $P_{X^n|U=u}$ indeed should be i.i.d.. This turns out to be a very important feature of the linear information coupling problems, since the problems are then optimized over finite dimensional spaces.

2.1 The Local Approximation

The key technique of our approach to solve the linear information coupling problems is to use a local approximation of the K-L divergence. Let P and Q be two distributions over the same alphabet $\mathcal{X}$, then $D(P\|Q) = \sum_x P(x) \log(P(x)/Q(x))$ can be viewed as a measure of distance between these two distributions. However, this distance

[2]This assumption can be proved to be "without loss of the optimality" for some cases [10]. In general, it requires a separate optimization, which is not the main issue addressed in this thesis. To that end, we also assume that the given marginal P_{X^n} has strictly positive entries.

measure is not symmetric, that is, $D(P\|Q) \neq D(Q\|P)$. The situation can be much simplified if P and Q are close. We assume that $Q(x) = P(x) + \epsilon J(x)$, for some small value ϵ, and a function $J : \mathcal{X} \mapsto \mathbb{R}$. Then, the KL divergence can be written, with the second order Taylor expansion, as

$$\begin{aligned} D(P\|Q) &= -\sum_x P(x) \log \frac{Q(x)}{P(x)} \\ &= -\sum_x P(x) \log \left(1 + \epsilon \cdot \frac{J(x)}{P(x)}\right) \\ &= \frac{1}{2}\epsilon^2 \cdot \sum_x \frac{1}{P(x)} J^2(x) + o(\epsilon^2). \end{aligned}$$

We think of J also as a column vector of dimension $|\mathcal{X}|$, and denote $\sum_x J^2(x)/P(x)$ as $\|J\|_P^2$, which is the weighted norm square of the perturbation vector J. It is easy to verify here that replacing the weights in this norm by $Q(x)$ only results in an $o(\epsilon^2)$ difference. That is, up to the first order approximation, the weights in the norm simply indicate the neighborhood of distributions where the divergence is computed. As a consequence, $D(P\|Q)$ and $D(Q\|P)$ are considered as equal up to the first order approximation.

For convenience of the notations, we define the *weighted perturbation vector* as

$$L(x) \triangleq \frac{1}{\sqrt{P(x)}} J(x), \quad \forall x \in \mathcal{X},$$

or in vector form $L \triangleq \left[\sqrt{P}^{-1}\right] J$, where $\left[\sqrt{P}^{-1}\right]$ represents the diagonal matrix with entries $\left\{\sqrt{P(x)}^{-1},\ x \in \mathcal{X}\right\}$. This allows us to write $\|J\|_P^2 = \|L\|^2$, where the last norm is simply the Euclidean norm.

With this definition of the norm on the perturbations of distributions, we can generalize to define the corresponding notion of inner products. Let $Q_i(x) = P(x) + \epsilon \cdot J_i(x)$, $\forall x, i = 1, 2$, we can define

$$\langle J_1, J_2 \rangle_P \triangleq \sum_x \frac{1}{P(x)} J_1(x) J_2(x) = \langle L_1, L_2 \rangle,$$

where $L_i = \left[\sqrt{P}^{-1}\right] J_i$, for $i = 1, 2$. From this, the notions of orthogonal perturbations and projections can be similarly defined. The point here is that we can view a neighborhood of distributions as a linear metric space, where each distribution Q is specified by the corresponding weighted perturbation L from P, and define notions of orthonormal basis and coordinates on it.

After establishing the local approximation of the K-L divergence, we can now study the single-letter version of the linear information coupling problem (2.4):

$$\max_{U \to X \to Y} I(U;Y), \tag{2.7}$$

$$\text{subject to: } I(U;X) \leq \delta, \tag{2.8}$$

$$\|P_{X|U=u} - P_X\|^2 = O(\delta), \ \forall u, \tag{2.9}$$

and observe how the local geometric structure helps us to visualize the solution. Here, we replace the notation δ in the constraint by $\frac{1}{2}\epsilon^2$, as its meaning is now clear. We assume that the distribution P_X is given as the operating point. The purpose of (2.7) is to design the distribution P_U and the conditional distributions $P_{X|U=u}$ to maximize the mutual information $I(U;Y)$, such that the constraint

$$I(U;X) = \sum_u P_U(u) \cdot D(P_{X|U}(\cdot|u) \| P_X) \leq \frac{1}{2}\epsilon^2, \tag{2.10}$$

is satisfied, and the marginal distribution $\sum_u P_U(u) P_{X|U=u} = P_X$. From the constraint (2.9), we can write the conditional distributions $P_{X|U=u}$ as perturbations of P_X. Written in vector form, we have $P_{X|U=u} = P_X + \epsilon \cdot J_u$, where J_u is the perturbation vector. With this notation and using the local approximation on $D(P_{X|U}(\cdot|u) \| P_X)$, the constraint (2.10) can be written as

$$\frac{1}{2}\epsilon^2 \sum_u P_U(u) \cdot \|J_u\|^2_{P_X} + o(\epsilon^2) \leq \frac{1}{2}\epsilon^2,$$

which is equivalent to $\sum_u P_U(u) \cdot \|J_u\|^2_{P_X} \leq 1$. Moreover, since $P_{X|U=u}$, for different u, need to be valid probability distributions and satisfy the marginal constraint, we have

$$\sum_x J_u(x) = 0, \text{ for all } u, \tag{2.11}$$

and

$$\sum_u P_U(u) J_u = 0. \tag{2.12}$$

Next, for each u, let $L_u = \left[\sqrt{P_X}^{-1}\right] J_u$ be the weighted perturbation vector. Now, we observe that in the output distribution space

$$\begin{aligned} P_{Y|U=u} &= W P_{X|U=u} = W P_X + \epsilon \cdot W J_u \\ &= P_Y + \epsilon \cdot W[\sqrt{P_X}] L_u, \end{aligned}$$

where the channel applied to an input distribution is simply written as the channel matrix W, with dimension $|\mathcal{Y}| \times |\mathcal{X}|$, multiplying the input distribution as a vector. At this point, we have reduced both the spaces of input and output distributions as linear spaces, and the channel acts as a linear transform between these two spaces. The linear information coupling problem (2.7) can be rewritten as, ignoring the $o(\epsilon^2)$ terms:

$$\begin{aligned} \max. \ & \sum_u P_U(u) \cdot \|W J_u\|^2_{P_Y}, \\ \text{subject to: } & \sum_u P_U(u) \cdot \|J_u\|^2_{P_X} = 1, \end{aligned}$$

or equivalently in terms of Euclidean norms,

$$\max. \ \sum_u P_U(u) \cdot \left\| \left[\sqrt{P_Y}^{-1}\right] W \left[\sqrt{P_X}\right] \cdot L_u \right\|^2, \tag{2.13}$$

$$\text{subject to: } \sum_u P_U(u) \cdot \|L_u\|^2 = 1. \tag{2.14}$$

In addition, from the definition of L_u, we can write (2.11) and (2.12) as

$$\sum_x \sqrt{P_X(x)} L_u(x) = 0, \text{ for all } u \tag{2.15}$$

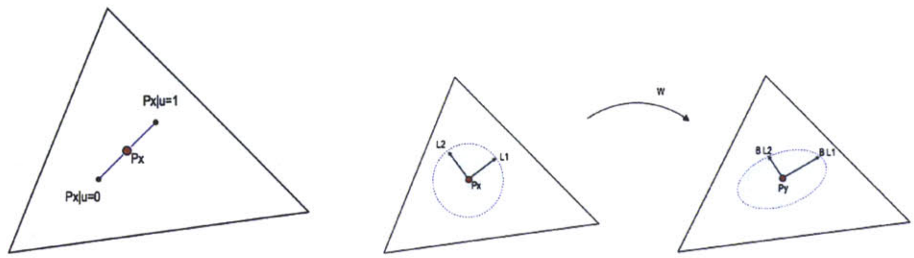
and

$$\sum_u P_U(u) L_u = 0. \tag{2.16}$$

We should notice that a choice of $P_{X|U=u}$ is equivalent to a choice of L_u, thus it is enough to just solve (2.13).

The problem (2.14) is a linear algebra problem. We need to find the joint distribution $U \to X \to Y$ by specifying P_U and the weighted perturbation vector L_u for each value of u, such that the constraints (2.15) and (2.16) on L_u are met, and also these perturbations are the most visible at the Y end, in the sense that multiplied by the channel matrix, WJ_u's have large norms. This can be readily solved by setting the weighted perturbation vectors L_u's to be along certain principal direction that satisfies the constraint (2.15), and has the largest output image under the matrix $B \triangleq \left[\sqrt{P_Y}^{-1}\right] W \left[\sqrt{P_X}\right]$.

The point here is that, with our local approximation and the linearized structure, for different values of u, the output norm square of the L_u's are simply added together averaged over $P_U(u)$. So, if there exists $u_1, u_2 \in \mathcal{U}$ such that $\|B \cdot L_{u_1}\|^2 > \|B \cdot L_{u_2}\|^2$, then we would like to align L_{u_2} to be along L_{u_1}, i.e. choosing a new vector $\hat{L}_{u_2} = \frac{\|L_{u_2}\|}{\|L_{u_1}\|} \cdot L_{u_1}$ to replace L_{u_2}. With this new choice $\hat{L}_{u_2}$, we increase the averaged output norm square (2.13), while keeping the constraint (2.14) to be satisfied. Therefore, we can see that the optimal solution of (2.13) happens when all vectors L_u are aligned along the principal direction that has the largest output image under the linear map B. Moreover, from the linearity of the map B, once we solve this principal direction, the choices of the cardinality of U and the corresponding $P_U(u)$ do not effect the optimal averaged output norm square, as long as the constraint (2.14) and (2.16) are satisfied. So, we can without loss of generality choose U as a uniformly binary



(a) (b)

Figure 2-1: (a) Choice of P_U and $P_{X|U}$ to maintain the marginal P_X. (b) Divergence Transition Map as a linear map between two spaces, with right and left singular vectors as orthonormal bases.

random variable, and $P_{X|U=0} = P_X + \epsilon \cdot L$, and $P_{X|U=1} = P_X - \epsilon \cdot L$, for some weighted perturbation vector L. This step is critical in the sense that it reduces the cardinality of U, and simplifies the overall problem as a readily solvable linear algebra problem. Figure 2-1(a) illustrates this idea from the geometric point of view. Then, the problem becomes

$$\max. \ \|B \cdot L\|^2, \tag{2.17}$$

$$\text{subject to: } \|L\|^2 = 1, \tag{2.18}$$

$$\sum_x \sqrt{P_X(x)} L(x) = 0. \tag{2.19}$$

We call this matrix B the *divergence transition matrix* (DTM), as it maps divergence in the space of input distributions to that of the output distributions.

Now, this is a linear algebra problem that aims to optimize (2.17) by a unit vector L subject to the linear constraint (2.19). To solve this problem, first note that if we ignore the linear constraint (2.19), the optimization of (2.17) is simply choosing L as the largest right (input) singular vector of B corresponding to the largest singular value. However, this choice may violate (2.19). In fact, the linear constraint (2.19) can be viewed as the orthogonality of the vector L and a vector $\underline{v}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$. Moreover, it is easy to see that $\underline{v}_0$ is a right singular vector of B corresponding to the singular value 1 and the left singular vector $\underline{w}_0 = \left[\sqrt{P_Y}, y \in \mathcal{Y}\right]^T$. Therefore, the linear constraint (2.19) restricts the weighted perturbation vector L in the subspace spanned by all the singular vectors of B except for $\underline{v}_0$. This implies that the optimal weighted perturbation vector L of (2.17) is the right singular vector of the matrix $B - \underline{w}_0\underline{v}_0^T$ corresponding to its largest singular value.

In addition, applying the data processing inequality, the following lemma shows that all singular values of B are upper bounded by 1.

Lemma 1. *Let the singular values of the DTM B be $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_m$, with the corresponding right singular vectors $\underline{v}_0, \underline{v}_1, \ldots, \underline{v}_m$, where $m = \min\{|\mathcal{X}|, |\mathcal{Y}|\} - 1$, then $\sigma_0 = 1$ and $\underline{v}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$.*

Proof. We only need to show that $\sigma_0 = 1$. First, observe that since $U \to X \to Y$ forms a Markov relation, from the data processing inequality, we have $I(U;Y) \leq I(U;X)$. This implies that for any weighted perturbation vector L that satisfies (2.19), the inequality $||B \cdot L||^2 \leq ||L||^2$ holds. Because L can be chosen as any singular vector of B other than $\underline{v}_0$, all singular values of B are upper bounded by 1, hence $\sigma_0 = 1$ is the largest singular value. □

From this lemma, we can see that the largest singular value of $B - \underline{w}_0\underline{v}_0^T$ is the second largest singular value of B, and the optimum of (2.17) is achieved by setting L to be along the right singular vector of B with the second largest singular value, i.e., $\underline{v}_1$.

We can visualize as in Figure 2-1(b) the orthonormal bases of the input and output spaces, respectively, according to the right and left singular vectors of B. The key point here is that while $I(U;X)$ measures how many bits of information is carried in X, depending on how the information is modulated, in terms of which direction the corresponding perturbation vector is, the information has different "visibility" at the receiver end. Our approach is to exploit the optimal perturbing direction so that the information can be conveyed to the receiver the most efficiently.

Example 1. In this example, we consider a very noisy ternary point-to-point channel with input symbols $\mathcal{X} = \{1, 2, 3\}$ and output symbols $\mathcal{Y} = \{1, 2, 3\}$ such that:

(i) The sub-channel between the input symbols $\{2, 3\}$ and the output symbols $\{2, 3\}$ is a binary symmetric channel (BSC) with crossover probability $\frac{1}{2} - \gamma$.

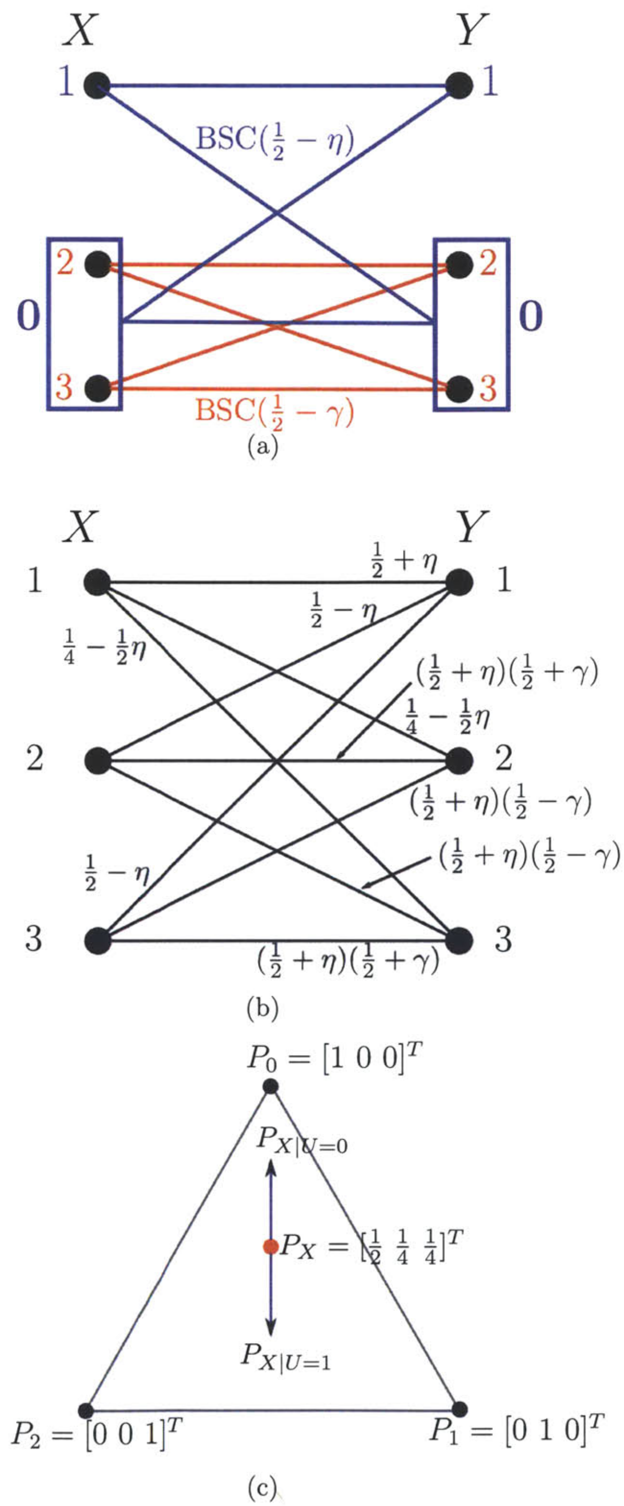


Figure 2-2: (a) The ternary point-to-point channel that is composed of two binary symmetric channels. (b) The channel transition probability of this ternary channel. (c) The optimal perturbation direction for the ternary channel to convey information to the receiver end. Here, the triangle represents all valid input distributions, and the vertices are the deterministic input distributions of the three input symbols.

(ii) If we employ the auxiliary input/output symbol $\mathbf{0}$ to represent the transmission/receiving of the input/output symbols 2 and 3, then the sub-channel between the input symbols $\{\mathbf{0}, 1\}$ and the output symbols $\{\mathbf{0}, 1\}$ is a BSC with crossover probability $\frac{1}{2} - \eta$.

This ternary channel is illustrated in Figure 6-5(b). Mathematically, the channel transition matrix of this ternary channel can be specified as

$$W = \begin{bmatrix} \frac{1}{2} + \eta & \frac{1}{2} - \eta & \frac{1}{2} - \eta \\ \frac{1}{4} - \frac{1}{2}\eta & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} + \gamma\right) & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} - \gamma\right) \\ \frac{1}{4} - \frac{1}{2}\eta & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} - \gamma\right) & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} + \gamma\right) \end{bmatrix},$$

which is illustrated in Figure 2-2(b). In addition, we assume that $1 \gg \eta \gg \gamma > 0$, so that this ternary channel is very noisy.

There are two modes that information can be transmitted through this channel. The first one is to modulate the message in the input symbols 1 and $\mathbf{0} = \{2, 3\}$ of the BSC $(\frac{1}{2} - \eta)$, and then this message can be decoded at the receiver end according to the output symbols 1 and $\mathbf{0}$. The second transmission mode is to modulate the message in the input symbols 2 and 3 of the BSC $(\frac{1}{2} - \gamma)$, and decode the message according to the output symbols 2 and 3. In this example, we employ our framework to study how information can be efficiently conveyed through this ternary channel, and the relation of efficient communication to these transmission modes.

To apply our approach, we fix the empirical distribution P_X as $[\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}]^T$, and the corresponding output distribution P_Y is $[\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}]^T$. Then, the DTM is

$$B = \begin{bmatrix} \frac{1}{2} + \eta & \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\eta & \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\eta \\ \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\eta & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} + \gamma\right) & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} - \gamma\right) \\ \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\eta & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} - \gamma\right) & \left(\frac{1}{2} + \eta\right)\left(\frac{1}{2} + \gamma\right) \end{bmatrix}.$$

For this DTM, the singular values are 1, 2η, and $(1 + 2\eta)\gamma$, with the corresponding right singular vectors $[\frac{1}{\sqrt{2}}\ \frac{1}{2}\ \frac{1}{2}]^T$, $[\frac{1}{\sqrt{2}}\ \frac{-1}{2}\ \frac{-1}{2}]^T$, and $[0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$. So, the second largest singular value is $\sigma_1 = 2\eta$, and the corresponding right singular vector is $\underline{v}_1 = [\frac{1}{\sqrt{2}} \; \frac{-1}{2} \; \frac{-1}{2}]^T$. Thus, the optimal perturbation vector (not weighted) is $[\frac{1}{2} \; \frac{-1}{4} \; \frac{-1}{4}]^T$, and the conditional distributions are $P_{X|U=0} = \left[\frac{1}{2} + \frac{1}{2}\epsilon \; \frac{1}{4} - \frac{1}{4}\epsilon \; \frac{1}{4} - \frac{1}{4}\epsilon\right]^T$ and $P_{X|U=1} = \left[\frac{1}{2} - \frac{1}{2}\epsilon \; \frac{1}{4} + \frac{1}{4}\epsilon \; \frac{1}{4} + \frac{1}{4}\epsilon\right]^T$. This can be geometrically visualized as in Figure 2-2(c).

Note that perturbing the input distribution along the directions $[\frac{1}{\sqrt{2}} \; \frac{-1}{2} \; \frac{-1}{2}]^T$, and $[0 \; \frac{1}{\sqrt{2}} \; \frac{-1}{\sqrt{2}}]^T$ corresponds to modulating and communicating the message through the BSC $(\frac{1}{2} - \eta)$, and BSC $(\frac{1}{2} - \gamma)$, respectively. So, our approach shows that the most efficient way to convey information through the channel is to couple the message into input sequences through the communication mode BSC $(\frac{1}{2} - \eta)$.

Remark 1. The above arguments imply that

$$I(U;Y) \leq \sigma_1^2 \cdot I(U;X), \tag{2.20}$$

where $\sigma_1 \leq 1$ is the second largest singular value of B. Thus, comparing to the data processing inequality $I(U;Y) \leq I(U;X)$, (2.20) can be viewed as a "strong data processing" inequality. Moreover, the solution of (2.17) tells how the equality can be achieved. Note that (2.20) comes from the local geometric structure, which requires the local constraint (2.9). One can obtain a stronger result than (2.20) that does not require the local constraint by replacing the σ_1^2 by the $s(X,Y)$ as we discussed in section 1.3.

Remark 2. In fact, these ideas are closely related to the method of information geometry [4], which studies the geometric structure of the space of probability distributions. In information geometry, the collection of probability distributions forms a manifold, and the K-L divergence behaves as the distance measure in this manifold. However, the K-L divergence is not symmetric, and this manifold is not flat, but has a rather complicated structure. On the other hand, our approach introduced in this chapter locally approximates this complicated manifold by a tangent hyperplane around P_X, which can be viewed as an Euclidean space. Moreover, the K-L divergence corresponds to the weighted Euclidean norm square in this linear space. For the linearized neighborhood around P_X, just like any other metric space, one can define many orthonormal bases. Here, we pick the orthonormal basis according to the SVD structure of the DTM B, which is particularly suitable as our goal is to study how much information can be coupled through this channel. This orthonormal basis illustrates the principle directions of conveying information to the receiver end under the channel map, and provides the insights of how to efficiently exploit the channel.

Remark 3. In many information theory problems, it is required to deal with the trade-off between multiple K-L divergences (mutual informations), which is a non-convex problem. Thus, finding global optimum for such problems is in general intrinsically intractable. In contrast, with our local approximation, the K-L divergence becomes a quadratic function. Therefore, the tradeoff between quadratic functions remains quadratic, which is much easier to deal with. In particular, our approach focuses on verifying the local optimality of the quadratic solutions, which is a natural thing to do, since the overall problem is not convex.

2.2 The Single-Letterization

The most important feature of the linear information coupling problem (2.4) is that the single-letterization is simple. To illustrate the idea, we first consider a 2-letter version of the point-to-point channel:

$$\begin{aligned} \max_{U \to X^2 \to Y^2} & \frac{1}{2} I(U; Y^2), \\ \text{subject to: } & \frac{1}{2} I(U; X^2) \le \delta, \\ & \frac{1}{2} \| P_{X^2|U=u} - P_{X^2} \|^2 = O(\delta), \ \forall u, \end{aligned} \tag{2.21}$$

Let P_X, P_Y, W, and B be the input and output distributions, channel matrix, and the DTM, respectively, for the single letter version of the problem. Then, the 2-letter problem has $P_X^{(2)} = P_X \otimes P_X$, $P_Y^{(2)} = P_Y \otimes P_Y$, and $W^{(2)} = W \otimes W$, where $\otimes$ denotes the Kronecker product, or also called as the tensor product. As a result, the new DTM is $B^{(2)} = B \otimes B$. Thus, the optimization in (2.21) has exactly the same form as in (2.7), where the only difference is that we need to find the SVD of $B^{(2)}$ instead of B. For that, we have the following well-known lemma:

Lemma 2. *Let $\underline{v}_i$ and $\underline{v}_j$ denote two right (or left) singular vectors of B with singular values σ_i and σ_j. Then, $\underline{v}_i \otimes \underline{v}_j$ is a right (or left) singular vector of $B^{(2)}$ and the corresponding singular value is $\sigma_i \cdot \sigma_j$.*

Recall that the largest singular value of B is $\mu_0 = 1$, with the right singular vector $\underline{v}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$, which corresponds to the direction orthogonal to the distribution simplex. This implies that the largest singular value of $B^{(2)}$ is also 1, corresponding to the singular vector $\underline{v}_0 \otimes \underline{v}_0$, which is again orthogonal to all valid choices of the weighted perturbation vectors.

The second largest singular value of $B^{(2)}$ is a tie between $\sigma_0 \cdot \sigma_1$ and $\sigma_1 \cdot \sigma_0$, with right singular vectors $\underline{v}_0 \otimes \underline{v}_1$ and $\underline{v}_1 \otimes \underline{v}_0$, where σ_1 is the second largest singular value of B, and $\underline{v}_1$ is the corresponding right singular vector. The optimal solution of (2.21) is thus the weighted perturbation vectors along the subspace spanned by these two vectors. This can be written as

$$P_{\underline{X}|U=u} = P_X \otimes P_X + \left[\sqrt{P_X \otimes P_X}\right] \cdot \left(\epsilon \underline{v}_0 \otimes \underline{v}_1 + \epsilon' \underline{v}_1 \otimes \underline{v}_0\right) \tag{2.22}$$

$$= \left(P_X + \epsilon' \left[\sqrt{P_X}\right] \underline{v}_1\right) \otimes \left(P_X + \epsilon \left[\sqrt{P_X}\right] \underline{v}_1\right) + O(\epsilon^2), \tag{2.23}$$

where (2.23) comes from noting that the vector $\underline{v}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$, and adding the appropriate cross (higher order) terms for factorization. Here, we assume that ϵ and ϵ' are of the same order, which makes the cross term $O(\epsilon^2)$. This means that up to the first order approximation, the optimal choice of $P_{X^2|U=u}$, for any value of u, has a product form, i.e., the two transmitted symbols in X^2 are conditionally independent given U. With a simple time-sharing argument, we can show that it is optimal to set $\epsilon = \epsilon'$. This implies that picking $P_{X^2|U=u}$ to be i.i.d. over the two symbols achieves the optimum, with the approximation in (2.23).

Finally, by considering the n^{th} Kronecker product, we can generalize this procedure to the single-letterization of the n-letter problem (2.4).

Remark 4. This proof of showing the single-letter optimality is simple. All we have used is the fact that the singular vectors of $B^{(2)}$ corresponding to the second largest singular value has a special form, $\underline{v}_0 \otimes \underline{v}_1$ or $\underline{v}_1 \otimes \underline{v}_0$. We can visualize this as follows. The space of 2-letter joint distributions $P_{X^2|U=u}$ has $(|\mathcal{X}|^2 - 1)$ dimensions. Around the i.i.d. marginal distribution $P_X \otimes P_X$, there is a $2 \cdot (|\mathcal{X}| - 1)$-dimensional subspace, such that the distributions in this subspace take the product form $Q_1 \otimes Q_2$, for some distributions Q_1 and Q_2 around P_X. These distributions can be written as perturbations from $P_X \otimes P_X$, with the weighted perturbations of the form $\underline{v}_0 \otimes \underline{v} + \underline{v}' \otimes \underline{v}_0$, for some $\underline{v}$ and $\underline{v}'$ orthogonal to $\underline{v}_0$. The above argument simply verifies that the optimal solution to (2.21), which is the singular vectors of the $B^{(2)}$ matrix, has this form. Generalizing to n-letter problems, our procedure reduces the dimensionality. Moreover, it turns out that this procedure can be applied to more general problems. In chapter 3 and 4, we will demonstrate that in quite a few other multi-terminal problems, the similar structure can be proved and used for single-letterization.

We would like to emphasize that the advantage of our approach is that it does not require any constructive proving technique, such as constructing auxiliary random variables. For any given problem, one can follow essentially the same procedure to find out the SVD structure of the corresponding DTM. The result either gives a proof of the local optimality of the single letter solutions or disproves it without any ambiguity.

Before moving to the more interesting multi-terminal problems, we discuss in the rest two sections that how the linear information coupling problems can be connected to the capacity problems, and also the relation between the linear information coupling problems and the Rényi maximal correlation. Readers, who are only interested in the application of our local approach to the multi-terminal problems, can directly turn to the chapter 3 and 4.

2.3 Capacity Achieving Layered Codes

In this section, we demonstrate that how the linear information coupling problem (2.4) can be a sub-problem of the capacity problem (2.2). The main idea here is that, after coupling a small piece of information U into input sequences by perturbing the empirical distribution P_X to $P_{X|U=u}$, we can keep coupling informations into the resulting input sequences by further perturbing the distributions $P_{X|U=u}$. Iteratively running this process, we can then come up with a layered coding scheme that couples a sequence of informations $U_1, U_2, \ldots$ into the input sequences, where the i-th layer of this coding scheme corresponds to the linear information coupling problem of the information U_i that operates on the distribution $P_{X|U_1=u_1,\ldots,U_{i-1}=u_{i-1}}$ resulted from the previous $i-1$ layers. In the following, we explore the detail of this layered coding scheme.

Let us start from the one-layer problem of this coding scheme. For a point-to-point channel with a transmitter X and a receiver Y, the goal of the one-layer problem is to efficiently transmit information through the Markov relation $U_1 \to X \to Y$, subjecting to the constraint $I(U_1;X) \le \frac{1}{2}\epsilon^2$, and $\|P_{X|U_1} - P_X\| = O(\epsilon)$. From the analyses of the linear information coupling problem, we know how to find the optimal $P^*_{X|U_1=u_1}$ and $P^*_{U_1}$ to achieve the solution[3]

$$r_1^* = \max_{U_1 \to X \to Y:\ I(U_1;X)\le \frac{1}{2}\epsilon^2,\ \|P_{X|U_1}-P_X\|=O(\epsilon)} I(U_1;Y) \tag{2.24}$$

Now, we propose the following coding scheme to explain the operational meaning of this solution. Suppose that there is a block[4] of $n_1 \cdot k_1$ i.i.d. P_X distributed input symbols $\underline{x}(1), \ldots, \underline{x}(k_1)$ generated at the transmitter, where $\underline{x}(i) \in \mathcal{X}^{n_1}$ represents a sub-block of n_1 input symbols $x_1(i), \ldots, x_{n_1}(i)$, for $1 \le i \le k_1$. Then, we "encode" a binary codeword $u_1(1), \ldots, u_1(k_1)$, with empirical distribution $P^*_{U_1}$, into this input symbol block by altering some of the symbols, such that the empirical distribution of each sub-block $\underline{x}(i)$ changes to $P^*_{X|U_1=u_1(i)}$. Note that the empirical distribution of the entire symbol block remains approximately the same as P_X. The receiver decodes this codeword according to different empirical output distributions of the k_1 sub-blocks. From (2.24), there exists binary block codes $u_1(1), \ldots, u_1(k_1)$ with rate $R_1^* = n_1 \cdot r_1^*$

[3]In this section, we implicitly assume that all the conditional distributions $P_{X|U}$ satisfy the local constraint, so that they are all close to P_X in the sense of ϵ.

[4]In this thesis all the "block length" and "number of sub-blocks" are assumed to be large.

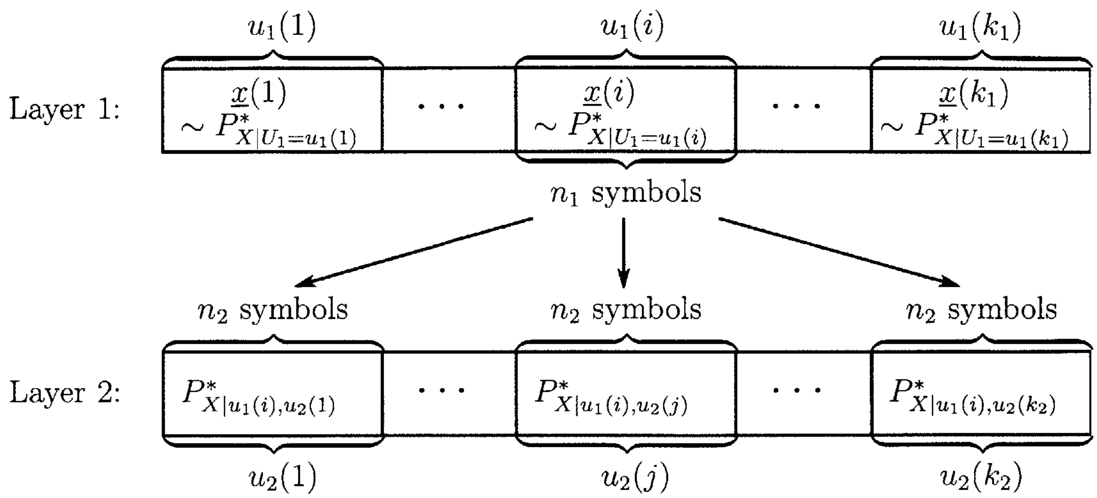


Figure 2-3: The empirical distribution of different sub-blocks in each layer after encoding.

bits/U_1 symbol, which can be reliably transmitted and decoded by using the above coding scheme. The empirical distributions of different blocks of input symbols, after this encoding procedure, are illustrated in Figure 2-3.

Now, we can add another layer to the one-layer problem. Theoretically, this is to consider a new set of linear information coupling problems

$$r_2^*(u_1) = \max_{(U_1,U_2)\to X\to Y:\ I(U_2;X|U_1=u_1)\le\frac{1}{2}\epsilon^2,\ \|P_{X|U_1=u_1,U_2}-P^*_{X|U_1=u_1}\|=O(\epsilon)} I(U_2;Y|U_1=u_1), \tag{2.25}$$

where the conditional distribution of X given $U_1 = u_1$ is specified as $P^*_{X|U_1=u_1}$. We can solve (2.25) with the same procedure as (2.24), and find the optimal solutions $P^*_{X|U_1=u_1,U_2=u_2}$ and $P^*_{U_2|U_1=u_1}$.

Then, we can encode this one more layer of codewords to the original layer with a similar coding scheme. To do this, we further divide each sub-block $\underline{x}(i)$ into k_2 small sub-blocks, and each of the small sub-block has n_2 symbols, where $n_2 \cdot k_2 = n_1$. Then, for a binary code $u_2(1), \ldots, u_2(k_2)$ with rate $R_2^*(u_1(i)) = n_2 \cdot r_2^*(u_1(i))$ bits/U_2 symbol, where the distribution of the bits in the codewords is $P^*_{U_2|U_1=u_1(i)}$, we encode the codewords into small sub-blocks of $\underline{x}(i)$ by exactly the same coding scheme as the one-layer problem. The transmission rate of this coding scheme over the entire input

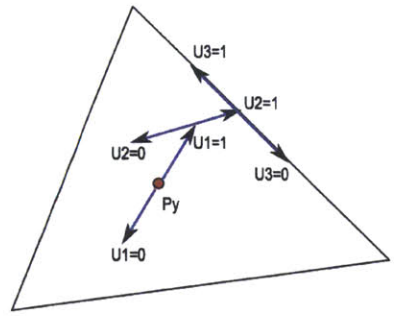


Figure 2-4: A layered approach for the coding problem.

symbol block $\underline{x}(1), \ldots, \underline{x}(k_1)$ is then

$$\sum_{u_1} r_2^*(u_1) P_{U_1}^*(u_1) = I(U_2; Y|U_1) \quad \text{bits/transmission.}$$

After this, the empirical distribution of the j-th small sub-block of $\underline{x}(i)$ changes to $P^*_{X|U_1=u_1(i),U_2=u_2(j)}$, which is illustrated in Figure 2-3. On the other hand, the empirical distribution of the entire $\underline{x}(i)$ remains approximately the same as $P^*_{X|U_1=u_1(i)}$. Thus, the decoding of the codewords $u_1(1), \ldots, u_1(k_1)$ of the first layer is not effected by adding the second layer, and can be proceeded as in the one-layer problem. The codewords of the second layer are then decoded after the first layer is decoded.

We can keep adding layers by recursively solving new linear information coupling problems, and sequentially applying the above layered coding scheme. Assume that there is a sequence of messages $U_1, U_2, ..., U_K$ that we want to encode. First, we can find a perturbation of the P_X distribution according to the value of U_1 by solving the corresponding linear information coupling problem. Then, by solving the new set of information coupling problems conditioned on each value of $U_1 = u_1$, we can find further perturbations of that according to the value of U_2, and so on. The corresponding perturbations in the output distribution space is illustrated in Figure 2-4.

In particular, we should notice that in this layered coding scheme, informations are encoded in a very large number of long symbol blocks. Thus, this scheme is in general not practical. However, studying this layered coding scheme still provides much theoretical insights on linear information coupling problems as we discuss in the following.

First, observing that the divergence transition map B depends on the neighborhood, in both the input and the output distribution spaces, through the weighting matrices $\left[\sqrt{P_X}\right]$ and $\left[\sqrt{P_Y}^{-1}\right]$. As we sequentially add layers, the neighborhood, and hence the weights, change from P_X, to $P_{X|U_1=u_1}$, to $P_{X|U_1=u_1,U_2=u_2,\ldots}$. Thus, the sequence of local problems we need to solve changes gradually, and the underlying manifold structure of the space of probability distributions has to be taken into consideration.

More importantly, as shown in Figure 2-4, the valid choices of distributions on the channel output must be in a convex region. For a given channel matrix W, whose column vectors are the conditional distributions of the output Y, conditioned on different values of the channel input $X = x$, the output distributions must belong to the convex region specified by these column vectors. As we add more layers, at some point the boundary of this convex region is reached. From which point, further layering is restricted to be along the hypersurface of this convex region. Conceptually, there is not much difference, since moving the output distributions on the hypersurface corresponds to not use a subset of the input alphabet. Hence, a local problem can in principle be written out with a reduced input alphabet. This can indeed be implemented in some special cases [10]. However, for general problems, especially multi-letter problems, specifying this high dimensional convex region and all its boundary constraints seems to be a combinatoric problem that forbids general analytical solutions.

Finally, while we only consider small perturbations in each step, eventually the perturbations reach the vertices of the simplex of valid distributions with probability 1. This implies that

$$\begin{aligned}&\lim_{K\to\infty} I(U_1,\ldots,U_K;X)\\&= H(X) - \lim_{K\to\infty} H(X|U_1,\ldots,U_K) = H(X),\end{aligned}$$

where $H(\cdot)$ is the entropy function. Thus, if we keep encoding layers in each step by recursively solving the local optimization problems, we can integral the local solutions in each step back to a global solution. This in fact provides a greedy solution to the global problem, and potentially demonstrates how the capacity can be obtained.

Example 2. In this example, we apply the layered coding scheme to the ternary channel that we considered in example 1, and show that the capacity can be achieved by this scheme. First, note that the channel capacity of the ternary channel in Figure 2-2(b) is $2\eta^2+\left(\frac{1}{2}+\eta\right)\gamma^2$, with the optimal input distribution $P_X = [\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}]^T$. Here, we ignore the higher order terms of $o(\eta^2)$ and $o(\gamma^2)$, since both η and γ are assumed to be small. From example 1, the optimal perturbation vector is $[\frac{1}{2}\ \frac{-1}{4}\ \frac{-1}{4}]^T$, and the corresponding conditional distributions are $P_{X|U_1=0} = \left[\frac{1}{2}+\frac{1}{2}\epsilon\ \frac{1}{4}-\frac{1}{4}\epsilon\ \frac{1}{4}-\frac{1}{4}\epsilon\right]^T$ and $P_{X|U_1=1} = \left[\frac{1}{2}-\frac{1}{2}\epsilon\ \frac{1}{4}+\frac{1}{4}\epsilon\ \frac{1}{4}+\frac{1}{4}\epsilon\right]^T$. To apply the layered coding scheme, we keep increasing the perturbation vector until the boundary is reached, i.e., increasing[5] ϵ to 1. Then, the conditional distribution $P_{X|U_1=0}$ reaches the vertex $[1\ 0\ 0]^T$, and $P_{X|U_1=1}$ reaches the boundary at $[0\ \frac{1}{2}\ \frac{1}{2}]^T$. This is shown in Figure 6-5(c). The achievable information rate by the first layer of perturbation is $I(U_1;Y) = \frac{1}{2}\epsilon^2(2\eta)^2 = 2\eta^2$.

Next, we perturb the conditional distribution $P_{X|U_1=1} = [0\ \frac{1}{2}\ \frac{1}{2}]^T$ along the boundary. This corresponds to a linear information coupling problem with reduced input alphabet, since the first input alpha "0" has no effect here. Therefore, the DTM of this problem has reduced dimension, and can be explicitly computed as

$$B_2 = \begin{bmatrix} \sqrt{\frac{1}{4}-\frac{1}{2}\eta} & \sqrt{\frac{1}{4}-\frac{1}{2}\eta} \\ \sqrt{\frac{1}{2}+\eta}\left(\frac{1}{2}+\gamma\right) & \sqrt{\frac{1}{2}+\eta}\left(\frac{1}{2}-\gamma\right) \\ \sqrt{\frac{1}{2}+\eta}\left(\frac{1}{2}-\gamma\right) & \sqrt{\frac{1}{2}+\eta}\left(\frac{1}{2}+\gamma\right) \end{bmatrix}.$$

The second largest singular value of this DTM is $\sigma_1 = \sqrt{2+4\eta}\cdot\gamma$, and the corresponding singular vector is $\underline{v}_1 = [0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$. Thus, the optimal perturbation vector is $[0\ \frac{1}{2}\ \frac{-1}{2}]^T$, and the conditional distributions are $P_{X|U_1=1,U_2=0} = \left[0\ \frac{1}{2}+\frac{1}{2}\epsilon\ \frac{1}{2}-\frac{1}{2}\epsilon\right]^T$

[5]Since we assume both η and γ are small, the local approximation of all divergence and mutual information of interests remains valid even if ϵ is not small. This is why we can increase ϵ here from a small number to 1 without violating the local approximation.

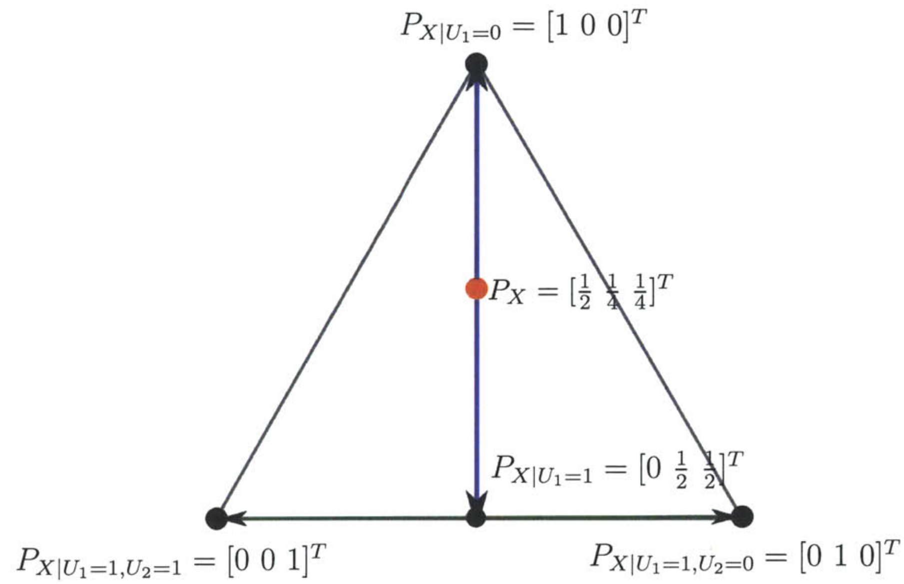


Figure 2-5: In the first layer, the input distribution P_X is perturbed to $P_{X|U_1=0}$ and $P_{X|U_1=1}$, where $P_{X|U_1=0}$ reaches one of the vertices, and $P_{X|U_1=1}$ reaches the boundary. In the second layer, the distribution $P_{X|U_1=1}$ is further perturbed to $P_{X|U_1=1,U_2=0}$ and $P_{X|U_1=1,U_2=1}$, where the rest two vertices are reached.

and $P_{X|U_1=1,U_2=1} = \left[0\ \frac{1}{2} - \frac{1}{2}\epsilon\ \frac{1}{2} + \frac{1}{2}\epsilon\right]^T$. Then, we keep increasing the perturbation vector until the two vertices $[0\ 1\ 0]^T$ and $[0\ 0\ 1]^T$ are reached with $\epsilon = 1$, as shown in Figure 6-5(c). The achievable information rate by the second layer of perturbation is

$$\begin{aligned} I(U_2; Y|U_1) &= I(U_2; Y|U_1 = 1) \cdot P(U_1 = 1) \\ &= \frac{1}{2}\epsilon^2(\sqrt{2 + 4\eta} \cdot \gamma)^2 \cdot \frac{1}{2} = \left(\frac{1}{2} + \eta\right)\gamma^2. \end{aligned}$$

After these two layers of perturbations, all the conditional distributions reach the vertices, and the total achievable information rate is $2\eta^2 + \left(\frac{1}{2} + \eta\right)\gamma^2$, which achieves the channel capacity of this ternary channel.

2.4 The Relation To Rényi Maximal Correlation

In this section, we show that the second largest singular value of the DTM is precisely the Rényi maximal correlation between random variables X and Y, where the marginal distributions P_X and P_Y are the given input and output distributions in the linear information coupling problem (2.7), and the transition probability kernel $P_{Y|X}$ is the channel W. Let us begin with the following definition.

Definition 1. [15] The *Rényi maximal correlation* $\rho_m(X,Y)$ between two random variables X and Y is defined by

$$\rho_m(X,Y) = \sup E\left[f(X)g(Y)\right], \tag{2.26}$$

where the supremum is over all Borel-measurable functions f and g such that

$$\begin{aligned} &E\left[f(X)\right] = E\left[g(Y)\right] = 0, \\ &E\left[f^2(X)\right] = E\left[g^2(Y)\right] = 1. \end{aligned} \tag{2.27}$$

The Rényi maximal correlation is a measure of dependence of random variables that is stronger and more general than the correlation coefficient, since it allows arbitrary zero-mean, unit-variance functions of X and Y. The Rényi maximal correlation is first introduced by Hirschfeld [12] and Gebelein [13] for discrete random variables and absolutely continuous random variables. Rényi [14, 15] compared the Rényi maximal correlation to other measures of dependence, and provided sufficient conditions for which the supremum of (2.26) is achieved. In particular, for discrete random variables X and Y, the sufficient conditions are met, and Rényi maximal correlation can be attained. Moreover, Rényi showed that if the function pair $(\hat{f}, \hat{g})$ achieves (2.26), then

$$\begin{aligned} E\left[\hat{g}(Y)|X\right] &= \rho_m(X,Y)\hat{f}(X), \\ E\left[\hat{f}(X)|Y\right] &= \rho_m(X,Y)\hat{g}(Y). \end{aligned} \tag{2.28}$$

Now, let $E\left[\cdot|\,X\right] : \mathcal{G}_Y \mapsto \mathcal{F}_X$ be the conditional expectation operator, where $\mathcal{F}_X$ and $\mathcal{G}_Y$ are the metric spaces of functions $f : \mathcal{X} \mapsto \mathbb{R}$ and $g : \mathcal{Y} \mapsto \mathbb{R}$, respectively, with the corresponding metric $\langle\cdot,\cdot\rangle_{P_X}$ and $\langle\cdot,\cdot\rangle_{P_Y}$. It is easy to see that

$$\begin{aligned}&\left\{\phi_x = \sqrt{P_X(x)}^{-1} \cdot \delta_x,\ x \in \mathcal{X}\right\},\\&\left\{\psi_y = \sqrt{P_Y(y)}^{-1} \cdot \delta_y,\ y \in \mathcal{Y}\right\},\end{aligned} \tag{2.29}$$

where

$$\delta_x(\tilde{x}) = \begin{cases} 1, & \text{if } \tilde{x} = x \\ 0, & \text{otherwise} \end{cases}, \quad \delta_y(\tilde{y}) = \begin{cases} 1, & \text{if } \tilde{y} = y \\ 0, & \text{otherwise} \end{cases},$$

are orthonormal bases for $\mathcal{F}_X$ and $\mathcal{G}_Y$. Thus, we can write the operator $E\left[\cdot|\,X\right]$ as an $|\mathcal{X}| \times |\mathcal{Y}|$ matrix E_{XY} with respect to orthonormal bases (2.29), and the (x, y) entry of this matrix is

$$\begin{aligned}E_{XY}(x,y) =& \langle\phi_x, E\left[\psi_y|X=x\right]\rangle_{P_X}\\=&\sum_{\tilde{x}\in\mathcal{X}} P_X(\tilde{x}) \cdot \left(\sqrt{P_X(\tilde{x})}^{-1}\delta_x(\tilde{x})\right) \cdot \left(\sum_{\tilde{y}\in\mathcal{Y}} \sqrt{P_Y(\tilde{y})}^{-1}\delta_y(\tilde{y}) P_{Y|X}\left(\tilde{y}|\tilde{x}\right)\right)\\=&\sqrt{P_Y(y)}^{-1} P_{Y|X}\left(y|x\right)\sqrt{P_X(x)}.\end{aligned}$$

Therefore, this matrix is precisely the transpose of the DTM. Consequently, there is a one-to-one correspondence between the singular vectors of the DTM and the singular functions of the conditioned expectation operator. In particular, note that $E\left[\mathbf{1}_Y|X\right] = \mathbf{1}_X$, where $\mathbf{1}_X$ and $\mathbf{1}_Y$ are constant functions with value 1 in $\mathcal{F}_X$ and $\mathcal{G}_X$, respectively. Thus, the constant function is a singular function of the conditional expectation operator, with singular vector 1. This corresponds to the first singular vector $\left[\sqrt{P_X(x)}, x \in \mathcal{X}\right]^T$ of the DTM. In addition, note that the zero-mean constraint (2.27) on functions in $\mathcal{F}_X$ is equivalent to the orthogonality to $\mathbf{1}_X$. Therefore, the rest singular functions of the conditioned expectation operator satisfy (2.27), and have a one-to-one correspondence to the singular vectors of the DTM other than the first one.

From (2.28), we know that $\hat{f}$ has to be the left singular function of the operator $E\left[\cdot|\,X\right]$ with respect to the largest singular value $\rho_m(X,Y)$, subject to the constraint that $\hat{f}$ is orthogonal to $\mathbf{1}_X$. Therefore, we have the following proposition.

Proposition 1. *The second largest singular value of the DTM is the Rényi maximal correlation* $\rho_m(X,Y)$. *Moreover, the functions* $\hat{f}$ *and* $\hat{g}$ *maximizing* (2.26) *can be obtained from the left and right singular vectors of the DTM with the second largest singular value, corresponding to the orthonormal bases* (2.29).

In particular, the relation of (2.7) and the Rényi maximal correlation $\rho_m(X,Y)$ was also shown in [11] and [26] by different approaches. With our local approximation approach, we can see that Proposition 1 can be obtained with simple linear algebra. Moreover, the geometric insight of the Rényi maximal correlation is also provided through our analyzation of the divergence transition matrix. In chapter 3 and 4, we will apply our local approach to multi-terminal communication systems, and the relation between the local geometry and the Rényi maximal correlation suggests the potential of generalizing the Rényi maximal correlation to the case of more than two random variables.

2.5 The Application Of Linear Information Coupling Problems To Lossy Source Coding

In this section, we illustrate how to apply our formulation of the linear information coupling problems, and the local geometric structure in the point-to-point problems, to the lossy source coding problems. The first attempt along this direction was from Tishby *et al.* [27], where the authors in [27] formulated the lossy source coding problems mathematically similar to our linear information coupling problems, except for the local constraints on $I(U;X)$. In fact, in [27], the authors assumed that $I(U;X)$ is a fixed number, which is not necessary small. With this assumption, they exploited the Lagrange multiplier method to develop an algorithm for finding optimal solutions. However, from this Lagrange multiplier approach, it is somehow difficult to obtain the insight of how information is exchanged between different terminals. Moreover, the Lagrange multiplier approach is extremely complicated when generalizing to the general networks, which can be considered as the result of the complicated manifold structures of the probability distribution spaces. On the other hand, as we introduced in this chapter, our local geometric approach is a powerful tool to simplify the complicated manifold structures of distributions. In the following, let us introduce how to apply our approach to study the lossy source coding problems.

First of all, thinking of now we have a source Y generated from some i.i.d. probability distribution P_Y. Suppose we can not directly observe Y, but can observe a noisy version X of Y, which can be viewed as passing the source Y through a transition kernel, or a channel $P_{X|Y}$. Then, with the noisy observation X, we want to infer or estimate the original source Y. Traditionally, we would like to find the sufficient statistic $T(X)$ from X, which tells all information about Y that one can say from observing X. However, in many application, such as the machine learning and image processing, both Y and X can come from some high dimensional graphical models. For example, the Y can be a set of images, and X can be the noisy observation of these images. Now, if one want to make some decisions about Y from observing X, such as classifying the images in a meaningful way, then the sufficient statistic can be quite complicated and hard to deal with. Instead of taking the sufficient statistic, we want to formulate a slightly different problem, where we only want to have an *insufficient statistic* U, but hope that it is efficient. Here is the mathematical formulation: given a source Y and the noisy observation X, we want to just say a few $\frac{1}{2}\epsilon^2$ words U about X, where ϵ^2 is assumed to be small, and hope that it can tell us as much about Y as possible. Then, this naturally corresponds to the optimization problem:

$$\max_{I(U;X)\leq\frac{1}{2}\epsilon^2} I(U;Y) \tag{2.30}$$

Again, we add the local constraint $\|P_{X|U=u} - P_X\| = O(\epsilon)$, so that the problem can be solved with our local geometric structure and the solution is $P_{X|U=u} - P_X = \epsilon^2 J_u$, where all the J_u's are along the scaled largest singular vector of the linear map between the space $\mathcal{P}_X$ and $\mathcal{P}_Y$. Moreover, from the discussion in section 2.1, we can simply take U as a binary random variable with $P_U(0) = P_U(1) = 1/2$, and let $P_{X|U=0} - P_X = \epsilon J$, and $P_{X|U=1} - P_X = -\epsilon J$, where J is the unit vector along the same direction as all the J_u's. Then, we can view this bit U as the most informative piece of information that X wants to say about Y.

Now, the next question is that, once we can observe a sequence of realizations $x_1, \ldots, x_n$ from X, how can we extract the most informative one bit U from this sequence? To answer this question, we can think of this as an inference problem, where the joint distribution P_{XU} of X and U is given as the optimal solution of (2.30). Then, we can use the maximum likelihood decision to estimate the value of U. To this end, we first compute the *empirical distribution* of $x_1, \ldots, x_n$, and denote this empirical distribution as $\hat{P}_X$. Then, the optimal decision rule is to compare the K-L divergences $D(\hat{P}_X \| P_{X|U=0})$ and $D(\hat{P}_X \| P_{X|U=1})$, which tells that $\hat{P}_X$ is more close to either $P_{X|U=0}$ or $P_{X|U=1}$. Mathematically, the decision rule is

$$U = \begin{cases} 0, & \text{if} \quad D(\hat{P}_X \| P_{X|U=1}) \geq D(\hat{P}_X \| P_{X|U=0}) \\ 1, & \text{if} \quad D(\hat{P}_X \| P_{X|U=1}) < D(\hat{P}_X \| P_{X|U=0}). \end{cases} \tag{2.31}$$

Moreover, with $P_{X|U=0} - P_X = \epsilon J$, and $P_{X|U=1} - P_X = -\epsilon J$, we can further simplify (2.31) by the local approximation to

$$U = \begin{cases} 0, & \text{if} \quad \sum_{i=1}^{n} \frac{J(x_i)}{P_X(x_i)} \geq 0 \\ 1, & \text{if} \quad \sum_{i=1}^{n} \frac{J(x_i)}{P_X(x_i)} < 0. \end{cases} \tag{2.32}$$

Now, from this decision rule, we can define the *score* function f_{score} for each observation x_i as

$$f_{\text{score}}(x_i) \triangleq \frac{J(x_i)}{P_X(x_i)}. \tag{2.33}$$

This is illustrated in Figure 2-6. Then, once we can observe a sequence of observations, we can extract the most informative bit by first computing the score of this sequence, which is the sum of the score of each observation, and then apply the decision rule as (2.32).

In fact, the most attractive feature of our score function is that for a sequence of observations, we can compute the score function of this sequence highly efficiently,

$$Y \xrightarrow{P_{X|Y}} X = x \xrightarrow{f_{\text{score}}} f_{\text{score}}(x)$$

Figure 2-6: The score function for each observation $X = x$.

because we only need to compute the score function for each observation separately and then add them up, but no need to compute a joint function for the entire sequence. While we will not discuss in detail, we would like to point out that this feature becomes very important when both X and Y becomes high dimensional graphical models. In those cases, our notion of score function provides efficient algorithms to extract the most informative piece of information from the noisy observations.

Finally, we would like to mention that this efficient insufficient statistic can be useful in different scientific areas, such as the stock market and the machine learning. In stock market, we can think of the source Y as the stock prices, and X is some side information. The goal is to extract one bit information U from X, which tells that we should either buy or sell a stock, so we would like this bit U to bring as much information about Y as possible. In particular, this problem was also studied in [11] by Erkip and Cover.

Moreover, in machine learning problem, we can think of Y as the hidden graphical model, and we can only observe the noisy observation X, and want to make some decisions about Y. The main difficulty of this is that both X and Y can be high dimensional models, and the complexity of making decisions can be extremely high. However, with the notion of the score we introduce in this section, we can simply obtain the most informative piece of information efficiently, and then make the decision according to this piece of information. Therefore, from these examples, we can see that the score function we develop in this section is in fact very useful for many problem, and local geometric structure turns out to be the core technique for studying these problems.

Chapter 3

The General Broadcast Channel

In this chapter, we apply the local approximation approach to general broadcast channels, and study the corresponding linear information coupling problems. We first illustrate our technique by considering the 2-user broadcast channel, and then extend to the K-user case. Finally, with the local geometry structure derived in this section, we will discuss the local optimality of the Marton's coding scheme on general broadcast channels.

3.1 The Linear Information Coupling Problems Of General Broadcast Channels

Now, let us start to discuss the geometry of general broadcast channels. First, a 2-user general broadcast channel with input $X \in \mathcal{X}$, and outputs $Y_1 \in \mathcal{Y}_1$, $Y_2 \in \mathcal{Y}_2$, is specified by the memoryless channel matrices W_1 and W_2. These channel matrices specify the conditional distributions of the output signals at two users, 1 and 2, as $W_i(y_i|x) = P_{Y_i|X}(y_i|x)$, for $i = 1, 2$. Let M_1, M_2, and M_0 be the two private messages and the common message, with rate R_1, R_2, and R_0, respectively. Then, using Fano's inequality, the multi-letter capacity region of the general broadcast channel is the set of rate tuple (R_0, R_1, R_2) such that

$$\begin{cases} R_0 & \leq \frac{1}{n}\min\{I(U;\underline{Y}_1), I(U;\underline{Y}_2)\}, \\ R_1 & \leq \frac{1}{n}I(V_1;\underline{Y}_1), \\ R_2 & \leq \frac{1}{n}I(V_2;\underline{Y}_2), \end{cases} \tag{3.1}$$

for some mutually independent random variables U, V_1, and V_2, such that $(U, V_1, V_2) \to \underline{X} \to \underline{Y}_1$ and $(U, V_1, V_2) \to \underline{X} \to \underline{Y}_2$, are both Markov chains. The signal vectors here all have the same dimension n. In principle, one should just optimize this rate region by finding the optimal coding distributions. However, since n can potentially be arbitrarily large, finding the structure of these optimal input distributions is necessary.

Now, we want to apply the local approximation technique we developed in section 2 to this broadcast channel problem. As a natural generalization from the point-to-point channel case (2.4), the linear information coupling problem of this 2-user broadcast channel is the characterization of the rate region:

$$\begin{cases} R_0 \leq \frac{1}{n}\min\{I(U;\underline{Y}_1), I(U;\underline{Y}_2)\} \\ R_1 \leq \frac{1}{n}I(V_1;\underline{Y}_1) \\ R_2 \leq \frac{1}{n}I(V_2;\underline{Y}_2) \end{cases} \tag{3.2}$$

$$\text{subject to:} \quad (U, V_1, V_2) \to \underline{X} \to (\underline{Y}_1, \underline{Y}_2), \ \frac{1}{n}I(U, V_1, V_2; \underline{X}) \leq \frac{1}{2}\epsilon^2,$$
$$\frac{1}{n}\|P_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)} - P_{\underline{X}}\| = O(\epsilon), \ \forall \ (u, v_1, v_2),$$

where U, V_1, V_2 are mutually independent random variables.

This rate region is the same as the capacity region (3.1) except for the local constraint $\frac{1}{n}I(U, V_1, V_2; \underline{X}) \leq \frac{1}{2}\epsilon^2$. This constraint can be interpreted as modulating all the common and private messages entirely as a thin layer of information into the input symbol sequence $\underline{X}$. Then, the goal of the linear information coupling problem (3.2) is to describe the boundary points of this region, by specifying the directions of perturbing $P_{\underline{X}}$, or the conditional distributions $P_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)}$ that have the marginal $P_{\underline{X}}$.

Note that the characterization of (3.2) involves the optimization over multiple rates R_0, R_1, and R_2, with respect to different messages M_0, M_1, and M_2. The following lemma seperates (3.2) to three sub-problems, so that we can apply the local geometric approach that was developed in section 2 to each individual sub-problem. The idea here is that, while the conditional distribution $P_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)}$ is perturbed from $P_{\underline{X}}$ by some vector J_{u,v_1,v_2} that is in general a joint function of U, V_1, and V_2, by the first order approximation, it is enough to only consider those perturbation vectors J_{u,v_1,v_2} that can be written as the linear combination of three vectors J_u, J_{v_1}, and J_{v_2}.

Lemma 3. *The rate region* (3.2) *is, up to the first order approximation, the same as the following rate region:*

$$\begin{cases} R_0 \leq \frac{1}{n}\min\{I(U;\underline{Y}_1), I(U;\underline{Y}_2)\} \\ R_1 \leq \frac{1}{n}I(V_1;\underline{Y}_1) \\ R_2 \leq \frac{1}{n}I(V_2;\underline{Y}_2) \end{cases} \tag{3.3}$$

$$\textit{subject to: } (U,V_1,V_2) \to \underline{X} \to (\underline{Y}_1,\underline{Y}_2),\ \frac{1}{n}I(U;\underline{X}) \leq \frac{1}{2}\epsilon_0^2,$$

$$\frac{1}{n}I(V_1;\underline{X}) \leq \frac{1}{2}\epsilon_1^2,\ \frac{1}{n}I(V_2;\underline{X}) \leq \frac{1}{2}\epsilon_2^2,\ \sum_{i=1}^{3}\epsilon_i^2 = \epsilon^2,$$

$$\frac{1}{n}\|P_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)} - P_{\underline{X}}\| = O(\epsilon),\ \forall\ (u,v_1,v_2),$$

Proof. Appendix A. □

Now, we can apply our technique in chapter 2 to this problem. In particular, for a tuple of $(\epsilon_0, \epsilon_1, \epsilon_2)$ with $\sum_{i=1}^{3}\epsilon_i^2 = \epsilon^2$, the optimization problem (3.3) reduces to three sub-problems: for $i = 1, 2$, the optimization problems for the private messages M_i

$$\begin{aligned} \max.\quad & \frac{1}{n}I(V_i;\underline{Y}_i) \\ \text{subject to:}\quad & V_i \to \underline{X} \to \underline{Y}_i,\ \frac{1}{n}I(V_i;\underline{X}) \leq \frac{1}{2}\epsilon_i^2, \\ & \frac{1}{n}\|P_{\underline{X}|V_i=v_i} - P_{\underline{X}}\| = O(\epsilon_i),\ \forall\ v_i, \end{aligned} \tag{3.4}$$

and the optimization problem for the common message M_0

$$\begin{aligned}
\max. \quad & \frac{1}{n}\min\left\{I(U;\underline{Y}_1), I(U;\underline{Y}_2)\right\} \qquad (3.5)\\
\text{subject to:} \quad & U \to \underline{X} \to (\underline{Y}_1, \underline{Y}_2),\ \frac{1}{n}I(U;\underline{X}) \leq \frac{1}{2}\epsilon_0^2\\
& \frac{1}{n}\|P_{\underline{X}|U=u} - P_{\underline{X}}\| = O(\epsilon_0),\ \forall\ u.
\end{aligned}$$

As in the point-to-point channel case, we assume that the input distribution of $\underline{X}$, as the operating point, is i.i.d. P_X. Hence, the output distributions of the two outputs Y_1 and Y_2 are also i.i.d. P_{Y_1} and P_{Y_2}. Moreover, we denote the conditional distributions as the perturbations from the marginal distribution: $P_{\underline{X}|U=u} = P_X^{(n)} + \sqrt{n}\epsilon_0 \cdot J_u$ and $P_{\underline{X}|V_i=v_i} = P_X^{(n)} + \sqrt{n}\epsilon_i \cdot J_{v_i}$ for $i = 1, 2$.

Then, the optimization problems (3.4) for private messages are the same as the linear information coupling problem for the point-to-point channel (2.4). Thus, by defining the single-letter DTM's $B_i \triangleq \left[\sqrt{P_{Y_i}}^{-1}\right] W_i \left[\sqrt{P_X}\right]$ for $i = 1, 2$, we can solve (3.4) with the same procedure as (2.4), and the single-letter solutions are optimal.

The optimization problem (3.5) is, however, quite different from the other two. Suppose that the weighted perturbation vector $L_u = \left[\sqrt{P_X^{(n)}}^{-1}\right] J_u$, the problem is to maximize

$$\min\left\{\sum_u P_U(u)\left\|B_1^{(n)}L_u\right\|^2, \sum_u P_U(u)\left\|B_2^{(n)}L_u\right\|^2\right\}, \qquad (3.6)$$

subject to

$$\sum_u P_U(u) \cdot \|L_u\|^2 = 1,$$

and also the constraints (2.15) and (2.16) that guarantee the validity of the weighted perturbation vector. Here, the $B_i^{(n)}$ is the n^{th} Kronecker product of the single-letter DTM B_i, for $i = 1, 2$.

Now, the core problem we want to address for (3.6) is that whether or not the single-letter version of (3.6) is optimal, and if it is optimal, how large of the cardinality

$|\mathcal{U}|$ is required. The point here is that, if we come up with a single-letter optimal solution of (3.6), but it requires the cardinality $|\mathcal{U}|$ to be large, which means that in contrast to the point-to-point case, we need to design multiple directions to achieve the optimal of (3.6), and this $|\mathcal{U}|$ is not just as a function of the cardinality $|\mathcal{X}|$ and $|\mathcal{Y}|$, then we would consider this solution as a multi-letter solution. So, we are not only interested in how many letters we need to use to achieve the optimality of (3.6), but also how large is the corresponding cardinality $|\mathcal{U}|$.

In order to answer these problems, we want to first choose U as a uniformly binary random variable, and consider the simpler problem

$$\lambda^{(n)} = \max_{L_u: \|L_u\|^2=1} \min\left\{ \left\| B_1^{(n)} L_u \right\|^2, \left\| B_2^{(n)} L_u \right\|^2 \right\}. \tag{3.7}$$

Specifically, we want to understand whether or not the single-letter version of (3.7) is optimal, i.e., $\lambda^{(1)} = \sup_n \lambda^{(n)}$. Note that the problem (3.7) is not equivalent to (3.6). However, it will be shown later in remark 5 that the single/multi-letter optimality of (3.6) can be immediately implied by the solutions of (3.7). Therefore, we will focus on solving (3.7) in the following.

First, note that different from the point-to-point problem (2.17), we need to choose the weighted perturbation vector L_u in (3.7) that have large images through two different linear systems simultaneously. In general, the tradeoff between two SVD structures can be rather messy problems. However, in this problem, for both $i = 1, 2$, $B_i^{(n)}$ have the special structure of being the Kronecker product of the single letter DTMs. Furthermore, both B_1 and B_2 have the largest singular value 1, corresponding to the same singular vector $\underline{v}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$. Although the rest of their SVD structures are not specified, the following theory characterizes the optimality of single-letter and finite-letter solutions for general cases.

Theorem 1. *Let B_i be the DTM of some DMC with respect to the same input distributions for $i = 1, 2, \ldots, k$, then for the linear information coupling problem*

$$\lambda^{(n)} = \max_{L_u: \|L_u\|^2=1, \langle L_u, \underline{v}_0^{(n)} \rangle = 0} \min_{1 \le i \le k} \left\{ \left\| B_i^{(n)} L_u \right\|^2 \right\}, \tag{3.8}$$

we have

$$\sup_n \lambda^{(n)} = \begin{cases} \lambda^{(1)} & \textit{if } k = 2, \\ \lambda^{(k)} & \textit{if } k > 2, \end{cases}$$

and for $k > 2$, there exists k-user broadcast channels such that

$$\lambda^{(k)} > \lambda^{(k-1)}.$$

In other words, the single-letter version of (3.8) *is optimal for the case with $k = 2$ receivers. When there are $k > 2$ receivers, single-letter version of* (3.8) *may not be optimal, but there still exists k-letter solutions that are optimal.*

The existence of optimal single-letter solutions for the 2 receivers case, and optimal k-letter solutions for the case of $k > 2$ receivers are presented in appendix B. It remains to show that when $k > 2$, to achieve the optimality of (3.8), the k-letter version of (3.8) is in general necessary for broadcast channels with k receivers. We illustrate here by an example that, when there are more than 2 receivers, i.i.d. distributions simply do not have enough degrees of freedom to be optimal in the tradeoff of more than 2 linear systems. Therefore, one has to design multi-letter product distributions to achieve the optimal. The following example, constructed with the geometric method, illustrates the key ideas.

Example 3 (The windmill channel). We consider a 3-user broadcast channel as shown in Figure 3-1(a). The input alphabet $\mathcal{X}$ is ternary, so that the perturbation vectors have 2 dimensions and can be easily visualized. Suppose that the empirical input distribution P_X is fixed as $[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]^T$, then the DTM B_1 for the first receiver Y_1 is

$$B_1 = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} & 1-\delta & \delta \\ \frac{1}{2} & \delta & 1-\delta \end{bmatrix} = \sigma_0 \underline{u}_0 \underline{v}_0^T + \sigma_1 \underline{u}_1 \underline{v}_1^T,$$

where $\sigma_0 = 1$ and $\sigma_1 = \sqrt{\frac{2}{3}}(1-2\delta)$ are the singular values of B_1, and $\underline{u}_0 = [\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}]^T$,

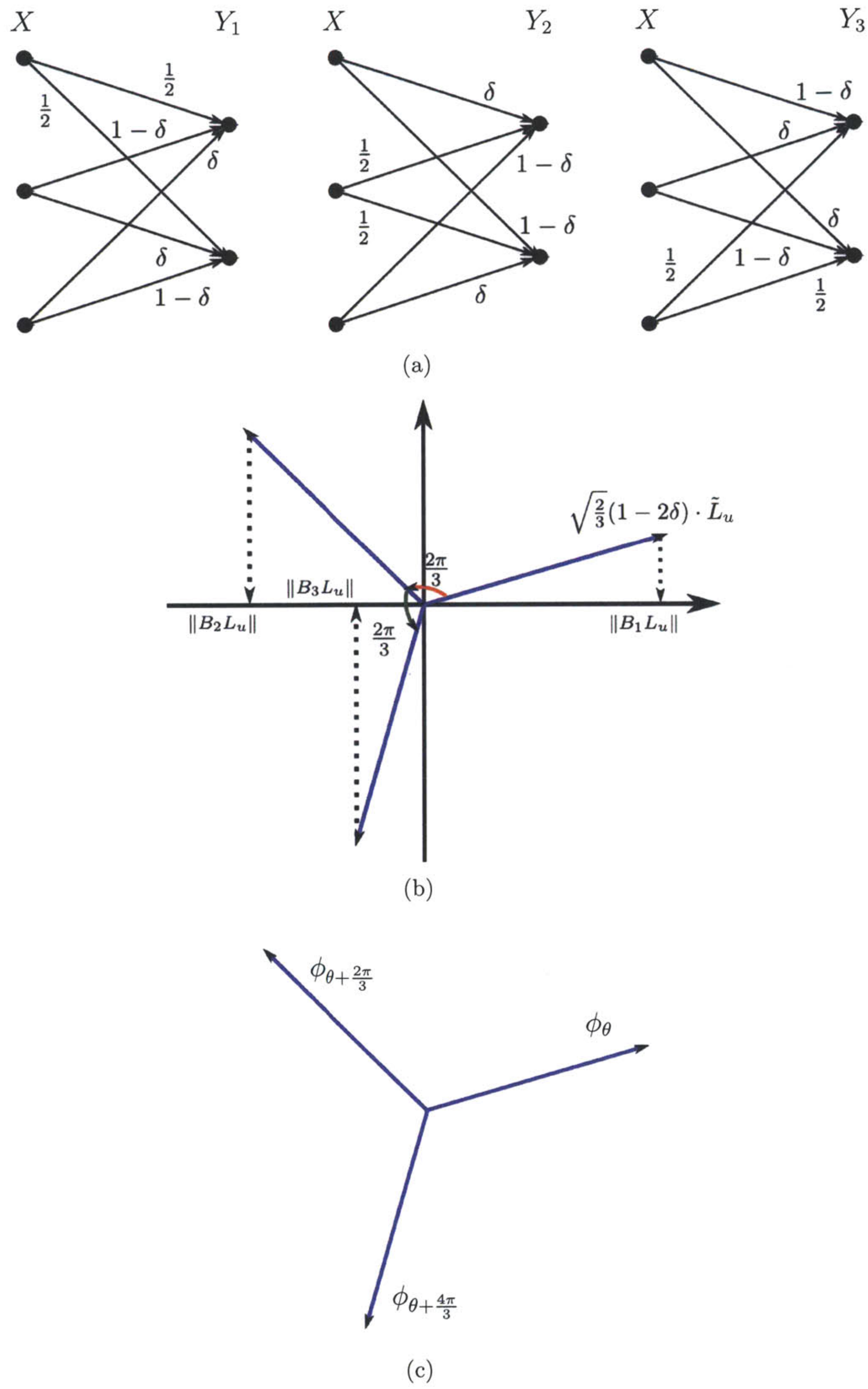


Figure 3-1: (a) A ternary input broadcast channel. (b) The lengths of the output images of L_u through matrices B_1, B_1, and B_1 are the scaled projection of $\tilde{L}_u$ and its rotated version to the horizontal axis. (c) The optimal perturbations over 3 time slots.

$\underline{u}_1 = [\frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$, $\underline{v}_0 = [\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}]^T$, and $\underline{v}_1 = [0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$ are the corresponding left and right singular vectors. Moreover, the DTM's B_2 and B_3 for receivers Y_2 and Y_3 can be written as

$$\begin{aligned} B_2 &= \sqrt{\frac{2}{3}} \begin{bmatrix} \delta & \frac{1}{2} & 1-\delta \\ 1-\delta & \frac{1}{2} & \delta \end{bmatrix} \\ &= \sigma_0 \underline{u}_0 \underline{v}_0^T + \sigma_1 \underline{u}_1 [1\ 0] \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} \begin{bmatrix} \underline{v}_1^T \\ \underline{v}_2^T \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} B_3 &= \sqrt{\frac{2}{3}} \begin{bmatrix} 1-\delta & \delta & \frac{1}{2} \\ \delta & 1-\delta & \frac{1}{2} \end{bmatrix} \\ &= \sigma_0 \underline{u}_0 \underline{v}_0^T + \sigma_1 \underline{u}_1 [1\ 0] \begin{bmatrix} \cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3} \\ \sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} \end{bmatrix} \begin{bmatrix} \underline{v}_1^T \\ \underline{v}_2^T \end{bmatrix}, \end{aligned}$$

where $\underline{v}_2 = [\frac{2}{\sqrt{6}}\ \frac{-1}{\sqrt{6}}\ \frac{-1}{\sqrt{6}}]^T$ is the vector such that $\underline{v}_0$, $\underline{v}_1$, and $\underline{v}_2$ form an orthonormal basis of $\mathbb{R}^3$. Since the weighted perturbation vector L_u has to be orthogonal to $\underline{v}_0$, or equivalently, a linear combination of $\underline{v}_1$ and $\underline{v}_2$, we can see that the lengths of the output images of a weighted perturbation vector L_u through B_1, B_2, and B_3 are the projection of the scaled version of $\tilde{L}_u = [\underline{v}_1^T\ \underline{v}_2^T]^T L_u$, as well as its rotated version with angle $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ to the horizontal axis. This is shown in Figure 3-1(b).

Now if we use single-letter inputs, it can be seen that for any L_u with $\|L_u\|^2 = 1$, $\min\{\|B_1 L_u\|^2, \|B_2 L_u\|^2, \|B_3 L_u\|^2\} \le \frac{1}{6} \cdot (1-2\delta)^2$. The problem here is that no matter what direction L_u takes, the three output norms are unequal, and the minimum one always limits the performance. However, if we use 3-letter input, and denote ϕ_θ as the perturbation vector such that

$$[\underline{v}_1^T\ \underline{v}_2^T]^T \cdot \left[\sqrt{P_X}^{-1}\right] \cdot \phi_\theta = [\cos\theta, \sin\theta]^T,$$

then we can take

$$P_{X^3|U=u} = (P_X + \epsilon\phi_\theta) \otimes (P_X + \epsilon\phi_{\theta+\frac{2\pi}{3}}) \otimes (P_X + \epsilon\phi_{\theta+\frac{4\pi}{3}})$$

for any value of θ, as shown in Figure 3-1(c). Intuitively, this input equalizes the three channels, and gives for all $i = 1, 2, 3$, $\|B_i^{(3)} L_u^{(3)}\|^2 = \frac{1}{3} \cdot (1 - 2\delta)^2$, which doubles the information coupling rate. Translating this solution to the coding language, it means that we take turns to feed the common information to each individual user. Noting that the solution is not a standard time-sharing input, and hence the performance is strictly out of the convex hull of the i.i.d. solutions. One can interpret this input as a repetition of the common message over three time-slots, where the information is modulated along equally rotated vectors. For this reason, we call this example the "windmill" channel. Additionally, it is easy to see that the construction of the windmill channel can be generalized to the cases of $k > 3$ receivers, where k-letter solutions is necessary.

Remark 5. Note that in (3.8), we let U be a binary random variable, and in this case, while there are optimal 3-letter solutions, the optimal single-letter solutions do not exist. However, one can in fact take U to be non-binary. For example, let $\mathcal{U} = \{0, 1, 2\}$ with $P_U(u) = 1/3$ for all u, and let

$$L_{U=i} = \left[\sqrt{P_X}^{-1}\right] \cdot \phi_{\theta+i\cdot\frac{2\pi}{3}},$$

for $i = 0, 1, 2$, then we can still achieve the information coupling rate $1/2$. Thus, there actually exists an optimal single-letter solution in (6.10) with cardinality $|\mathcal{U}| = 3$. On the other hand, we can see from this example that when there are k receivers, it requires cardinality $|\mathcal{U}| = k$ for obtaining optimal single-letter solutions. This implies that (6.10) is in general not single-letter optimal unless we allow the cardinality of $\mathcal{U}$ to be proportional to the number of receivers. In fact, this example shows that finding a single weighted perturbation vector with a large image at the outputs of all 3 channels is difficult. The tension between these 3 linear systems requires more degrees of freedom in choosing the perturbations, or in other words, the way that common information is modulated. Such more degrees of freedom can be provided either by using multi-letter solutions or have larger cardinality bounds. This effect is not captured by the conventional single-letterization approach.

Theorem 1 significantly reduces the difficulty of solving the multi-letter optimization problem (3.5). The remaining is to find the optimal weighted perturbation vector L_u for the single-letter version of (3.5), if the number of receivers $k = 2$, or the k-letter version, if $k > 2$. These are finite dimensional convex optimization problems [16], which can be readily solved.

Remark 6. Suppose that the supremum of (3.7) is σ_0^2, and the second largest singular values of B_1 and B_2 are σ_1 are σ_2, respectively. Since the tradeoff between two systems can not exceed the optimum of each individual system, we have

$$\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \leq \sigma_0^2 \leq \min\{\sigma_1^2, \sigma_2^2\}, \tag{3.9}$$

where the lower bound comes from a simple time-sharing argument. This inequality shows that, with the same amount of perturbations, less amount of common messages can be created to both receivers than the private messages. Moreover, with our approach, we can explicitly computing all these σ_i's, and therefore quantify the difficulty of generating common messages at receiver ends than private messages.

3.2 The K-User Broadcast Channel

The extension of our approach to the general broadcast channels with $K > 2$ users is straightforward. A K-user broadcast channel with input $X \in \mathcal{X}$, and outputs $Y_i \in \mathcal{Y}_i$, is specified by the memoryless channel matrices W_i, for $i = 1, 2, \ldots, K$. These channel matrices specify the conditional distributions of the output signals at each user as $W_i(y|x) = P_{Y_i|X}(y|x)$, for $i = 1, 2, \ldots, K$. We denote $\mathcal{S}$ as the set of all nonempty subsets of $\{1, 2, \ldots, K\}$. Then, for all $\mathcal{I} = \{i_1, i_2, \ldots, i_k\} \in \mathcal{S}$, users $i_1, i_2, \ldots, i_k$ receive the common message $M_{\mathcal{I}}$, with rate $R_{\mathcal{I}}$. Then, using auxiliary random variables $U_{\mathcal{I}}$, and the notation $U_{\mathcal{S}} \triangleq (U_{\mathcal{I}},\ \mathcal{I} \in \mathcal{S})$, the linear information coupling problem of the K-User broadcast channel is the optimization of the rate region

$$\begin{aligned} & R_{\mathcal{I}} \leq \frac{1}{n} \min_{i \in \mathcal{I}} \left\{ I(U_{\mathcal{I}}; \underline{Y}_i) \right\},\ \forall \mathcal{I} \in \mathcal{S} \\ \text{subject to:}\quad & U_{\mathcal{S}} \to \underline{X} \to (\underline{Y}_1, \ldots, \underline{Y}_K), \\ & \frac{1}{n} I(U_{\mathcal{I}}; \underline{X}) \leq \frac{1}{2}\epsilon_{\mathcal{I}}^2,\ \forall \mathcal{I} \in \mathcal{S},\ \sum_{\mathcal{I} \in \mathcal{S}} \epsilon_{\mathcal{I}}^2 = \epsilon^2, \\ & \frac{1}{n} \| P_{\underline{X}|U_{\mathcal{S}}=u_{\mathcal{S}}} - P_{\underline{X}} \| = O(\epsilon),\ \forall u_{\mathcal{S}} \in \mathcal{U}_{\mathcal{S}}. \end{aligned} \tag{3.10}$$

Following the same procedure as the 2-user broadcast channel case, the optimization problem (3.10) can be reduced to some sub-problems. For each message $M_{\mathcal{I}}$ that is common to receivers $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$, the corresponding optimization problem is

$$\begin{aligned} \max.\quad & \frac{1}{n} \min_{i \in \mathcal{I}} \left\{ I(U_{\mathcal{I}}; \underline{Y}_i) \right\} \\ \text{subject to:}\quad & U_{\mathcal{I}} \to \underline{X} \to (\underline{Y}_{i_1}, \ldots, \underline{Y}_{i_k}),\ \frac{1}{n} I(U_{\mathcal{I}}; \underline{X}) \leq \frac{1}{2}\epsilon_{\mathcal{I}}^2 \\ & \frac{1}{n} \| P_{\underline{X}|U_{\mathcal{I}}=u_{\mathcal{I}}} - P_{\underline{X}} \| = O(\epsilon_{\mathcal{I}}),\ \forall\ u_{\mathcal{I}} \in \mathcal{U}_{\mathcal{I}}. \end{aligned} \tag{3.11}$$

Defining the DTM's $B_i = \left[\sqrt{P_{Y_i}}^{-1}\right] W_i \left[\sqrt{P_X}\right]$, for $i = 1, 2, \ldots, K$, and the weighted perturbation vectors $L_{u_{\mathcal{I}}} = \left[\sqrt{P_{\underline{X}}}^{-1}\right] J_{u_{\mathcal{I}}}$, for all $\mathcal{I} \in \mathcal{S}$, the optimization problem (3.11) becomes

$$\max_{L_{u_{\mathcal{I}}} : \|L_{u_{\mathcal{I}}}\|^2 = 1} \min_{i \in \mathcal{I}} \left\{ \left\| B_i^{(n)} L_{u_{\mathcal{I}}} \right\|^2 \right\}. \tag{3.12}$$

Finally, from Theorem 1, the multi-letter optimization problem (3.12) can be reduced to just a finite dimensional convex optimization problem.

3.3 The Marton's Coding Scheme For General Broadcast Channel

The currently best inner bound of the general broadcast channel was due to Marton [8]. The coding technique used by Marton is the foundation of many of the results on multi-terminal problems we have today. In this subsection, our goal is to study the local optimality of Marton's rate region under the local geometric structure.

Marton's achievable rate region is the following single-letter expression. For auxiliary random variables U, V_1, and V_2, which satisfy the Markov relation $(U, V_1, V_2) \to X \to (Y_1, Y_2)$, a rate tuple is achievable if

$$\left\{\begin{array}{rcl} R_0 & \leq & \min\{I(U;Y_1), I(U;Y_2)\} \\ R_0 + R_1 & \leq & I(U, V_1; Y_1) \\ R_0 + R_2 & \leq & I(U, V_2; Y_2) \\ R_0 + R_1 + R_2 & \leq & \min\{I(U;Y_1), I(U;Y_2)\} + I(V_1;Y_1|U) + I(V_2;Y_2|U) - I(V_1;V_2|U). \end{array}\right. \tag{3.13}$$

In fact, this achievable rate region is a generalization from a slightly different one, which was first proposed by Cover [5], van der Meulen [6], and then improved by Hajek and Pursley [7] (CMHP region):

$$\left\{\begin{array}{rcl} R_0 & \leq & \min\{I(U;Y_1), I(U;Y_2)\} \\ R_0 + R_1 & \leq & I(U, V_1; Y_1) \\ R_0 + R_2 & \leq & I(U, V_2; Y_2), \\ R_0 + R_1 + R_2 & \leq & \min\{I(U;Y_1), I(U;Y_2)\} + I(V_1;Y_1|U) + I(V_2;Y_2|U). \end{array}\right. \tag{3.14}$$

where V_1 and V_2 are independent conditioned on U. The gist of the rate region (3.14) is to use superposition code and i.i.d. $P_{X|UV_1V_2}$ to obtain an achievable rate region. We can see that the only difference between Marton's rate region and the CMHP region is that Marton allows V_1 and V_2 to be dependent to each other, conditioned on U. Therefore, Marton's rate region (3.13) is optimized over a larger set of probability

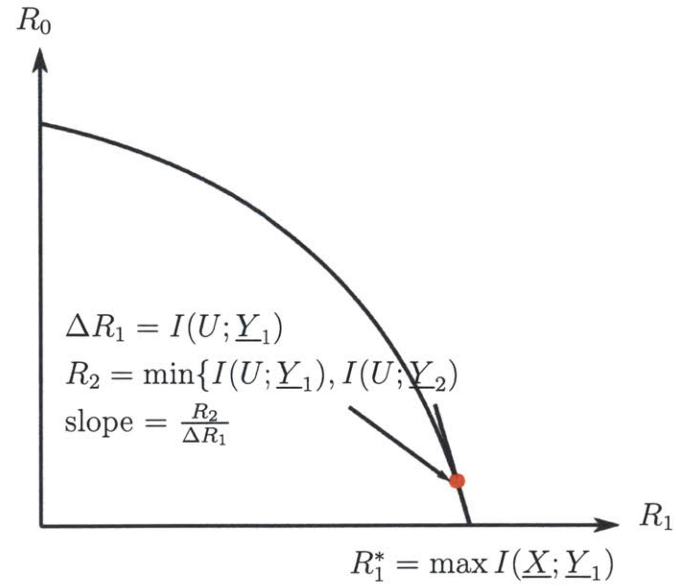


Figure 3-2: The slope around R_1^* on the capacity region.

distributions, and can potentially be better than (3.14). On the other hand, with the relaxation on the dependency of U, V_1, and V_2, there happens to be a minus term $I(V_1; V_2|U)$ on the achievable sum rate.

In this section, we aim to apply our local geometric approach to study the local optimality of the Marton's rate region. Our goal is to characterize the slope around a boundary point on the multi-letter capacity region, and then verify if this slope is achievable by Marton's region with i.i.d. $P_{X|UV_1V_2}$. Specifically, suppose that (R_1, R_2, R_0) is the rate tuple of the private and common information rates for a 2-user broadcast channel, we want to focus on the slope of the boundary point $(R_1^*, 0, 0)$ on the multi-letter capacity region along the $R_1 - R_0$ plane, where $R_1^* = \max_{P_{\underline{X}}} I(\underline{X};\underline{Y}_1)$. This slope l can be written in the multi-letter form

$$l = \max_{U \to \underline{X} \to \underline{Y}:\ \frac{1}{n} I(U;\underline{X}) \leq \frac{1}{2}\epsilon^2} \frac{\min\{I(U;\underline{Y}_1), I(U;\underline{Y}_2)\}}{I(U;\underline{Y}_1)}. \tag{3.15}$$

Now, if we further restrict all the conditional distributions $P_{\underline{X}|U=u}$ are close to $P_{\underline{X}}$, for all $u \in \mathcal{U}$, in the sense of (1.3), then (3.15) can be simplified by our local geometric approach to

$$l = \max_{\|L\|^2=1} \frac{\min\left\{\|B_1^{(n)}L\|^2, \|B_2^{(n)}L\|^2\right\}}{\|B_1^{(n)}L\|^2}, \tag{3.16}$$

where B_1 and B_2 are the DTM's for this broadcast channel.

Now, let us show that (3.16) has single-letter optimal solutions. First, it is clearly that $l \leq 1$. If there exists a weighted perturbation vector L such that $\|B_1^{(n)}L\|^2 \leq \|B_2^{(n)}L\|^2$, then this L achieves $l = 1$. From Theorem 1, there exist optimal single-letter solutions in this case. Otherwise, if for any L, $\|B_1^{(n)}L\|^2 > \|B_2^{(n)}L\|^2$, then

$$l = \max_{\|L\|^2=1} \frac{\|B_2^{(n)}L\|^2}{\|B_1^{(n)}L\|^2} = \max_{\|L'\|^2=1} \|\hat{B}^{(n)}L'\|^2, \tag{3.17}$$

where $L' = B_1^{(n)}L$, and $\hat{B} = B_2 B_1^{-1}$. Following the same argument as section 2.2, there exist optimal single-letter solutions for (3.17). Therefore, we know that (3.16) has optimal single-letter solutions, and (3.15) with the local constraint can be simplified to the single-letter form

$$l = \max_{U \to X \to Y:\ I(U;X) \leq \frac{1}{2}\epsilon^2,\ \|P_{X|U} - P_X\| = O(\epsilon)} \frac{\min\{I(U;Y_1), I(U;Y_2)\}}{I(U;Y_1)}. \tag{3.18}$$

Note that (3.18) can be achieved in Marton's rate region by choosing $V_1 = X$, $V_2 =$ const., with the i.i.d. input distribution. Thus, Marton's coding scheme is locally optimal around the point $(R_1^*, 0, 0)$, provided that all the conditional distributions $P_{\underline{X}|U=u}$ are assumed to be local.

Moreover, while Marton's achievable rate region is only for two-user case, her coding scheme can be generalized to the K-user broadcast channels, for $K > 2$, to obtain single-letter achievable rate regions. Similar to the 2-user case, we can verify the single-letter optimality of the Marton's coding scheme on the slope around a corner point on the capacity region. What is interesting here is that with our local geometric structure, we can show that Marton's single-letter region does not achieve the optimal slope on the multi-letter capacity region, unless we allow the cardinalities of the auxiliary random variables increase with the number of receivers. For example, consider the degrade broadcast channel $X \to Y' \to (Y_1, Y_2, Y_3)$, and let R' be the private information rate of Y', and R_0 is the common information rate to all receivers. Then, the optimal slope around the corner point $(R', 0)$ along the $R' - R_0$ plane can be written as

$$l = \max_{\frac{1}{n} I(U;\underline{X}) \le \frac{1}{2}\epsilon^2} \frac{\min \{I(U;\underline{Y}_1), I(U;\underline{Y}_2), I(U;\underline{Y}_3)\}}{I(U;\underline{Y}')}. \tag{3.19}$$

Again, if we assume all the conditional distributions $P_{\underline{X}|U}$ are local, i.e., $\frac{1}{n}\|P_{\underline{X}|U} - P_{\underline{X}}\| = O(\epsilon)$, then from Theorem 1, we know that either multi-letter solutions or larger cardinality of $\mathcal{U}$ is required to achieve the optimum of (3.19). On the other hand, Marton's rate region can only achieve the slope of the single-letter version of (3.19). Therefore, under the local constraint, in order to use Marton's coding scheme to achieve the slope (3.19), we need to allow the cardinality of $\mathcal{U}$ to increase with the number of receivers.

In fact, Nair and El Gamal showed in [9] that Marton's coding scheme is indeed not globally optimal for a certain type of three-user broadcast channel. With our local geometry approach, by Theorem 1, we can visualize why Marton's coding scheme may potentially fail to be locally optimal, and how coding over multiple letters helps in taking turns communicating to different users.

Chapter 4

The Multiple Access Channel with Common Sources

In addition to the broadcast channel, the multiple access channel (MAC) is another fundamental multi-terminal communication model that has been widely investigated in information theory [1], [17], [18]. In this chapter, we aim to develop new insights of this communication model by our geometric approach. Here, we consider the set-up, where the transmitters can not only have the knowledge of their own private sources, but each subset of transmitters also share the knowledge of certain common source. All these private and common sources are assumed to be independent with each other. The goal is to efficiently communicate all the private and common sources to the receiver. To illustrate how our technique can be applied to this problem, we first consider the 2-transmitter multiple access channel with one common source.

4.1 The Linear Information Coupling Problems Of Multiple Access Channels

Suppose that the 2-transmitter multiple access channel has the inputs $X_1 \in \mathcal{X}_1$, $X_2 \in \mathcal{X}_2$, and the output $Y \in \mathcal{Y}$. The memoryless channel is specified by the channel matrix W, where $W(y|x_1, x_2) = P_{Y|X_1,X_2}(y|x_1, x_2)$ is the conditional distribution of the output signals. We want to communicate three messages M_1, M_2, and M_0 to the receiver Y with rate R_1, R_2, and R_0, where M_1 and M_2 are observed privately by transmitters 1 and 2, respectively, and M_0 is the common source for both transmitters. Then, the multi-letter capacity region of this problem is the set of rate tuples (R_0, R_1, R_2) such that

$$\begin{cases} R_0 \leq \frac{1}{n} I(U; \underline{Y}), \\ R_1 \leq \frac{1}{n} I(V_1; \underline{Y}), \\ R_2 \leq \frac{1}{n} I(V_2; \underline{Y}), \end{cases} \tag{4.1}$$

for some mutually independent random variables U, V_1, and V_2, such that $U \to (\underline{X}_1, \underline{X}_2) \to \underline{Y}$, $V_1 \to \underline{X}_1 \to \underline{Y}$, and $V_2 \to \underline{X}_2 \to \underline{Y}$ are all Markov chains. Again, we assume that the signal vectors here all have the same dimension n. The corresponding linear information coupling problem is then to characterize the following rate region:

$$\begin{cases} R_0 \leq \frac{1}{n} I(U; \underline{Y}) \\ R_1 \leq \frac{1}{n} I(V_1; \underline{Y}) \\ R_2 \leq \frac{1}{n} I(V_2; \underline{Y}) \end{cases} \tag{4.2}$$

$$\begin{aligned} \text{subject to:} \quad & U \to (\underline{X}_1, \underline{X}_2) \to \underline{Y}, \; V_1 \to \underline{X}_1 \to \underline{Y}, \; V_2 \to \underline{X}_2 \to \underline{Y}, \\ & \frac{1}{n} I(U, V_1, V_2; \underline{X}_1, \underline{X}_2) \leq \frac{1}{2}\epsilon^2, \\ & \frac{1}{n} \| P_{\underline{X}_1, \underline{X}_2 | (U, V_1, V_2) = (u, v_1, v_2)} - P_{\underline{X}_1, \underline{X}_2} \| = O(\epsilon), \; \forall \; (u, v_1, v_2). \end{aligned}$$

Here, both $P_{\underline{X}_1, \underline{X}_2 | (U, V_1, V_2) = (u, v_1, v_2)}$ and $P_{\underline{X}_1, \underline{X}_2}$ are $|\mathcal{X}_1| \times |\mathcal{X}_2|$ dimensional vectors, and the norm is the Euclidean norm. Now, following similar arguments as Lemma 3 in the broadcast channel case, the rate region (4.2) is, up to the first order approximation, the same as

$$\begin{cases} R_0 \leq \frac{1}{n} I(U; \underline{Y}) \\ R_1 \leq \frac{1}{n} I(V_1; \underline{Y}) \\ R_2 \leq \frac{1}{n} I(V_2; \underline{Y}) \end{cases} \tag{4.3}$$

subject to: $U \to (\underline{X}_1, \underline{X}_2) \to \underline{Y},\ V_1 \to \underline{X}_1 \to \underline{Y},\ V_2 \to \underline{X}_2 \to \underline{Y},$

$$\frac{1}{n} I(U; \underline{X}_1, \underline{X}_2) \leq \frac{1}{2}\epsilon_0^2,\ \frac{1}{n} I(V_1; \underline{X}_1) \leq \frac{1}{2}\epsilon_1^2,\ \frac{1}{n} I(V_2; \underline{X}_2) \leq \frac{1}{2}\epsilon_2^2,\ \sum_{i=0}^{2} \epsilon_i^2 = \epsilon^2,$$

$$\frac{1}{n} \| P_{\underline{X}_1, \underline{X}_2 | (U, V_1, V_2) = (u, v_1, v_2)} - P_{\underline{X}_1, \underline{X}_2} \| = O(\epsilon),\ \forall\ (u, v_1, v_2).$$

The optimization of the rate region (4.3) can be reduced to three sub-problems: the optimization problems for the private sources M_1 and M_2:

$$\begin{aligned} \max.\quad & \frac{1}{n} I(V_i; \underline{Y}) \\ \text{subject to:}\quad & V_i \to \underline{X}_i \to \underline{Y},\ \frac{1}{n} I(V_i; \underline{X}_i) \leq \frac{1}{2}\epsilon_i^2, \\ & \frac{1}{n} \| P_{\underline{X}_i | V_i = v_i} - P_{\underline{X}_i} \| = O(\epsilon_i),\ \forall\ v_i. \end{aligned} \tag{4.4}$$

for $i = 1, 2$, and the optimization problem for the common source M_0:

$$\begin{aligned} \max.\quad & \frac{1}{n} I(U; \underline{Y}) \\ \text{subject to:}\quad & U \to (\underline{X}_1, \underline{X}_2) \to \underline{Y},\ \frac{1}{n} I(U; \underline{X}_1, \underline{X}_2) \leq \frac{1}{2}\epsilon_0^2, \\ & \frac{1}{n} \| P_{\underline{X}_1, \underline{X}_2 | U = u} - P_{\underline{X}_1, \underline{X}_2} \| = O(\epsilon_0),\ \forall\ u \end{aligned} \tag{4.5}$$

To apply our approach, we assume that the joint distribution of $\underline{X}_1$ and $\underline{X}_2$ is i.i.d. P_{X_1, X_2} as the operating point, then the corresponding output distribution at the receiver is also i.i.d. P_Y. The conditional distributions are written as vectors that are perturbed from the marginal distributions: $P_{\underline{X}_i | V_i = v_i} = P_{X_i}^{(n)} + \sqrt{n}\epsilon_i \cdot J_{v_i}$, for $i = 1, 2$, and $P_{\underline{X}_1, \underline{X}_2 | U = u} = P_{X_1, X_2}^{(n)} + \sqrt{n}\epsilon_0 \cdot J_{0,u}$. For the problems (4.4) and (4.5), note that $P_{\underline{Y} | V_i = v_i} = W_i^{(n)} \cdot P_{\underline{X}_i | V_i = v_i}$, for $i = 1, 2$, where $W_i^{(n)}$ is the n^{th} Kronecker product of the matrix

$$W_i(y|x_i) \triangleq \sum_{x_{3-i}\in\mathcal{X}_{3-i}} W(y|x_1,x_2)P_{X_{3-i}}(x_{3-i}). \tag{4.6}$$

Then, defining the DTM's $B_i \triangleq \left[\sqrt{P_Y}^{-1}\right] W_i \left[\sqrt{P_{X_i}}\right]$ for $i=1,2$, we can solve (4.4) and (4.5) with the same procedure as the point-to-point case.

To solve the common source problem (4.5), first observe that U is the only common information that transmitters 1 and 2 can share, so $\underline{X}_1 \to U \to \underline{X}_2$ forms a Markov relation, and

$$P_{\underline{X}_1,\underline{X}_2|U=u} = P_{\underline{X}_1|U=u} \otimes P_{\underline{X}_2|U=u}. \tag{4.7}$$

Furthermore, we can also write the conditional distributions $P_{\underline{X}_i|U=u}$ as the perturbation form $P_{\underline{X}_i} + \sqrt{n}\epsilon_0 J_{i,u}$ for some perturbation vector $J_{i,u}$, for $i=1,2$. Then, (4.7) becomes

$$\begin{aligned} P_{\underline{X}_1,\underline{X}_2|U=u} &= P_{\underline{X}_1|U=u} \otimes P_{\underline{X}_2|U=u} = \left(P_{\underline{X}_1} + \sqrt{n}\epsilon_0 J_{1,u}\right) \otimes \left(P_{\underline{X}_2} + \sqrt{n}\epsilon_0 J_{2,u}\right) \\ &= P^{(n)}_{X_1} \otimes P^{(n)}_{X_2} + \sqrt{n}\epsilon_0 \cdot J_{1,u} \otimes P^{(n)}_{X_2} + \sqrt{n}\epsilon_0 \cdot P^{(n)}_{X_1} \otimes J_{2,u} + O(\epsilon^2). \end{aligned} \tag{4.8}$$

Therefore, compare (4.8) to $P_{\underline{X}_1,\underline{X}_2|U=u} = P^{(n)}_{X_1,X_2} + \sqrt{n}\epsilon_0 \cdot J_{0,u}$, we know that $J_{0,u}$ has a special form $J_{1,u} \otimes P^{(n)}_{X_2} + P^{(n)}_{X_1} \otimes J_{2,u}$, and $P^{(n)}_{X_1,X_2}$ is equal to $P^{(n)}_{X_1} \otimes P^{(n)}_{X_2}$ up to the first order approximation. This special structure of $J_{0,u}$ comes from the Markov relation $\underline{X}_1 \to U_0 \to \underline{X}_2$. Conceptually, this simplifies the problem in the way that it significantly reduces the dimension of the space that we need to search for the optimal perturbation vector $J_{0,u}$.

Now, with (4.8), the conditional output distribution can be written as

$$
\begin{aligned}
& P_{\underline{Y}|U=u} \\
&= W^{(n)} \cdot P_{\underline{X}_1,\underline{X}_2|U=u} \\
&= W^{(n)} \cdot (P_{X_1}^{(n)} \otimes P_{X_2}^{(n)} + \epsilon_0 \cdot J_{1,u} \otimes P_{X_2}^{(n)} + \epsilon_0 \cdot P_{X_1}^{(n)} \otimes J_{2,u}) + O(\epsilon^2) \\
&= P_{\underline{Y}} + \epsilon_0 W_1^{(n)} J_{1,u} + \epsilon_0 W_2^{(n)} J_{2,u} + O(\epsilon^2) \\
&= P_Y^{(n)} + \epsilon_0 \left[W_1^{(n)} \; W_2^{(n)} \right] \cdot \begin{bmatrix} J_{1,u} \\ J_{2,u} \end{bmatrix} + O(\epsilon^2) \\
&= P_Y^{(n)} + \epsilon_0 W_{0,n} \cdot J_u + O(\epsilon^2),
\end{aligned}
$$

where $W_{0,n} \triangleq \left[W_1^{(n)} \; W_2^{(n)} \right]$ is the concatenation of the two matrices $W_1^{(n)}$ and $W_2^{(n)}$, and $J_u \triangleq \left[J_{1,u}^T \; J_{2,u}^T \right]^T$ is the concatenation of the two perturbation vectors $J_{1,u}$ and $J_{2,u}$. Then, the problem (4.5) becomes

$$
\begin{aligned}
\max. \quad & \sum_u P_U(u) \cdot \|W_{0,n} J_u\|^2_{P_Y^{(n)}} \\
\text{subject to:} \quad & \sum_u P_U(u) \cdot \left(\|J_{1,u}\|^2_{P_{X_1}^{(n)}} + \|J_{2,u}\|^2_{P_{X_2}^{(n)}} \right) = 1.
\end{aligned}
$$

Furthermore, if we define the DTM

$$
B_{0,n} \triangleq \left[\sqrt{P_Y^{(n)}}^{-1} \right] W_{0,n} \begin{bmatrix} \sqrt{P_{X_1}^{(n)}} & \mathrm{O} \\ \mathrm{O} & \sqrt{P_{X_2}^{(n)}} \end{bmatrix} = \left[B_1^{(n)} \; B_2^{(n)} \right],
$$

and the weighted perturbation vector

$$
L_u = \begin{bmatrix} \left[\sqrt{P_{X_1}^{(n)}}^{-1} \right] \cdot J_{1,u} \\ \left[\sqrt{P_{X_2}^{(n)}}^{-1} \right] \cdot J_{2,u} \end{bmatrix}, \tag{4.9}
$$

then we can write the problem as

$$\begin{aligned} \max. \quad & \sum_u P_U(u) \cdot \|B_{0,n} L_u\|^2 \\ \text{subject to:} \quad & \sum_u P_U(u) \cdot \|L_u\|^2 = 1. \end{aligned}$$

Following the same arguments as the point-to-point case, we can choose U as a uniformly binary random variable without loss of generality, and the problem can be simplified to the n-letter problem

$$\max. \ \|B_{0,n} L\|^2. \tag{4.10}$$

$$\text{subject to: } \|L\|^2 = 1, \tag{4.11}$$

$$\langle \underline{v}_{i,0}^{(n)}, L_i \rangle = 0, \ i = 1, 2, \tag{4.12}$$

where $\underline{v}_{i,0} = \left[\sqrt{P_{X_i}}, x_i \in \mathcal{X}_i\right]^T$, and L_1 is the vector formed by the first $n \cdot |\mathcal{X}_1|$ entries of L, and L_2 is the vector formed by the last $n \cdot |\mathcal{X}_2|$ entries of L, i.e., $L = [L_1^T \ L_2^T]^T$. Here, the constraint (4.12) comes from (4.9) and the assumption that $J_{1,u}$ and $J_{2,u}$ are both perturbation vectors.

Again, our goal is to determine wether or not the multi-letter problem (4.10) has single-letter optimal solutions. In fact, the problem (4.10) is quite similar to the multi-letter point-to-point problem that we have solved in section 2.2, except for two fundamental differences. First, note that the vector $\left[(\underline{v}_{1,0}^{(n)})^T \ (\underline{v}_{2,0}^{(n)})^T\right]^T$ is the right singular vector of $B_{0,n}$ corresponding to its largest singular value. So, the constraint (4.12) implies that L has to be not only orthogonal to this singular vector, but its two components L_1 and L_2 have to be individually orthogonal to $\underline{v}_{1,0}^{(n)}$ and $\underline{v}_{2,0}^{(n)}$. This is a stronger constraint than (2.19) in the point-to-point case. Second, the matrix $B_{0,n}$ in the n-letter problem is not the n-Kronecker product of the matrix $B_{0,1}$ in the corresponding single-letter problem.

Therefore, in order to apply the same procedure as the point-to-point case, we need to prove two things. First, for the single-letter version of (4.10), we need to show that the right singular vector of the $B_{0,1}$ corresponding to its second largest singular value satisfies the constraint (4.12). Second, we need to show that the second largest singular value of $B_{0,n}$ is the same as $B_{0,1}$, and the corresponding singular vector has the product form just like the point-to-point case.

These two issues are addressed at Theorem 2 in section 4.2, which shows that (4.10) is single-letter optimal and the optimum of the single-letter version of this problem is achieved by the singular vector of the DTM corresponding to the second largest singular value. Therefore, like the point-to-point case, we only need to deal with the single-letter problem here, and the optimal solutions can be readily solved. In the following, we simply denote $B_{0,1}$ as B_0.

Remark 7. Suppose that the second largest singular values of B_0, B_1, and B_2 are σ_0, σ_1, and σ_2, respectively. Note that σ_0, σ_1, and σ_2 satisfy the inequality

$$\max\{\sigma_1^2, \sigma_2^2\} \leq \sigma_0^2 \leq \sigma_1^2 + \sigma_2^2. \tag{4.13}$$

where the lower bound of (4.13) is obvious, and the upper bound is due to (4.10) and for any valid weighted perturbation vector $L = [L_1^T \; L_2^T]^T$,

$$\|B_0 L\|^2 \leq (\|B_1 L_1\| + \|B_2 L_2\|)^2 \leq \sigma_1^2 + \sigma_2^2. \tag{4.14}$$

Here, the first inequality of (4.14) is the triangle inequality, and the second inequality is from the Cauchy-Schwarz inequality. The inequality (4.13) tells that, by perturbing the same amount, we can convey more common sources to the receiver end than the private sources. This is not surprising, since both transmitters have the knowledge of the common source, they can cooperate and obtain the coherent combining gain, or also known as the beam-forming gain. Moreover, we can quantify this gain by explicitly computing the parameters σ_i's.

In fact, these kinds of quantification can be potentially useful in analyzing complicated communication networks. Note that the common sources for multiple nodes in a communication network are usually generated by preceding nodes as the common messages in some broadcast channels. Therefore, when optimizing the network

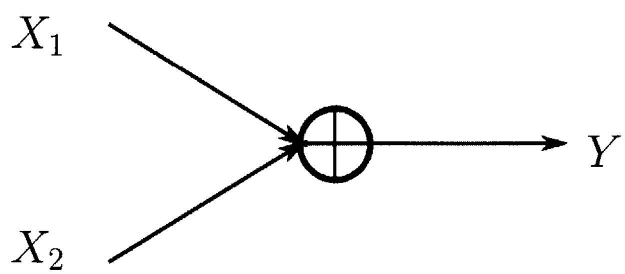


Figure 4-1: The binary addition channel with two binary inputs X_1 and X_2, and the ternary output $Y = X_1 + X_2$.

throughput, there exists a tradeoff relation between generating common messages to multiple node (c.f. remark 6) and the corresponding coherent combining gain. Our approach in fact provides a quantitive way to describe these kinds of tradeoff relations.

Example 4. Consider the binary adder channel as shown in Figure 4-1, where X_1 and X_2 are both binary inputs, and the tenary output $Y = X_1 + X_2$ (the arithmetic addition, not modulo 2). The empirical distribution of both P_{X_1} and P_{X_2} are fixed as $[\frac{1}{2}\ \frac{1}{2}]^T$, and the corresponding output distribution is $[\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4}]^T$. The DTM's for transmitter 1 and 2 are

$$B_1 = B_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus, the DTM for the common source is

$$B_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The second largest singular value of B_0 is 1 with right singular vector $[\frac{1}{2}\ \frac{-1}{2}\ \frac{1}{2}\ \frac{-1}{2}]^T$. In comparison, the second largest singular of both B_1 and B_2 are $\frac{1}{\sqrt{2}}$ with right singular vector $[\frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$. Therefore, the cooperation between two transmitters by the common source provides a 3dB coherent combining gain.

Remark 8. In fact, the concept of the coherent combining gain can be readily understood in the Gaussian noise additive multiple access channels, where the gain comes from lying the transmission signals of the two transmitters along the same direction. However, for the discrete memoryless MAC, it is in general not clear how the coherent combing gain should be defined. The technique we developed in this section provides a way to characterize this gain. Moreover, our results also provide the insights of how to manage the interference between transmitters, which can potentially be useful in analyzing the information flow of more complicated networks.

4.2 The K-Transmitter Multiple Access Channel And The Deterministic Model

In this section, we extend our development to the K-transmitter multiple access channel with common sources. The K-transmitter multiple access channel has K inputs $X_i \in \mathcal{X}_i$, for $i = 1, 2, \ldots, K$, and one output $Y \in \mathcal{Y}$, with the memoryless channel matrix W. The channel matrix specifies the conditional distributions of the output signals at the receiver end as $W(y|x_1, x_2, \ldots, x_K) = P_{Y|X_1,X_2,\ldots X_K}(y|x_1, x_2, \ldots, x_K)$. Let $\mathcal{S}$ be the set of all nonempty subsets of $\{1, 2, \ldots, K\}$. Then, each subset of transmitters $i_1, i_2, \ldots, i_k$ observe the common source $M_\mathcal{I}$ with rate $R_\mathcal{I}$, where $\mathcal{I} = \{i_1, i_2, \ldots, i_k\} \in \mathcal{S}$. Using auxiliary random variables $U_\mathcal{I}$, the linear information coupling problem of the K-transmitter multiple access channel with common sources is the optimization of the rate region

$$\begin{aligned} & R_\mathcal{I} \leq \frac{1}{n} I(U_\mathcal{I}; \underline{Y}), \ \forall \mathcal{I} \in \mathcal{S} \\ \text{subject to:} \quad & U_\mathcal{I} \to (\underline{X}_{i_1}, \ldots, \underline{X}_{i_k}) \to \underline{Y}, \ \forall \mathcal{I} \in \mathcal{S}, \\ & \frac{1}{n} I(U_\mathcal{I}; \underline{X}_{i_1}, \ldots, \underline{X}_{i_k}) \leq \frac{1}{2}\epsilon_\mathcal{I}^2, \ \sum_{\mathcal{I} \in \mathcal{S}} \epsilon_\mathcal{I}^2 = \epsilon^2, \\ & \frac{1}{n} \| P_{\underline{X}_1,\ldots,\underline{X}_K|U_\mathcal{S}=u_\mathcal{S}} - P_{\underline{X}_1,\ldots,\underline{X}_K} \| = O(\epsilon), \ \forall \ u_\mathcal{S} \in U_\mathcal{S}. \end{aligned} \tag{4.15}$$

With the same procedure as before, (4.15) can be reduced to some sub-problems:

for each source $M_{\mathcal{I}}$ common to transmitters $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$, the corresponding optimization problem is

$$\begin{aligned} \max . \quad & \frac{1}{n} I(U_{\mathcal{I}}; \underline{Y}) \\ \text{subject to:} \quad & U_{\mathcal{I}} \to (\underline{X}_{i_1}, \ldots, \underline{X}_{i_k}) \to \underline{Y}, \ \frac{1}{n} I(U_{\mathcal{I}}; \underline{X}_{\mathcal{I}}) \leq \frac{1}{2}\epsilon_{\mathcal{I}}^2, \\ & \frac{1}{n} \| P_{\underline{X}_{i_1}, \ldots, \underline{X}_{i_k} | U_{\mathcal{I}} = u_{\mathcal{I}}} - P_{\underline{X}_{i_1}, \ldots, \underline{X}_{i_k}} \| = O(\epsilon_{\mathcal{I}}), \ \forall \ u_{\mathcal{I}} \in U_{\mathcal{I}}. \end{aligned} \tag{4.16}$$

Then, the DTM's for the sources observed by transmitters $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$ is $B_{\mathcal{I},n} = \left[B_{i_1}^{(n)} \ \ldots \ B_{i_k}^{(n)} \right]$ for $\mathcal{I} \in \mathcal{S}$, where $B_i \triangleq \left[\sqrt{P_Y}^{-1} \right] W_i \left[\sqrt{P_{X_i}} \right]$ for $i \in \mathcal{I}$. Here, similar to (4.6), the matrix W_i is defined as

$$W_i(y|x_i) \triangleq \sum_{x_j \in \mathcal{X}_j, j \neq i} W(y|x_1, \ldots, x_K) \cdot \left(\sum_{x_i \in \mathcal{X}_i} P_{X_1, \ldots, X_L}(x_1, \ldots, x_K) \right). \tag{4.17}$$

Moreover, let the perturbations on the marginal input distributions be $P_{\underline{X}_{i_j} | U_{\mathcal{I}} = u_{\mathcal{I}}} = P_{X_{i_j}}^{(n)} + \sqrt{n}\epsilon_{\mathcal{I}} \cdot J_{i_j, u_{\mathcal{I}}}$, for $1 \leq j \leq k$. The weighted perturbations are defined as $L_{u_{\mathcal{I}}} = \left[L_{i_1}^T \ \ldots \ L_{i_k}^T \right]^T$, where $L_{i_j} = \left[\sqrt{P_{X_{i_j}}^{(n)}}^{-1} \right] J_{i_j, u_{\mathcal{I}}}$, for $1 \leq j \leq k$.

Then, following the same arguments as the 2-transmitter case, the multi-letter optimization problem (4.16) becomes

$$\max_{L_{u_{\mathcal{I}}} : \| L_{u_{\mathcal{I}}} \|^2 = 1} \| B_{\mathcal{I},n} L_{u_{\mathcal{I}}} \|^2 . \tag{4.18}$$

The following Theorem shows the single-letter optimality of (4.18), and the optimal weighted perturbation vector is again the singular vector of the second largest singular value.

Theorem 2. *For a multiple access channel with transmitters* $1, 2, \ldots, k$, *let* B_i *be the corresponding DTM's. Then, the second largest singular value of* $B_{0,n} = \left[B_1^{(n)} \ \ldots \ B_k^{(n)} \right]$ *is the same as* $B_0 \triangleq B_{0,1}$. *Moreover, let* $L_u = \left[L_1^T \ \ldots \ L_k^T \right]^T$ *be the singular vector of* B_0 *with the second largest singular value, where* L_i *is an* $|\mathcal{X}_i|$*-dimensional vector. Then,* L_i *is orthogonal to* $\left[\sqrt{P_{X_i}}, x_i \in \mathcal{X}_i \right]^T$, *for all* $1 \leq i \leq k$.

Proof. We need to proof two statements: (i) the second largest singular value of $B_{0,n} = \begin{bmatrix} B_1^{(n)} & \dots & B_k^{(n)} \end{bmatrix}$ is the same as B_0, and (ii) L_i is orthogonal to $\left[\sqrt{P_{X_i}}, x_i \in \mathcal{X}_i\right]^T$, for all $1 \leq i \leq k$.

First, let us prove the statement (i). Suppose that $\sigma_{1,n}$ is the second largest singular value of $B_{0,n}$, then we want to show that $\sigma_{1,n} = \sigma_{1,1}$, for all $n > 1$. Let us first show that $\sigma_{1,2} = \sigma_{1,1}$. Observe that $\left[\sqrt{P_Y}, y \in \mathcal{Y}\right]^T$ is the left singular vector of B_0, corresponding to the largest singular value $\sqrt{k}$, thus we can assume that the singular values of B_0 are $\sigma_{0,1} = \sqrt{k} \geq \sigma_{1,1} \geq \sigma_{2,1} \geq \dots \geq \sigma_{m,1}$, and the corresponding left singular vectors are $\underline{w}_0 = \left[\sqrt{P_Y}, y \in \mathcal{Y}\right]^T, \underline{w}_1, \underline{w}_2, \dots, \underline{w}_m$, where $m \triangleq \min\left\{\sum_{i=1}^k |\mathcal{X}_i|, |\mathcal{Y}|\right\} - 1$. Note that for all $0 \leq i \leq m$,

$$
\begin{aligned}
(\underline{w}_0 \otimes \underline{w}_i)^T \cdot \left(B_{0,2}B_{0,2}^T\right) &= \left(\underline{w}_0^T \otimes \underline{w}_i^T\right) \cdot \left(B_1B_1^T \otimes B_1B_1^T + B_2B_2^T \otimes B_2B_2^T\right) \\
&= \underline{w}_0^T \otimes \left[\underline{w}_i^T \cdot \left(B_1B_1^T + B_2B_2^T\right)\right] \\
&= \underline{w}_0^T \otimes \left[\underline{w}_i^T \cdot \left(B_0B_0^T\right)\right] = \sigma_{i,1}^2 \cdot \underline{w}_0^T \otimes \underline{w}_i^T,
\end{aligned}
$$

therefore, $\underline{w}_0 \otimes \underline{w}_i$ is a singular vector of $B_{0,2}$, with singular value $\sigma_{i,1}$. Similarly, $\underline{w}_j \otimes \underline{w}_0$ is a singular vector of $B_{0,2}$, with singular value $\sigma_{j,1}$, for all $0 \leq j \leq m$. Hence, in order to show that $\sigma_{1,2} = \sigma_{1,1}$, we only need to show that for any unit vector $\underline{w} \in \text{span}\left\{\underline{w}_i \otimes \underline{w}_j,\ 1 \leq i,j \leq m\right\}$, $\|\underline{w}^T \cdot B_{0,2}\| \leq \sigma_{1,2}$. To this end, note that from Lemma 2, we have $\|\underline{w}^T \cdot (B_0 \otimes B_0)\| \leq \sigma_{1,2}^2$, therefore

$$
\begin{aligned}
\|\underline{w}^T \cdot B_{0,2}\|^2 &\leq \|\underline{w}^T \cdot B_{0,2}\|^2 + \|\underline{w}^T \cdot (B_1 \otimes B_2)\|^2 + \|\underline{w}^T \cdot (B_2 \otimes B_1)\|^2 \\
&= \|\underline{w}^T \cdot (B_0 \otimes B_0)\|^2 \leq \sigma_{1,2}^4 \leq \sigma_{1,2}^2.
\end{aligned}
$$

Thus, we have $\sigma_{1,2} = \sigma_{1,1}$. With the same arguments, we can show that for any positive integer N, $\sigma_{1,2^N} = \sigma_{1,1}$. Since $\sigma_{1,n}$ is non-decreasing with n, this implies that $\sigma_{1,n} = \sigma_{1,1}$, for all n.

Now, let us prove the statement (ii). For simplicity, we denote $\underline{v}_{i,0} = \left[\sqrt{P_{X_i}}, x_i \in \mathcal{X}_i\right]^T$, and $\underline{v} = \left[\underline{v}_{1,0}^T \ \dots \ \underline{v}_{k,0}^T\right]^T$, then $\underline{v}$ is the singular vector of B_0, corresponding to the largest singular value $\sqrt{k}$. Suppose that $\langle L_i, \underline{v}_{i,0}\rangle = \mathbb{I}_i$, since L_u is orthogonal to $\underline{v}$, we have $\sum_{i=1}^{k} \mathbb{I}_i = 0$. If there exits a j such that $\mathbb{I}_j \neq 0$, then define the vector $\tilde{L}_u = \left[\tilde{L}_1^T \ \dots \ \tilde{L}_k^T\right]^T$, where $\tilde{L}_i = (L_i - \mathbb{I}_i \cdot \underline{v}_i)/\sqrt{1-\mathbb{I}}$, and $\mathbb{I} = \sum_{i=1}^{k} \mathbb{I}_i^2 > 0$. This definition of $\tilde{L}_u$ is valid because $\mathbb{I} < \sum_{i=1}^{k} \|L_i\|^2 = 1$. Then, it is easy to verify that $\|\tilde{L}_u\| = 1$, and $\tilde{L}_u$ is orthogonal to $\underline{v}$. Moreover,

$$
\begin{aligned}
B_0 \cdot \tilde{L}_u &= \sqrt{1-\mathbb{I}}^{-1} \cdot \left(B_0 \cdot L_u - \sum_{i=1}^{k} \mathbb{I}_i \cdot \left(B_i \cdot \underline{v}_{i,0}\right) \right) \\
&= \sqrt{1-\mathbb{I}}^{-1} \cdot \left(B_0 \cdot L_u - \left(\sum_{i=1}^{k} \mathbb{I}_i \right) \cdot \underline{w}_0 \right) \\
&= \sqrt{1-\mathbb{I}}^{-1} \cdot \left(B_0 \cdot L_u\right),
\end{aligned}
$$

where $\underline{w}_0 = \left[\sqrt{P_Y}, y \in \mathcal{Y}\right]^T$. Therefore, $\|B_0 \cdot \tilde{L}_u\| > \|B_0 \cdot L_u\|$, since $\mathbb{I} > 0$. This contradicts to the assumption that L_u is the singular vector of B_0, corresponding to the second largest singular value. Thus, L_i is orthogonal to $\left[\sqrt{P_{X_i}}, x_i \in \mathcal{X}_i\right]^T$, for all $1 \leq i \leq k$.

$\square$

Chapter 5

Interference Channels and The Deterministic Model

In fact, the quantifications (3.9) and (4.13) in chapter 3 and 4 suggest some important insights in studying the network communication problems. First from (4.13), we know that transmitting sources that are known by multiple transmitters is more advantageous as the transmitters can cooperate with each other to create coherent combining gains. Therefore, in order to increase the communication rates of a network, it is motivated to create common sources between transmitters. On the other hand, in a communication network, these common sources are generated as the common messages in some broadcast channels from other nodes. From (3.9), we know that it consumes more network resources, such as time, frequency band, or transmission power, to generate such common messages than the private messages. Hence, there is a tradeoff relation between the cost of generating the common messages and the coherent combining gain in transmitting common sources. With the framework developed in the previous sections, we want to investigate the structure of this tradeoff relation and optimize the communication rates of the networks. To this end, our next step is to study the interference channels in this chapter, since the interference channel is the simplest channel model that includes the notion of both common sources and common messages.

5.1 The Linear Information Coupling Problems Of Interference Channels

For an interference channel with transmitters X_1, X_2, and receivers Y_1, Y_2, we model the common sources for transmitters and the common messages to receivers by considering the transmission of nine types of messages U_{ij}, for $i, j = 0, 1, 2$. Here, the first index i represents that the message is private to the transmitter X_1, X_2, and common to both transmitters, for $i = 1$, 2, and 0, and the next index is similarly defined for the receivers. For example, the message U_{10} is the private source of X_1 that is transmitted to both receivers as the common message. Then, the linear information coupling problem for the interference channel is:

$$\begin{aligned} & R_{ij} \leq I(U_{ij}; \underline{Y}_j), \ j \neq 0, \ \forall i, \\ & R_{i0} \leq \min\left\{I(U_{i0}; \underline{Y}_1), I(U_{i0}; \underline{Y}_2)\right\}, \ j = 0, \ \forall i, \end{aligned} \tag{5.1}$$

subject to the constraints:

$$\begin{aligned} & \frac{1}{n} I(U_{ij}; \underline{X}_i) \leq \delta_{ij}, \ i \neq 0, \ \forall j, \\ & \frac{1}{n} I(U_{0j}; \underline{X}_1, \underline{X}_2) \leq \delta_{0j}, \ \forall j, \\ & \sum_{i,j=0,1,2} \delta_{ij} = \delta, \\ & \frac{1}{n} \| P_{\underline{X}_1, \underline{X}_2 | \{U_{ij}:\ i,j=0,1,2\}} - P_{\underline{X}_1, \underline{X}_2} \|^2 = O(\delta) \end{aligned}$$

Here, we employ δ and δ_{ij} to indicate $\frac{1}{2}\epsilon^2$ and $\frac{1}{2}\epsilon_{ij}^2$ for the convenience of notation.

Following similar local geometric approaches as the broadcast channel and multiple access channel cases, we can show that the multi-letter problem (5.1) has optimal single-letter solutions[1]. Therefore, we only need to focus on the single-letter version

[1]In fact, similar to the broadcast channels, for interference channels with k receivers, the corresponding linear information coupling problems always have optimal k-letter solutions.

of (5.1). With the same procedure as before, (5.1) can be reduced to

$$R_{ij} = \delta_{ij}\sigma_{ij}^2, \text{ for } i, j = 0, 1, 2, \sum_{i,j=0,1,2} \delta_{ij} \leq \delta, \tag{5.2}$$

where σ_{ij}^2's are the channel parameters that model the ability of the channel in transmitting different kinds of messages, and can be computed in a similar manner as previous chapters:

$$\sigma_{ij}^2 = \begin{cases} \sigma_{\mathsf{smax}}^2(B_{ij}), & i \neq 0, j \neq 0; \\ \max_{\mathbf{v}_i} \min\{||B_{i1}\mathbf{v}_i||^2, ||B_{i2}\mathbf{v}_i||^2\}, & i \neq 0, j = 0; \\ \sigma_{\mathsf{smax}}^2([B_{1j}\ B_{2j}]), & i = 0, j \neq 0, \\ \max_{\mathbf{u}} \min\left\{||[B_{11}\ B_{21}]\mathbf{u}||^2, ||[B_{12}\ B_{22}]\mathbf{u}||^2\right\} & i = 0, j = 0. \end{cases}$$

Here, B_{ij} indicates the DTM with respect to $W_{Y_j|X_i}$, and $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{u})$ are unit-norm vectors, such that $\mathbf{v}_1$ and the first $|\mathcal{X}_1|$ entries of $\mathbf{u}$ are orthogonal to $\left[\sqrt{P_{X_1}},\ x_1 \in \mathcal{X}_1\right]^T$, and $\mathbf{v}_2$ and the last $|\mathcal{X}_2|$ entries of $\mathbf{u}$ are orthogonal to $\left[\sqrt{P_{X_2}},\ x_2 \in \mathcal{X}_2\right]^T$. Note that the σ_{ij}^2's here also satisfy the inequalities similar to (3.9) and (4.13):

$$\begin{aligned} \frac{\sigma_{11}^2\sigma_{12}^2}{\sigma_{11}^2 + \sigma_{12}^2} &\leq \sigma_{10}^2 \leq \min\{\sigma_{11}^2, \sigma_{12}^2\}, \\ \frac{\sigma_{21}^2\sigma_{22}^2}{\sigma_{21}^2 + \sigma_{22}^2} &\leq \sigma_{20}^2 \leq \min\{\sigma_{21}^2, \sigma_{22}^2\}, \\ \frac{\sigma_{01}^2\sigma_{02}^2}{\sigma_{01}^2 + \sigma_{02}^2} &\leq \sigma_{00}^2 \leq \min\{\sigma_{01}^2, \sigma_{02}^2\}, \\ \max\{\sigma_{11}^2, \sigma_{21}^2\} &\leq \sigma_{01}^2 \leq \sigma_{11}^2 + \sigma_{21}^2, \\ \max\{\sigma_{12}^2, \sigma_{22}^2\} &\leq \sigma_{02}^2 \leq \sigma_{12}^2 + \sigma_{22}^2. \end{aligned} \tag{5.3}$$

These inequalities demonstrate both the difficulties of transmitting common messages, and the coherent combing gains of transmitting common sources, in an interference channel. The following example shows the achievability of some of the inequalities.

Example 5. Consider a quaternary-inputs binary-outputs IC where the channel transition probability $P(y_1|x_1x_2)$ is

$$P(0|x_1x_2) = \begin{cases} \frac{1}{3}(2-\alpha), & x_1x_2 = (00, 01, 02, 10, 11, 12); \\ \alpha, & x_1x_2 = (03, 13, 23, 33); \\ \frac{1}{3}(4-5\alpha), & x_1x_2 = (20, 21, 22, 32); \\ \frac{1}{3}(-2+7\alpha), & x_1x_2 = (30, 31), \end{cases}$$

$$P(1|x_1x_2) = 1 - P(0|x_1x_2), \ \forall (x_1, x_2),$$

and $P(y_2|x_1x_2)$ is

$$P(0|x_1x_2) = \begin{cases} \frac{1}{3}(2-\alpha), & x_1x_2 = (22, 23, 20, 32, 33, 30); \\ \alpha, & x_1x_2 = (21, 31, 01, 11); \\ \frac{1}{3}(4-5\alpha), & x_1x_2 = (02, 03, 00, 10); \\ \frac{1}{3}(-2+7\alpha), & x_1x_2 = (12, 13), \end{cases}$$

$$P(1|x_1x_2) = 1 - P(0|x_1x_2), \ \forall (x_1, x_2).$$

To have valid probability distributions, similarly we assume that $\frac{2}{7} \leq \alpha \leq \frac{5}{7}$. Suppose that both P_{X_1} and P_{X_2} are fixed as $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]^T$. The probability transition matrix W_{ij} w.r.t $P_{Y_j|X_i}$ is then computed as

$$W_{11} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1-\alpha & \alpha \\ \frac{1}{2} & \frac{1}{2} & \alpha & 1-\alpha \end{bmatrix},$$

$$W_{21} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1-\alpha & \alpha \\ \frac{1}{2} & \frac{1}{2} & \alpha & 1-\alpha \end{bmatrix},$$

$$W_{12} = \begin{bmatrix} 1-\alpha & \alpha & \frac{1}{2} & \frac{1}{2} \\ \alpha & 1-\alpha & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$W_{22} = \begin{bmatrix} 1-\alpha & \alpha & \frac{1}{2} & \frac{1}{2} \\ \alpha & 1-\alpha & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

This gives $B_{ij} = \frac{\sqrt{2}}{\sqrt{3}} W_{ij}$. Performing similar computations as those in previous chapters, we can get

$$\begin{aligned}
\sigma_{11}^2 &= \sigma_{12}^2 = \sigma_{21}^2 = \sigma_{22}^2 = \frac{1}{2}(1-2\alpha)^2, \\
\sigma_{10}^2 &= \sigma_{20}^2 = \frac{1}{4}(1-2\alpha)^2, \\
\sigma_{01}^2 &= \sigma_{02}^2 = (1-2\alpha)^2, \\
\sigma_{00}^2 &= \frac{1}{2}(1-2\alpha)^2.
\end{aligned}$$

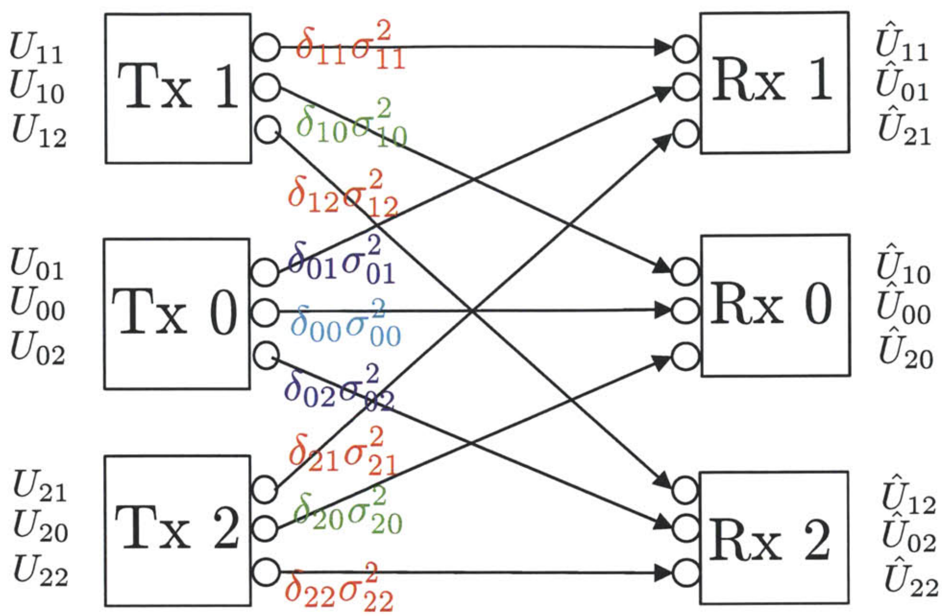
This example is an extreme case where sending common message to both receivers is the most difficult as possible, while sending the common source by the cooperation between transmitters is the easiest due to the maximally-achieved beamforming gain. Note that $4\sigma_{10}^2 = 2\sigma_{11}^2 = \sigma_{01}^2$, thus implying that $(\sigma_{10}^2, \sigma_{20}^2)$ achieve the lower bounds in (5.3), while $(\sigma_{01}^2, \sigma_{02}^2)$ achieve the upper bounds in (5.3).

5.2 δ-Efficiency Region

For the convenience of presentation, in the rest of this thesis, we call the optimized rate regions of the linear information coupling problems as the *δ-efficiency regions*, as they characterize the first order efficiency of transmitting informations. Then, the δ-efficiency regions of the broadcast channel, multiple access channel, and the interference channel can be written respectively as:

$$\begin{aligned}
\mathcal{C}_{\mathsf{BC}} &= \bigcup_{\delta_1+\delta_2+\delta_0\le\delta} \left\{(R_1,R_2,R_0) : R_i \le \delta_i\sigma_i^2, i\in[0:2]\right\}, \\
\mathcal{C}_{\mathsf{MAC}} &= \bigcup_{\delta_1+\delta_2+\delta_0\le\delta} \left\{(R_1,R_2,R_0) : R_i \le \delta_i\sigma_i^2, i\in[0:2]\right\}, \\
\mathcal{C}_{\mathsf{IC}} &= \bigcup_{\sum_{ij}\delta_{ij}\le\delta} \left\{(R_{11},R_{10},\cdots,R_{22}) : R_{ij} \le \delta_{ij}\sigma_{ij}^2,\ i,j\in[0:2]\right\}.
\end{aligned}$$

where the σ_{ij}^2's in each region can be computed in the same way as before. Note that while we restrict δ to be small when deriving the δ-efficiency region, once the region itself has been derived, the δ becomes simply a scaling factor. Therefore, we



$$\mathcal{C}_{\mathsf{IC}} = \bigcup_{\sum_{ij}\delta_{ij}\leq 1} \left\{(R_{11}, R_{10}, \cdots, R_{22}) : R_{ij} \leq \delta_{ij}\sigma_{ij}^2\right\}$$

Figure 5-1: A deterministic model for interference channels. The transmitter Tx k transmits the messages U_{kj}, and the receiver Rx k decodes the messages U_{ik}.

can normalize the δ-efficiency region with respect to δ by replacing the δ to 1 in this region. This normalization still keeps the critical characterizations of the δ-efficiency region, and the unit of the normalized region is then δ nats. In the following, we will just focus on the normalized δ-efficiency region, and still call it as the δ-efficiency region without ambiguity. Now, with these notations and terminologies, we can start to explore the details of our deterministic model.

5.3 The Deterministic Model

The δ-efficiency region of the interference channel leads us to model an interference channel as a deterministic model with nine bit-pipes, each having the capacity of $\delta_{ij}\sigma_{ij}^2$. Unlike traditional wired networks, the capacities of these bit-pipes are flexible: $\delta_{ij}\sigma_{ij}^2$ can change depending on different allocations of $\{\delta_{ij}\}$ subject to $\sum_{i,j}\delta_{ij} \leq 1$.

Fig. 5-1 shows a pictorial representation of our deterministic model for an interference channel. The idea of this deterministic model is that, for $k = 1, 2$, a physical transmitter X_k serves two purposes: transmitting its private sources, and cooperating with the other transmitter to transmit the common sources. Therefore, we can model X_1 and X_2 by *virtual* transmitters Tx 1, 2, and Tx 0, such that Tx k intends to send the private sources U_{kj}, for $k = 1, 2$, and Tx 0 intends to send the common sources U_{0j}. Similarly, the physical receiver Y_1 and Y_2 can be modeled by virtual receivers Rx 1, 2, and Rx 0, such that Rx k wishes to decode the private messages U_{ik}, for $k = 1, 2$, and Rx 0 wishes to decode the common messages U_{i0}. By presenting the virtual transmitters and receivers, the interference channel is modeled with three transmitters and receivers, where each transmitter transmits its individual type of sources, and each receiver decodes its individual type of messages. Then, with this model, the message U_{ij} is transmitted from the Tx i to the Rx j. In addition, the circles here indicate bit-pipes intended for the transmission of different types of messages. For instance, the top circle indicates a bit-pipe for transmitting the private message w.r.t Rx 1. Note that the circles also denote that the messages are transmitted through parallel channels without interfering with each other. Therefore, the circles should be viewed as nine pairs, where each pair represents one of the parallel links.

In this deterministic model, from (5.3), the largest among all σ_{ij}^2's is either σ_{01}^2 or σ_{02}^2. So, to optimize the total throughput through this interference channel, we will just let either δ_{01} or δ_{02} be 1, and deactivate other links. In other words, the optimal strategy to convey information through this interference channel is to transmit it as the common source for both transmitters to the receiver ends as the private message. However, this deterministic model is less interesting in the single-hop case, because the single-hop interference channel does not capture the difficulty of generating this common source before transmitting it to the receiver ends cooperatively. On the other hand, in general multi-hop layered networks, this kind of cost has to be taken into account, since the common source of transmitters in one layer is generated from the previous layer as the common message. This creates an issue of planning which kinds of common messages should be generated in a given network to optimize the throughput. It turns out that the deterministic model we developed in this section becomes a powerful tool in studying this issue for multi-hop layered networks, which will be demonstrated in the next chapter.

Chapter 6

Multi-hop Layered Networks

In this chapter, for the simplicity of presentation, we consider the general layered networks with two users in each layer, while our approach can be extended to more general cases without any difficulty. For the two-user L-layered network, the ℓ-th layer is an interference channel with input symbols $\mathcal{X}_1^{(\ell)}$, $\mathcal{X}_2^{(\ell)}$, and output symbols $\mathcal{Y}_1^{(\ell)}$, $\mathcal{Y}_2^{(\ell)}$, and the channel matrix $W^{(\ell)}_{Y_1Y_2|X_1X_2}$ that specifies the transition probability. As shown in Figure 6-1, each user $i^{(\ell)}$ in the ℓ-th intermediate layer of this network is composed of a receiver $Y_i^{(\ell)}$ and a transmitter $X_i^{(\ell+1)}$.

To simplify the problem, we assume a decode-and-forward scheme [20], such that the received signals of each layer are decoded as a part of the messages, which are then forwarded to the next layer. With the decode-and-forward scheme, each layer can abstracted as the deterministic model like what we did for the interference channel, and a concatenation of these layers constitutes the deterministic model of the multi-

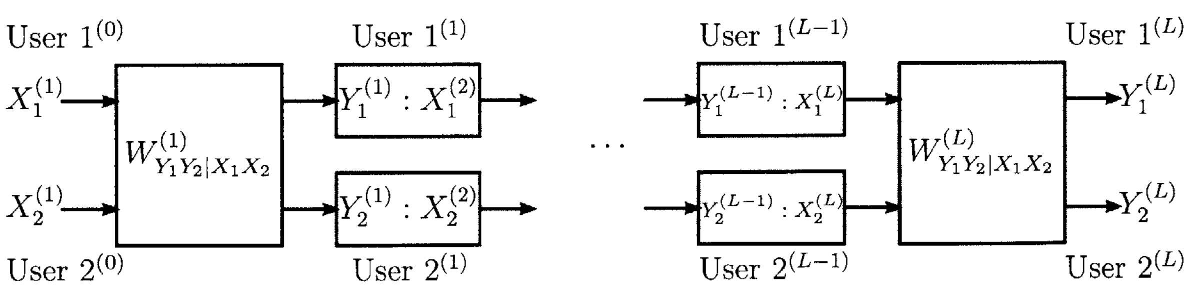


Figure 6-1: The L-layered network.

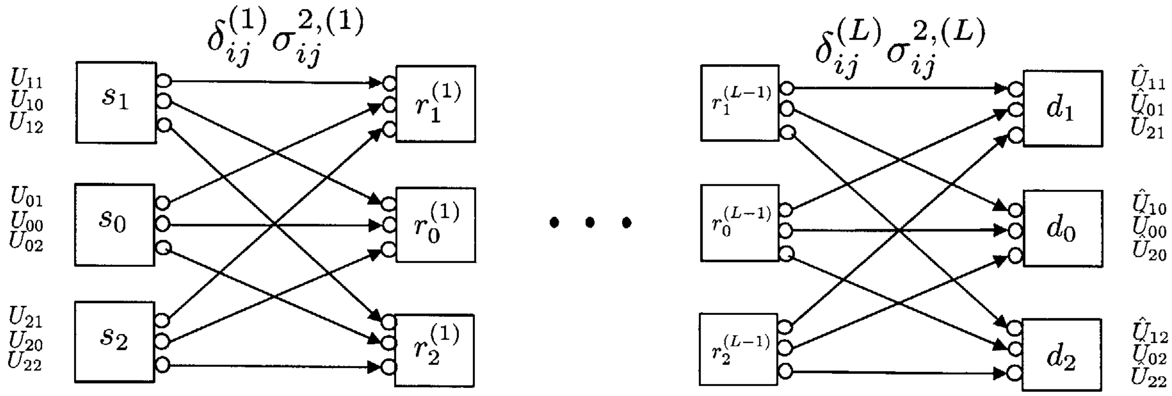


Figure 6-2: A deterministic model for multi-hop interference networks. Here, $\sigma_{ij}^{(\ell)}$, for $i, j = 0, 1, 2$ represent the channel parameters of the ℓ-th layer. We use S_i to represent both the transmitter and receiver for every layer, for $i = 0, 1, 2$.

hop layered network. Then, the deterministic model of the L-layered network can be constructed as Fig. 6-2. Here, we use node s_i to represent the transmitter i in the first layer, and node d_i to represent the receiver i in the last layer, and node $r_i^{(\ell)}$ to represent both the transmitter i and receiver i in the ℓ-th intermediate layer, for $i = 0, 1, 2$, and $\ell = 1, 2, \ldots, L-1$. In addition, the channel of layer ℓ consists of 9 bit-pipes, each having the capacity of $\delta_{ij}^{(\ell)}\sigma_{ij}^{2,(\ell)}$, for $i, j = 0, 1, 2$, and $\ell \in [1 : L]$, and the corresponding constraint for δ_{ij}'s is, normalized with respect to the number of layers,

$$\sum_{\ell=1}^{L}\sum_{i=0}^{2}\sum_{j=0}^{2}\delta_{ij}^{(\ell)} \leq L. \tag{6.1}$$

Importantly, in this thesis, we focus on the routing capacity for simplicity, i.e., we do not allow for network coding [21]. Then, for each set of $\delta_{ij}^{(\ell)}$ that satisfies (6.1), we can obtain a layered network with fixed capacity $\delta_{ij}^{(\ell)}\sigma_{ij}^{2,(\ell)}$ in each link (i, j) in the ℓ-th layer. This becomes the traditional information routing problem, and we can find the capacity region for these 9 types of messages U_{ij} in this network. Therefore, the δ-efficiency region of this layered network is the union of these capacity regions over all sets of $\delta_{ij}^{(\ell)}$ satisfying (6.1). The following Theorem characterizes the δ-efficiency region of a multi-hop layered network.

Theorem 3. *Consider a two-source two-destination multi-hop layered network illustrated in Fig. 6-2. Assume that 9 messages U_{ij}'s are mutually independent. Under the assumption of* (6.1)*, the δ-efficiency region is then*

$$\mathcal{C}_{\mathsf{LN}} = \bigcup_{\sum \delta_{ij} \leq L} \left\{ (R_{11}, R_{10}, \cdots, R_{22}) : R_{ij} \leq \delta_{ij} \sigma_{ij}^2 \right\}, \tag{6.2}$$

where

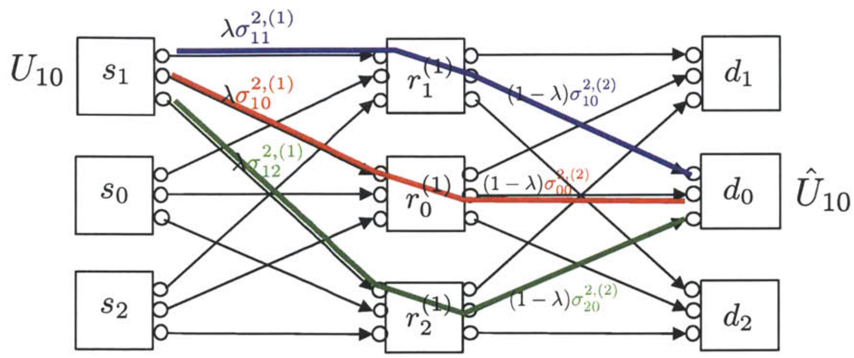
$$\sigma_{ij} = \frac{1}{L} \max_{q \in [1:3^{L-1}]} M(\mathcal{P}_{ij}^{(q)}).$$

Here $M(\mathcal{P}_{ij}^{(q)})$ denotes the harmonic mean of the elements in the set $\mathcal{P}_{ij}^{(q)}$, which consists of the capacities of the links along the q-th information-flow path from virtual source i to virtual destination j.

Proof. Unlike single-hop networks, in multi-hop networks, each link can be used for multiple purposes, i.e., $\delta_{ij}^{(\ell)}$ can be the sum of the network resources consumed for the multiple-message transmission. For conceptual simplicity, we introduce message-oriented notations δ_{ij}'s, each indicating the sum of the $\delta_{ij}^{(\ell)}$'s which contribute to delivering the message U_{ij}. The constraint of $\sum \delta_{ij}^{(\ell)} \leq L$ then leads to $\sum \delta_{ij} \leq L$. Here the key observation is that the tradeoff between the 9-message rates is decided only by the constraint of $\sum \delta_{ij} \leq L$, i.e., given a fixed allocation of δ_{ij}'s, the 9 sub-problems are independent each other.

Now let us fix δ_{ij}'s subject to the constraint, and consider the message U_{ij}. Note that there are 3^{L-1} possible paths for the transmission of this message, the problem is reduced to finding the most efficient path that maximizes R_{ij}, as well as finding a corresponding resource allocation for the links along the most efficient path.

We illustrate the idea of solving this problem through an example in Fig. 6-3. Consider the delivery of U_{10}. In the case of $L = 2$, we have three possible paths $(\mathcal{P}_{10}^{(1)}, \mathcal{P}_{10}^{(2)}, \mathcal{P}_{10}^{(3)})$, identified by blue, red and green paths. The key point here is that the maximum rate for each path is simply a harmonic mean of the link capacities along the path, normalized by the number of layers. To see this, consider the top



$$\sigma_{10}^2 = \frac{1}{2}\max\left\{M(\sigma_{11}^{2,(1)}, \sigma_{10}^{2,(2)}), M(\sigma_{10}^{2,(1)}, \sigma_{00}^{2,(2)}), M(\sigma_{12}^{2,(1)}, \sigma_{20}^{2,(2)})\right\}$$

Figure 6-3: The maximum rate for U_{10} when $L = 2$. In this example, we have three possible paths for sending U_{10} as shown in the figure. For each path, the maximum rate is computed as a harmonic mean of the link capacities along the path, normalized by the number of layers. Therefore, σ_{10}^2 is given as above.

blue path $\mathcal{P}_{10}^{(1)}$ consisting of two links with capacities of $\sigma_{11}^{2,(1)}$ and $\sigma_{10}^{2,(2)}$, i.e., $\mathcal{P}_{10}^{(1)} = \{\sigma_{11}^{2,(1)}, \sigma_{10}^{2,(2)}\}$. Suppose that δ_{ij} is allocated such that the λ fraction is assigned to the first link and the remaining $(1-\lambda)$ fraction is assigned to the second link. The rate is then computed as $\min\{\lambda\sigma_{11}^{2,(1)}, (1-\lambda)\sigma_{10}^{2,(2)}\}$. Note that this can be maximized as $\frac{\sigma_{11}^{2,(1)}\sigma_{10}^{2,(2)}}{\sigma_{11}^{2,(1)}+\sigma_{10}^{2,(2)}} = \frac{1}{2}M(\sigma_{11}^{2,(1)}, \sigma_{10}^{2,(2)})$. Therefore, the maximum rate is

$$\sigma_{10}^2 = \frac{1}{2}\max\left\{M(\sigma_{11}^{2,(1)}, \sigma_{10}^{2,(2)}), M(\sigma_{10}^{2,(1)}, \sigma_{00}^{2,(2)}), M(\sigma_{12}^{2,(1)}, \sigma_{20}^{2,(2)})\right\}.$$

We can easily show that for an arbitrary L-layer case, the maximum rate for each path is the normalized harmonic mean. This completes the proof. □

Remark 9. Theorem 3 leads to a Viterbi-type algorithm to search for the optimal path. Instead of searching for all possible paths with complexity $O(3^L)$, note that (3) is equivalent to finding a path from the first layer to the last layer, where the inverse sum of $\sigma_{i_k i_{k+1}}^{2,(k)}$ is minimized. Thus, we can take $1/\sigma_{i_k i_{k+1}}^{2,(k)}$ as the cost, and run the Viterbi algorithm [22] to find the path with minimal total cost, and the complexity is reduced to $O(L)$.

In addition, Theorem 3 immediately provides the solution of the optimal total throughput of this network, which is the following Corollary.

Corollary 1. *Consider a layered network illustrated in Fig. 6-2, the optimal sum rate (δ-sum efficiency) under the constraint* (6.1) *is*

$$C_{\mathsf{sum}} = \max_{i_1, i_2, \ldots, i_{L+1} \in [0:2]} M(\sigma^{2,(1)}_{i_1 i_2}, \sigma^{2,(2)}_{i_2 i_3}, \ldots, \sigma^{2,(L)}_{i_L i_{L+1}}), \tag{6.3}$$

where $M(x_1, \cdots, x_n)$ denotes the harmonic mean

$$M(x_1, \cdots, x_n) \triangleq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

Moreover, the optimal $i_1, i_2, \ldots, i_{L+1}$ of (3) *can be obtained by a Viterbi algorithm [22] with complexity $O(L)$.*

6.1 Multi-hop Networks With Identical Layers

While Theorem 3 indicates how to find the optimal communication strategy in polynomial time for general layered networks, it is sometimes more useful to understand the "patterns" or structures of the optimal communication schemes for large scale networks. For example, if the channel parameters are only available locally, then the communication patterns can be helpful in the designs. In this section, we investigate this issue by considering the L-layered networks, for $L \to \infty$, with identical layers, such that $\sigma^{2,(\ell)}_{ij} = \sigma^2_{ij}$ for all ℓ, as shown in Fig. 6-2. The following Theorem specifies the fundamental communication modes of the optimal communication strategies for the layered networks with identical layers.

Theorem 4 (Identical layers). *Consider a layered network illustrated in Fig. 6-2, where $\sigma^{2,(\ell)}_{ij} = \sigma^2_{ij}, \forall \ell$, and $L \to \infty$. Then, the optimal sum rate (δ-sum efficiency) is*

$$\begin{aligned} C_{\mathsf{sum}} = \max\{&\sigma^2_{11}, \sigma^2_{00}, \sigma^2_{22}, M(\sigma^2_{10}, \sigma^2_{01}), M(\sigma^2_{20}, \sigma^2_{02}), M(\sigma^2_{12}, \sigma^2_{21}), \\ &M(\sigma^2_{10}, \sigma^2_{02}, \sigma^2_{21}), M(\sigma^2_{20}, \sigma^2_{01}, \sigma^2_{12})\}, \end{aligned} \tag{6.4}$$

where $M(x_1, \cdots, x_n)$ denotes the harmonic mean.

Proof. **(Converse proof)** First observe that we use the routing-only scheme to pass information through the network. Thus, for any optimal communication scheme, we have the inflow equals to outflow for every node in the intermediate layers, i.e., for all k and ℓ,

$$\sum_{i=0}^{2} \delta_{ik}^{(l-1)} \sigma_{ik}^2 = \sum_{j=0}^{2} \delta_{kj}^{(l)} \sigma_{kj}^2. \tag{6.5}$$

Moreover, the total throughput of the network is $\sum_{k,j} \delta_{kj}^{(1)} \sigma_{kj}^2$. Now, for a network with L layers, let us define a tuple of $\delta_{ij}^{(l)}$ as a γ-*scheme*, if

$$\sum_{k=0}^{2} \left| \sum_{j=0}^{2} \delta_{kj}^{(1)} \sigma_{kj}^2 - \sum_{i=0}^{2} \delta_{ik}^{(L)} \sigma_{ik}^2 \right| = \gamma,$$

and $C_{\mathsf{sum},\gamma}^{(L)}$ is the optimal achievable throughput among all γ-schemes. Note that it suffices to only consider γ-schemes that satisfy (6.5). For those γ-schemes, from the definition of γ-schemes, we have

$$\gamma \leq 2C_{\mathsf{sum},\gamma}^{(L)} \leq 2\hat{\sigma}^2,$$

where $\hat{\sigma}^2 \triangleq \max_{i,j} \sigma_{ij}^2$. Thus, we can restrict ourselves to consider only the γ-schemes with $\gamma \leq 2\hat{\sigma}^2$, and then we have

$$C_{\mathsf{sum}} = \lim_{L\to\infty} \max_{\gamma \leq 2\hat{\sigma}^2} C_{\mathsf{sum},\gamma}^{(L)}. \tag{6.6}$$

Now, for any γ-scheme $\delta_{ij}^{(l)}$ of a network with L layers that achieves $C_{\mathsf{sum},\gamma}^{(L)}$ and satisfies (6.5), we consider the tuple $\tilde{\delta}_{ij}^{(1)}$ for $i,j = 0,1,2$, where

$$\tilde{\delta}_{ij}^{(1)} = \frac{1}{L} \sum_{l=1}^{L} \delta_{ij}^{(l)}.$$

Then, we have

$$\begin{aligned}
&\sum_{k=0}^{2}\left|\sum_{j=0}^{2}\tilde{\delta}_{kj}^{(1)}\sigma_{kj}^2-\sum_{i=0}^{2}\tilde{\delta}_{ik}^{(1)}\sigma_{ik}^2\right|\\
=&\frac{1}{L}\sum_{k=0}^{2}\left|\sum_{j=0}^{2}\sum_{l=1}^{L}\delta_{kj}^{(l)}\sigma_{kj}^2-\sum_{i=0}^{2}\sum_{l=1}^{L}\delta_{ik}^{(l)}\sigma_{ik}^2\right|\\
=&\frac{1}{L}\sum_{k=0}^{2}\left|\sum_{l=1}^{L}\sum_{j=0}^{2}\delta_{kj}^{(l)}\sigma_{kj}^2-\sum_{l=2}^{L+1}\sum_{i=0}^{2}\delta_{ik}^{(l-1)}\sigma_{ik}^2\right|\\
=&\frac{1}{L}\sum_{k=0}^{2}\left|\sum_{j=0}^{2}\delta_{kj}^{(1)}\sigma_{kj}^2-\sum_{i=0}^{2}\delta_{ik}^{(L)}\sigma_{ik}^2\right|=\frac{\gamma}{L}.
\end{aligned}$$

Therefore, $\tilde{\delta}_{ij}^{(1)}$ is a (γ/L)-scheme for a network with one layer. Moreover, from (6.5), for the γ-scheme $\delta_{ij}^{(l)}$, the inflow and outflow of all layers are the same. So, the total throughput of the (γ/L)-scheme $\tilde{\delta}_{ij}^{(1)}$ is

$$\sum_{k=0}^{2}\sum_{j=0}^{2}\tilde{\delta}_{kj}^{(1)}\sigma_{kj}^2=\frac{1}{L}\sum_{l=1}^{L}\sum_{k=0}^{2}\sum_{j=0}^{2}\delta_{kj}^{(l)}\sigma_{kj}^2=\sum_{k=0}^{2}\sum_{j=0}^{2}\delta_{kj}^{(1)}\sigma_{kj}^2=C_{\mathsf{sum},\gamma}^{(L)}.$$

Hence, this scheme achieves $C_{\mathsf{sum},\gamma}^{(L)}$. This implies that $C_{\mathsf{sum},\gamma}^{(L)}\le C_{\mathsf{sum},\frac{\gamma}{L}}^{(1)}$. Combining with (6.6), we have

$$C_{\mathsf{sum}}\le\lim_{L\to\infty}\max_{\gamma\le 2\hat{\sigma}^2}C_{\mathsf{sum},\frac{\gamma}{L}}^{(1)}=C_{\mathsf{sum},0}^{(1)}.$$

Hence, C_{sum} is upper-bounded by the solution to the following optimization problem:

$$\begin{aligned}
&C_{\mathsf{sum}}\le\max_{\delta_{ij}}\sum_{i,j}\delta_{ij}\sigma_{ij}^2\\
&\text{s.t.}\ \sum_{i,j}\delta_{ij}\le 1,\ \ \delta_{ij}\ge 0\,\forall i,j,\ \sum_{i=0}^{2}\delta_{ik}\sigma_{ik}^2=\sum_{j=0}^{2}\delta_{kj}\sigma_{kj}^2,\ k\in[0:2].
\end{aligned}$$

Note that the objective indicates the total amount of information that flows into the destinations. The three equality constraints in the above can be equivalently written as two equality constraints:

$$\begin{aligned}\delta_{01} &= \left(\frac{\sigma_{10}^2}{\sigma_{01}^2}\right)\delta_{10} + \left(\frac{\sigma_{20}^2}{\sigma_{01}^2}\right)\delta_{20} - \left(\frac{\sigma_{02}^2}{\sigma_{01}^2}\right)\delta_{02}\\ \delta_{12} &= \left(\frac{\sigma_{20}^2}{\sigma_{12}^2}\right)\delta_{20} + \left(\frac{\sigma_{21}^2}{\sigma_{12}^2}\right)\delta_{21} - \left(\frac{\sigma_{02}^2}{\sigma_{12}^2}\right)\delta_{02}.\end{aligned} \tag{6.7}$$

Noting that all of the δ_{ij}'s are non-negative, we take a careful look at the minus terms associated with δ_{02}. This leads us to consider two cases: (1) $\delta_{02} = 0$; (2) $\delta_{02} \neq 0$.

The first is an easy case. For $\delta_{02} = 0$, the problem can be simplified into:

$$\begin{aligned}\max_{\delta_{ij}} &\sum_{i=0}^{2} \delta_{ii}\sigma_{ii}^2 + (2\delta_{10}\sigma_{10}^2 + 3\delta_{20}\sigma_{20}^2 + 2\delta_{21}\sigma_{21}^2) :\\ \text{s.t.} &\sum_{i=0}^{2} \delta_{ii} + \left(1 + \frac{\sigma_{10}^2}{\sigma_{01}^2}\right)\delta_{10} + \left(1 + \frac{\sigma_{21}^2}{\sigma_{12}^2}\right)\delta_{21}\\ &+ \left(1 + \frac{\sigma_{20}^2}{\sigma_{01}^2} + \frac{\sigma_{20}^2}{\sigma_{12}^2}\right)\delta_{20} \le 1, \quad \delta_{ij} \ge 0, \ \forall i,j.\end{aligned}$$

This LP problem is straightforward. Due to the linearity, the optimal solution must be setting only one δ_{ij} as a non-trivial maximum value while making the other allocations zeros. Hence, we obtain:

$$C_{\mathsf{sum}} \le \max\left\{\sigma_{11}^2, \sigma_{00}^2, \sigma_{22}^2, M(\sigma_{10}^2, \sigma_{01}^2), M(\sigma_{12}^2, \sigma_{21}^2), M(\sigma_{20}^2, \sigma_{01}^2, \sigma_{12}^2)\right\}. \tag{6.8}$$

Here, the fourth term $M(\sigma_{10}^2, \sigma_{01}^2)$, for example, is obtained when $\delta_{10} = \frac{1}{1+\sigma_{10}^2/\sigma_{01}^2}$ and $\delta_{ij} = 0$ for $(i,j) \neq (1,0)$. The last term $M(\sigma_{20}^2, \sigma_{01}^2, \sigma_{12}^2)$ corresponds to the case when $\delta_{20} = \frac{1}{1+\sigma_{20}^2/\sigma_{01}^2+\sigma_{20}^2/\sigma_{12}^2}$ and $\delta_{ij} = 0$ for $(i,j) \neq (2,0)$.

We next consider the second case of $\delta_{02} \neq 0$. First note that since δ_{01} and δ_{12} are nonnegative, by (6.7), we get

$$\begin{aligned}\delta_{02} &\le \left(\frac{\sigma_{20}^2}{\sigma_{02}^2}\right)\delta_{20} + \left(\frac{\sigma_{10}^2}{\sigma_{02}^2}\right)\delta_{10},\\ \delta_{02} &\le \left(\frac{\sigma_{20}^2}{\sigma_{02}^2}\right)\delta_{20} + \left(\frac{\sigma_{21}^2}{\sigma_{02}^2}\right)\delta_{21}.\end{aligned}$$

The key point here is that in general LP problems, whenever $\delta_{02} \neq 0$, the optimal solution occurs when δ_{02} is the largest as possible and the above two inequalities are balanced:

$$\delta_{02} = \left(\frac{\sigma_{20}^2}{\sigma_{02}^2}\right)\delta_{20} + \left(\frac{\sigma_{10}^2}{\sigma_{02}^2}\right)\delta_{10},$$
$$\left(\frac{\sigma_{10}^2}{\sigma_{02}^2}\right)\delta_{10} = \left(\frac{\sigma_{21}^2}{\sigma_{02}^2}\right)\delta_{21}.$$

Therefore, for $\delta_{02} \neq 0$, the problem can be simplified into:

$$\max_{\delta_{ij}} \sum_{i=0}^{2} \delta_{ii}\sigma_{ii}^2 + (3\delta_{10}\sigma_{10}^2 + 2\delta_{20}\sigma_{20}^2):$$
$$\text{s.t.} \quad \sum_{i=0}^{2} \delta_{ii} + \left(1 + \frac{\sigma_{10}^2}{\sigma_{02}^2} + \frac{\sigma_{10}^2}{\sigma_{21}^2}\right)\delta_{10} + \left(1 + \frac{\sigma_{20}^2}{\sigma_{02}^2}\right)\delta_{20}\delta_{ij} \geq 0, \ \forall i, j.$$

This LP problem is also straightforward. Using the linearity, we can get:

$$C_{\mathsf{sum}} \leq \max\left\{\sigma_{11}^2, \sigma_{00}^2, \sigma_{22}^2, M(\sigma_{20}^2, \sigma_{02}^2), M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)\right\}. \tag{6.9}$$

By (6.8) and (6.9), we complete the converse proof.

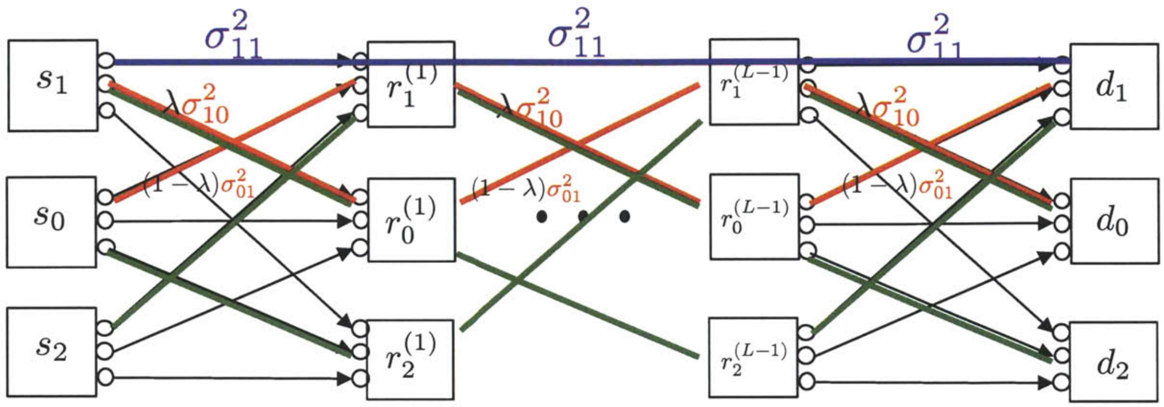
For the achievability, note that $\sigma_{ii} = M(\sigma_{ii})$, so all 8 modes in (6.4) can be written in the form $M(\sigma_{i_1i_2}, \sigma_{i_2i_3}, \ldots, \sigma_{i_ki_1})$, for $k = 1, 2, 3$, and $i_1, \ldots, i_k$ are mutually different. Then, for $k = 1, 2, 3$, $n \in [1 : k]$, and $\ell \in [1 : L]$, the $M(\sigma_{i_1i_2}, \sigma_{i_2i_3}, \ldots, \sigma_{i_ki_1})$ can be achieved by setting

$$\delta_{i_ni_{n+1}}^{(\ell)} = \delta_{i_ni_{n+1}} = \frac{M(\sigma_{i_1i_2}, \sigma_{i_2i_3}, \ldots, \sigma_{i_ki_1})}{k\sigma_{i_ni_{n+1}}}, \tag{6.10}$$

and deactivating all other links by setting their δ_{ij}'s to zero. We assume that in (6.10), when $n = k$, $\delta_{i_ki_{k+1}}$ denotes $\delta_{i_ki_1}$. It is easy to verify that the assignment of (6.10) satisfies the constraint (6.1), thus we prove the achievability. □

Theorem 4 implies that the optimal communication scheme is from one of the eight communication modes in (6.4). Fig. 6-4 illustrates the communication schemes that achieves modes σ_{00}^2, $M(\sigma_{12}^2, \sigma_{21}^2)$, and $M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)$, where other modes can be

$$\sigma_{ij}^{2,(\ell)} = \sigma_{ij}^2, \ \ell = 1, 2, \cdots, \infty$$



$$C_{\mathsf{sum}} = \max\{\sigma_{11}^2, \sigma_{00}^2, \sigma_{22}^2, M(\sigma_{10}^2, \sigma_{01}^2), M(\sigma_{20}^2, \sigma_{02}^2), M(\sigma_{12}^2, \sigma_{21}^2) M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2), M(\sigma_{20}^2, \sigma_{01}^2, \sigma_{12}^2)\}.$$

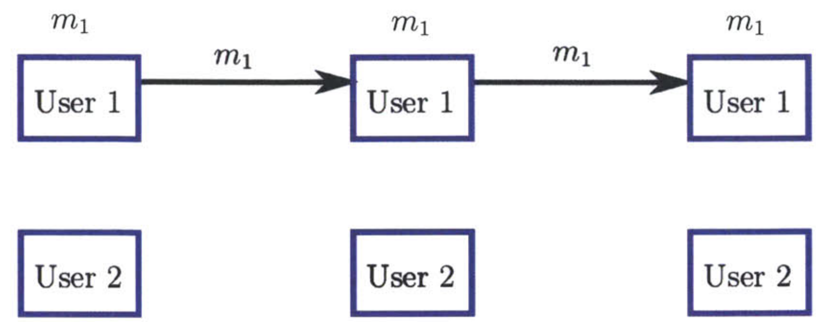
Figure 6-4: Sum efficiency of multi-hop interference networks with identical layers.

achieved similarly. For example, the mode $M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)$ is achieved by using links $1-0$, $0-2$, and $2-1$, such that

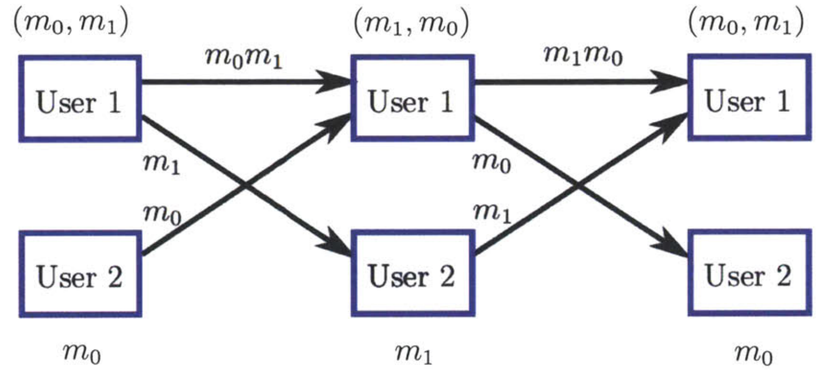
$$\delta_{10}\sigma_{10}^2 = \delta_{02}\sigma_{02}^2 = \delta_{21}\sigma_{21}^2 = \frac{M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)}{3},$$

and other $\delta_{ij} = 0$. Then, the information flow for each layer and the sum rate are all $M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)$.

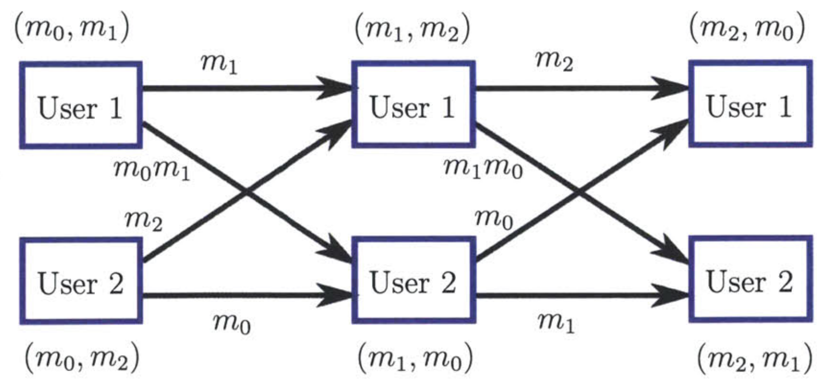
More interestingly, in order to achieve (6.4), it requires the cooperation between users, and rolling the knowledges of different part of messages between users layer by layer. We demonstrate this by considering the communication scheme that achieves $M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)$ as an example. Suppose that at the first layer, the node s_i has the knowledge of information m_i, for $i = 0, 1, 2$. Since s_0 is the virtual node that represents the common message of both users, user 1 knows messages (m_0, m_1), and user 2 knows (m_0, m_2). Then, to achieve $M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)$, user 1 broadcasts its private message m_1



(a)



(b)



(c)

Figure 6-5: The rolling of different pieces of information between users layer by layer for the optimal communication scheme that achieves (a) σ_{11}^2 (b) $M(\sigma_{10}^2, \sigma_{01}^2)$ (c) $M(\sigma_{10}^2, \sigma_{02}^2, \sigma_{21}^2)$.

to both users in the next layer, and both users in the first layer cooperate to transmit their common message to user 2 in the next layer as the private message. Thus, in the second layer user 1 decodes messages (m_1, m_2) and user 2 decodes (m_1, m_0). Similarly, in the third layer, user 1 decodes (m_2, m_0) and user 2 decodes (m_2, m_1), and then loop back. This effect is shown by Fig. 6-5(c). Therefore, according to the values of channel parameters, Theorem 4 demonstrates the optimal communication mode, and hence indicates what kind of common messages should be generated to achieve the optimal sum rate.

Remark 10. The generalization of our development in this chapter to arbitrary M-source K-destination networks is straightforward. In the most general setting, we have $(2^M - 1)$ virtual sources, $(2^K - 1)$ virtual destinations, and $(2^M - 1)(2^K - 1)$ messages. For example, in the case of $(M, K) = (3, 3)$,

$$\text{virtual sources: } s_1, s_2, s_3, s_{12}, s_{13}, s_{23}, s_{123},$$

$$\text{virtual destinations: } d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_{123},$$

where, for instance, s_{12} indicates a virtual terminal that sends messages accessible by sources 1 and 2; and d_{12} denotes a virtual terminal that decodes messages intended for destinations 1 and 2. Under this model, we have $7 \times 7 = 49$ messages, denoted by $U_{\mathcal{S},\mathcal{D}}$, where $\mathcal{S}, \mathcal{D} \subseteq \{1, 2, 3\} (\neq \varnothing)$, each indicating a message which is accessible by the set $\mathcal{S}$ of sources, and is intended for the set $\mathcal{D}$ of destinations. For this network, we can then obtain 49-dimensional δ-efficiency regions and δ-sum efficiency, as we did in Theorems 3 and 4. This generalization can also be carried to the case of networks with feedback, which will be developed in the next chapter.

Chapter 7

Multi-hop Layered Networks With Feedback

We next explore the role of feedback under our local geometric approach. As the previous chapter, we employ the decode-and-forward scheme for both forward and feedback transmissions, under which decoded messages at each node (instead of analog received signals) are fed back to the nodes in preceding layers. In this model, one can view feedback as bit-pipe links added on top of a deterministic channel. With this assumption on the feedback, we can easily see that the feedback does not provide any gain on the δ-efficiency region of the broadcast channels and multiple access channels. However, in the following, we will show that feedback can indeed provide certain gain on the δ-efficiency region of the multi-hop layered networks. Let us start by considering the interference channels with feedback.

7.1 Interference Channels

Proposition 2. *Consider the deterministic model of interference channels illustrated in Fig. 5-1. Assume that decoded messages at each receiver are fed back to all the transmitters. Let* δ_{ij} *be the network resource consumed for delivering the message*

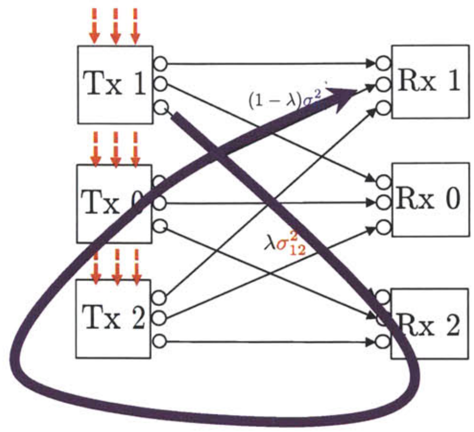


Figure 7-1: An alternative way to deliver the common message of U_{10}. One alternative way is to take a route: virtual-Tx 1 $\rightarrow$ virtual-Rx 2 $\stackrel{\text{feedback}}{\longrightarrow}$ virtual-Tx 0 $\rightarrow$ virtual-Rx 1. The message is clearly Rx-common, as it is delivered to both virtual-Rxs. We can optimize the allocation to the two links to obtain the rate of $\frac{1}{2}M(\sigma_{12}^2, \sigma_{01}^2)$.

U_{ij}, and assume $\sum \delta_{ij} \leq 1$. The feedback δ-efficiency region is then

$$\mathcal{C}_{\mathsf{IC}}^{\mathsf{fb}} = \bigcup_{\sum \delta_{ij} \leq 1} \left\{ (R_{11}, \cdots, R_{22}) : R_{k0} \leq \delta_{k0}\sigma_{k0}^{2,\mathsf{fb}},\ k \neq 0,\ R_{ij} \leq \delta_{ij}\sigma_{ij}^2,\ (i,j) \neq (1,0),(2,0) \right\},$$

where

$$\begin{aligned} \sigma_{10}^{2,\mathsf{fb}} &= \max\left\{ \sigma_{10}^2, \frac{M(\sigma_{12}^2, \sigma_{01}^2)}{2}, \frac{M(\sigma_{11}^2, \sigma_{02}^2)}{2} \right\}, \\ \sigma_{20}^{2,\mathsf{fb}} &= \max\left\{ \sigma_{20}^2, \frac{M(\sigma_{21}^2, \sigma_{02}^2)}{2}, \frac{M(\sigma_{22}^2, \sigma_{01}^2)}{2} \right\}. \end{aligned} \tag{7.1}$$

Proof. Fix δ_{ij}'s subject to the constraint. First, consider the transmission of U_{ij} when $(i,j) \neq (1,0),(2,0)$. In this case, the maximum rate can be achieved by using the Tx i-to-Rx j link. Hence, $R_{ij} \leq \delta_{ij}\sigma_{ij}^2$.

On the other hand, in sending U_{10}, we may have better alternative paths. One alternative way is to take a route as shown in Fig. 7-1: Tx 1 $\rightarrow$ Rx 2 $\stackrel{\text{feedback}}{\longrightarrow}$ Tx 0 $\rightarrow$ virtual-Rx 1. The message is clearly Rx-common, as it is delivered to both virtual-Rxs. Suppose that the network resource δ_{10} is allocated such that the λ fraction is assigned to the σ_{12}^2-capacity link and the remaining $(1-\lambda)$ fraction is assigned to the σ_{01}^2-capacity link. The rate is then $\min\{\lambda\sigma_{12}^2, (1-\lambda)\sigma_{01}^2\}$, which can be maximized as $\frac{1}{2}M(\sigma_{12}^2, \sigma_{01}^2)$. The other alternative path is: virtual-Tx 1 $\rightarrow$ virtual-Rx 1 $\xrightarrow{\text{feedback}}$ virtual-Tx 0 $\rightarrow$ virtual-Rx 2. With this route, we can achieve $\frac{1}{2}M(\sigma_{11}^2, \sigma_{02}^2)$. Therefore, we can obtain $\sigma_{10}^{2,\mathsf{fb}}$ as claimed. Similarly we can get the claimed $\sigma_{20}^{2,\mathsf{fb}}$. $\square$

With this theorem, we find that in contrast to broadcast channels and multiple access channels, feedback can provide an increase on the δ-efficiency region of the interference channels. Here is an example.

Example 6. Consider the same interference channel as in Example 5 but which includes feedback links from all receivers to all transmitter. We obtain the same σ_{ij}'s except the following two:

$$\sigma_{10}^{2,\mathsf{fb}} = \sigma_{20}^{2,\mathsf{fb}} = \frac{1}{3}(1-2\alpha)^2 \geq \frac{1}{4}(1-2\alpha)^2 = \sigma_{10}^2 = \sigma_{20}^2.$$

Note that $\frac{\sigma_{10}^{2,\mathsf{fb}}}{\sigma_{10}^2} = \frac{4}{3}$ when $\alpha \neq \frac{1}{2}$, implying a 33% gain w.r.t R_{10}. $\square$

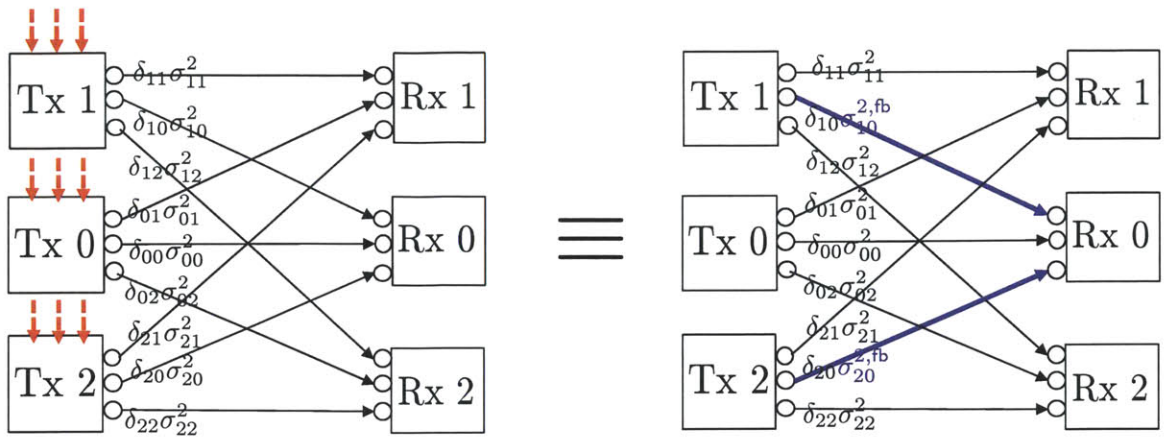
Here, the gain comes from the fact that feedback provides better alternative paths. The nature of the gain coincides with that in [23, 19]. Also the feedback gain is multiplicative, which is qualitatively similar to the gain in the two-user Gaussian interference channels [19]. Note that feedback is useful in increasing individual rates such as R_{10} or R_{20}, but it does not increase the sum-efficiency though:

$$C_{\mathsf{sum}}^{\mathsf{fb}} = \max_{i,j}\{\sigma_{ij}^2, \sigma_{10}^{2,\mathsf{fb}}, \sigma_{20}^{2,\mathsf{fb}}\} = \max\{\sigma_{01}^2, \sigma_{02}^2\} = C_{\mathsf{sum}}.$$

Remark 11. Note that with Proposition 2, one can simply model an interference channel with feedback as a nonfeedback interference channel, where the channel parameters $(\sigma_{10}^2, \sigma_{20}^2)$ are replaced by the $(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{20}^{2,\mathsf{fb}})$ in (7.1). See Fig. 7-2.

7.2 Multi-hop Layered Networks

For multi-hop layered networks, we investigate two feedback models: (1) full-feedback model, where the decoded messages at each node is fed back to the nodes in all the preceding layers; (2) layered-feedback model, where the feedback is available only



$$\mathcal{C}_{\mathsf{IC}}^{\mathsf{fb}} = \bigcup_{\sum_{ij}\delta_{ij}\le 1} \left\{(R_{11}, R_{10}, \cdots, R_{22}) : R_{10} \le \delta_{10}\sigma_{10}^{2,\mathsf{fb}}, R_{20} \le \delta_{20}\sigma_{20}^{2,\mathsf{fb}}, R_{ij} \le \delta_{ij}\sigma_{ij}^2\right\}$$

Figure 7-2: Interference channels with feedback. A feedback IC can be interpreted as a nonfeedback IC where $(\sigma_{10}^2, \sigma_{20}^2)$ are replaced by the $(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{20}^{2,\mathsf{fb}})$ in (7.1).

to the nodes in the immediately preceding layer. The following Theorem shows that these two types of feedbacks in fact provide the same amount of gain on the δ-efficiency region of the multi-hop layered networks.

Theorem 5. *Consider a two-source two-destination multi-hop layered networks illustrated in Fig. 6-2. Assume that $\delta_{ij}^{(\ell)}$'s satisfy the constraint of* (6.1)*. Then, the feedback δ-efficiency region of the full-feedback model is the same as that of the layered-feedback model, and is given by*

$$\mathcal{C}_{\mathsf{IN}}^{\mathsf{fb}} = \bigcup_{\sum \delta_{ij} \le L} \left\{(R_{11}, R_{10}, \cdots, R_{22}) : R_{ij} \le \delta_{ij}\sigma_{ij}^2\right\}, \tag{7.2}$$

where

$$\sigma_{ij}^2 = \frac{1}{L} \max_{1\le q\le 3^{L-1}} M(\mathcal{P}_{ij}^{\mathsf{fb},(q)}).$$

Here the elements of the set $\mathcal{P}_{ij}^{\mathsf{fb},(q)}$ are with respect to a translated network where

$(\sigma_{10}^{2,(\ell)}, \sigma_{20}^{2,(\ell)})$ *are replaced by* $(\sigma_{10}^{2,(\ell),\mathsf{fb}}, \sigma_{20}^{2,(\ell),\mathsf{fb}})$ *for each layer* $\ell \in [1:L]$*:*

$$\begin{aligned}
\sigma_{10}^{2,(\ell),\mathsf{fb}} &= \max\left\{\sigma_{10}^{2,(\ell)}, \frac{M(\sigma_{12}^{2,(\ell)}, \sigma_{01}^{2,(\ell)})}{2}, \frac{M(\sigma_{11}^{2,(\ell)}, \sigma_{02}^{2,(\ell)})}{2}\right\}, \\
\sigma_{20}^{2,(\ell),\mathsf{fb}} &= \max\left\{\sigma_{20}^{2,(\ell)}, \frac{M(\sigma_{21}^{2,(\ell)}, \sigma_{02}^{2,(\ell)})}{2}, \frac{M(\sigma_{22}^{2,(\ell)}, \sigma_{01}^{2,(\ell)})}{2}\right\}.
\end{aligned} \tag{7.3}$$

Proof. First we will prove the equivalence between the full-feedback and layered-feedback models. We introduce some notations. Let $X_i[t]$ be the transmitted signal of virtual source s_i at time t; let $X_i^{(\ell)}[t]$ be the transmitted signal of node $r_i^{(\ell)}$ at time t; and let $X^{(\ell)}[t] = \left[X_1^{(\ell)}[t], X_0^{(\ell)}[t], X_2^{(\ell)}[t]\right]$, where $\ell \in [1:L-1]$. Define $X^{t-1} = \{X[j]\}_{j=1}^{t-1}$. Let $Y_i^{(\ell)}[t]$ be the received signal of node $r_i^{(\ell)}$ at time t, and let $Y^{(\ell)}[t] = \left[Y_1^{(\ell)}[t], Y_0^{(\ell)}[t], Y_2^{(\ell)}[t]\right]$, where $\ell \in [1:L]$. Let $U_i = [U_{i1}, U_{i0}, U_{i2}]$. We use the notation $A \stackrel{f}{=} B$ to indicate that A is a function B.

Under the full-feedback model, we then get

$$\begin{aligned}
X_i[t] &\stackrel{f}{=} (U_i, \{Y^{(\ell),t-1}\}_{\ell=1}^{L}) \\
&\stackrel{f}{=} (U_i, Y^{(1),t-1}, X^{(1),t-1}) \\
&\stackrel{f}{=} (U_i, Y^{(1),t-1}, \{Y^{(\ell),t-2}\}_{\ell=2}^{L}) \\
&\stackrel{f}{=} (U_i, Y^{(1),t-1}, X^{(1),t-2}) \\
&\quad\vdots \\
&\stackrel{f}{=} (U_i, Y^{(1),t-1}, X^{(1)}[1]) \\
&\stackrel{f}{=} (U_i, Y^{(1),t-1})
\end{aligned} \tag{7.4}$$

where the second step follows from the fact that in deterministic layered networks, $\{Y^{(\ell),t-1}\}_{\ell=2}^{L}$ is a function of $X^{(1),t-1}$; the third step follows from the fact that $X^{(1),t-1} \stackrel{f}{=} (Y^{(1),t-2}, \{Y^{(\ell),t-2}\}_{\ell=2}^{L})$; and the second last step is due to iterating the previous steps $(t-3)$ times.

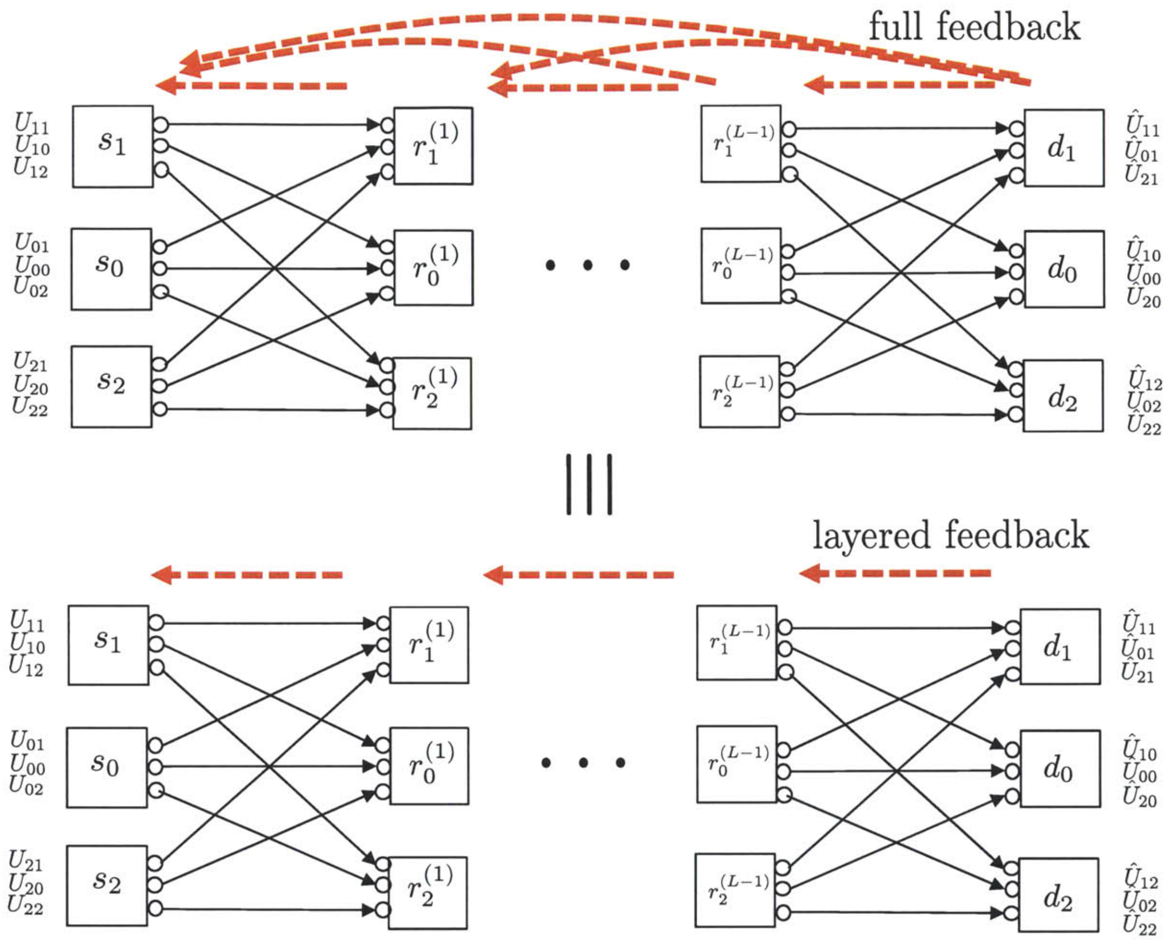


Figure 7-3: Network equivalence. The feedback efficiency region of the full-feedback model is the same as that of the layered-feedback model.

Using similar arguments, we can also show that for $\ell \in [1 : L-1]$,

$$
\begin{aligned}
X_i^{(\ell)}[t] &\stackrel{f}{=} (Y_i^{(\ell),t-1}, \{Y^{(j),t-1}\}_{j=\ell+1}^{L}) \\
&\stackrel{f}{=} (Y_i^{(\ell),t-1}, Y^{(\ell+1),t-1}, X^{(\ell+1),t-1}) \\
&\stackrel{f}{=} (Y_i^{(\ell),t-1}, Y^{(\ell+1),t-1}, \{Y^{(j),t-2}\}_{j=\ell+2}^{L}) \\
&\stackrel{f}{=} (Y_i^{(\ell),t-1}, Y^{(\ell+1),t-1}, X^{(\ell+1),t-2}) \qquad (7.5) \\
&\quad \vdots \\
&\stackrel{f}{=} (Y_i^{(\ell),t-1}, Y^{(\ell+1),t-1}, X^{(\ell+1)}[1]) \\
&\stackrel{f}{=} (Y_i^{(\ell),t-1}, Y^{(\ell+1),t-1}).
\end{aligned}
$$

The functional relationships of (7.4) and (7.5) imply that any rate point in the full-feedback δ-efficiency region can also be achieved in the layered-feedback δ-efficiency region. This proves the equivalence of the two feedback models. See Fig. 7-3.

We next focus on the capacity characterization under the layered-feedback model. The key idea is to employ Proposition 2, thus translating each layer with feedback into an equivalent nonfeedback layer, where $(\sigma_{10}^{2,(\ell)}, \sigma_{20}^{2,(\ell)})$ are replaced by $(\sigma_{10}^{2,(\ell),\mathsf{fb}}, \sigma_{20}^{2,(\ell),\mathsf{fb}})$ in (7.3). We can then apply Theorem 3 to obtain the claimed δ-efficiency region. $\square$

7.3 Multi-hop Networks With Identical Layers

Proposition 3. *Consider a two-source two-destination symmetric multi-hop layered network where $\sigma_{ij}^{(\ell)} = \sigma_{ij}, \forall \ell$ and $L = \infty$. For both full-feedback and layered-feedback models, the feedback δ-sum efficiency is the same as*

$$
\begin{aligned}
C_{\mathsf{sum}}^{\mathsf{fb}} = \max\big\{&\sigma_{11}^2, \sigma_{00}^2, \sigma_{22}^2, \\
&M(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{01}^2), M(\sigma_{20}^{2,\mathsf{fb}}, \sigma_{02}^2), M(\sigma_{12}^2, \sigma_{21}^2) \\
&M(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{02}^2, \sigma_{21}^2), M(\sigma_{20}^{2,\mathsf{fb}}, \sigma_{01}^2, \sigma_{12}^2)\Big\},
\end{aligned}
\tag{7.6}
$$

where $(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{20}^{2,\mathsf{fb}})$ are of the same formulas as those in (7.1).

Proof. The proof is immediate from Theorems 4, 5, and Proposition 2. First, with Theorem 5, it suffices to focus on the layered-feedback model. We then employ Proposition 2 to translate each layer with the layered feedback into an equivalent nonfeedback layer with the replaced parameters $(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{20}^{2,\mathsf{fb}})$. We can then use Theorem 4 to obtain the desired δ-sum efficiency. $\square$

Unlike single-hop networks, in multi-hop networks, the δ-sum efficiency can increase with feedback. Here is an example.

Example 7. Consider a two-source two-destination symmetric multi-hop layered network, where each layer is the interference channel shown in Fig. 7-4. The transmitter 1 has two binary inputs X_1' and X_1'', and the transmitter 2 has one binary input. The

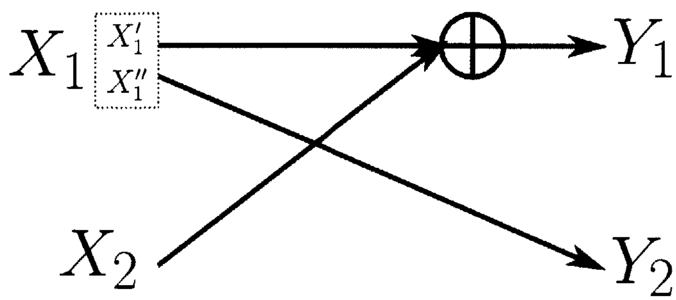


Figure 7-4: The input X_1 is composed of two binary inputs X_1' and X_1'', and the input X_2 is binary. The output $Y_1 = X_1' \oplus X_2$, and the output $Y_2 = X_1''$.

output Y_1 is equal to $X_1' \oplus X_2$ and the output Y_2 is equal to X_1''. Suppose that P_{X_2} is fixed as $[0.1585, 0.8415]$, and $P_{X_1} = P_{X_1'X_1''}$ is fixed as

$$P_{X_1'X_1''} = \begin{cases} 0.095, & X_1'X_1'' = (00, 01); \\ 0.405, & X_1'X_1'' = (10, 11). \end{cases}$$

Then, we have

$$\begin{aligned} (\sigma_{11}^2, \sigma_{12}^2, \sigma_{10}^2) &= (0.35, 1, 0.26), \\ (\sigma_{21}^2, \sigma_{22}^2, \sigma_{20}^2) &= (0.25, 0, 0), \\ (\sigma_{01}^2, \sigma_{02}^2, \sigma_{00}^2) &= (0.6, 1, 0.375). \end{aligned}$$

This is a valid example, as the above parameters satisfy (5.3). From Theorem 4, the nonfeedback δ-sum efficiency is computed as $C_{\mathsf{sum}} = M(\sigma_{12}^2, \sigma_{21}^2) = 0.4$. On the other hand, $(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{20}^{2,\mathsf{fb}}) = (0.375, 0.2)$ and from Proposition 3, the feedback δ-sum efficiency is computed as $C_{\mathsf{sum}}^{\mathsf{fb}} = M(\sigma_{10}^{2,\mathsf{fb}}, \sigma_{01}^2) = 0.4615$, thus showing a 15.4% improvement. $\square$

We also find some classes of symmetric multiple networks where feedback provides no gain in δ-sum efficiency.

Corollary 1. *Consider a two-source two-destination symmetric multi-hop layered* *network, where*

$$\lambda := \sigma_{11}^2 = \sigma_{12}^2 = \sigma_{21}^2 = \sigma_{22}^2,$$
$$\mu := \sigma_{10}^2 = \sigma_{20}^2,$$
$$\sigma := \sigma_{01}^2 = \sigma_{02}^2,$$
$$\sigma_{00}^2.$$

Assume that the parameters of $(\lambda, \mu, \sigma, \sigma_{00}^2)$ *satisfy* (5.3). *We then get:*

$$C_{\mathsf{sum}} = C_{\mathsf{sum}}^{\mathsf{fb}} = \max\{\lambda, \sigma_{00}^2, M(\mu, \sigma), M(\mu, \lambda, \sigma)\}.$$

Proof. Theorem 4 immediately yields $C_{\mathsf{sum}} = \max\{\lambda, \sigma_{00}^2, M(\mu, \sigma), M(\mu, \lambda, \sigma)\}$. From Proposition 3, we get:

$$C_{\mathsf{sum}}^{\mathsf{fb}} = \max\left\{C_{\mathsf{sum}}, M\left(\frac{M(\lambda,\sigma)}{2}, \sigma\right), M\left(\frac{M(\lambda,\sigma)}{2}, \sigma, \lambda\right)\right\}.$$

Note that

$$M\left(\frac{M(\lambda,\sigma)}{2}, \sigma\right) = \lambda\left(\frac{2\sigma}{2\lambda+\sigma}\right) \leq \lambda,$$

where the inequality comes from $\sigma \leq 2\lambda$ due to (5.3). Similarly we can show that $M\left(\frac{M(\lambda,\sigma)}{2}, \sigma, \lambda\right) \leq \lambda$. Therefore, $C_{\mathsf{sum}}^{\mathsf{fb}} = C_{\mathsf{sum}}$. □

Chapter 8

Conclusion

In this thesis, we developed a local geometric approach, which approximates the K-L divergence by a squared Euclidean metric in an Euclidean space. Under the local approximation, we constructed the coordinates and the notion of orthogonality for the probability distribution spaces. With this approach, we can solve a certain class of information theory problems, which we call the linear information coupling problems, for communication networks. We also showed that the single-letterization of multi-terminal problems can be simplified as some linear algebra problems, which can be solved in a systematic way. Applying our approach to the general broadcast channels, the transmission of the common message can be formulated as the tradeoff between multiple linear systems. In order to achieve the optimal tradeoff, it is required to either code over multiple letters with the number of letters proportional to the number of receivers, or allowing the cardinality of the auxiliary random variables growing with the receivers. For the multiple access channel with common sources, there exists some coherent combing gains due to the cooperation between transmitters, and we can evaluate this gain quantitively by using our technique.

The development for single-hop communication channels can be generalized to multi-hop communication networks, such as layered networks. The key observation is that, with our approach, we in fact quantitively provide the cost of generating common information, and characterize how beneficial the common source can be. Therefore, we can formulate a new type of information flow optimization problem according to this local geometric approach. We implement this idea to the multi-hop layered networks by constructing a new type of deterministic model that captures the critical channel parameters by our approach. For the networks with general layers, we propose a Viterbi algorithm to find the communication schemes that provides the optimal throughputs. Furthermore, for networks with identical layers, we reduce the optimal communication schemes to eight fundamental communication modes. We also characterize how information are exchanging in each communication mode. Finally, we explore the role of feedback in the network communication problems under this local geometric framework. It turns out that the feedback can provide certain multiplicative gain in the network throughputs. With our approach we can also specify the communication schemes to optimally utilize the feedback gain. In general, our results tell that how users in networks should cooperate with each other to optimize the network throughputs. This provides the insights in understanding the network architectures and the designing of communication systems.

Appendix A

Proof of Lemma 3

Since U, V_1, and V_2 are mutually independent, we have

$$\begin{aligned} I(U, V_1, V_2; \underline{X}) &= I(U; \underline{X}) + I(V_1; \underline{X}|U) + I(V_2; \underline{X}|U, V_1) \\ &\geq I(U; \underline{X}) + I(V_1; \underline{X}) + I(V_2; \underline{X}). \end{aligned}$$

Thus, the rate region (3.2) is belong to (3.3). On the other hand, for any rate tuple (R_0, R_1, R_2) in (3.3) achieved by some mutually independent U, V_1, and V_2, with $\frac{1}{n} I(U; \underline{X}) = \frac{1}{2}\epsilon_0^2$, and $\frac{1}{n} I(V_i; \underline{X}) = \frac{1}{2}\epsilon_i^2$ for $i = 1, 2$, where $\sum_{i=1}^{3} \epsilon_i^2 \leq \epsilon^2$, we assume that the conditional distributions achieving this rate tuple have the perturbation forms $P_{\underline{X}|U=u} = P_{\underline{X}} + \sqrt{n}\epsilon_0 \cdot J_u$, and $P_{\underline{X}|V_i=v_i} = P_{\underline{X}} + \sqrt{n}\epsilon_i \cdot J_{v_i}$, for $i = 1, 2$. Then, it is easy to verify that

$$Q_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)} = P_{\underline{X}} + \sqrt{n}\epsilon_0 J_u + \sqrt{n}\epsilon_1 J_{v_1} + \sqrt{n}\epsilon_2 J_{v_2}$$

is a valid conditional distribution with marginal $P_{\underline{X}}$. Therefore, using $Q_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)}$ as the conditional distribution, the mutual information

$$\begin{aligned} \frac{1}{n} I(U, V_1, V_2; \underline{X}) &= \frac{1}{2} \sum_{u,v_1,v_2} P_U(u) P_{V_1}(v_1) P_{V_2}(v_2) \cdot \|\epsilon_0 J_u + \epsilon_1 J_{v_1} + \epsilon_2 J_{v_2}\|_{P_X}^2 + o(\epsilon^2) \\ &= \frac{1}{2}\left(\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2\right) + o(\epsilon^2) \leq \frac{1}{2}\epsilon^2 + o(\epsilon^2), \qquad \text{(A.1)} \end{aligned}$$

where (A.1) is resulted from the definition of the perturbation vectors:

$$\sum_{u} P_U(u) J_u = 0, \quad \sum_{v_i} P_{V_i}(v_i) J_{v_i} = 0, \text{ for } i = 1, 2,$$

and

$$\sum_{u} P_U(u) \left\| J_u \right\|_{P_X}^2 = 1,$$
$$\sum_{v_i} P_{V_i}(v_i) \left\| J_{v_i} \right\|_{P_X}^2 = 1, \text{ for } i = 1, 2.$$

Hence, we can take $Q_{\underline{X}|(U,V_1,V_2)=(u,v_1,v_2)}$ as the conditional distribution in (3.2), and obtain a rate tuple that is equal to (R_0, R_1, R_2), up to the first order approximation.

Appendix B

Proof of Theorem 1

In this appendix, we prove the following two statements: (i) there exist k-letter optimal solutions for k-user broadcast channels, and (ii) for 2-user broadcast channels, there exist single-letter optimal solutions.

First, let us prove statement (i). Assume that $B_1, \ldots, B_k$ are DTM's of the k-user broadcast channel, and denote $\underline{v}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$, and $m = |\mathcal{X}| - 1$. We want to prove that the multi-letter problem

$$\max_{L \in \mathbb{R}^{n \cdot |\mathcal{X}|} : \|L\|^2 = 1, \langle L, \underline{v}_0^{(n)} \rangle = 0} \min_{1 \leq i \leq k} \left\{ \left\| B_i^{(n)} L \right\|^2 \right\}, \tag{B.1}$$

has k-letter optimal solutions, where $\underline{v}_0^{(n)}$ is the n^{th} Kronecker product of $\underline{v}_0$. The key step is to show the following stronger result:

Lemma 4 (optimal solutions have product forms). *For any $n \geq k$, there exist $|\mathcal{X}|$-dimensional vectors $\underline{v}_1^*, \underline{v}_2^*, \ldots, \underline{v}_k^*$, which satisfy $\sum_{i=1}^k \|\underline{v}_i^*\|^2 = 1$, and $\langle \underline{v}_0, \underline{v}_i^* \rangle = 0$, for $i = 1, 2, \ldots, k$, such that*

$$L_n = \underline{v}_0^{(n-k)} \otimes \sum_{i=1}^{k} \left(\underline{v}_0^{(i-1)} \otimes \underline{v}_i^* \otimes \underline{v}_0^{(k-i)} \right)$$

is an optimal solution of (B.1).

Once we can prove Lemma 4, following the same arguments as section 2.2, we can construct optimal k-letter solutions. Now, to prove Lemma 4, we need to first establish the following Lemma 5, which illustrates the required degree of freedom for describing the optimal tradeoff between multiple linear systems.

Lemma 5. *Assume that* $\Theta_i = \operatorname{diag}\{\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,M}\}$*, for* $i = 1, 2, \ldots, k$*, if the optimization problem*

$$\begin{aligned} \max. \quad & \|\Theta_k \cdot \underline{c}\|^2 \\ \text{subject to:} \quad & \|\underline{c}\|^2 = 1, \text{ and } \|\Theta_i \cdot \underline{c}\|^2 = \lambda_i, \\ & \text{for } i = 1, 2, \ldots, k-1, \end{aligned} \tag{B.2}$$

has global optimal solutions, then there is a global optimal solution $\underline{c}^* = [c_1^* \; c_2^* \; \ldots \; c_M^*]^T$ *with at most* k *nonzero entries.*

Proof of Lemma 5. Let us assume that $\underline{c}^* = [c_1^* \; c_2^* \; \ldots \; c_M^*]^T$ is the global optimal solution of (B.2) with the least number of nonzero entries l. If $l > k$, then without loss of generality, we can assume that $c_i \neq 0$ for $i \leq l$, and $c_i = 0$ for $i > l$. For $i = 0, 1, \ldots, k$, define the vector $\underline{\theta}_{i,l} = \left[\theta_{i,1}^2 \; \theta_{i,2}^2 \; \ldots \; \theta_{i,l}^2\right]^T \in \mathbb{R}^l$, and $\underline{\theta}_{0,l} = [1 \; \ldots \; 1]^T \in \mathbb{R}^l$, then the null space $\operatorname{Null}\left(\underline{\theta}_{i,l}\right) \subset \mathbb{R}^l$ has dimension at least $l-1$, and the space of the intersecting of null spaces $\cup_{i=0}^{k-1} \operatorname{Null}\left(\underline{\theta}_{i,l}\right)$ has dimension at least $l - k > 0$. Let the nonzero vector $\underline{d} = [d_1 \; d_2 \; \ldots \; d_l]^T \in \cup_{i=0}^{k-1} \operatorname{Null}\left(\underline{\theta}_{i,l}\right)$, and $\underline{c}_t = [c_{1,t} \; \ldots \; c_{l,t} \; 0 \; \ldots \; 0]^T \in \mathbb{R}^m$, where $c_{i,t} = \sqrt{c_i^{*2} + t \cdot d_i}$. Since $\underline{c}^*$ is a global optimal, we have $\frac{\partial \|\Theta_k \cdot \underline{c}_t\|^2}{\partial t} = 0$, which implies that $\langle \underline{\theta}_{k,l}, \underline{d} \rangle = 0$. Thus, $\underline{c}_{t^*}$ is also a global optimal solution, where $t^* \triangleq \max\{t : c_i^{*2} + t \cdot d_i \geq 0, \; \forall 1 \leq i \leq l\}$. However, $\underline{c}_{t^*}$ has at most $l - 1$ nonzero entries, which contradicts to the assumption of $\underline{c}^*$. Therefore, $\underline{c}^*$ has at most k nonzero entries. $\square$

This immediately implies the following Corollary.

Corollary 2. *Assume that* $\Theta_i = \operatorname{diag}\{\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,M}\}$*, for* $i = 1, 2, \ldots, k$*, then the optimization problem*

$$\max_{\|\underline{c}\|^2 = 1} \min_{1 \leq i \leq k} \left\{\|\Theta_i \cdot \underline{c}\|^2\right\} \tag{B.3}$$

has a global optimal solution $\underline{c}^* = [c_1^* \ c_2^* \ \ldots \ c_M^*]^T$ *with at most* k *nonzero entries.*

Now, let us prove Lemma 4. We demonstrate here the proof of the case $k = 2$. The general cases can be proved by the same arguments

Proof of Lemma 4. Let $\sigma_0 = 1, \sigma_1, \ldots, \sigma_m$ and $\mu_0 = 1, \mu_1, \ldots, \mu_m$ be the singular values of the DTM's B_1 and B_2, respectively, and the corresponding right singular vectors are $\underline{v}_0, \underline{v}_1, \ldots, \underline{v}_m$ and $\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_m$, where $m \triangleq |\mathcal{X}| - 1$, and $\underline{v}_0 = \underline{u}_0 = \left[\sqrt{P_X}, x \in \mathcal{X}\right]^T$. Moreover, we assume that $\underline{v}_i = \sum_{k=0}^{m} \phi_{ik} \underline{u}_k$, where $\Phi_0 = [\phi_{ij}]_{i,j=0}^{m}$ is an $|\mathcal{X}|$-by-$|\mathcal{X}|$ unitary matrix. Then, we have $\phi_{00} = 1$, $\phi_{i0} = \phi_{0i} = 0$, for all $i > 0$. Suppose that

$$\begin{aligned} L &= \sum_{\substack{i_1, \ldots, i_n = 0 \\ (i_1, \ldots, i_n) \neq (0, \ldots, 0)}}^{m} \alpha_{i_1 \cdots i_n} \cdot \left(\underline{v}_{i_1} \otimes \cdots \otimes \underline{v}_{i_n}\right) \\ &= \sum_{j_1, \ldots, j_n = 0}^{m} \beta_{j_1 \cdots j_n} \cdot \left(\underline{u}_{j_1} \otimes \cdots \otimes \underline{u}_{j_n}\right), \end{aligned}$$

is an optimal solution of (B.1), where

$$\beta_{j_1 \cdots j_n} = \sum_{\substack{i_1, \ldots, i_n = 0 \\ (i_1, \ldots, i_n) \neq (0, \ldots, 0)}}^{m} \alpha_{i_1 \cdots i_n} \phi_{i_1 j_1} \cdots \phi_{i_n j_n}.$$

Then, since $\|L\| = 1$, we have

$$\sum_{\substack{i_1, \ldots, i_n = 0 \\ (i_1, \ldots, i_n) \neq (0, \ldots, 0)}}^{m} \|\alpha_{i_1 \cdots i_n}\|^2 = \sum_{j_1, \ldots, j_n = 0}^{m} \|\beta_{j_1 \cdots j_n}\|^2 = 1.$$

Now, let us define

$$\mathcal{S}_k = \{(i_1, \ldots, i_n) : 0 \leq i_a \leq m, \ \forall \ 1 \leq a \leq n, i_k \neq 0, \ i_{k+1} = \cdots = i_n = 0\},$$

$$\mathcal{T}_k = \left\{(i_1, \ldots, i_k) : 0 \leq i_a \leq m, \ \forall \ 1 \leq a \leq k\right\},$$

with $\mathcal{T}_0 = \emptyset$, $\mathcal{T} = \cup_{k=0}^{n-1} \mathcal{T}_k$, and

$$\xi : \mathcal{T} \mapsto \{1, 2, \ldots, M\}$$

is a bijective map, where $M = |\mathcal{T}|$. Then,

$$\begin{aligned}
\left\| B_1^{(n)} \cdot L \right\|^2 &= \sum_{\substack{i_1,\ldots,i_n=0 \\ (i_1,\ldots,i_n)\neq(0,\ldots,0)}}^{m} \alpha_{i_1\cdots i_n}^2 \sigma_{i_1}^2 \cdots \sigma_{i_n}^2 \\
&\leq \sum_{k=1}^{n} \sum_{(i_1,\ldots,i_n)\in\mathcal{S}_k} \alpha_{i_1\cdots i_n}^2 \sigma_{i_k}^2 \\
&= \sum_{k=1}^{n} \sum_{(i_1,\ldots,i_{k-1})\in\mathcal{T}_{k-1}} \sum_{i_k=1}^{m} \alpha_{i_1\cdots i_{k-1} i_k 0\ldots 0}^2 \sigma_{i_k}^2 \\
&= \sum_{k=1}^{n} \sum_{(i_1,\ldots,i_{k-1})\in\mathcal{T}_{k-1}} \|\Sigma \cdot \underline{\alpha}_{\xi(i_1,\ldots,i_{k-1})}\|^2 \\
&= \sum_{i=1}^{M} \|\Sigma \cdot \underline{\alpha}_i\|^2,
\end{aligned}$$

where $\Sigma = \operatorname{diag}\{\sigma_1, \ldots, \sigma_m\}$, and $\underline{\alpha}_{\xi(i_1,\ldots,i_{k-1})}$ is defined as

$$\underline{\alpha}_{\xi(i_1,\ldots,i_{k-1})} = \left[\alpha_{i_1\cdots i_{k-1}10\cdots0} \; \alpha_{i_1\cdots i_{k-1}20\cdots0} \cdots \alpha_{i_1\cdots i_{k-1}m0\cdots0}\right]^T.$$

Moreover,

$$\begin{aligned}
&\left\| B_2^{(n)} \cdot L \right\|^2 \\
&= \sum_{j_1,\ldots,j_n=0}^{m} \beta_{j_1\cdots j_n}^2 \mu_{j_1}^2 \cdots \mu_{j_n}^2 \\
&= \sum_{j_1,\ldots,j_n=0}^{m} \mu_{j_1}^2 \cdots \mu_{j_n}^2 \cdot \left(\sum_{k=1}^{n} \sum_{(i_1,\ldots,i_n)\in\mathcal{S}_k} \alpha_{i_1\cdots i_n} \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \right)^2 \\
&= \sum_{k=1}^{n} \sum_{j_1,\ldots,j_n=0}^{m} \mu_{j_1}^2 \cdots \mu_{j_n}^2 \cdot \left(\sum_{(i_1,\ldots,i_n)\in\mathcal{S}_k} \alpha_{i_1\cdots i_n} \phi_{i_1 j_1} \cdots \phi_{i_n j_n} \right)^2, \qquad \text{(B.4)}
\end{aligned}$$

where (B.4) is because for any $1 \leq k_1 < k_2 \leq n$,

$$\begin{aligned}
&\sum_{j_1,\dots,j_n=0}^{m} \left(\sum_{(i_1,\dots,i_n)\in\mathcal{S}_{k_1}} \alpha_{i_1\cdots i_n}\phi_{i_1j_1}\cdots\phi_{i_nj_n} \right) \cdot \left(\sum_{(i_1,\dots,i_n)\in\mathcal{S}_{k_2}} \alpha_{i_1\cdots i_n}\phi_{i_1j_1}\cdots\phi_{i_nj_n} \right) \mu_{j_1}^2\cdots\mu_{j_n}^2 \\
&= \sum_{j_1,\dots,j_n=0}^{m} \sum_{\substack{(i_1,\dots,i_{k_1},0\dots,0)\in\mathcal{S}_{k_1} \\ (i'_1,\dots,i'_{k_2},0\dots,0)\in\mathcal{S}_{k_2}}} \alpha_{i_1\cdots i_{k_1}0\cdots 0}\alpha_{i'_1\cdots i'_{k_2}0\cdots 0} \cdot \left(\phi_{i_1j_1}\phi_{i'_1j_1}\right)\cdots \underbrace{\left(\phi_{0j_{k_2}}\phi_{i'_{k_2}j_{k_2}}\right)}_{=0,\ \text{since } i'_{k_2}\neq 0} \mu_{j_1}^2\cdots\mu_{j_n}^2 \\
&=0.
\end{aligned}$$

Then, let $\Omega = \operatorname{diag}\{\mu_1,\dots,\mu_m\}$ and $\Phi = [\phi_{ij}]_{i,j=1}^m$, for each k, we have

$$\begin{aligned}
&\sum_{j_1,\dots,j_n=0}^{m} \left(\sum_{(i_1,\dots,i_n)\in\mathcal{S}_k} \alpha_{i_1\cdots i_n}\phi_{i_1j_1}\cdots\phi_{i_nj_n} \right)^2 \mu_{j_1}^2\cdots\mu_{j_n}^2 \\
&\leq \sum_{j_1,\dots,j_n=0}^{m} \left(\sum_{(i_1,\dots,i_n)\in\mathcal{S}_k} \alpha_{i_1\cdots i_n}\phi_{i_1j_1}\cdots\phi_{i_nj_n} \right)^2 \mu_{j_k}^2 \\
&= \sum_{j_1,\dots,j_n=0}^{m} \sum_{(i_1,\dots,i_n)\in\mathcal{S}_k} \sum_{(i'_1,\dots,i'_n)\in\mathcal{S}_k} \alpha_{i_1\cdots i_n}\alpha_{i'_1\cdots i'_n} \cdot \left(\phi_{i_1j_1}\phi_{i'_1j_1}\right)\cdots\left(\phi_{i_nj_n}\phi_{i'_nj_n}\right)\mu_{j_k}^2 \\
&= \sum_{j_k=0}^{m} \sum_{(i_1,\dots,i_{k-1})\in\mathcal{T}_{k-1}} \left(\sum_{i_k=1}^{m} \alpha_{i_1\cdots i_k 0\cdots 0}\phi_{i_kj_k} \right)^2 \mu_{jk}^2 \qquad \text{(B.5)} \\
&= \sum_{(i_1,\dots,i_{k-1})\in\mathcal{T}_{k-1}} \|\Omega\Phi^T \underline{\alpha}_{\xi(i_1,\dots,i_{k-1})}\|^2,
\end{aligned}$$

where (B.5) is because Φ is a unitary matrix, and

$$\sum_{j_r=0}^{m} \phi_{i_rj_r}\phi_{i'_rj_r} = \begin{cases} 1 & \text{if } i_r = i'_r \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, (B.4) becomes

$$\left\| B_2^{(n)} \cdot L \right\|^2 \leq \sum_{k=1}^{n} \sum_{(i_1,\dots,i_{k-1})\in\mathcal{T}_{k-1}} \|\Omega\Phi^T \underline{\alpha}_{\xi(i_1,\dots,i_{k-1})}\|^2 = \sum_{i=1}^{M} \|\Omega\Phi^T\underline{\alpha}_i\|^2.$$

Now, let us define $\Theta_1 = \text{diag}\{\theta_{1,1}, \ldots, \theta_{1,M}\}$, and $\Theta_2 = \text{diag}\{\theta_{2,1}, \ldots, \theta_{2,M}\}$ as

$$\theta_{1,i} = \begin{cases} \frac{\|\Sigma \underline{\alpha}_i\|}{\|\underline{\alpha}_i\|} & \text{if } \|\underline{\alpha}_i\| \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\theta_{2,i} = \begin{cases} \frac{\|\Omega \Phi^T \underline{\alpha}_i\|}{\|\Phi^T \cdot \underline{\alpha}_i\|} & \text{if } \|\Phi^T \cdot \underline{\alpha}_i\| \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then, from Corollary 2, there exists an optimal solution $\underline{c}^*$ of the optimization problem

$$\max_{\underline{c} \in \mathbb{R}^M : \|\underline{c}\|^2 = 1} \min\left\{\|\Theta_1 \cdot \underline{c}\|^2, \|\Theta_2 \cdot \underline{c}\|^2\right\}, \tag{B.6}$$

with at most two nonzero entries. Let the i_1-th entry $c^*_{i_1}$ and the i_2-th entry $c^*_{i_2}$ of $\underline{c}^*$ are nonzero. Note that $\|\underline{\alpha}_i\| = \|\Phi^T \cdot \underline{\alpha}_i\|$, for all $1 \leq i \leq M$, and

$$\sum_{i=1}^{M} \|\underline{\alpha}_i\|^2 = \sum_{\substack{i_1, \ldots, i_n = 0 \\ (i_1, \ldots, i_n) \neq (0, \ldots, 0)}}^{m} \|\alpha_{i_1 \cdots i_n}\|^2 = 1,$$

thus the vector $\underline{\alpha} = [\|\underline{\alpha}_1\| \; \|\underline{\alpha}_2\| \; \ldots \; \|\underline{\alpha}_M\|]^T$ has unit norm. This implies that

$$\begin{aligned}
\min\left\{\|\Theta_1 \cdot \underline{c}^*\|^2, \|\Theta_2 \cdot \underline{c}^*\|^2\right\} &\geq \min\left\{\|\Theta_1 \cdot \underline{\alpha}\|^2, \|\Theta_2 \cdot \underline{\alpha}\|^2\right\} \\
&= \min\left\{\sum_{i=1}^{M} \|\Sigma \underline{\alpha}_i\|^2, \sum_{i=1}^{M} \|\Omega \Phi^T \underline{\alpha}_i\|^2\right\} \\
&= \min\left\{\|B_1^{(n)} \cdot L\|^2, \|B_2^{(n)} \cdot L\|^2\right\}.
\end{aligned}$$

Now, let us take vectors $\underline{v}_1^* = c^*_{i_1} \cdot \sum_{j=1}^{m} \frac{\underline{\alpha}_{i_1}(j)}{\|\underline{\alpha}_{i_1}\|} \underline{v}_j$, and $\underline{v}_2^* = c^*_{i_2} \cdot \sum_{j=1}^{m} \frac{\underline{\alpha}_{i_2}(j)}{\|\underline{\alpha}_{i_2}\|} \underline{v}_j$, where $\underline{\alpha}_{i_1}(j)$ and $\underline{\alpha}_{i_2}(j)$ are the j-th entries of $\underline{\alpha}_{i_1}$ and $\underline{\alpha}_{i_2}$, respectively. Then, the vector $L_2 = \underline{v}_0^{(n-2)} \otimes (\underline{v}_1^* \otimes \underline{v}_0 + \underline{v}_0 \otimes \underline{v}_2^*)$ satisfies $\|L_2\| = 1$, and

$$\begin{aligned}
\min\left\{\|B_1^{(n)} \cdot L_2\|^2, \|B_2^{(n)} \cdot L_2\|^2\right\} &= \min\left\{\|\Theta_1 \cdot \underline{c}^*\|^2, \|\Theta_2 \cdot \underline{c}^*\|^2\right\} \\
&\geq \min\left\{\|B_1^{(n)} \cdot L\|^2, \|B_2^{(n)} \cdot L\|^2\right\}.
\end{aligned}$$

Since L is an optimal solution of (B.1), L_2 is also an optimal solution, with the product form. $\square$

Now, let us prove statement (ii), i.e., when $k = 2$, (B.1) has optimal single-letter solutions. To this end, we assume that $L_2 = \underline{v}_1 \otimes \underline{v}_0 + \underline{v}_0 \otimes \underline{v}_2$ is an optimal solution, where $\underline{v}_1 = \sum_{j=1}^{m} a_j \underline{v}_j$, $\underline{v}_2 = \sum_{j=1}^{m} b_j \underline{v}_j$, and $\|\underline{v}_1\|^2 + \|\underline{v}_1\|^2 = 1$. Let $\underline{a} = [a_1 \ \ldots \ a_m]^T$ and $\underline{b} = [b_1 \ \ldots \ b_m]^T$, then,

$$\min \left\{ \|B_1^{(2)} \cdot L_2\|^2, \|B_2^{(2)} \cdot L_2\|^2 \right\} = \min \left\{ \|\Sigma \underline{a}\|^2 + \|\Sigma \underline{b}\|^2, \|\Omega \Phi^T \underline{a}\|^2 + \|\Omega \Phi^T \underline{b}\|^2 \right\}$$

Our goal is to show that there exists a unit vector $\underline{c} = [c_1 \ \ldots \ c_m]^T$ such that

$$\begin{cases} \|\Sigma \underline{c}\|^2 \geq \|\Sigma \underline{a}\|^2 + \|\Sigma \underline{b}\|^2, \\ \|\Omega \Phi^T \underline{c}\|^2 \geq \|\Omega \Phi^T \underline{a}\|^2 + \|\Omega \Phi^T \underline{b}\|^2. \end{cases} \tag{B.7}$$

For $m = 2$, we consider the vectors $\underline{c}_1 = [\sqrt{a_1^2 + b_1^2} \ \ \sqrt{a_2^2 + b_2^2}]^T$, and $\underline{c}_2 = [\sqrt{a_1^2 + b_1^2} \ \ -\sqrt{a_2^2 + b_2^2}]^T$. Obviously, $\|\Sigma \underline{c}_1\|^2 = \|\Sigma \underline{c}_2\|^2 = \|\Sigma \underline{a}\|^2 + \|\Sigma \underline{b}\|^2$. Note that

$$\begin{aligned} &\left(\|\Omega \Phi^T \underline{a}\|^2 + \|\Omega \Phi^T \underline{b}\|^2 - \|\Omega \Phi^T \underline{c}_1\|^2\right) \cdot \left(\|\Omega \Phi^T \underline{a}\|^2 + \|\Omega \Phi^T \underline{b}\|^2 - \|\Omega \Phi^T \underline{c}_2\|^2\right) \\ = &- 4 \left(a_1 b_2 - b_1 a_2\right)^2 \cdot \left(\phi_{11} \phi_{21} \mu_1^2 + \phi_{12} \phi_{22} \mu_2^2\right)^2 \leq 0 \end{aligned}$$

Therefore, at least one of $\underline{c}_1$ and $\underline{c}_2$ satisfies (B.7).

For $m > 2$, let $A = [\underline{a} \ \underline{b}]$, and consider the SVD of matrices $\Sigma A = U_1^T \Sigma_1 V_1$, and $\Omega \Phi^T A = U_2^T \Sigma_2 V_2$, where V_i is a 2×2 unitary matrix, and Σ_i is a diagonal matrix, for $i = 1, 2$. Moreover, we denote $V_1 = [\varphi_1 \ \varphi_2]$, and $V_2 = [\psi_1 \ \psi_2]$, where φ_i and ψ_i are all two dimensional unit vectors. Then, $\|\Sigma \underline{a}\|^2 = \|\Sigma_1 \varphi_1\|^2$, $\|\Sigma \underline{b}\|^2 = \|\Sigma_1 \varphi_2\|^2$, and $\|\Omega \Phi^T \underline{a}\|^2 = \|\Sigma_2 \left(V_2 V_1^{-1}\right) \psi_1\|^2$, $\|\Omega \Phi^T \underline{b}\|^2 = \|\Sigma_2 \left(V_2 V_1^{-1}\right) \psi_2\|^2$. Since $V_2 V_1^{-1}$ is a unitary matrix, there exists a two dimensional unit vector $\underline{c}'$, such that

$$\begin{cases} \|\Sigma_1 \underline{c}'\|^2 \geq \|\Sigma_1 \varphi_1\|^2 + \|\Sigma_1 \varphi_2\|^2, \\ \|\Sigma_2 \left(V_2 V_1^{-1}\right) \underline{c}'\|^2 \geq \|\Sigma_2 \left(V_2 V_1^{-1}\right) \psi_1\|^2 + \|\Sigma_2 \left(V_2 V_1^{-1}\right) \psi_2\|^2. \end{cases}$$

Now, taking $\underline{c} = AV_1^{-1}\underline{c}'$, then $||\Sigma\underline{c}||^2 = ||\Sigma_1\underline{c}'||^2$, and $||\Omega\Phi^T\underline{c}||^2 = ||\Sigma_2\left(V_2V_1^{-1}\right)\underline{c}'||^2$, which implies that $\underline{c}$ satisfies (B.7). Thus, the unit vector $L_1 = \sum_{i=1}^{m} c_i\underline{v}_i$ satisfies $||L_1||^2 = 1$, and

$$\min\left\{||B_1L_1||^2, ||B_2L_1||^2\right\} \geq \min\left\{||B_1^{(2)} \cdot L_2||^2, ||B_2^{(2)} \cdot L_2||^2\right\}.$$

Therefore, L_1 is a single-letter optimal solution for (B.1).

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