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ABSTRACT: The Hodge numbers of generic elliptically fibered Calabi-Yau threefolds over toric base surfaces fill out the "shield" structure previously identified by Kreuzer and Skarke. The connectivity structure of these spaces and bounds on the Hodge numbers are illuminated by considerations from F-theory and the minimal model program. In particular, there is a rigorous bound on the Hodge number $h_{21} \leq 491$ for any elliptically fibered Calabi-Yau threefold. The threefolds with the largest known Hodge numbers are associated with a sequence of blow-ups of toric bases beginning with the Hirzebruch surface $\mathbb{F}_{12}$ and ending with the toric base for the F-theory model with largest known gauge group.

Contents

1. Introduction

Since Calabi-Yau threefolds were first identified as key geometries for superstring compactification to four dimensions [1], the classification of manifolds of this type has been widely studied by string theorists and mathematicians (See [2] for an introduction to the subject.) It is still not known whether there are a finite number of distinct topological classes of Calabi-Yau threefolds, or if the Hodge numbers of such manifolds are bounded. Toric geometry provides a powerful tool for describing certain classes of Calabi-Yau manifolds. Using a construction of Batyrev [3], Kreuzer and Skarke [4] produced a comprehensive list of some 473.8 million examples of families of Calabi-Yau manifolds associated with four-dimensional reflexive polytopes. These examples include manifolds with 30,108 distinct pairs of Hodge numbers. Graphed on a scatter plot, these Hodge numbers take the famous "shield" shape (Figure 1). While other classes of Calabi-Yau manifolds have since been constructed (see for example [6, 7], [8] for a review), they give Hodge numbers that fit within this same general shape. The boundary of the set of allowed Hodge numbers has not yet been explained in any systematic way.

Recently, D. Morrison and the author used an alternative approach to construct a large class of Calabi-Yau threefolds [9, 10]. Motivated by F-theory considerations, we systematically analyzed the set of surfaces that can support elliptically fibered Calabi-Yau threefolds. These base surfaces are all connected in a complicated network through transitions associated with blowing up and down points in the surface. From the mathematics of the minimal model program, all such bases (aside from the trivial example of the Enriques surface, which is connected in a more complicated way) can be found by blowing up a series of points on $\mathbb{P}^2$ or the Hirzebruch surfaces $\mathbb{F}_m$ [11, 12]. By analyzing the intersection structure of irreducible effective divisors on the base surface, in [9] we identified specific
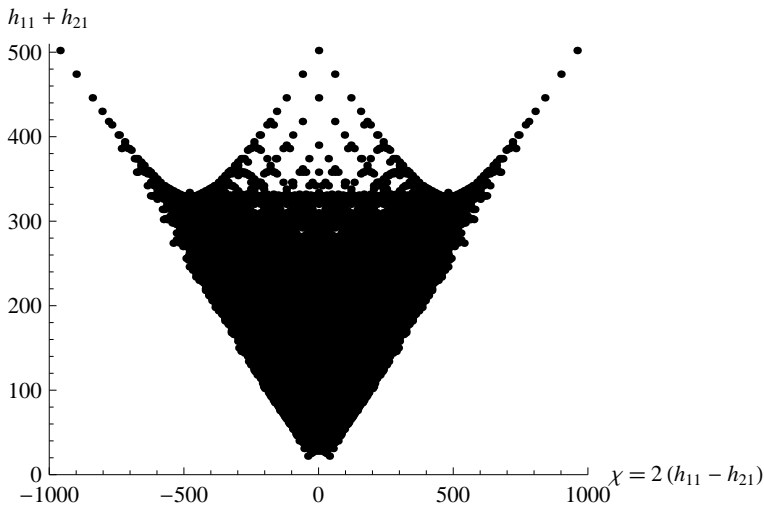
Figure 1: The 30,108 distinct Hodge numbers of the 473.8 million Calabi-Yau threefolds identified by Kreuzer and Skarke. Data from [5].

geometric structures ("non-Higgsable clusters" of divisors) that characterize the geometry of elliptically fibered Calabi-Yau threefolds. In particular, in [10] we used this approach to construct all smooth toric bases that support elliptic fibrations with section where the total space is a Calabi-Yau threefold. There are 61,539 such toric bases. While the approach taken in [10] is also based in toric geometry, the analysis is simplified from that of work such as [4] by the focus on the geometry of the base. The analysis of [9] is also applicable to a systematic analysis of non-toric bases and Calabi-Yau threefolds.

In this paper we consider the Hodge structure of generic Calabi-Yau threefolds over the 61,539 bases constructed in [10]. We find that the Hodge numbers h_{11}, h_{21} of these manifolds are distributed throughout the "shield" region identified in [4]. The geometry of the base surfaces connects all these Calabi-Yau manifolds into a connected web, in accordance with the conjecture of Reid [13]. Furthermore, the structure of the bases gives a clear geometric understanding of the outer boundary of the shield region. The top boundary of the region of known allowed Hodge numbers roughly follows a specific trajectory of blow-ups of the base $\mathbb{F}_{12}$ that terminates in the Calabi-Yau associated with the F-theory model having the largest gauge group among models in this class. Although the analysis in this paper focuses on generic elliptic fibrations over toric bases, a much larger class of elliptically fibered Calabi-Yau manifolds can be realized by tuning Weierstrass moduli, as in F-theory constructions, to produce singular elliptic fibrations that are then resolved. Again, though we focus primarily here on toric bases, the methods of [9] suggest that similar results may hold for elliptically fibered Calabi-Yau threefolds over non-toric bases.

In Section 2 we describe the Hodge structure of the generic elliptic fibrations over the bases identified in [10]. We characterize the bounds on the region of allowed Hodge numbers in Section 3. In Section 4 we make some brief comments on extensions of this analysis to non-generic elliptic fibrations, non-toric bases, and non-elliptically fibered threefolds.

Cluster	gauge algebra	H_{charged}	Possible connected clusters
(-12)	$\mathfrak{e}_8$	0	(-2, -2, -3) or below
(-8)	$\mathfrak{e}_7$	0	(-2, -3, -2) or below
(-7)	$\mathfrak{e}_7$	28	(-2, -3, -2) or below
(-6)	$\mathfrak{e}_6$	0	(-3) or below
(-5)	$\mathfrak{f}_4$	0	(-3, -2, -2) or below
(-4)	$\mathfrak{so}(8)$	0	(-4) or below
(-3, -2, -2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	8	any cluster
(-3, -2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	8	(-8) or below
(-3)	$\mathfrak{su}(3)$	0	(-6) or below
(-2, -3, -2)	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	16	(-8) or below
(-2, -3)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	8	(-5) or below
(-2, -2, -3)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	8	(-5) or below
(-2, -2, …, -2)	no gauge group	0	any cluster

Table 1: Allowed clusters and connections between clusters by -1 curves in a toric surface that can be used as the base of an elliptic fibration. For each cluster, the table indicates the resulting contribution to the gauge algebra, the charged matter content, and the set of clusters that can follow the first cluster after a -1 curve, where "or below" refers to the order of clusters in this table. Note that the clusters $(-3, -2, -2)$ and $(-3, -2)$ are ordered; for example, a -12 can be connected by a -1 curve to the final -2 of the cluster $(-3, -2, -2)$ but not to the -3 curve. For clarity these clusters are listed in both directions.

2. Hodge structure

In [9, 10] we classified the 61,539 smooth toric compact complex surfaces that can support an elliptically fibered Calabi-Yau threefold. We do not repeat the analyses of those papers here, but follow the notation of those papers and briefly summarize some of the salient results here. Each toric base is described by a fan associated with a closed loop of k divisors $D_1, \ldots D_k$ with self-intersection $D_i \cdot D_i = -n_i$ and nonvanishing intersection between adjacent divisors $D_i \cdot D_{i+1} = D_k \cdot D_1 = 1$. The sequence of self-intersection numbers can only contain specific subsequences ("clusters") of self-intersections -2 or below. Only certain combinations of clusters can be connected by -1 curves; a list of allowed clusters and connections appears in Table 1

The allowed toric base surfaces are all realized by blowing up a succession of points on either $\mathbb{P}^2$ or $\mathbb{F}_m, m \leq 12$. Some of the 61,539 bases are not strictly toric in that they arise from toric bases with curves of self-intersection $-9, -10$, or -11 that are blown up at non-toric points to give -12 curves, but these bases have very similar behavior to the true toric bases and we include these in the class of toric bases for the purposes of this paper.

In [10], the toric bases were analyzed in the context of F-theory [14, 15, 12] compactifications to 6 dimensions. Though the results that we describe here for elliptically fibered Calabi-Yau manifolds are independent of the physics of F-theory, the tools and perspective provided by F-theory and the minimal model program for classification of surfaces are very helpful in illuminating the structure of these elliptic fibrations. In the six-dimensional

supergravity theory produced by an F-theory compactification on an elliptically fibered Calabi-Yau threefold, gravitational anomaly cancellation relates the numbers of tensor, vector, and hypermultiplet fields in the 6D theory through [16, 17, 18] (see [19] for a review of 6D supergravity theories, anomalies, and F-theory compactifications).

$$H - V = 273 - 29T \,. \tag{2.1}$$

For each base, the numbers of these types of fields, as well as the rank r of the gauge group are determined from the toric data. In particular, $T = k - 3$, where k is the number of curves in the fan of the base, and the number of neutral hypermultiplets is related to the number of Weierstrass monomials, after taking proper account of automorphisms of the base and degrees of freedom associated with -2 curves not carrying a gauge group. The gauge group and charged matter content are determined by the set of clusters as described in Table 1, with for example a single -12 curve corresponding to an $\mathfrak{e}_8$ component in the gauge algebra. The toric data also allows for a determination of the Hodge numbers h_{11}, h_{21} of the generic elliptically fibered threefold X over each base B, where

$$h_{11}(X) = r + T + 2 = r + k - 1 \tag{2.2}$$

and

$$h_{21}(X) = H_{\text{neutral}} - 1 = 272 + V - 29T - H_{\text{charged}} \,. \tag{2.3}$$

in terms of the fields in the corresponding supergravity model from F-theory.

The 7524 distinct Hodge number combinations for the elliptically fibered Calabi-Yau threefolds over the 61,539 toric bases are plotted in Figure 2. These Hodge numbers exhibit the same "shield" pattern seen in the Kreuzer and Skarke data. In fact, the Hodge data from toric bases is a proper subset of the set of Hodge numbers from the Kreuzer and Skarke list. It is not surprising that the generic elliptic fibrations over toric bases all appear in the Kreuzer and Skarke list. What is perhaps more surprising is that the smaller list from the set of toric bases extends through the same general range of Hodge numbers realized on the larger list, and in particular contains all of the larger combinations of Hodge numbers near the upper limits of the range found by Kreuzer and Skarke. The list of threefolds from toric bases does not extend as close to the origin, however, in the region of small Hodge numbers. Note that while the Hodge number pairs appearing in the Kreuzer and Skarke list are invariant under mirror symmetry, which exchanges h_{11}, h_{21}, the Hodge numbers for fibrations over toric bases do not have this symmetry. In many cases there are several distinct families of Calabi-Yau manifolds in Kreuzer and Skarke's list that have a given set of Hodge numbers. The fibration structure of these manifolds can be analyzed, for example by using the PALP software package [20], to identify which toric hypersurface model corresponds to a fibration over a given toric base.

As mentioned above, all 61,539 of the Calabi-Yau threefolds constructed in this fashion are connected through extremal transitions associated with blowing up and down points on the base. In F-theory these transitions are associated with tensionless string transitions [21, 12]. The number of blow-ups from $\mathbb{P}^2$, or one less than the number of blow-ups from $\mathbb{F}_m$ corresponds to the number of tensor multiplets T in the corresponding F-theory model. The
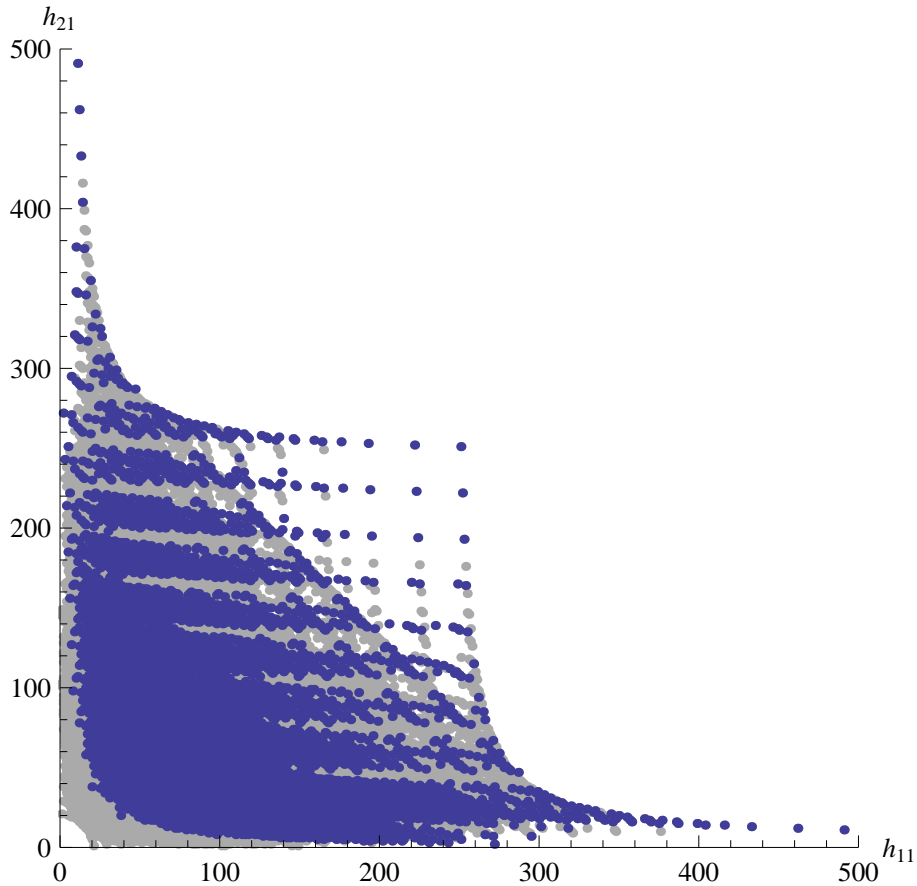
Figure 2: The 7524 distinct Hodge number pairs for generic elliptically fibered Calabi-Yau threefolds over toric bases (dark/blue data points). Plot axes are Hodge numbers h_{11}, h_{21}. Kreuzer-Skarke Hodge pairs are shown in background in light gray for comparison.

parameter $T = k-3$ is useful in characterizing the complexity of the base. The models with $T = 0, 1$ are $\mathbb{P}^2$ and the Hirzebruch surfaces, and all lie on the left-hand side of the diagram, ranging from $\mathbb{F}_0, \mathbb{F}_1$, and $\mathbb{F}_2$, which all have Hodge numbers $(h_{11}, h_{21}) = (3, 243)$, and $\mathbb{P}^2$ with Hodge numbers $(2, 272)$ to $\mathbb{F}_{12}$ with Hodge numbers $(11, 491)$. As more points are blown up, T increases, as does the rank of the gauge group, so h_{11} monotonically increases. At the same time, h_{21} monotonically decreases along any blow-up sequence. The change in h_{21} denotes the number of free parameters that must be tuned in the Weierstrass model over a given base to effect a blow-up. Note that the monotonic increase in h_{11} and decrease in h_{21} is true for any sequence of blow-up operations on the base, whether or not the base is toric.

3. Bounds

The shape of the upper bound on Hodge numbers in the "shield" configuration has been noted in previous work, but, as far as the author of this paper knows, never explained.

From the point of view of elliptic fibrations over toric bases, however, the upper boundary on allowed Hodge numbers has a simple and relatively clear interpretation.

We begin by considering the elliptically fibered Calabi-Yau threefolds with the largest values of h_{21}. Since h_{21} decreases with any blow-up of the base (toric or not), the largest values of this parameter will appear for bases with a minimal value of T. These are the Hirzebruch surfaces $\mathbb{F}_m$, which (except for $\mathbb{F}_1$) do not contain -1 curves that can be further blown down. From (2.3) it is clear that the largest value of h_{21} will appear when the gauge group is largest. This occurs for the base $\mathbb{F}_{12}$, where the Hodge numbers associated with the corresponding Calabi-Yau threefold are $(11, 491)$, since the non-Higgsable gauge algebra in the corresponding 6D F-theory model is $\mathfrak{e}_8$ with $V = 248$, and there is no charged matter. This gives a rigorous bound $h_{21} \leq 491$ for any Calabi-Yau threefold that admits an elliptic fibration (independent of whether the base is toric). Indeed, $(11, 491)$ is the Hodge number pair with the largest value of h_{21} in both the toric base and Kreuzer-Skarke lists, suggesting that this bound may hold even outside the class of elliptically fibered Calabi-Yau threefolds.

Going to the other corner of the allowed region, we consider threefolds with the largest values of h_{11}. As described in [10], the largest value of T arising from a threefold associated with a toric base is $T = 193$. The associated 6D gravity theory has a gauge algebra containing 17 $\mathfrak{e}_8$ summands, 16 $\mathfrak{f}_4$ summands, and 32 $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ summands, and was originally identified in [22, 23]. The chain of self-intersections of the divisors in the toric base is essentially 16 repeated copies of the pattern

$$\ldots, -12, -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1, -12, \ldots \qquad (3.1)$$

with a 0 self-intersection curve connecting to the two ends. The actual toric base has -11's in the next-to-last positions on each side that must be blown up at additional non-toric points, as discussed in [10]. We denote this sequence by the shorthand

$$(-12 // -11 // (-12)^{13} // -11 // -12, 0) \qquad (3.2)$$

where the double slash denotes a connection between the adjacent curves by the sequence of curves connecting the two -12's in the pattern (3.1), and $(-12)^{13}$ indicates 13 12's connected by 12 copies of this pattern. The Hodge numbers associated with the generic threefold over this surface are $(491, 11)$, so this Calabi-Yau manifold should be related to the generic threefold over $\mathbb{F}_{12}$ by mirror symmetry, as noted in [22]. The maximum value of $h_{11} = 491$ is compatible with mirror symmetry and the bound $h_{21} \leq 491$ given above for elliptically fibered Calabi-Yau threefolds.

Now, we consider the shape of the upper boundary of the shield region, which in Figure 2 describes the maximum value of h_{21} that can be realized as a function of h_{11}. For threefolds over toric bases, the sequence of extreme values along this curve are determined in a simple way by the sequence of blow-ups, starting from the base $\mathbb{F}_{12}$, that maximize the increase in h_{11} as h_{21} decreases, culminating in the base (3.2). In the F-theory picture, this trajectory is followed by increasing the dimension of the gauge group (minus the number of charged matter fields) as quickly as possible when blowing up points on the base. In situations not involving an increase in the gauge group or change in matter content, a

blow-up on the base trades 29 neutral hypermultiplet moduli for one tensor multiplet, as can be seen from (2.1). For example, blowing up one, two, three, or four toric points on the base $\mathbb{F}_{12}$ cannot produce a new gauge group or matter and must lead to changes in the Hodge numbers of $\Delta h_{11} = +1, \Delta h_{21} = -29$. Some of the configurations in the sequence of toric bases realized by blowing up points on $\mathbb{F}_{12}$ are listed in Table 1, along with the resulting Hodge numbers. For the first four blow-ups, there is no way to increase the gauge group, so h_{21} drops by 29 for each blow-up and h_{11} increases by one. The first three pairs of Hodge numbers $(11, 491), (12, 462)$, and $(13, 433)$ in this sequence have the largest values of h_{21} not only in the toric base data set but also in the Kreuzer and Skarke data set (Figure 3). This explains the slope of -29 of the outer curve of the bounding region at the tip. Note that there are two distinct toric constructions in the Kreuzer and Skarke data set with Hodge numbers $(12, 462)$, and four with $(13, 433)$. Using PALP, it is easy to check that, for example, one of the $(12, 462)$ cases corresponds to a non-generic elliptic fibration over $\mathbb{F}_{12}$ as expected. On the fifth blow-up, it is possible to produce a base associated with a gauge group $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ and non-Higgsable matter. This gives a threefold with Hodge numbers $(19, 355)$. At this point the slope of the bounding curve becomes less steep, as further gauge groups can be added with additional blow-ups.

From the threefold with Hodge numbers $(13, 433)$ there is a chain of threefolds over toric bases that roughly follows the upper boundary of the region of allowed Hodge numbers in the Kreuzer-Skarke database between $h_{11} = 13$ and $h_{11} = 150$. The bases associated with these threefolds are realized by further blow-ups of $\mathbb{F}_{12}$ that maximize the size of the gauge group (with matter subtracted). Note that in this range the threefolds with toric bases are not uniformly at the absolute boundary of the allowed region. The Kreuzer-Skarke database includes some Hodge pairs that are slightly above those realized by generic threefolds over toric bases in this region. For example, the Kreuzer-Skarke data includes a threefold with Hodge numbers $(14, 416)$ though no toric base gives a generic threefold with these Hodge numbers (though $(416, 14)$ does appear in the toric base data, which is not mirror symmetric). We return to this example in Section 4.1

Near the region of the central peak in the "shield," the data from generic threefolds over toric bases again contains all the Hodge pairs realized in the Kreuzer-Skarke database (see Figure 4). In particular, for $h_{11} > 160$ and $h_{21} > 180$, all the points on the upper boundary of the region are realized by the sequence of blow-ups of $\mathbb{F}_{12}$ mentioned above. Part of this sequence of bases is given in Table 2, where two separate sequences of blow-ups connect the bases with $T = 60, 65, 75$ and $T = 60, 64, 75$. Note that the bases in the sequence with $T = 64, 75, 86, 97$ are connected in order by combinations of 11 blow-ups at each stage. When, for example, the point at the intersection of the -10 and following -1 curves is blown up in the base with $T = 75$, the sequence becomes $-11, -1, -2, -2, -2, -3, \ldots$. As discussed in [9] this base does not support an elliptic fibration, the point between the -2 and -3 curves leaves the Kodaira classification and must be blown up, leading to further blow-ups. In the toric language this is easy to see from the dual polytope to the toric fan, as discussed in [10]; the additional blow-ups are along curves where sections $f \in -4K, g \in -6K$ must vanish to degrees 4, 6 once the first blow up is performed. The "point" of the central peak is associated with Hodge numbers $(251, 251)$. The associated
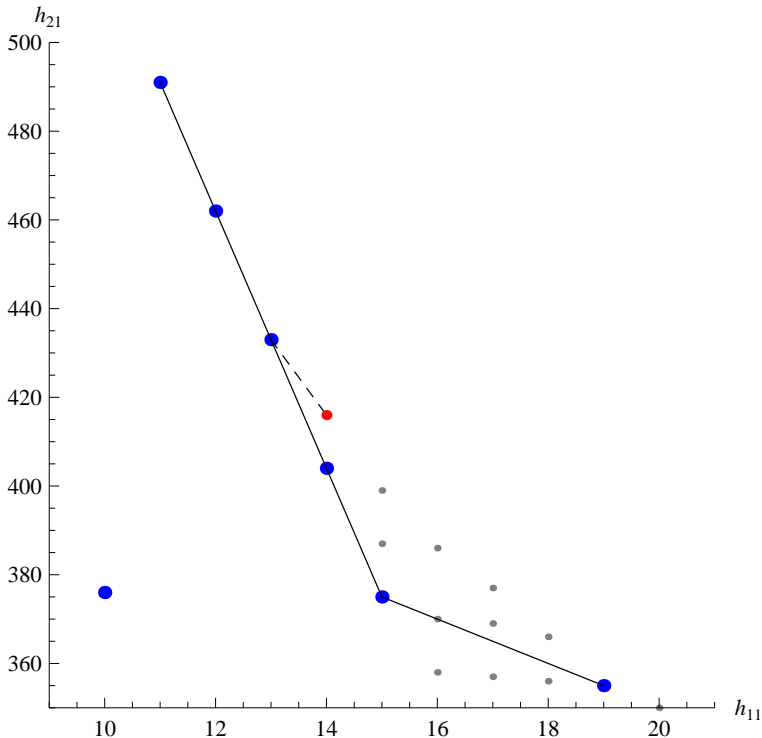
Figure 3: The "tip" of the shield region containing known Calabi-Yau threefold Hodge numbers with large h_{21}. Large (blue) dots represent generic threefolds over toric bases, small (gray) dots are from Kreuzer-Skarke list. Dots connected with a solid line represent the line of toric bases connected by blow-up transitions closest to the shield boundary. Dotted line and mid-size (red) dot represents a threefold realized by tuning a Weierstrass model over the toric base corresponding to Hodge numbers $(13, 433)$. (Many other Hodge pairs from the Kreuzer-Skarke list can be realized in a similar fashion by tuning Weierstrass moduli, though only one example is depicted.)

toric base has self-intersections $(-12// - 11//(-12)^7, 0, 6)$, and, as for $\mathbb{F}_{12}$, the next few blow ups cannot increase the gauge group so decrease h_{21} by 29 while increasing h_{11} by 1. It is interesting to note that these apparently very different geometric steps — blowing up 11 points simultaneously in the bases with Hodge numbers $(164, 254), \ldots, (222, 252)$, and blowing down single curves in the bases with Hodge numbers $(252, 222), \ldots, (254, 164)$ — are dual through mirror symmetry[1]. Understanding the relationship between these transitions may give some new insights into mirror symmetry.

Continuing down the chain, after several blow-ups the gradient again becomes less steep, leading to a final sequence of bases carrying elliptically fibered threefolds with maximal values of h_{11}. Again, the sequence is the mirror image of the initial descent with gradient -29, and involves 11 blow ups at each of the final steps.

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[1]Though as mentioned above the toric base data does not necessarily give mirror symmetric Calabi-Yau constructions, the corresponding geometries in the Kreuzer and Skarke database can be identified from the Hodge numbers and polytope sizes, and are mirror symmetric.

T	curve self-intersection numbers	h_{11}, h_{21}
1	(-12, 0, 12, 0)	(11, 491)
2	(-12, -1, -1, 11, 0)	(12, 462)
3	(-12, -1, -2, -1, 10, 0)	(13, 433)
4	(-12, -1, -2, -2, -1, 9, 0)	(14, 404)
5	(-12, -1, -2, -2, -2, -1, 8, 0)	(15, 375)
6	(-12, -1, -2, -2, -3, -1, -2, 8, 0)	(19, 355)
⋮	⋮	⋮
60	$(-8, -1, -2, -3, -1, -5, -1, -3, -2, -2, -1, -11//12^4, 0, 6)$	(159, 255)
64	$(9//11//12^4, 0, 6)$	(164, 254)
65	$(-8, -1, -2, -3, -2, -1, -8, -1, -2, -3, -1, -5,$ $-1, -3, -2, -2, -1, (-12)^5, 0, 6)$	(176, 254)
75	$(-10// -11//(-12)^5, 0, 6)$	(193, 253)
86	$(-11// -11//(-12)^6, 0, 6)$	(222, 252)
97	$(-12// -11//(-12)^7, 0, 6)$	(251, 251)
98	$(-12// -11//(-12)^7, -1, -1, 5)$	(252, 222)
99	$(-12// -11//(-12)^7, -1, -2, -1, 4)$	(253, 193)
100	$(-12// -11//(-12)^7, -1, -2, -2, -1, 3)$	(254, 164)
⋮	⋮	⋮
171	$(-11// -11//(-12)^{11}// -11// -11, 0)$	(433, 13)
182	$(-12// -11//(-12)^{12}// -11// -11, 0)$	(462, 12)
193	$(-12// -11//(-12)^{13}// -11// -12, 0)$	(491, 11)

Table 2: Some of the toric bases that arise in the sequence of blow-ups from $\mathbb{F}_{12}$ to the maximal model with $T = 193$, and the Hodge numbers of the generic elliptically fibered Calabi-Yau threefold over these bases. This sequence of threefolds runs along the upper boundary of the "shield" region of known Calabi-Yau threefold Hodge numbers.

4. Extensions

4.1 Tuning models over toric bases

The threefolds we have considered here are the generic Weierstrass models over each toric base. This gives only a very small fraction of the full set of threefolds that can be realized as elliptic fibrations over toric bases. As has been widely studied in the context of F-theory, by tuning the Weierstrass coefficients over any given base the degree of vanishing of the discriminant over given curves can be enhanced. This produces a singular elliptic fibration that must be resolved to get a smooth Calabi-Yau. In the F-theory picture, these singularities in the discriminant locus give rise to nonabelian gauge group factors in the 6D supergravity theory that can be identified through the Kodaira classification and the Tate algorithm [24, 12, 25, 26, 27]. In [10] we gave an explicit description of the basis of monomials for Weierstrass models over any of the 61,539 toric bases. In general, from the F-theory point of view tuning monomials to get a higher degree of vanishing of f, g over a
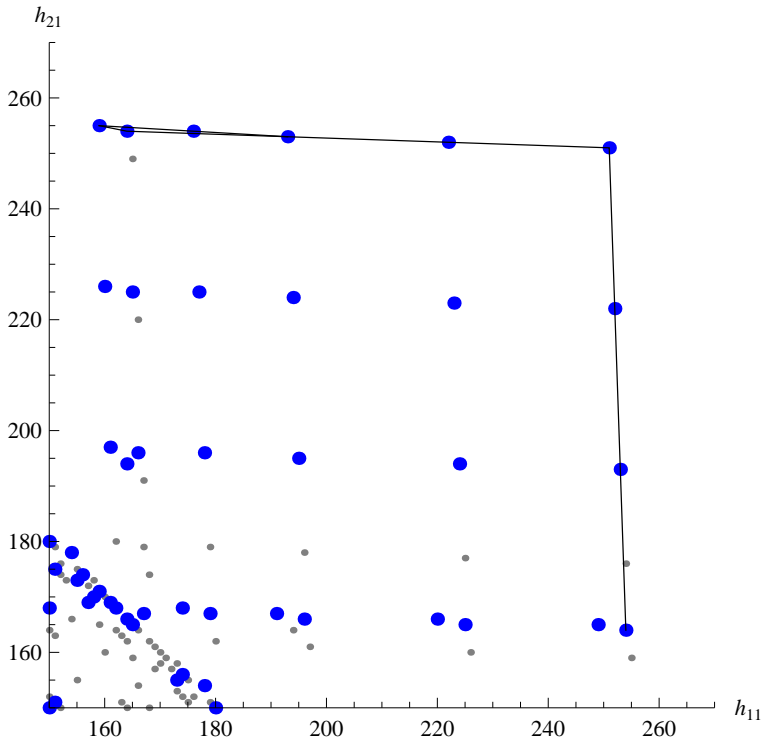
Figure 4: The region around the central point of the "shield" region of known allowed Hodge numbers for Calabi-Yau threefolds. Large (blue) dots represent generic threefolds over toric bases, small (gray) dots are from Kreuzer-Skarke list. Dots connected with a solid line represent the line of toric bases connected by blow-up transitions closest to the shield boundary.

given curve will also give rise to charged matter fields associated with the intersection of that curve with other curves and with the rest of the discriminant locus [25, 28, 29].

For the Hodge numbers of the smooth resolved Calabi-Yau, the effect of tuning Weierstrass moduli to enhance the vanishing of the discriminant locus over certain curves decreases h_{21}. The appearance of gauge groups in the corresponding F-theory model increases h_{11}, and the appearance of matter decreases h_{21} further. Thus, for each of the 61,539 generic Calabi-Yau threefolds discussed in the main part of this paper there is a large family of additional threefolds with larger h_{11} and smaller h_{21}. Anomaly cancellation conditions in 6D supergravity strongly constrain the combinations of gauge groups and matter content that can be realized for F-theory models associated with elliptic fibrations over any given base [30, 31, 32]. In particular, the number of moduli available in h_{21} provides a bound on the complexity of the models that can be realized over any given base through (2.3). In [33], it was conjectured that by tuning Weierstrass moduli any combination of gauge groups over specific divisors and matter content compatible with anomaly cancellation conditions could be realized in F-theory, up to the limit imposed by the number of degrees of freedom in neutral hypermultiplets (h_{21}). We are not aware of any exceptions to this conjecture at this time. A systematic study of F-theory models produced by tuning Weierstrass coefficients in this way to produce $SU(N)$ gauge groups was carried out over

Hirzebruch bases $\mathbb{F}_m, m = 0, 1, 2$ in [33], and over $\mathbb{P}^2$ in [34]. A systematic analysis of the $T = 0$ 6D supergravity theories corresponding to models over $\mathbb{P}^2$ with $SU(N)$ gauge groups was carried out in [35], and the possible matter representations compatible with anomaly cancellation were identified in this case. In general, there is no systematic classification of codimension two singularities in F-theory models corresponding to different matter representations [34, 36, 37]. Braun has systematically identified some 100,000 toric hypersurface Calabi-Yau threefolds in the Kreuzer-Skarke database that are elliptic fibrations over $\mathbb{P}^2$ [38]. These include many of the $SU(N)$ models studied in [35, 34], as well as a wide range of other models.

In principle, it should be possible to systematically construct a tremendous number of different elliptically fibered Calabi-Yau threefolds by tuning Weierstrass moduli to achieve different F-theory models over the 61,539 toric bases. The Hodge numbers of these threefolds can easily be computed through (2.2) and (2.3). It is possible that many of these threefolds will not have a construction through the Batyrev method and will not be contained in the Kreuzer-Skarke database. On the other hand, because of the Hodge number structure and reduction in h_{21} when tuning moduli, these models are likely to lie within the shield region.

A full exploration of Calabi-Yau threefolds associated with these tuned Weierstrass models is left for future work, but we mention a few specific examples here. In [33] we carried out an explicit construction of F-theory models on $\mathbb{F}_m$ for small values of m with $SU(N)$ gauge groups on the divisor Σ having self-intersection $\Sigma \cdot \Sigma = -m$, and explicitly computed the degrees of freedom used in this construction. The simplest class of cases are $SU(N)$ models on $\mathbb{F}_2$. Tuning the Weierstrass model to realize this gauge group on Σ requires fixing $N^2 - 1$ moduli, and from anomaly considerations there must be $2N$ charged hypermultiplets in the fundamental representation of $SU(N)$. The Hodge numbers for the resolved elliptically fibered threefolds in these cases are then $(2 + N, 242 - N^2)$. While most of these Hodge pairs do not appear in the spectrum associated with toric bases, they appear in the Kreuzer-Skarke data up to $N = 15$, which is precisely the upper limit expected from anomaly conditions [33]. It is natural to expect that these models in the Kreuzer-Skarke database are the resolved elliptic fibrations over $\mathbb{F}_2$ with enhanced gauge symmetry in the F-theory picture, since the Hodge numbers are relatively sparse in that region. In fact, $(4, 238)$ (the $N = 2$ case) is the point with $h_{11} = 4$ having the largest value of h_{21}, and $(5, 233)$ (the $N = 3$ case) has the largest h_{21} for $h_{11} = 5$ aside from the $(5, 251)$ point associated with $\mathbb{F}_3$. Direct computation with PALP confirms that, for example, the unique example with Hodge numbers $(4, 238)$ is a tuned Weierstrass model over the base $\mathbb{F}_2$.

Another simple class of tuned Weierstrass models with $SU(N)$ gauge groups are those associated with elliptic fibrations over $\mathbb{P}^2$, where the $SU(N)$ is realized over a degree one curve. These models were constructed from the supergravity and F-theory points of view in [35, 34]. The corresponding resolved Calabi-Yau threefolds have $(h_{11}, h_{21}) = (1 + N, 271 - N(45 - N)/2)$ for $N \geq 3$, and $(3, 231)$ for $N = 2$. These Hodge numbers appear in the Kreuzer-Skarke database for N from 2 up to the maximum expected of 24. These models were identified in the Kreuzer-Skarke data by Braun [39], who has explicitly analyzed the corresponding toric models and determined that many of them have

a fibration structure matching precisely with the resolved Calabi-Yau manifolds associated with enhanced symmetry on $\mathbb{P}^2$ bases.

Finally, we can use the appearance of enhanced symmetry with tuning of moduli to explain some of the points in the Kreuzer-Skarke database that lie slightly above the trajectory of generic threefolds over toric bases described in the previous section. While in general tuning moduli leads to a decrease in h_{21} that keeps the Hodge data below this curve, near the boundary there are cases when tuning a small gauge group gives a larger value of h_{21}. One place where such configurations are easy to identify is in the regions where the gradient of the shield boundary is steepest. This occurs, for example, near the point $(11, 491)$ associated with the $\mathbb{F}_{12}$ base, where the gradient is -29. Tuning a gauge group on the $+12$ curve on $\mathbb{F}_{12}$ produces large amounts of charged matter, and cannot give points near the boundary. There are also no single blow-ups of $\mathbb{F}_{12}$ that can be tuned to give models with enhanced gauge symmetry and large h_{21}. Blowing up $\mathbb{F}_{12}$ twice, however, gives the toric base described by the sequence of curves with self-intersections $(-12, -1, -2, -1, 10, 0)$; the generic threefold over this base has Hodge numbers $(13, 433)$. A gauge group $SU(2)$ on the second -1 curve in the F-theory picture can be shown to have 10 fundamental matter representations, so that the resolved Calabi-Yau has Hodge numbers $(14, 416)$. This is precisely the first point observed above in the Kreuzer-Skarke data that lies above the trajectory of maximal blow-ups of $\mathbb{F}_{12}$ roughly outlining the upper boundary of the Hodge shield, as shown in Figure 3.

All of the tuned models just described involve gauge groups on toric divisors on the bases in the F-theory picture. There are also more complicated ways of tuning a gauge group. For example, as analyzed in [35, 34] for elliptic fibrations over the base $\mathbb{P}^2$, gauge groups can be tuned over a divisor described by a degree b curve in the base. As b increases, the possible matter representations become more exotic, and the complexity of the corresponding resolved Calabi-Yau threefolds grows. For example, for $b = 2$ the most generic model with an $SU(N)$ gauge group has $48 - 4N$ fundamental and 6 anti-symmetric representations. The Hodge numbers for the corresponding threefolds will be $(1 + N, 271 + 2N^2 - 45N)$. Again, these numbers appear in the Kreuzer-Skarke database for N from 3 up to (and beyond) the expected bound of 12. Because this is a fairly populated region, without further analysis it is unclear whether the corresponding Calabi-Yau's are the associated elliptic fibrations; it would be interesting to understand these and related models further. In general the Calabi-Yau threefolds arising from tuning gauge groups over non-toric divisors may have no convenient toric description, and may not appear in the Kreuzer-Skarke list. As another example, from anomaly analysis it seems that there should be a 6D model possible over $\mathbb{P}^2$ with gauge group $SU(4)$, 64 fundamental matter fields, and one matter field in the **20** "box" representation [35]. No F-theory model is known for this type of matter representation, which would correspond to an exotic codimension two singularity on a singular divisor of degree $b = 4$. An F-theory model of this type would correspond to a resolved Calabi-Yau with Hodge numbers smaller than any that appear in the Kreuzer-Skarke database — the smallest Hodge numbers that appear there are $(14, 14)$. So either this model cannot be realized in F-theory, or it is a representative of a class of Calabi-Yau threefolds not in the Kreuzer-Skarke list. In addition to exotic

matter representations for nonabelian gauge groups, it is also possible to tune the generic Weierstrass model over a given base to get extra sections in the elliptic fibration enhancing the rank of the Mordell-Weil group. In the F-theory picture this produces additional $U(1)$ gauge group factors that contribute to h_{11} [12]. Since these factors are nonlocal, they are more difficult to describe in F-theory (and less constrained by 6D anomalies [40]), though some classes of elliptic fibrations with extra sections can be characterized systematically [41, 37, 42]. We have not considered elliptic fibrations with multiple sections here, but these would provide an important direction for systematically expanding on this work.

4.2 Non-toric bases

While we have focused in this paper on toric bases, a similar analysis can be carried out for non-toric bases. Some of the results described above provide bounds on elliptically fibered Calabi-Yau threefolds independent of whether the base is toric, such as the upper bound $h_{21} \leq 491$ and the identification of the possible Hodge number combinations for large $h_{21} > 400$. More work would be needed, however, beyond the analysis of Weierstrass tunings discussed in the previous section, to produce a complete list of Hodge pairs possible for elliptically fibered Calabi-Yau threefolds including non-toric bases. The analysis of [9] places constraints on the kinds of clusters that can appear and their connection structure even on non-toric bases. This leads to bounds on the gauge groups allowed in F-theory models and constrains the set of allowed possibilities. A simple example of a class of non-toric bases that is explored in [10] corresponds to bases containing a number n of -4 curves and no other curves of self-intersection -3 or less. In the F-theory context these correspond to gravity theories with gauge group $SO(8)^n$ and no matter. Simple constraints on possible configurations bound $n \leq 20$, and a stronger bound closer to $n \leq 12$ is probably possible. Some models with $T = 9 + n$ are given in [10], where the divisor structure on the base contains a set of closed loops with alternating $-1, -4$ curves. The resolved Calabi-Yau threefolds elliptically fibered over these bases will have $h_{11} = 11 + 5n, h_{21} = 11 - n$. These Hodge numbers appear in the Kreuzer and Skarke list for $n = 3, \ldots, 9$. In general, many more non-toric bases than toric bases support elliptically fibered Calabi-Yau threefolds, and a systematic analysis of the possibilities, particularly for bases carrying large non-Higgsable gauge groups in the F-theory picture, might expand the story presented here in interesting directions. In general, however, it seems likely that including non-toric bases will not produce Calabi-Yau threefolds with Hodge numbers that go significantly outside the region spanned by the threefolds elliptically fibered over toric bases. In particular, it seems unlikely that non-toric bases can give elliptically fibered threefolds with significantly larger Hodge numbers than those found here. The reason that it may be difficult to realize large Hodge numbers follows similar lines to the heuristic arguments in [10] arguing that it is difficult to find a base associated with a T value much larger than $T = 193$. The basic idea is that the only way to get large T (or large h_{11}) is to incorporate many $\mathfrak{e}_8$ summands in the gauge algebra. But this requires divisors of self-intersection -12 in the base, and such divisors can only be connected to clusters of the form $-2, -2, -3$. The optimal known base (the $T = 193$ model in Table 2) involves a linear such chain — basically 16 copies of the periodic sequence (3.1). Adding loops or extra branching to the network of intersecting

irreducible effective curves does not seem to provide structure that could increase T or h_{11} significantly beyond the $T = 193$ case. Furthermore, bases at large T, like Calabi-Yau threefolds with large $h_{11} + h_{21}$, become sparse, at least in the toric constructions known, so it is harder to find structures that would push the bounds very far. Thus, it seems likely that while systematically including non-toric bases will dramatically increase the range of possible Calabi-Yau threefold constructions, this will not significantly modify the bounds of the region of allowed Hodge numbers.

Note that both for the additional models found by tuning Weierstrass parameters on toric bases discussed in the previous section, and for threefolds fibered over non-toric bases, all these elliptically fibered threefolds will be connected through blowing up and blowing down points in the bases. Thus, this very large set of threefolds are connected in a network.

4.3 Non-elliptically fibered threefolds

The fact that the number of birational equivalence classes of elliptically fibered Calabi-Yau threefolds is finite was proven some time ago [43]. A simple argument for this conclusion from the minimal model/Weierstrass picture is given in [32]. So it is not surprising that there are bounds to the Hodge numbers possible for elliptically fibered Calabi-Yau threefolds. While it will be interesting to further analyze the precise bounds on the elliptically fibered class of spaces, the most interesting questions involve the more general set of Calabi-Yau threefolds without the restriction to elliptic fibrations. The fact that the boundary of the region of allowed Hodge numbers seems to be the same for the sampling of toric hypersurface models considered by Kreuzer and Skarke and for the more constrained set of elliptic fibrations over toric bases suggests that the bounds on Hodge numbers may be universal and apply to non-elliptically fibered Calabi-Yau threefolds in general.

While the approach taken here does not suggest any completely general approach to bounding the Hodge numbers for arbitrary Calabi-Yau threefolds, it does suggest one approach which may lead to bounds at least for the set of threefolds connected by extremal transitions. The transitions we have described here are associated with blowing up and down points on the base of an elliptic fibration. These correspond in the threefolds to transitions that connect manifolds of different topology in more complicated ways [12], such as the conifold transition. It may be enlightening to consider the combinatorial structure of how the triple intersection product and Mori cone of threefolds connected by blow-up transitions in the base are related. This may suggest a more general set of rules for transitions that would allow for a generalization of the bounds considered here to more general Calabi-Yau threefolds. A related approach has recently been used to construct novel Calabi-Yau threefolds with small Hodge numbers using conifold-type transitions [7].

4.4 Elliptically fibered Calabi-Yau fourfolds

A similar exploration of elliptically fibered Calabi-Yau fourfolds through toric 3D base manifolds may reveal structure analogous to the "shield" pattern for Hodge numbers of CY fourfolds. While the set of transitions between threefold bases is more complicated, in the toric context the mathematics of Mori theory is well understood and it should be

possible to systematically explore the space of toric threefold bases, as outlined briefly in [10]. Work in this direction is currently underway.

References

[1] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, "Vacuum Configurations for Superstrings," Nucl. Phys. B **258**, 46 (1985).

[2] M. Gross, D. Huybrechts, and D. Joyce, *Calabi-Yau Manifolds and Related Geometries*, Springer-Verlag, Berlin (2003).

[3] V. Batyrev, Duke Math. Journ. **69**, 349 (1993); "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties," J. Alg. Geom. **3**, 493 (1994) `alg-geom/9310003`.

[4] M. Kreuzer and H. Skarke, "Complete classification of reflexive polyhedra in four-dimensions," Adv. Theor. Math. Phys. **4**, 1209 (2002) `hep-th/0002240`.

[5] M. Kreuzer and H. Skarke, data available online at `http://hep.itp.tuwien.ac.at/ kreuzer/CY/`

[6] V. Batyrev and M. Kreuzer, "Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions," `arXiv:0802.3376 [math.AG]`.

[7] P. Candelas and R. Davies, "New Calabi-Yau Manifolds with Small Hodge Numbers," Fortsch. Phys. **58**, 383 (2010) `arXiv:0809.4681 [hep-th]`.

[8] R. Davies, "The Expanding Zoo of Calabi-Yau Threefolds," Adv. High Energy Phys. **2011**, 901898 (2011) `arXiv:1103.3156 [hep-th]`.

[9] D. R. Morrison and W. Taylor, "Classifying bases for 6D F-theory models," `arXiv:1201.1943 [hep-th]`.

[10] D. R. Morrison and W. Taylor, "Toric bases for 6D F-theory models," `arXiv:1204.0283`

[11] A. Grassi, "On minimal models of elliptic threefolds," Math. Ann. **290** (1991) 287–301.

[12] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – II," Nucl. Phys. B **476**, 437 (1996) `arXiv:hep-th/9603161`.

[13] M. Reid, "The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible," Math. Annalen **278** 329-334 (1987).

[14] C. Vafa, "Evidence for F-Theory," Nucl. Phys. B **469**, 403 (1996) `arXiv:hep-th/9602022`.

[15] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – I," Nucl. Phys. B **473**, 74 (1996) `arXiv:hep-th/9602114`.

[16] M. B. Green, J. H. Schwarz and P. C. West, "Anomaly Free Chiral Theories In Six-Dimensions," Nucl. Phys. B **254**, 327 (1985).

[17] A. Sagnotti, "A Note on the Green-Schwarz mechanism in open string theories," Phys. Lett. B **294**, 196 (1992) `arXiv:hep-th/9210127`.

[18] J. Erler, "Anomaly Cancellation In Six-Dimensions," J. Math. Phys. **35**, 1819 (1994) `arXiv:hep-th/9304104`.

[19] W. Taylor, "TASI Lectures on Supergravity and String Vacua in Various Dimensions," `arXiv:1104.2051 [hep-th]`.

[20] M. Kreuzer and H. Skarke, "PALP: A Package for analyzing lattice polytopes with applications to toric geometry," Comput. Phys. Commun. **157**, 87 (2004) `math/0204356 [math-sc]`.

[21] N. Seiberg and E. Witten, "Comments on String Dynamics in Six Dimensions," Nucl. Phys. B **471**, 121 (1996) `arXiv:hep-th/9603003`.

[22] P. Candelas, E. Perevalov, G. Rajesh, "Toric geometry and enhanced gauge symmetry of F theory / heterotic vacua," Nucl. Phys. **B507**, 445-474 (1997). `hep-th/9704097`.

[23] P. S. Aspinwall, D. R. Morrison, "Point - like instantons on K3 orbifolds," Nucl. Phys. **B503**, 533-564 (1997). `hep-th/9705104`.

[24] K. Kodaira, "On compact analytic surfaces. II, III," Ann. Math. **77** 563 (1963); Ann. Math. **78** 1 (1963).

[25] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, "Geometric singularities and enhanced gauge symmetries," Nucl. Phys. B **481**, 215 (1996) `arXiv:hep-th/9605200`.

[26] S. Katz, D. R. Morrison, S. Schafer-Nameki and J. Sully, "Tate's algorithm and F-theory," JHEP **1108**, 094 (2011) `arXiv:1106.3854 [hep-th]`.

[27] A. Grassi and D. R. Morrison, "Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds," `arXiv:1109.0042 [hep-th]`.

[28] S. H. Katz and C. Vafa, "Matter from geometry," Nucl. Phys. B **497**, 146 (1997) `arXiv:hep-th/9606086`.

[29] A. Grassi, D. R. Morrison, "Group representations and the Euler characteristic of elliptically fibered Calabi-Yau threefolds", J. Algebraic Geom. 12 (2003), 321-356 `arXiv:math/0005196`.

[30] V. Kumar and W. Taylor, "String Universality in Six Dimensions," `arXiv:0906.0987 [hep-th]`.

[31] V. Kumar and W. Taylor, "A bound on 6D $\mathcal{N} = 1$ supergravities," JHEP **0912**, 050 (2009) `arXiv:0910.1586 [hep-th]`.

[32] V. Kumar, D. R. Morrison and W. Taylor, "Global aspects of the space of 6D $\mathcal{N} = 1$ supergravities," JHEP **1011**, 118 (2010) `arXiv:1008.1062 [hep-th]`.

[33] V. Kumar, D. R. Morrison and W. Taylor, "Mapping 6D $\mathcal{N} = 1$ supergravities to F-theory," JHEP **1002**, 099 (2010) `arXiv:0911.3393 [hep-th]`.

[34] D. R. Morrison and W. Taylor, "Matter and singularities," `arXiv:1106.3563 [hep-th]`

[35] V. Kumar, D. Park and W. Taylor, "6D supergravity without tensor multiplets," `arXiv:1011.0726 [hep-th]`.

[36] M. Esole and S. -T. Yau, "Small resolutions of SU(5)-models in F-theory," `arXiv:1107.0733` `[hep-th]`.

[37] M. Esole, J. Fullwood and S. -T. Yau, "D_5 elliptic fibrations: non-Kodaira fibers and new orientifold limits of F-theory," `arXiv:1110.6177` `[hep-th]`.

[38] V. Braun, "Toric Elliptic Fibrations and F-Theory Compactifications," `arXiv:1110.4883` `[hep-th]`.

[39] V. Braun, *private communication*.

[40] D. S. Park and W. Taylor, "Constraints on 6D Supergravity Theories with Abelian Gauge Symmetry," `arXiv:1110.5916` `[hep-th]`.

[41] A. Klemm, B. Lian, S. S. Roan, S. -T. Yau, "Calabi-Yau fourfolds for M theory and F theory compactifications," Nucl. Phys. **B518**, 515-574 (1998). `hep-th/9701023`.

[42] D. S. Park, "Anomaly Equations and Intersection Theory," `arXiv:1111.2351` `[hep-th]`.

[43] M. Gross, "A finiteness theorem for elliptic Calabi–Yau threefolds," Duke Math. Jour. **74**, 271 (1994).