The Spacetime of Double Field Theory:
Review, Remarks, and Outlook

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Abstract

We review double field theory (DFT) with emphasis on the doubled spacetime and its generalized coordinate transformations, which unify diffeomorphisms and b-field gauge transformations. We illustrate how the composition of generalized coordinate transformations fails to associate. Moreover, in dimensional reduction, the $O(d,d)$ T-duality transformations of fields can be obtained as generalized diffeomorphisms. Restricted to a half-dimensional subspace, DFT includes 'generalized geometry', but is more general in that local patches of the doubled space may be glued together with generalized coordinate transformations. Indeed, we show that for certain T-fold backgrounds with non-geometric fluxes, there are generalized coordinate transformations that induce, as gauge symmetries of DFT, the requisite $O(d,d;\mathbb{Z})$ monodromy transformations. Finally we review recent results on the α' extension of DFT which, reduced to the half-dimensional subspace, yields intriguing modifications of the basic structures of generalized geometry.

Contents

1 Introduction

The underlying symmetry of gravity is given by the diffeomorphism group or the group of general coordinate transformations. Accordingly, the basic dynamical variables are tensor fields, including the metric tensor g_{ij} that describes the spacetime geometry. This holds true in the target space description of string theory, where the metric is augmented by an antisymmetric tensor b_{ij} (the b-field) and a scalar ϕ (dilaton), as well as various p-forms, depending on the string theory considered. The universal spacetime low-energy action for the massless fields common to all oriented closed string theories then reads

$$S = \int d^D x \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] , \qquad (1.1)$$

where $D = 26$ or $D = 10$, and H denotes the three-form field strength of the b-field. This action admits the diffeomorphism invariance of gravity and an abelian gauge symmetry of the b-field, but it does not display any 'higher' symmetry that would explain the special role of the b-field and dilaton, as opposed to any other matter fields that can be coupled to gravity. In closed string theory, however, the field content is uniquely determined and closely related to the T-duality group $O(d, d; \mathbb{Z})$ that is well known to emerge [1] when the theory is put on a torus background T^d. In fact, for the low-energy theory of the massless fields the symmetry becomes a continuous $O(d, d; \mathbb{R})$, henceforth written as just $O(d, d)$. The components of g_{ij} and b_{ij} along the torus transform into each other according to the Buscher rules [2], and it is only the particular action (1.1) that is compatible with T-duality. This naturally leads one to wonder if there is a way to make these features manifest at the level of a spacetime action such as (1.1). In this paper we will review 'double field theory' that provides such a formulation [3–7] and has been the focus of much recent attention. For earlier attempts see [8–11], and the concluding section for a more detailed guide to the literature.

In double field theory (DFT) an action as (1.1) can be formulated in an $O(D, D)$ covariant fashion by organizing g, b and ϕ into new field variables that are $O(D, D)$ tensors [7]. Specifically, the fundamental fields are given by a symmetric $O(D, D)$ matrix

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{pmatrix} , \qquad (1.2)$$

and an $O(D, D)$ singlet dilaton d related to ϕ via $e^{-2d} = \sqrt{-g} e^{-2\phi}$, where $M, N = 1, \ldots, 2D$ are fundamental $O(D, D)$ indices. The matrix (1.2) is naturally viewed as a metric on a doubled space with $2D$ coordinates $X^M = (\tilde{x}_i, x^i)$, with corresponding derivatives $\partial_M = (\tilde{\partial}^i, \partial_i)$, where x^i are the usual spacetime coordinates. The additional coordinates $\tilde{x}_i$, dual to winding modes of the closed string, are known to be present in the full string theory on a torus, as seen when formulated as a second-quantized string field theory [12,13]. In DFT all coordinates are doubled for any background, while imposing a constraint that effectively renders half of them 'inactive'. More precisely, we impose the constraint

$$\eta^{MN} \partial_M \partial_N \equiv \partial^M \partial_M = 0 , \qquad \eta^{MN} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} , \qquad (1.3)$$

where η_{MN} denotes the $O(D,D)$ invariant metric. The vanishing of $\partial^M \partial_M$ when acting on arbitrary fields and parameters is the weak form of the constraint and can be identified with the level matching constraint of closed string theory. When, in addition, $\partial^M \partial_M$ is constrained to vanish for all products of fields and gauge parameters, we have the strong version of the constraint. The strong version goes beyond the level-matching constraint of closed string theory and implies that the fields depend only on half of the (doubled) coordinates. When DFT is applied to a background with d abelian isometries, like a torus T^d, and fields are independent of these d coordinates, we can, however, realize the full $O(d,d)$ symmetry geometrically by the use of $\tilde{x}$-dependent coordinate transformations.

The usual gauge transformations of the metric and b-field, i.e., diffeomorphisms generated by a vector ξ^i and abelian gauge transformations $\delta_{\tilde{\xi}} b_{ij} = \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i$ can be lifted to $O(D,D)$ covariant gauge transformations of $\mathcal{H}_{MN}$ and the dilaton d, with parameter $\xi^M = (\tilde{\xi}_i, \xi^i)$,

$$
\begin{aligned}
\delta_\xi \mathcal{H}_{MN} &= \xi^P \partial_P \mathcal{H}_{MN} + \left(\partial_M \xi^P - \partial^P \xi_M\right) \mathcal{H}_{PN} + \left(\partial_N \xi^P - \partial^P \xi_N\right) \mathcal{H}_{MP} , \\
\delta_\xi \left(e^{-2d}\right) &= \partial_M \left(\xi^M e^{-2d}\right) ,
\end{aligned}
\tag{1.4}
$$

where indices are raised and lowered with η_{MN} defined in (1.3). When specializing to the components of $\mathcal{H}_{MN}$ in (1.2) and setting $\tilde{\partial}^i = 0$ these transformations reduce to the standard gauge transformations. Let us stress that also the gauge transformations for $\tilde{\partial}^i \neq 0$ are well motivated from string theory. In fact, the gauge transformations to cubic order have been derived from closed string field theory [4], and there is a unique way to make them background independent as transformations of g_{ij} and b_{ij} [6]. Application to (1.2) then uniquely leads to the above gauge transformations of $\mathcal{H}$ and d.

There is a unique scalar $\mathcal{R}$ written in terms of second derivatives of $\mathcal{H}_{MN}$ and d that indeed transforms as a scalar under (1.4), i.e., $\delta_\xi \mathcal{R} = \xi^M \partial_M \mathcal{R}$. Since e^{-2d} transforms as a density, the scalar $\mathcal{R}$ can be used to write a manifestly gauge invariant action

$$
S_{\mathrm{DFT}} = \int d^{2D} X \, e^{-2d} \, \mathcal{R}(\mathcal{H}, d) .
\tag{1.5}
$$

Setting $\tilde{\partial}^i = 0$ and writing it in terms of g, b and ϕ the above action reduces to (1.1). Thus, once formulated in terms of the right dynamical objects and geometrical structures, the two-derivative part of the spacetime action of bosonic string theory is unique, quite in contrast to the original formulation. Again, for $\tilde{\partial}^i \neq 0$, this action is also well motivated from string theory in that expanded to cubic order around a constant toroidal background, it coincides with the closed string field theory action of the massless fields to that order.

In this article we will elaborate on the geometrical implications of DFT. It is mainly a review, but we also give some new results. Specifically, we show in sec. 4 that even the strongly constrained DFT allows for backgrounds that are globally well-defined in the doubled geometry, but not in standard differential geometry, in this sense going beyond conventional supergravity. We will discuss the role of generalized coordinate transformations on the doubled space and why they require some notion of generalized manifold. To explain this point recall that in general relativity we begin with a conventional manifold for which coordinates on overlapping patches are related by the usual general coordinate transformations. These, in turn, can be written infinitesimally as Lie derivatives generated by the vector parameter ξ^i. In contrast, the gauge

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transformations (1.4) are not given by Lie derivatives on the doubled space but rather represent 'generalized Lie derivatives' $\widehat{\mathcal{L}}_\xi$, so that we have $\delta_\xi \mathcal{H}_{MN} = \widehat{\mathcal{L}}_\xi \mathcal{H}_{MN}$. If follows immediately that we cannot view (1.4) as infinitesimal diffeomorphisms on the doubled space and, therefore, the doubled space needs to be viewed as a suitably generalized manifold that is 'patched together' by some generalized coordinate transformations.

A closely related observation is that the formulation of DFT requires the *constant* $O(D,D)$ metric η_{MN} defined in (1.3). For a conventional manifold there is no coordinate independent sense in which a metric can take the constant form in (1.3). Put differently, the doubled manifold would be of a rather special 'flat' form, allowing for preferred coordinate systems for which η_{MN} is constant. In contrast, the generalized Lie derivative in DFT does leave η_{MN} invariant, $\widehat{\mathcal{L}}_\xi \eta_{MN} = 0$, implying that the notion of coordinate transformations on the doubled space should be generalized in such a way that η_{MN} is actually invariant.

In a recent paper, two of us gave a proposal for such generalized coordinate transformations that meet all consistency conditions tested so far. A generalized vector V_M transforms under $X \to X'$ as [14]:

$$V'_M(X') = \mathcal{F}_M{}^N V_N(X) , \tag{1.6}$$

where

$$\mathcal{F}_M{}^N = \frac{1}{2} \left(\frac{\partial X^P}{\partial X'^M} \frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P} \frac{\partial X^N}{\partial X'^P} \right) , \tag{1.7}$$

and indices on coordinates are raised and lowered with η_{MN}. Similarly, an arbitrary generalized tensor transforms tensorially, each index being rotated by $\mathcal{F}$. Even though these transformations do not describe conventional general coordinate transformations on the $2D$-dimensional space they *do* encode arbitrary general coordinate transformations on D-dimensional isotropic subspaces of the doubled space, that is, spaces for which tangent vectors are null in the metric η. The components of $\mathcal{H}_{MN}$ in (1.2) then transform conventionally as tensors, without imposing any constraints on the geometry encoded by the D-dimensional metric g_{ij}. As we will review, at the same time the action of $\mathcal{F}$ on η_{MN} is such that it is left invariant, as required. For fields depending only on x, b-field gauge transformations are generalized coordinate transformations $\tilde{x}_i \to \tilde{x}_i - \tilde{\xi}_i(x)$ that mix x and $\tilde{x}$ coordinates.

DFT is a framework that is flexible enough to encode all that is contained in the usual spacetime action (1.1). In particular, the presence of the 'flat' $O(D,D)$ metric η_{MN} does not imply that the spacetime metric g_{ij} (encoded by $\mathcal{H}_{MN}$) is also flat or, for that matter, restricted at all. In fact, DFT is also flexible enough to encode the framework of generalized geometry, which has been developed in pure mathematics [15–18]. This geometry does not change the underlying manifold M (it is not doubled). The tangent bundle $T(M)$, however, is replaced by $T(M) \oplus T^*(M)$ and structures such as the Courant bracket are defined on this extended bundle. Before the advent of DFT, however, it appears that generalized geometry was not developed to the extent that invariant curvatures and thus actions such as (1.5) could be defined. Generalized geometry is manifestly contained in DFT in that we may solve the strong constraint (1.3) by setting, say, $\tilde{\partial}^i = 0$, after which the components (V_i, V^i) of a generalized vector V^M acquire a definite interpretation as vector (V^i) and one-form (V_i), thereby encoding an element of $T \oplus T^*$. Moreover, the generalized Lie derivatives of vectors are given by the action of the so-called Dorfman bracket, and their closure is governed by the Courant bracket.

In this way, to zeroth order in α', DFT may be viewed as the first implementation of generalized geometry at the level of the full spacetime action. However, as we will also review, taking α' to be non-zero the generalized Lie derivative of DFT acquires α' corrections that even on the half-dimensional subspace modify the defining structures of generalized geometry [19].

Apart from implementing the generalized geometry program and thus reformulating the usual spacetime theory in terms of a geometry that is better adapted to the T-duality properties of string theory, we will show how even at the two-derivative level DFT is yet more general, naturally encoding 'non-geometric' backgrounds. The most direct way to describe backgrounds that are not captured by ordinary supergravity is to try to relax the strong constraint (1.3) so that solutions may depend locally both on x and $\tilde{x}$. In fact, in the full closed string field theory on a torus background only the weaker level-matching constraint is required. It is a subtle question whether the constraint can be relaxed for the massless fields only (or to zeroth order in α') or for different backgrounds, but there has been progress exhibiting relaxed constraints in massive deformations of type IIA [20], generalized Scherk-Schwarz compactifications [21, 22], and now even more generally in [23]. We will be assuming the strong constraint throughout the paper. We will see that even in the strongly constrained DFT there are still 'non-geometric' field configurations in DFT that are globally well-defined when patched with the generalized coordinate transformations (1.6). These examples are closely related to the idea of 'T-folds', where one allows 'patching by $O(d, d)$ transformations'. We believe, however, that a more precise formulation of this idea, in the sense of a natural extension of the general coordinate transformations of differential geometry, was lacking. We will argue that (1.6) fills the gap.

For the convenience of the reader we summarize in the following the main messages of the review part:

- DFT can be viewed, in particular, as the physical implementation of the 'generalized geometry' concepts of Hitchin-Gualtieri. In particular, it encodes the familiar low-energy limits of string theory in generality: the backgrounds are not restricted to tori. DFT and 'generalized geometry' are not really 'alternative' approaches.

- The gauge transformations of the low-energy limit of string theory are unified in DFT in the form of *generalized coordinate transformations*. These treat (arbitrary) diffeomorphisms of the D-dimensional subspace and b-field gauge transformations on the same footing, in addition to encoding $O(d, d)$ transformations for certain configurations. Generalized coordinate transformations are quite different from ordinary diffeomorphisms: they compose in a non-standard manner, such that the composition is non-associative.

- DFT encodes also α' corrections and thereby goes beyond two-derivative approximations and also beyond 'generalized geometry' in that even on the half-dimensional subspace the basic structures of generalized geometry are α'-deformed.

In addition to the review parts, we will also present a number of new results, which we highlight in the following:

- We analyze the structure of *simultaneous* diffeomorphisms and b-field gauge transformations to illustrate the unusual properties of generalized coordinate transformations

(non-standard composition and non-associativity).

- We discuss in the context of DFT the beloved chain of geometric and non-geometric backgrounds originating from the constant H-flux on a three-torus. All 'gluing conditions' that make these spaces well-defined can be treated in a uniform manner as generalized coordinate transformations. In particular, the quantization condition on the H-flux has a natural geometric interpretation in terms of the periodicity conditions of the dual torus. Moreover, the non-geometric Q-flux background is well-defined in DFT thanks to generalized coordinate transformations that rotate x into $\tilde{x}$ coordinates.

- We consider a background that is 'truly non-geometric' in the sense that it is not T-dual to a geometric one. In particular, it contains H, f, and Q flux simultaneously. Nevertheless, it is globally well-defined in DFT thanks to large generalized coordinate transformations that act on the fields as the relevant $O(d, d, \mathbb{Z})$ monodromies. In this way, even strongly constrained DFT appears to go beyond supergravity in that it contains backgrounds ('T-folds') that cannot be made globally well-defined in conventional geometry.

This article is organized as follows. In sec. 2 we discuss the generalized coordinate transformations (1.6) of DFT. We show how they contain conventional diffeomorphisms and b-field gauge transformations. The unusual composition properties of generalized coordinate transformations arise because the algebra of infinitesimal transformations is governed by the Courant- or C-bracket rather than the Lie bracket. We make this plain by studying composition for the explicit example of simultaneous general coordinate and b-field gauge transformations and show that, as transformations of fields, they are associative, but become non-associative at the level of the corresponding coordinate transformation. This is a form of non-associativity that may have a counterpart in closed string field theory, whose gauge algebra is of L_∞ type [13], rather than a strict Lie algebra. In section 3 we discuss the relation of $O(d, d)$ to generalized coordinate transformations. In sec. 4 we discuss explicit examples of non-geometric spaces that are not globally well-defined, but which can be patched by generalized coordinate transformations. For these examples the transformations take the form of $O(d, d)$ transformations viewed as transformations of fields but, intriguingly, do not act in the naive $O(d, d)$ representation on the (doubled) coordinates. In sec. 5 we discuss the issues related to the relaxation of the strong constraint. Finally, in sec. 6, we review the recently introduced α'-extended geometry [19], which requires an intriguing generalization of most geometric concepts, e.g., the Lie derivatives, inner product, etc. We close by giving a summary and outlook.

2 Generalized coordinate transformations in DFT

We introduce in this section the notion of generalized coordinate transformations in DFT and discuss special cases, such as conventional diffeomorphisms and b-field gauge transformations, and the subtleties of their geometrical interpretation. Specifically, we display the type of non-associativity that emerges when combined diffeomorphisms and b-field gauge transformations are viewed as generalized coordinate transformations.

2.1 Diffeomorphisms and b-field gauge transformations

Generalized coordinate transformations $X^M \to X'^M$ act on tensors as in (1.6). Thus, on the generalized metric (1.2) we have

$$\mathcal{H}'_{MN}(X') = \mathcal{F}_M{}^K \mathcal{F}_N{}^L \mathcal{H}_{KL}(X) , \qquad (2.1)$$

where we recall

$$\mathcal{F}_M{}^N = \frac{1}{2}\left(\frac{\partial X^P}{\partial X'^M} \frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P} \frac{\partial X^N}{\partial X'^P} \right) . \qquad (2.2)$$

Let us first review how conventional general coordinate transformations are included in (2.1). Assume that the strong constraint is solved by having all fields depend only on x, not $\tilde{x}$, and consider the transformation

$$x^i \to x^{i\prime} = x^{i\prime}(x) , \qquad \tilde{x}'_i = \tilde{x}_i . \qquad (2.3)$$

It can be easily seen that for this special transformation the two terms in (2.2) actually give the same contribution. Thus,

$$\mathcal{H}'_{MN}(X') = \frac{\partial X^P}{\partial X'^M} \frac{\partial X'_P}{\partial X_K} \frac{\partial X^Q}{\partial X'^N} \frac{\partial X'_Q}{\partial X_L} \mathcal{H}_{KL}(X) . \qquad (2.4)$$

Specializing to the component $\mathcal{H}^{ij} = g^{ij}$ and employing the usual index splitting $V^M = (V_i , V^i)$ we obtain

$$\mathcal{H}'^{ij}(x') = \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i} \frac{\partial x'^p}{\partial x^k} \frac{\partial \tilde{x}_q}{\partial \tilde{x}'_j} \frac{\partial x'^q}{\partial x^l} \mathcal{H}^{kl}(x) = \delta^i_p \frac{\partial x'^p}{\partial x^k} \delta^j_q \frac{\partial x'^q}{\partial x^l} \mathcal{H}^{kl}(x) = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} \mathcal{H}^{kl}(x) , \quad (2.5)$$

using in the second step that $\tilde{x}'_i = \tilde{x}_i$. This is the conventional general coordinate transformation of a contravariant 2-tensor $\mathcal{H}^{ij}$. Thus, as required, DFT correctly reproduces the usual diffeomorphisms acting on the (inverse) metric g^{ij}. Similarly, it is easy to see that all other components of $\mathcal{H}_{MN}$ transform such that they give rise to the usual diffeomorphisms acting on the component fields g_{ij} and b_{ij}.

Let us next compare with the generalized coordinate transformation of the $O(D, D)$ metric η_{MN}, which in components reads more precisely

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} . \qquad (2.6)$$

The transformation takes the same form as in (2.4), with $\mathcal{H}$ replaced by η. As in (2.5), the diagonal components η^{ij} and η_{ij} transform tensorially and therefore, being zero, they remain zero. Also the off-diagonal components transform tensorially. Indeed, again from (2.4) we find

$$\eta'^i{}_j = \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i} \frac{\partial x'^p}{\partial x^k} \frac{\partial x^q}{\partial x'^j} \frac{\partial \tilde{x}'_q}{\partial \tilde{x}_l} \eta^k{}_l = \delta^i_p \frac{\partial x'^p}{\partial x^k} \frac{\partial x^q}{\partial x'^j} \delta^l_q \delta^k{}_l = \frac{\partial x'^i}{\partial x^q} \frac{\partial x^q}{\partial x'^j} = \delta^i{}_j = \eta^i{}_j . \quad (2.7)$$

Thus η is invariant under general coordinate transformations. This is how it is consistent in DFT to have a *constant* metric without restriction on the group of general coordinate transformations and therefore without restriction on allowed spacetime geometries.

Next consider the the b-field gauge transformations, which take the form

$$b'_{ij} = b_{ij} + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i . \tag{2.8}$$

They are encoded in (2.4), via the generalized coordinate transformation

$$\tilde{x}'_i = \tilde{x}_i - \tilde{\xi}_i(x) , \qquad x^{i\prime} = x^i , \tag{2.9}$$

acting on a generalized metric that depends only on x. The details were given explicitly in section 3.1 of [14] and thus we do not repeat them here. Note that this transformation leaves x invariant but mixes $\tilde{x}$ with x.

In double field theory we can also consider generalized coordinates transformations of the form

$$x'^i = x^i - \xi^i(\tilde{x}) , \qquad \tilde{x}'_i = \tilde{x}_i , \tag{2.10}$$

i.e. they leave $\tilde{x}$ invariant but mix x with $\tilde{x}$. Provided the fields are now assumed to depend on $\tilde{x}$, not x, these transformations satisfy the strong constraint eq. (1.3). We can parametrize the $O(D,D)$ matrix $\mathcal{H}$ in a different way using a new metric $\tilde{g}^{ij}$ and a bi-vector field β^{ij} as follows:

$$\mathcal{H}_{MN} = \begin{pmatrix} \tilde{g}^{ij} - \beta^{ik}\tilde{g}_{kl}\beta^{lj} & -\beta^{ik}\tilde{g}_{kj} \\ \tilde{g}_{ik}\beta^{kj} & \tilde{g}_{ij} \end{pmatrix} . \tag{2.11}$$

Then one can show that the transformation (2.10) acts like a gauge transformation on β,

$$\beta^{ij\prime} = \beta^{ij} + \tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i , \tag{2.12}$$

and hence it is called beta gauge transformation. As before, for certain backgrounds, the beta gauge transformations become a b-field gauge transformation or diffeomeorphisms in another T-duality frame, respectively. We finally note that we may also evaluate the DFT action for the fields in (2.11), but still depending on the usual x coordinates. In this case the action reduces to that discussed in relation to non-geometric fluxes in [25, 26, 34].

We close this subsection by discussing a form of 'trivial' gauge transformations that leave the fields invariant. Consider a generalized coordinate transformation of the form

$$X'^M = X^M - \partial^M \chi , \tag{2.13}$$

for some function χ. We view this as an exact transformation. It was shown in [14] that for an exact coordinate transformation of the form $X'^M = X^M - \zeta^M(X)$ the associated $\mathcal{F}$ can be written in terms of the matrix $a_M{}^N = \partial_M \zeta^N$ as

$$\mathcal{F} = 1 + a - a^t + \sum_{n=2}^{\infty} \left(a^n - \frac{1}{2}a^{n-1}a^t \right) . \tag{2.14}$$

The strong constraint implies that $a^t a = 0$, for any transformation. Specializing to the transformation (2.13) we obtain $a_M{}^N = \partial_M \partial^N \chi$, which implies $a = a^t$ and that all powers of a vanish by the strong constraint. It then follows from (2.14) that $\mathcal{F} = 1$ and so the generalized coordinate transformation induced by (2.13), say on the generalized metric, reads

$$\mathcal{H}'(X') = \mathcal{F} \mathcal{H}(X) \mathcal{F}^t = \mathcal{H}(X) . \tag{2.15}$$

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We next observe that

$$\mathcal{H}'(X') \;=\; \mathcal{H}'(X - \vec{\partial}\chi) \;=\; \mathcal{H}'(X) - \partial^M\chi\,\partial_M\mathcal{H}'(X) + \cdots \;=\; \mathcal{H}'(X)\,, \qquad (2.16)$$

where we used the strong constraint that implies that all χ dependent terms vanish. Therefore, fields are strictly invariant under these trivial gauge transformations:

$$\mathcal{H}'(X) \;=\; \mathcal{H}(X)\,. \qquad (2.17)$$

An example is given by fields that are independent of $\tilde{x}$ and a shift $\tilde{x}'_i = \tilde{x}_i - \partial_i\chi$, with a function χ depending only on x. From (2.9) we identify this as a b-field gauge transformation with exact one-form gauge parameter $\tilde{\xi}_i = \partial_i\chi$. We thus recover the well-known 'gauge symmetry of gauge symmetries' for the b-fields gauge transformations. Similarly, for fields depending only on $\tilde{x}$ and a shift $x'^i = x^i - \tilde{\partial}^i\chi$, with a function χ depending only on $\tilde{x}$, we get a trivial β gauge transformation (2.12). Apart from these straightforward examples there are more subtle trivial coordinate transformations for which χ may be a function of both some x and $\tilde{x}$. This redundancy in the gauge transformations of DFT will become important below, c.f. sec. 4, where only some representative among the "equivalent" coordinate transformations is compatible with certain topological restrictions.

2.2 Composition of generalized coordinate transformations

We now turn to a discussion of the general composition of the transformations (2.1), which is different from that of ordinary diffeomorphisms. To explain this point it is convenient to introduce an alternative, 'active' form of the transformations as the exponential of the infinitesimal transformations governed by generalized Lie derivatives. Thus, consider the transformation

$$V'_M(X) \;=\; \exp\left(\widehat{\mathcal{L}}_\xi\right) V_M(X)\,. \qquad (2.18)$$

Note that here both sides depend on X, not X'. The question now is whether there is an associated generalized coordinate transformation $X \to X' = X - \xi(X) + \cdots$ so that this implies the same field transformation as (2.1). This turns out to be a technically rather non-trivial problem. We have confirmed in [14] that, in an expansion in ξ, the two transformations agree if

$$X'^M \;=\; e^{-\Theta^K(\xi)\partial_K}X^M\,, \qquad (2.19)$$

where

$$\Theta^K(\xi) \;=\; \xi^K + \delta_3^K(\xi) + \mathcal{O}(\xi^5)\,, \qquad \text{with} \qquad \delta_3^K(\xi) \equiv \frac{1}{12}(\xi\xi^L)\partial^K\xi_L\,, \qquad (2.20)$$

using the short-hand notation $\xi = \xi^P\partial_P$. For ordinary diffeomorphisms in standard geometry one would simply have $\Theta^M = \xi^M$. The extra term in Θ carries the index on a derivative. This implies, via the strong constraint, that on a scalar the action of such a transformation is like that of an ordinary diffeomorphism. This is as it should be, since the generalized Lie derivative coincides with the ordinary Lie derivative for the case of a scalar. A closed form of $\Theta(\xi)$ is not known, nor the geometrical significance of this function.

With the equivalent form (2.18) of finite transformations we can immediately analyze composition. Consider the consecutive action of two exponentials, with parameters ξ_1 and ξ_2, respectively,

$$e^{\widehat{\mathcal{L}}_{\xi_1(X)}} e^{\widehat{\mathcal{L}}_{\xi_2(X)}} = e^{\widehat{\mathcal{L}}_{\xi_{12}^c(X)}} . \tag{2.21}$$

The resulting transformation, indicated on the right hand side, has a parameter that can be computed with the Baker-Campbell-Hausdorff (BCH) relation. Indeed, using the commutator of generalized Lie derivatives,

$$\left[\widehat{\mathcal{L}}_{\xi_1}, \widehat{\mathcal{L}}_{\xi_2}\right] = \widehat{\mathcal{L}}_{[\xi_1,\xi_2]_C} , \tag{2.22}$$

with the C-bracket

$$\left[\xi_1, \xi_2\right]_C^M \equiv \xi_1^N \partial_N \xi_2^M - \frac{1}{2} \xi_{1N} \partial^M \xi_2^N - (1 \leftrightarrow 2) , \tag{2.23}$$

the result reads

$$\xi_{12}^c = \xi_2 + \xi_1 + \frac{1}{2} \left[\xi_2, \xi_1\right]_C + \frac{1}{12} \left(\left[\xi_2, \left[\xi_2, \xi_1\right]_C\right]_C + \left[\xi_1, \left[\xi_1, \xi_2\right]_C\right]_C\right) + \dots . \tag{2.24}$$

If we had the Lie bracket rather than the C-bracket in here, this formula would encode the familiar group structure of diffeomorphisms: acting with two diffeomorphisms is equivalent to acting with one that is simply the composition of maps of the first two. Here, however, we have the C-bracket, which in turn implies that acting with two generalized coordinate transformations is equivalent to a third, but this third one is not given by the direct composition of maps, thereby reflecting a novel group structure.

This observation has a curious consequence because the C-bracket (2.23) has a non-trivial Jacobiator. This leads to a form of non-associativity, as we now explain. Acting on fields, symmetry transformations are always associative. Thus we must have

$$\left(e^{\widehat{\mathcal{L}}_{\xi_1(X)}} e^{\widehat{\mathcal{L}}_{\xi_2(X)}}\right) e^{\widehat{\mathcal{L}}_{\xi_3(X)}} = e^{\widehat{\mathcal{L}}_{\xi_1(X)}} \left(e^{\widehat{\mathcal{L}}_{\xi_2(X)}} e^{\widehat{\mathcal{L}}_{\xi_3(X)}}\right) . \tag{2.25}$$

To verify this, we use the notation $\xi^c(\xi_2, \xi_1) = \xi_{12}^c$ for the parameter in (2.24) and note that the above requires

$$\exp\left(\widehat{\mathcal{L}}_{\xi^c(\xi_3, \xi^c(\xi_2, \xi_1))}\right) = \exp\left(\widehat{\mathcal{L}}_{\xi^c(\xi^c(\xi_3, \xi_2), \xi_1)}\right) . \tag{2.26}$$

A straightforward computation shows, however, that the gauge parameters differ,

$$\xi^c\left(\xi_3, \xi^c(\xi_2, \xi_1)\right) = \xi^c\left(\xi^c(\xi_3, \xi_2), \xi_1\right) - \frac{1}{6} J(\xi_1, \xi_2, \xi_3) + \mathcal{O}(\xi^4) , \tag{2.27}$$

where

$$J(\xi_1, \xi_2, \xi_3) = \left[\xi_1, \left[\xi_2, \xi_3\right]_C\right]_C + \text{cyclic} , \tag{2.28}$$

is the C-bracket Jacobiator. This Jacobiator is actually a trivial parameter: $J^M = \partial^M N$, where N is the Nijenhuis tensor defined by

$$N(\xi_1, \xi_2, \xi_3) = \frac{1}{6} \left(\left\langle \left[\xi_1, \xi_2\right]_C, \xi_3\right\rangle + \text{cyclic}\right) . \tag{2.29}$$

Generalized Lie derivatives with trivial gauge parameters vanish. The effective parameters in (2.27) differ by a trivial term, as one can convince oneself that the higher order terms

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are also Jacobiators. As a result (2.26) holds. Therefore, as field transformations the gauge transformations are perfectly associative.

Intriguingly, this does not hold as transformations of coordinates. In order to see this consider three consecutive coordinate transformations defined by the maps m_i with $i = 1, 2, 3$:

$$
\begin{aligned}
m_1 &: \ X \to X', & X' &= e^{-\Theta(\xi_1)(X)} X, \\
m_2 &: \ X' \to X'', & X'' &= e^{-\Theta(\xi_2)(X')} X', \\
m_3 &: \ X'' \to X''', & X''' &= e^{-\Theta(\xi_3)(X'')} X''.
\end{aligned}
\tag{2.30}
$$

The map $m_{21} = m_2 \star m_1$ that defines the generalized diffeomorphism associated with the action of m_1 followed by m_2 is given by

$$
m_{21} : \quad X \to X'' \qquad X'' = e^{-\Theta(\xi^c(\xi_2, \xi_1))(X)} X.
\tag{2.31}
$$

Having three successive transformations they can be implemented in two different ways resulting in maps m_α and m_β:

$$
\begin{aligned}
m_\alpha &= m_3 \star (m_2 \star m_1) : \ X \to X'''_\alpha & X'''_\alpha &= \exp\left[-\Theta(\xi^c(\xi_3, \xi^c(\xi_2, \xi_1))) \right] X, \\
m_\beta &= (m_3 \star m_2) \star m_1 : \ X \to X'''_\beta & X'''_\beta &= \exp\left[-\Theta(\xi^c(\xi^c(\xi_3, \xi_2), \xi_1)) \right] X.
\end{aligned}
\tag{2.32}
$$

To cubic order in parameters

$$
\begin{aligned}
\Theta(\xi^c(\xi_3, \xi^c(\xi_2, \xi_1))) &= \xi^c(\xi_3, \xi^c(\xi_2, \xi_1)) + \delta_3(\xi_1 + \xi_2 + \xi_3) + \mathcal{O}(\xi^4), \\
\Theta(\xi^c(\xi^c(\xi_3, \xi_2), \xi_1)) &= \xi^c(\xi^c(\xi_3, \xi_2), \xi_1) + \delta_3(\xi_1 + \xi_2 + \xi_3) + \mathcal{O}(\xi^4).
\end{aligned}
\tag{2.33}
$$

It then follows from (2.27) that

$$
\Theta\big(\xi^c\big(\xi_3, \xi^c(\xi_2, \xi_1)\big)\big) = \Theta\big(\xi^c\big(\xi^c(\xi_3, \xi_2), \xi_1\big)\big) - \frac{1}{6} J(\xi_1, \xi_2, \xi_3) + \mathcal{O}(\xi^4).
\tag{2.34}
$$

We therefore have

$$
\begin{aligned}
X'''_\alpha(X) &= \exp\left[-\Theta\big(\xi^c\big(\xi^c(\xi_3, \xi_2), \xi_1\big)\big) + \tfrac{1}{6} J(\xi_1, \xi_2, \xi_3) + \mathcal{O}(\xi^4) \right] X \\
&= \exp\left[-\Theta\big(\xi^c\big(\xi^c(\xi_3, \xi_2), \xi_1\big)\big) \right] \exp\left[\tfrac{1}{6} J(\xi_1, \xi_2, \xi_3) \right] X + \mathcal{O}(\xi^4) X \\
&= \exp\left[-\Theta\big(\xi^c\big(\xi^c(\xi_3, \xi_2), \xi_1\big)\big) \right] \big(X + \tfrac{1}{6} J(\xi_1, \xi_2, \xi_3) X \big) + \mathcal{O}(\xi^4) X \\
&= X'''_\beta(X) + \tfrac{1}{6} J(\xi_1, \xi_2, \xi_3) X + \mathcal{O}(\xi^4) X \\
&= X'''_\beta(X) + \tfrac{1}{6} \vec{\partial} N(\xi_1, \xi_2, \xi_3) + \mathcal{O}(\xi^4) X,
\end{aligned}
\tag{2.35}
$$

where we used $J = \partial^K N \, \partial_K = (\vec{\partial} N)^K \partial_K$. In components we write

$$
X'''^K_\alpha(X) = X'''^K_\beta(X) + \tfrac{1}{6} \partial^K N(\xi_1, \xi_2, \xi_3) + \mathcal{O}(\xi^4) X.
\tag{2.36}
$$

This is the failure of associativity of the $\star$ composition of generalized diffeomorphisms, calculated to leading order. Only the Jacobiator contributes to this order. The anomalous term δ_3 in $\Theta(\xi)$ cancelled out. In the next subsection we will illustrate this fact by inspecting *simultaneous* general coordinate and *b*-field gauge transformations.

2.3 Simultaneous diffeomorphisms and b-field gauge transformations

For definiteness we take the solution of the strong constraint for which all fields depend only on x. Then we can still consider the generalized coordinate transformation

$$x^{i\prime} = x^{i\prime}(x) , \qquad \tilde{x}'_i = \tilde{x}_i - \zeta_i(x) , \tag{2.37}$$

corresponding to a simultaneous general coordinate and b-field gauge transformation. As the gauge parameters depend only on x this is consistent with the strong constraint. In absence of abelian isometries these are essentially the only residual transformations compatible with this solution of the strong constraint. We compute

$$\frac{\partial X'_M}{\partial X_P} = \begin{pmatrix} \frac{\partial x^{i\prime}}{\partial x^p} & \frac{\partial \tilde{x}'_i}{\partial x^p} \\ \frac{\partial x^{i\prime}}{\partial \tilde{x}_p} & \frac{\partial \tilde{x}'_i}{\partial \tilde{x}_p} \end{pmatrix} \equiv \begin{pmatrix} \Lambda_p{}^i(x) & -\partial_p \zeta_i \\ 0 & \delta^p{}_i \end{pmatrix} , \tag{2.38}$$

where we treated P is a row index and M as a column index, and we introduced the notation

$$\Lambda_p{}^i(x) \equiv \frac{\partial x^{i\prime}}{\partial x^p} . \tag{2.39}$$

The inverse matrix is then given by

$$\frac{\partial X_Q}{\partial X'_M} = \begin{pmatrix} \frac{\partial x^q}{\partial x^{i\prime}} & \frac{\partial \tilde{x}_q}{\partial x^{i\prime}} \\ \frac{\partial x^q}{\partial \tilde{x}'_i} & \frac{\partial \tilde{x}_q}{\partial \tilde{x}'_i} \end{pmatrix} , \tag{2.40}$$

where

$$\frac{\partial x^q}{\partial x^{i\prime}} = (\Lambda^{-1})_i{}^q , \quad \frac{\partial x^q}{\partial \tilde{x}'_i} = 0 , \quad \frac{\partial \tilde{x}_q}{\partial \tilde{x}'_i} = \delta_q{}^i , \quad \frac{\partial \tilde{x}_q}{\partial x^{i\prime}} = (\Lambda^{-1})_i{}^p \partial_p \zeta_q . \tag{2.41}$$

With (2.38) and (2.41) we now can compute the various components of the matrix $\mathcal{F}$ defined in (2.2), using the usual splitting $^M = (_i , {}^i)$,

$$\mathcal{F}_i{}^j = (\Lambda^{-1})_i{}^j , \qquad \mathcal{F}^i{}_j = \Lambda_j{}^i , \qquad \mathcal{F}^{ij} = 0 ,$$

$$\mathcal{F}_{ij} = \frac{1}{2}\Big((\Lambda^{-1})_i{}^p \Lambda_j{}^q \partial_p \zeta_q - \partial_j \zeta_i + (\Lambda^{-1})_i{}^p (\partial_p \zeta_j - \partial_j \zeta_p)\Big) . \tag{2.42}$$

We note that in this case the two terms in $\mathcal{F}$ are different and thus both needed. As a consistency check one may verify $\mathcal{F} \in O(D,D)$,

$$\mathcal{F}_M{}^P \mathcal{F}^N{}_P = \delta_M{}^N . \tag{2.43}$$

Specializing (2.1) to components we have, for instance,

$$g'^{ij}(x') = \mathcal{H}'^{ij}(x') = \mathcal{F}^i{}_k \mathcal{F}^j{}_l \mathcal{H}^{kl} = \Lambda_k{}^i \Lambda_l{}^j g^{kl} = \frac{\partial x^{i\prime}}{\partial x^k} \frac{\partial x^{j\prime}}{\partial x^l} g^{kl} , \tag{2.44}$$

i.e., the metric still transforms with a standard general coordinate transformation. For the transformation of the b-field we have to inspect an off-diagonal component,

$$-g'^{ip} b'_{pj} = \mathcal{H}'^i{}_j = \mathcal{F}^{iK} \mathcal{F}_j{}^L \mathcal{H}_{KL} = \mathcal{F}^i{}_k \mathcal{F}_j{}^L \mathcal{H}^k{}_L = \mathcal{F}^i{}_k \mathcal{F}_j{}^l \mathcal{H}^k{}_l + \mathcal{F}^i{}_k \mathcal{F}_{jl} \mathcal{H}^{kl}$$

$$= \Lambda_k{}^i \Big[(\Lambda^{-1})_j{}^l \mathcal{H}^k{}_l + \frac{1}{2}\Big((\Lambda^{-1})_j{}^p \Lambda_l{}^q \partial_p \zeta_q - \partial_l \zeta_j + (\Lambda^{-1})_j{}^p (\partial_p \zeta_l - \partial_l \zeta_p)\Big)\mathcal{H}^{kl}\Big] \tag{2.45}$$

$$= -\Lambda_k{}^i g^{kl}\Big[(\Lambda^{-1})_j{}^p b_{lp} - \frac{1}{2}(\Lambda^{-1})_j{}^p \partial_p \zeta_q \Lambda_l{}^q + \frac{1}{2} \partial_l \zeta_j - (\Lambda^{-1})_j{}^p \partial_{[p} \zeta_{l]}\Big] ,$$

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where we use the (anti)symmetrization convention $[ab] = \frac{1}{2}(ab - ba)$. Writing $g'^{ip}\, b'_{pj} = \Lambda_k{}^i \Lambda_l{}^p g^{kl}\, b'_{pj}$ the above expression quickly yields the following transformation for b:

$$b'_{ij} \;=\; (\Lambda^{-1})_i{}^k\,(\Lambda^{-1})_j{}^l\,\big(b_{kl} + \partial_{[k}\,\zeta_{l]}\big) \;+\; \tfrac{1}{2}\big((\Lambda^{-1})_i{}^k\partial_k\zeta_j - (\Lambda^{-1})_j{}^k\partial_k\zeta_i\big)\,. \tag{2.46}$$

In the last term Λ^{-1} just transforms ∂ into ∂' and so the result can also be written as

$$b'_{ij} \;=\; \frac{\partial x^k}{\partial x'^i}\,\frac{\partial x^l}{\partial x'^j}\,\big(b_{kl} + \partial_{[k}\,\zeta_{l]}\big) + \partial'_{[i}\,\zeta_{j]}\,. \tag{2.47}$$

This can be viewed as a b-field gauge transformation, followed by a diffeomorphism, followed by another b-field gauge transformation, with the same parameter, but performed in the new coordinate system. An alternative writing is obtained by recalling that $\partial_{[k}\,\zeta_{l]}$ transforms as a two-form, so that we also have

$$b'_{ij}(x') \;=\; \frac{\partial x^k}{\partial x'^i}\,\frac{\partial x^l}{\partial x'^j}\,b_{kl}(x) + \frac{1}{2}\Big(\partial'_i\big(\zeta'_j(x') + \zeta_j(x)\big) - \partial'_j\big(\zeta'_i(x') + \zeta_i(x)\big)\Big)\,, \tag{2.48}$$

where $\zeta'_i(x') = \frac{\partial x^k}{\partial x'^i}\zeta_k(x)$ is the coordinate transformed one-form parameter. Thus, the resulting transformation is a general coordinate transformation together with a b-field gauge transformation with respect to a parameter that is the average of the original one and the coordinate transformed one. Note that for a trivial coordinate transformation both terms agree; in particular a trivial transformation is given by $x'^i = x^i$, $\tilde{x}'_i = \tilde{x}_i - \partial_i\chi$.

The particular combination (2.48) is forced on us by the original $O(D,D)$ covariant form of generalized coordinate transformations, which in turn is compatible with the C-bracket. As we discussed above, the C-bracket has a non-trivial Jacobiator, leading to the non-associativity of coordinate transformations. We will illustrate in the next subsection the unusual composition law for the coordinate transformations underlying the transformations (2.48) and the non-associativity of successive compositions.

2.4 Composition, Courant-bracket and non-associativity

Let us now ask the question how the simultaneous general coordinate and b-field gauge transformations compose. We will see that they do not compose in the naive sense of coordinate maps, i.e., the 'group structure' will be non-trivial. To this end let us consider the consecutive action of two generalized coordinate transformations,

$$\begin{aligned}
m_1 \;:\; & & x'^i \;&=\; x'^i(x)\,, & \tilde{x}'_i \;&=\; \tilde{x}_i - \zeta_{1i}(x)\,,\\
m_2 \;:\; & & x''^i \;&=\; x''^i(x')\,, & \tilde{x}''_i \;&=\; \tilde{x}'_i - \zeta'_{2i}(x')\,,
\end{aligned} \tag{2.49}$$

where we denoted the transformations by m_1 and m_2 for later use. We note that we have chosen a notation with a $'$ on ζ_2 that is in principle redundant, because there is no independent definition of a $\zeta_2(x)$, but it is convenient in order to remind us with respect to which coordinate systems the parameters are originally defined. The first transformation in (2.49) leads by use of (2.48) to

$$b'_{ij}(x') \;=\; \frac{\partial x^k}{\partial x'^i}\,\frac{\partial x^l}{\partial x'^j}\,b_{kl}(x) + \frac{1}{2}\Big(\partial'_i\big(\zeta'_{1j}(x') + \zeta_{1j}(x)\big) - \partial'_j\big(\zeta'_{1i}(x') + \zeta_{1i}(x)\big)\Big)\,. \tag{2.50}$$

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Similarly, the second transformations leads to

$$b''_{ij}(x'') = \frac{\partial x'^p}{\partial x''^i} \frac{\partial x'^q}{\partial x''^j} b'_{pq}(x') + \frac{1}{2}\left(\partial''_i\left(\zeta''_{2j}(x'') + \zeta'_{2j}(x')\right) - \partial''_j\left(\zeta''_{2i}(x'') + \zeta'_{2i}(x')\right)\right). \quad (2.51)$$

Inserting now (2.50) in the first term in here we get

$$\begin{aligned}\frac{\partial x'^p}{\partial x''^i} \frac{\partial x'^q}{\partial x''^j} b'_{pq}(x') &= \frac{\partial x^k}{\partial x''^i} \frac{\partial x^l}{\partial x''^j} b_{kl}(x) + \frac{1}{2}\frac{\partial x'^p}{\partial x''^i} \frac{\partial x'^q}{\partial x''^j}\partial'_p\left(\zeta'_{1q}(x') + \zeta_{1q}(x)\right) - (i \leftrightarrow j) \\ &= \frac{\partial x^k}{\partial x''^i} \frac{\partial x^l}{\partial x''^j} b_{kl}(x) + \frac{1}{2}\left(\partial''_i\zeta''_{1j}(x'') + \frac{\partial x'^q}{\partial x''^j}\partial''_i\zeta_{1q}(x) - (i \leftrightarrow j)\right),\end{aligned} \quad (2.52)$$

where we used again that $\partial'_{[p}\zeta'_{q]}$ transforms as a 2-form, and we used the chain rule in the last term. This last term can be written as

$$\frac{1}{2}\frac{\partial x'^q}{\partial x''^j}\partial''_i\zeta_{1q}(x) = \frac{1}{2}\partial''_i\left(\frac{\partial x'^q}{\partial x''^j}\zeta_{1q}(x)\right) - \frac{1}{2}\frac{\partial^2 x'^q}{\partial x''^i\partial x''^j}\zeta_{1q}(x). \quad (2.53)$$

The last term is symmetric in i, j and thus drops out in (2.52). Thus we have

$$\frac{\partial x'^p}{\partial x''^i} \frac{\partial x'^q}{\partial x''^j} b'_{pq}(x') = \frac{\partial x^k}{\partial x''^i} \frac{\partial x^l}{\partial x''^j} b_{kl}(x) + \frac{1}{2}\partial''_i\left(\zeta''_{1j}(x'') + \frac{\partial x'^q}{\partial x''^j}\zeta_{1q}(x)\right) - (i \leftrightarrow j). \quad (2.54)$$

Using this now in (2.51) we obtain

$$b''_{ij}(x'') = \frac{\partial x^k}{\partial x''^i} \frac{\partial x^l}{\partial x''^j} b_{kl}(x) + \frac{1}{2}\partial''_i\left(\zeta''_{1j}(x'') + \frac{\partial x'^q}{\partial x''^j}\zeta_{1q}(x) + \zeta''_{2j}(x'') + \zeta'_{2j}(x')\right) - (i \leftrightarrow j). \quad (2.55)$$

This is the final form of the consecutive action of (2.49) on the b-field.

Let us now see how this compares to the transformation associated with the naive composition of the generalized coordinate transformations. To this end we have to view this transformation as a single generalized coordinate transformation $X \to X''$. According to (2.48) such a transformation acts as

$$b''_{ij}(x'') = \frac{\partial x^k}{\partial x''^i} \frac{\partial x^l}{\partial x''^j} b_{kl}(x) + \frac{1}{2}\partial''_i\left(\zeta''_{12j}(x'') + \zeta_{12j}(x)\right) - (i \leftrightarrow j), \quad (2.56)$$

where

$$\tilde{x}''_i = \tilde{x}_i - \zeta_{12i}(x), \quad (2.57)$$

for some effective parameter $\zeta_{12i}(x)$. Next we have to compare this parameter with the one that would emerge from direct composition of the two transformations (2.49), which we denote by ϑ_{12}. This parameter is easily computed,

$$\tilde{x}''_i = \tilde{x}'_i - \zeta'_{2i}(x') = \tilde{x}_i - \zeta'_{2i}(x') - \zeta_{1i}(x) \equiv \tilde{x}_i - \vartheta_{12i}(x), \quad (2.58)$$

and thus

$$\vartheta_{12i}(x) = \zeta'_{2i}(x') + \zeta_{1i}(x). \quad (2.59)$$

In here, of course, we have to think of x' as a function of x according to (2.49), in order for both sides to be functions of x. Replacing ζ_{12} by ϑ_{12} in (2.56) yields

$$b''_{ij}(x'') = \frac{\partial x^k}{\partial x''^i} \frac{\partial x^l}{\partial x''^j} b_{kl}(x) + \frac{1}{2}\partial''_i\left(\zeta''_{1j}(x'') + \zeta_{1j}(x) + \frac{\partial x^k}{\partial x''^j}\zeta'_{2k}(x') + \zeta'_{2j}(x')\right) - (i \leftrightarrow j). \quad (2.60)$$

Comparing with (2.55) it is evident that this differs, for generic values of ζ_1 and ζ_2, in the second term from the actual transformation $b \to b''$. This is, of course, what we expected, since we saw above that composition is governed by the C-bracket, which differs from the Lie bracket governing ordinary composition of diffeomorphisms when one considers both a non-zero diffeomorphism and b-field gauge parameter.

It would be convenient to have a closed expression for the effective parameter ζ_{12} in terms of ζ_1 and ζ_2 in order to investigate the unconventional rules of composition in more detail. One can, of course, determine ζ_{12} to arbitrary order in an derivative expansion by using the BCH formula and the C-bracket. However, so far we did not find a simple closed expression for ζ_{12}, but one may hope that a better analytic understanding of the composition is possible. It is likely that this would require a better parametrization of the transformations (2.49).

We close this section by discussing why generalized coordinate transformations of the type (2.49) are necessarily non-associative as transformations of coordinates when both the diffeomorphism and the b-field gauge parameter are non-trivial. This follows from the specific form of the Jacobiator of the C-bracket,

$$J(U,V,W) \;=\; \big[[U,V]_C, W\big]_C + \big[[V,W]_C, U\big]_C + \big[[W,U]_C, V\big]_C \,, \tag{2.61}$$

which is given by

$$J(U,V,W) \;=\; \frac{1}{6}\partial^M\Big(\langle [U,V]_C, W\rangle + \langle [V,W]_C, U\rangle + \langle [W,U]_C, V\rangle\Big)\,, \tag{2.62}$$

c.f. (2.29) above. From the structure of the C-bracket it follows that this vanishes for $\tilde{\partial}^i = 0$ when all three arguments have only vector parts, i.e., when $U_i = 0$, etc. In contrast, for non-zero vector and one-form contributions this is non-zero. In particular, the composition of three transformations with non-zero diffeomorphism and b-field gauge parameter leads to a non-trivial Jacobiator. Thus, if we add to the two transformations m_1 and m_2 in (2.49) a third transformation m_3 and if we denote the composition of generalized coordinate transformations by $\star$, it follows that generically

$$m_3 \star (m_2 \star m_1) \;\neq\; (m_3 \star m_2) \star m_1 \,. \tag{2.63}$$

In other words, combined diffeomorphisms and b-field gauge transformations are non-associative when viewed as coordinate transformations on the doubled space. On the contrary, as we explained above, these transformations are perfectly associative when acting on fields.[1]

3 $O(d,d)$ as generalized coordinate transformations

In this section we discuss the realization of particular $O(D,D)$ transformations as generalized coordinate transformations of DFT. It turns out that in presence of d commuting isometries, i.e., for field configurations that are independent of d coordinates, the $O(d,d)$ subgroup can be viewed as part of the generalized coordinate transformations. The action on the coordinates is, however, different from the naive $O(d,d)$ action but nevertheless reproduces the required field

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[1]Further comments on the diffeomorphisms of DFT can be found in [24].

transformations. In order to understand this somewhat unexpected feature, in the first subsection we discuss again, but from a different perspective, how the generalized diffeomorphisms of DFT differ from ordinary diffeomorphisms. In the second subsection we discuss explicitly how to realize $O(d,d)$ transformations as generalized coordinate transformations.

3.1 The role of $O(D,D)$ and its invariant metric

We have seen above that the presence of a gauge invariant flat metric η_{MN} does not impose constraints on the geometry of the D-dimensional subspace of the doubled space since the gauge transformations (or coordinate transformations) are governed by generalized Lie derivatives (or generalized coordinate transformations). Here we will briefly elucidate this point from yet another perspective.[2] We explain that generalized diffeomorphisms cannot be viewed as ordinary diffeomorphisms in a doubled space subject to constraints such as, e.g., the condition that they preserve the constant metric η_{MN}.

Naively, one may have tried to implement a doubled geometry with $O(D,D)$ metric η_{MN} as follows. Start from a conventional $2D$-dimensional manifold $\mathcal{M}_{2D}$, thus governed by the conventional diffeomorphism group $\mathrm{Diff}(\mathcal{M}_{2D})$, but impose the additional condition that it respects the $O(D,D)$ metric η_{MN}. This would be in complete analogy to symplectic geometry, where one starts with a $2D$-dimensional manifold, but then imposes the constraint that a symplectic form is left invariant. This in turn reduces the diffeomorphism group to the (still infinite-dimensional) group of symplectomorphisms. Applying the same strategy to the $O(D,D)$ metric we would require

$$\delta_\xi \eta_{MN} \;=\; \mathcal{L}_\xi \eta_{MN} \;=\; \xi^K \partial_K \eta_{MN} + \partial_M \xi^K \eta_{KN} + \partial_N \xi^K \eta_{KM} \;=\; 0 \,. \tag{3.1}$$

Using η_{MN} to raise and lower indices this condition becomes

$$\partial_M \xi_N + \partial_N \xi_M \;=\; 0 \,. \tag{3.2}$$

This is the usual Killing equation on a flat space, whose general solution is $\xi_M = a_M + \Lambda_{MN} X^N$, with a_M and $\Lambda_{MN} = -\Lambda_{NM}$ constant, corresponding to translations and rigid $SO(D,D)$ transformations. Thus, the diffeomorphism group is reduced according to

$$\mathrm{Diff}(\mathcal{M}_{2D}) \;\rightarrow\; \mathrm{ISO}(D,D) \,. \tag{3.3}$$

If we now apply such a transformation to another tensor, say a vector V_M, we can use condition (3.2) and write

$$\begin{aligned}
\delta_\xi V_M \;&=\; \xi^K \partial_K V_M + \partial_M \xi^K V_K \\
&=\; \xi^K \partial_K V_M + \frac{1}{2}\big(\partial_M \xi^K - \partial^K \xi_M\big) V_K \,.
\end{aligned} \tag{3.4}$$

This differs from the generalized Lie derivative $\widehat{\mathcal{L}}_\xi V_M = \xi^K \partial_K V_M + \big(\partial_M \xi^K - \partial^K \xi_M\big) V_K$ due to the factor of one-half in the last term. This shows quite clearly that the generalized diffeomorphisms cannot be interpreted as conventional diffeomorphisms on the doubled space. In fact,

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[2]OH thanks Axel Kleinschmidt for discussions on this point.

while the generalized Lie derivatives close according to the modified C-bracket, by construction the similarly looking transformations (3.4) must still close according to the conventional Lie bracket. To verify this we compute the commutator

$$
\begin{aligned}
\left[\delta_{\xi_1}, \delta_{\xi_2}\right] V_M &= \left[\xi_2, \xi_1\right]^K \partial_K V_M + \frac{1}{2}\left(\partial_M \left[\xi_2, \xi_1\right]^K - \partial^K \left[\xi_2, \xi_1\right]_M\right) V_K \\
&\quad - \frac{1}{4}\left(\left(\partial_M \xi_{2K} + \partial_K \xi_{2M}\right)\left(\partial^K \xi_1^L + \partial^L \xi_1^K\right) - (1 \leftrightarrow 2)\right) V_L \,,
\end{aligned}
\tag{3.5}
$$

with the conventional Lie bracket

$$
\left[\xi_1, \xi_2\right]^K = \xi_1^L \partial_L \xi_2^K - \xi_2^L \partial_L \xi_1^K \,.
\tag{3.6}
$$

The first line in (3.5) gives the expected transformation $\delta_{\xi_{12}} V_M$ with $\xi_{12} = [\xi_2, \xi_1]$, while in the second line only terms with the symmetrized derivative of the parameters survived, which are zero by (3.2). Thus, the gauge transformations (3.4) close according to the usual Lie bracket.

In summary, if we work with a conventional $2D$-dimensional manifold with metric η_{MN}, diffeomorphisms are restricted to $\mathrm{ISO}(D, D)$ transformations to preserve the metric, and their closure is governed by the usual Lie bracket. These transformations do not contain the full D-dimensional diffeomorphism group but only its rigid subgroup $GL(D, \mathbb{R}) \subset \mathrm{ISO}(D, D)$. If we work instead with a *generalized* $2D$-dimensional manifold with metric η_{MN}, we impose the strong constraint on all fields and gauge parameters. Generalized diffeomorphisms preserve the constant form of η_{MN} because generalized Lie derivatives satisfy $\widehat{\mathcal{L}}_\xi \eta_{MN} = 0$ for all ξ^M. Generalized Lie derivatives close with an algebra governed by the C-bracket. The strong constraint restricts the possible generalized diffeomorphisms but allows arbitrary conventional diffeomorphisms of the D-dimensional subspace, thereby not posing any constraints on the physical spacetime.

As we will discuss below, for backgrounds with commuting isometries, e.g., for torus backgrounds T^d, generalized diffeomorphisms include the T-duality $O(d, d)$ transformations, but with gauge parameters that differ from the naive $O(d, d)$ ansatz. Specifically, in a particular form of the generalized coordinate transformations (related to others by a trivial gauge transformation, c.f. eq. (2.13) above) it differs by factors of one-half, which are related to those encountered above.

3.2 $O(d, d)$ and dimensional reduction

We now discuss how for special field configurations $O(d, d)$ transformations result from generalized coordinate transformations. This is relevant for configurations for which the fields are independent of a subset of d coordinates. This condition on the background holds if we have, for example, d commuting isometries, taken to mean that all fields have zero Lie derivatives along d vector fields that have zero Lie brackets. The reduction, called strict dimensional reduction, is different from Kaluza-Klein compatification, where massive modes arise from coordinate dependence along the extra dimensions.

Specifically, we split the coordinates into (x^μ, y^α), $\alpha = 1, \ldots, d$, and assume that metric and b-field are independent of y^α. We then focus on the $O(d, d)$ action on the field components

along the coordinate directions y^α, which we may combine into a $d \times d$-dimensional matrix,

$$\mathcal{E}_{\alpha\beta} = g_{\alpha\beta} + b_{\alpha\beta} \,. \tag{3.7}$$

The $O(d,d)$ transformations are then encoded in the usual covariant rotation of the generalized metric or, equivalently, by

$$\mathcal{E}' = g_{O(d,d)}\mathcal{E} = (A\mathcal{E} + B)(C\mathcal{E} + D)^{-1} \,. \tag{3.8}$$

Here $g_{O(d,d)}$ is an $O(d,d)$ group element of the form

$$g_{O(d,d)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,, \tag{3.9}$$

with d-dimensional matrices A, B, C, D satisfying

$$A^t C + C^t A = 0 \,, \quad B^t D + D^t B = 0 \,, \quad A^t D + C^t B = I \,. \tag{3.10}$$

Our task is now to realize the required $O(d,d)$ transformations as generalized coordinate transformations in DFT. Throughout this section we will not assume any topological conditions for the background configurations and so we ignore the question whether the generalized coordinate transformations to be given below are 'globally well-defined'. In other words, we assume the 'internal' space on which $O(d,d)$ acts to be non-compact, with fields independent of these coordinates, which is the case relevant for a strict dimensional reduction. The physically more relevant case of toroidal backgrounds requires some care for the quantization conditions and will be discussed in detail in the next section.

Before we consider the relevant special subgroups of $O(d,d)$ let us first discuss the action of a generic $O(d,d)$ element $h^M{}_N$ viewed as the coordinate transformation of the (doubled) internal coordinates $Y^M = (\tilde{y}_\alpha, y^\alpha)$

$$Y'^M = h^M{}_N Y^N \,, \quad \text{or} \quad Y' = hY \,. \tag{3.11}$$

A straightforward computation, whose details can be found in [14], shows that the above coordinate transformation leads to an $\mathcal{F}$ of the form

$$\mathcal{F}^M{}_N = \left[(h^{-1})^2\right]^M{}_N \,. \tag{3.12}$$

This means that as far as the rotation of field components is concerned, it acts as the square of the expected matrix, while for the transformation of the coordinate argument it acts as expected. But since the fields are assumed to be independent of Y this allows us to modify the transformations in order to fix the field transformations. We now turn to the various special subgroups for which we will see that the coordinate transformations can be adapted, e.g. by taking the square root, so as to induce the required field transformation.

(i) $GL(d)$ transformations
These are given in terms of the subgroup defined by $A, D \in GL(d)$ with

$$D = (A^t)^{-1} \,, \quad B = C = 0 \,. \tag{3.13}$$

The action of this $GL(d)$ subgroup of $O(d,d)$ rotates both the y coordinates and their duals $\tilde{y}$. We can consider, however, the transformation that rotates the y's but not the $\tilde{y}$'s:

$$y' = (A^t)^{-1} y \,. \tag{3.14}$$

These are trivially contained in the generalized coordinate transformations of DFT. An equivalent transformation at the level of fields, and thus related to the above by a 'trivial' transformation, rotates y and $\tilde{y}$ with the square root of the $O(d,d)$ matrix (assuming this element is in the component connected to the identity).

(ii) Shifts in the b-field

Constant shifts in the b-field are given by the following matrices:

$$A = D = I \,, \quad C = 0 \quad \rightarrow \quad B^t = -B \,. \tag{3.15}$$

These transformations can be viewed as generalized coordinated transformations (2.9) that implement b-field gauge transformation (2.8),

$$\tilde{y}'_\alpha = \tilde{y}_\alpha - \tilde{\xi}_\alpha(y) \,, \qquad \tilde{\xi}_\alpha(y) = \frac{1}{2} B_{\alpha\beta} y^\beta \,. \tag{3.16}$$

Note that there is a additional factor of one-half when comparing this generalized coordinate transformation with the $O(d,d)$ transformation induced by (3.15) acting on the coordinates as an $O(d,d)$ vector. This factor of one-half should not come as a surprise, for it corresponds precisely to the same factor encountered in sec. 3.1 when comparing conventional coordinate transformations on a doubled space with the generalized coordinate transformations of DFT. There is no contradiction in the appearance of this factor one-half, because the fields do not depend on $\tilde{y}$, and thus their arguments are not transformed, with the result that the field transformations are exactly as required by the $O(d,d)$ action. Let us note that when considering the double torus, a transformations with these factors of one-half may be incompatible with the periodicity conditions. We will see, however, that in these cases there is an alternative coordinate transformation, with the same action on fields, i.e., related by a trivial gauge transformation, that *is* compatible with the torus identifications.

(iii) Shifts in β

These are the transformations that are conjugate to (ii), i.e.

$$A = D = I \,, \quad B = 0 \quad \rightarrow \quad C^t = -C \,. \tag{3.17}$$

This transformation corresponds to constants shifts of the β field defined in (2.11). It can again be viewed as a generalized coordinated transformation, this time in the form (2.10). In order to see this recall that in (3.12) we obtained for the naive coordinate transformation the square of the $O(d,d)$ matrix. Thus, while generally the $O(d,d)$ transformations cannot be viewed as generalized coordinate transformations, they do allow for such an interpretation in case of group elements connected to the identity and for backgrounds with abelian isometries (for which the fields do not depend on the corresponding coordinates). For then we can simply take the square root of the $O(d,d)$ element so that the rotation of field components is as required. Specialized to (3.17) this amounts to replacing the $O(d,d)$ matrix according to

$$h^M{}_N(C) = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ C^{\alpha\beta} & \delta^\alpha{}_\beta \end{pmatrix} \quad \Rightarrow \quad (\sqrt{h})^M{}_N(C) = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ \frac{1}{2} C^{\alpha\beta} & \delta^\alpha{}_\beta \end{pmatrix} \,. \tag{3.18}$$

Equivalently, this amounts to choosing in (2.10)

$$\xi^\alpha(\tilde{y}) = \frac{1}{2}C^{\alpha\beta}\tilde{y}_\beta \,, \tag{3.19}$$

which by construction induces the expected β gauge transformation. Note that the parameter now depends on $\tilde{y}_\alpha$, which is compatible with the strong constraint since we assumed that there are isometries along the dual y^α directions. Note also that there is an additional factor of one-half when comparing this transformation with the corresponding element of $O(d,d)$, because we needed to take the square root. This, again, is perfectly consistent and to be expected in view of the discussion in sec. 3.1, but in order to be compatible with the torus identifications we will eventually adopt again a modified generalized coordinate transformation without factors of one-half.

(iv) Factorized T-duality along all directions
Finally, for the so-called factorized T-duality along all directions one may choose an $O(d,d)$ matrix of the form

$$A = D = 0 \,, \quad C = B = I \,, \tag{3.20}$$

which exchanges x coordinates and $\tilde{x}$ coordinates.[3] In this case we have to distinguish between d odd and d even. In fact, for d odd we are dealing with an $O(d,d)$ element that cannot be continuously connected with the identity element of $O(d,d)$. In contrast, for d even we can find a continuous path connecting the identity with an $O(d,d)$ transformation exchanging x and $\tilde{x}$. Only in the latter case can we associate such a T-duality transformation with a generalized coordinate transformation in generality, as we will now discuss, giving a counter-example for $d = 1$.

Let us now investigate the simplest example, $d = 1$, in which case the most general (constant) coordinate transformation takes the form

$$Y' = \begin{pmatrix} \tilde{y}' \\ y' \end{pmatrix} = QY \,, \qquad Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \,. \tag{3.21}$$

Note that for generality here we do not restrict to an $O(2,2)$ matrix, because viewed as a coordinate transformation a priori we may employ a general, invertible transformation. The transformation matrix $\mathcal{F}$ is then given by

$$\mathcal{F} = \frac{1}{2}\left((Q^{-1})^t\eta Q\eta + \eta Q\eta(Q^{-1})^t\right) = \frac{1}{ad-bc}\begin{pmatrix} d^2 - bc & 0 \\ 0 & a^2 - bc \end{pmatrix} \,. \tag{3.22}$$

In order for this transformation to describe the overall factorized T-duality inversion it would have to be of an off-diagonal form, such as

$$\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{3.23}$$

This follows since in a generalized coordinate transformation $\mathcal{F}$ acts on the generalized metric $\mathcal{H}$ in the same way as an $O(d,d)$ element and so would have to be of the (off-diagonal) matrix

---

[3]A factorized T duality is a 'genuine' T-duality transformation in the sense that it is an $O(d,d)$ transformation that does not belong to the 'geometric subgroup' $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{\frac{1}{2}d(d-1)}$ that originates from special general coordinate transformations and b-field gauge transformations.

form of a factorized T-duality. There is, however, clearly no solution for a, b, c, d that satisfies this condition.

We now turn to the case that d is even, focusing on $d = 2$. For this case an explicit continuous family of $O(2, 2)$ matrices connecting the identity with a genuine T-duality transformation was constructed in appendix A.3 in the second reference of [27]. This family is defined by

$$h(\alpha) \;=\; \exp\left[\alpha\left(T^{14} + T^{12} + T^{32} + T^{34}\right)\right], \qquad \alpha \in \left[0, \frac{\pi}{2}\right], \tag{3.24}$$

where $(T^{MN})^K{}_L = 2\eta^{K[M}\delta^{N]}{}_L$ denote the $O(d, d)$ generators in the fundamental representation. For $\alpha = 0$ this gives the identity matrix, while for $\alpha = \frac{\pi}{2}$ we obtain

$$h(\tfrac{\pi}{2}) \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3.25}$$

which is the action of two T-dualities, in directions 1 and 2. Since we saw in eq. (3.12) that in order to reproduce a genuine T-duality transformation as a generalized coordinate transformation we have to take the *square root* of the $O(d, d)$ matrix this is what we have to do for (3.25). Since this transformations is connected to the identity this is straightforward and we obtain

$$\left(h(\tfrac{\pi}{2})\right)^{\frac{1}{2}} \;=\; h(\tfrac{\pi}{4}) \;=\; \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}. \tag{3.26}$$

By construction, the corresponding generalized coordinate transformation $X \to X' = h(\tfrac{\pi}{4})X$ induces the same $O(d, d)$ transformation *on fields* as (3.25). We will use (3.26), in an alternative form with an identical action on fields, to discuss a (genuinely) non-geometric background.

4   T-folds as well-defined backgrounds in DFT

In this section we discuss particular spacetimes in DFT and their patching conditions as generalized coordinate transformations. Conventional spacetime manifolds are generally encoded in DFT via coordinate patchings that involve only the x, not the $\tilde{x}$, for which the generalized coordinate transformations of DFT reduce to the usual coordinate transformations of conventional differential geometry. In contrast, for certain special toroidal backgrounds we can allow for patching conditions that involve the $\tilde{x}$, in which case the coordinate transformations are related but not identical to $O(d, d)$ transformations. This will be illustrated with various examples.

4.1   Generalities

We start by discussing generalized string backgrounds, which are configurations that are defined using generalized coordinate transformations for patching together different coordinate charts

instead of the standard diffeomorphisms used for differentiable Riemannian manifolds. Known examples are T-folds [28, 29], namely non-geometric string backgrounds with so-called Q-fluxes that are locally Riemannian spaces but fail to be so globally because field configurations around non-trivial homology cycles are glued using T-duality transformations rather than b-field gauge transformations or diffeomorphisms. In addition to their essential role for the definition of T-folds, T-duality transformations often relate geometrical and non-geometrical backgrounds to each other. For example, a three-dimensional torus with constant H-field is T-dual along one direction to a geometrical twisted torus, which after a T-duality transformation along a second direction is T-dual to a three-dimensional T-fold with non-geometric Q-flux. Performing one more T-duality transformation along the third direction one obtains a non-geometric R-flux background, leading to the well-known chain [30]

$$H_{abc} \xrightarrow{T_x} f^a{}_{bc} \xrightarrow{T_y} Q_c{}^{ab} \xrightarrow{T_z} R^{abc} . \tag{4.1}$$

Here, $f^a{}_{bc} = -2e_{[b}{}^m e_{c]}{}^n \partial_m e_n{}^a$ is the 'geometric flux' related to the Levi-Civita spin connection. The R-flux background is usually thought not to be a Riemannian space, even locally. Using DFT and dual coordinates, however, one can begin to make sense of the background associated with an R-flux space.

The geometric interpretation of the new fluxes Q and R, in contrast to H and f, is not clear a priori in terms of the usual 10-dimensional supergravities. However, as the non-geometric fluxes are obtained through $O(D, D)$ transformations we can use DFT to formulate an action for non-geometric string backgrounds, which is on equal footing with the original NS action in eq. (1.1). To be more specific, as explained in [25, 26, 31–34], one can perform certain field redefinitions of the metric g and the b-field, which have the form of $O(D, D)$ transformations, and which lead to new background variables, namely a metric $\tilde{g}_{ij}$, a bivector β^{ij} instead of the b-field, and a dilaton $\tilde{\phi}$. Alternatively, we can take a different parametrization of the fundamental generalized metric in terms of a metric and a bivector rather than a 2-form, see (2.11). The action written in terms of these variables schematically takes the form

$$S = \int d^D x \, d^D \tilde{x} \, \sqrt{-\tilde{g}} e^{-2\tilde{\phi}} \left[\tilde{\mathcal{R}} + 4(\partial \tilde{\phi})^2 - \frac{1}{4} Q^2 + \cdots \right] , \tag{4.2}$$

where the dots represent terms that vanish when we set the differential operators $\beta^{ij}\partial_j$ and $\tilde{\partial}^i$ equal to zero. The non-geometric Q-flux (replacing H) is given by

$$Q_i{}^{jk} = \partial_i \beta^{jk} . \tag{4.3}$$

This action is invariant under beta gauge transformations of the form (2.12). Although this is not manifest in the above form, one can show that Q plays a natural role as (part of) a new connection $\tilde{\nabla}^i$ for the winding derivatives [25], see also [34]. More precisely, the Q^2 term in the above action then becomes part of a second (dual) Einstein-Hilbert term involving winding derivatives. Moreover, as discussed in [25], the full DFT action written in this form also encodes R-flux contributions, which read

$$R^{ijk} = 3\left(\tilde{\partial}^{[i} \beta^{jk]} + \beta^{p[i} \partial_p \beta^{jk]} \right) . \tag{4.4}$$

This is a tensor under the 'beta gauge transformations' (2.12) parametrized by ξ^i. Its leading term is the complete T-dual of the H-flux and not visible in conventional geometry with only

x coordinates. Its subleading term, however, is visible in conventional supergravity written in terms of the new variables $\tilde{g}_{ij}$ and β^{ij}. We finally note that one can also show [35] that non-geometric objects like Q-branes are globally well-defined solutions of the action (4.2), in analogy to their T-dual counterparts, the NS-5-branes, which are well-defined solutions of the original action (1.1).

In the following we will discuss some aspects of non-geometric backgrounds and their corresponding $O(D, D)$ monodromy transformations. We will show that these can be seen as generalized coordinate transformations, which define the generalized patching conditions of the non-geometric background spaces. In particular, we will discuss a background that is genuinely non-geometric in the sense that it is not T-dual to a geometric background, but which can nevertheless be consistently defined in DFT by virtue of generalized coordinate transformations that take the form of factorized T-dualities.

The $D = (d + d')$-dimensional backgrounds $\mathcal{M}^{d+d'}$ to be considered in the following can be described in a convenient uniform manner: they all take the form, at least locally, of a fibration of a d-dimensional torus T_f^d over a d'-dimensional base $\mathcal{B}^{d'}$:

$$T_f^d \hookrightarrow \mathcal{M}^{d+d'} \to \mathcal{B}^{d'} \, . \tag{4.5}$$

In our examples, the base space will be the one-dimensional circle S^1.

We will show that all backgrounds to be defined below are globally well-defined in DFT according to the following universal picture. The coordinates are split into x^1, x^2 for the 2-torus (augmented by the dual coordinates $\tilde{x}_1, \tilde{x}_2$ for the doubled torus) and the coordinate z, with the identification $z \sim z + 2\pi$. We consider backgrounds whose metric and b-field depend only on z. In order to show that such a background is globally well-defined we have to verify that the metric and b-field at $z = 0$ and $z = 2\pi$ are gauge equivalent and so can be consistently 'glued together'. In standard supergravity this is the case if they are related by a diffeomorphism or a b-field gauge transformation. However, DFT also allows for genuinely non-geometric backgrounds, for which generalized coordinate transformations are required that take the form of genuine T-duality transformations. More specifically, we will show that in each case there is a generalized coordinate transformation of the doubled torus coordinates $(x^1, x^2, \tilde{x}_1, \tilde{x}_2)$ so that

$$\mathcal{H}'(g', b')(z = 2\pi) \; = \; \mathcal{H}(g, b)(z = 0) \, , \tag{4.6}$$

as depicted graphically in Figure 1.

While for conventional backgrounds the generalized diffeomorphism used in the gluing acts only on (x^1, x^2) and so reduces to a conventional diffeomorphism, for the non-geometric backgrounds a generalized diffeomorphism is required that acts non-trivially on the full doubled coordinates. Before we discuss this in detail, we review in the next subsection the relation between the needed $O(d, d)$ transformations and generalized diffeomorphisms.

We now illustrate the above general results with various examples motivated by non-geometric string compactifications and non-geometric fluxes. For comparison with the literature on the subject it is instructive to give these transformations also in terms of the Kähler
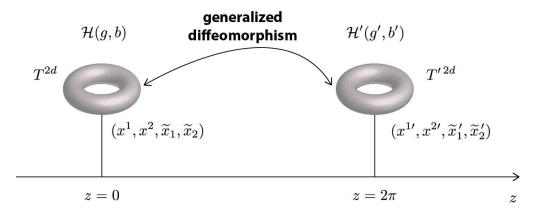
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Figure 1: Construction of a background field configuration for a double-torus fibration over a circle $z \sim z + 2\pi$. A generalized diffeomorphism of the double torus induces the desired identifications of fields at $z = 0$ and $z = 2\pi$.

parameter ρ and the complex structure τ of the torus, which are given by

$$\rho \;=\; -b_{12} + i\,V\,, \qquad \tau \;=\; \frac{g_{12}}{g_{11}} + i\,\frac{V}{g_{11}}\,, \tag{4.7}$$

where V is the volume of the 2-torus. The T-duality group $O(2,2)$ acts on ρ and τ according to the isomorphism

$$SO(2,2) \;\cong\; SL(2)_\tau \times SL(2)_\rho\,, \tag{4.8}$$

with the usual $SL(2)$ action on τ and ρ

$$\tau' \;=\; \frac{a\tau + b}{c\tau + d}\,, \qquad \rho' \;=\; \frac{a'\rho + b'}{c'\rho + d'}\,. \tag{4.9}$$

The explicit embedding for the $SL(2)$ parameters into an $O(D,D)$ matrix as (3.9) reads [39],

$$A = a' \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad B = b' \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}\,, \quad C = c' \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}\,, \quad D = d' \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\,. \tag{4.10}$$

4.2 3-torus with constant H-flux

We first consider a three-dimensional target space, which we take to be a flat torus along the directions $x^i = (x^1, x^2, x^3 = z)$, with an H-flux $H_3 = \bar{H} dx^1 \wedge dx^2 \wedge dz$, where $\bar{H}$ is a constant. The H flux is quantized according to [36][4]

$$\frac{1}{(2\pi)^2 \alpha'} \int H_3 \;\in\; \mathbb{Z}\,. \tag{4.11}$$

For now we use conventions for which the length dimensions of fields, constants and coordinates is given by

$$[g] \;=\; [b] \;=\; 0\,, \quad [\alpha'] \;=\; L^2\,, \quad [x] \;=\; L\,, \quad [\bar{H}] \;=\; \frac{1}{L}\,, \tag{4.12}$$

---

[4]This holds for conventions in which the world-sheet string action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left(\sqrt{h} h^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j\, g_{ij} + \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j\, b_{ij} \right)\,.$$

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where x collectively denotes the coordinates x^i and L is a length scale. The metric takes the constant form $g_{ij} = \delta_{ij}$ and the coordinates are identified according to $x^i \sim x^i + 2\pi R_i$. Therefore the flux integral in (4.11) is given by

$$\int H_3 = \int \bar{H} d^3 x = (2\pi)^3 R_1 R_2 R_3 \bar{H} , \tag{4.13}$$

so that the quantization condition (4.11) becomes

$$\frac{2\pi R_1 R_2 R_3 \bar{H}}{\alpha'} \in \mathbb{Z} . \tag{4.14}$$

The b-field, in a particular gauge, can be written as

$$b \equiv b_{12} = -b_{21} = \bar{H} z , \tag{4.15}$$

with all other components set to zero.

Next, we investigate the question whether this background is globally well-defined. To this end we have to compare the field configurations at $z = 0$ and $z = 2\pi R_3$. While g is constant, we find that for b

$$b(2\pi R_3) - b(0) = 2\pi R_3 \bar{H} . \tag{4.16}$$

This can be compensated by a b-field gauge transformation acting on the fields living in a neighborhood of $z = 2\pi R_3$. A possible choice is:

$$\tilde{\xi}_1 = 2\pi R_3 \bar{H} x^2 , \quad \tilde{\xi}_2 = 0 \quad \rightarrow \quad b' = b - 2\pi R_3 \bar{H} , \tag{4.17}$$

so that after this gauge transformation

$$b'(2\pi R_3) = b(2\pi R_3) - 2\pi R_3 \bar{H} = b(0) , \tag{4.18}$$

and the space is globally well-defined despite the apparent lack of periodicity in z.

As reviewed earlier, the above b-field gauge transformation can also be realized as a generalized coordinate transformation which, noting (4.17), is written as

$$\begin{aligned} x^{1'} &= x^1 , \\ x^{2'} &= x^2 , \\ \tilde{x}'_1 &= \tilde{x}_1 - 2\pi R_3 \bar{H} x^2 , \\ \tilde{x}'_2 &= \tilde{x}_2 . \end{aligned} \tag{4.19}$$

Therefore this background possesses patching conditions that are naturally viewed as DFT generalized coordinate transformations, mixing x and $\tilde{x}$ coordinates and thus treating the required general coordinate and b-field gauge transformations on the same footing.

Let us now show that the above coordinate transformations on the double torus in fact require the quantization of the H-flux. For this we take the $\tilde{x}_1$ circle to have the radius $\tilde{R}_1$ given by T-duality:

$$\tilde{R}_1 = \frac{\alpha'}{R_1} . \tag{4.20}$$

Then, in (4.19), if we shift $x^2 \to x^2 + 2\pi R_2$ we must get the same point on the dual torus. This requires that the resulting shift in $\tilde{x}_1$ be some integer multiple n of $2\pi \tilde{R}_1$:

$$(2\pi R_3)(2\pi R_2)\,\bar{H} \;=\; n\,2\pi\frac{\alpha'}{R_1} \quad \to \quad 2\pi\,R_1 R_2 R_3\,\frac{1}{\alpha'}\,\bar{H} \;=\; n\,, \tag{4.21}$$

which is precisely the quantization condition (4.14) of the flux. We could have realized the same duality transformation by the coordinate transformation $\tilde{x}_2' = \tilde{x}_2 + 2\pi R_3 \bar{H} x^1$, which by the same argument gives the flux quantization upon taking the dual circle along $\tilde{x}_2$ to have radius $\tilde{R}_2 = \frac{\alpha'}{R_2}$. We have thus arrived at a purely geometric perspective on the flux quantization condition. Let us note, finally, that for this background DFT is not strictly needed as the patching may also be done in conventional supergravity using the allowed b-field gauge transformations. This will change for some examples below.

We next discuss the above example for slightly different conventions, which will be more convenient below. Specifically, we will work with dimensionless coordinates so that, e.g., the metric becomes dimensionful, and the radii no longer enter the coordinates but the metric. To this end we perform the coordinate transformation $x^i \to \hat{x}^i = \frac{x^i}{R_i} \in [0, 2\pi]$, dropping the hats shortly. This leads to a metric that is given in terms of the radii R_i by

$$g \;=\; \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}. \tag{4.22}$$

The new b-field after this coordinate transformation ($\hat{b} \equiv \hat{b}_{12}$) is given by

$$\hat{b} \;=\; R_1 R_2\, b \;=\; R_1 R_2\, \bar{H}\, z \;=\; R_1 R_2 R_3\, \bar{H}\, \hat{z} \;\equiv\; \alpha' H\, \hat{z}\,, \tag{4.23}$$

where we introduced

$$H \;\equiv\; \frac{R_2 R_2 R_3 \bar{H}}{\alpha'} \quad \to \quad 2\pi H \in \mathbb{Z}\,, \tag{4.24}$$

and rewrote the quantization condition (4.14). In these conventions the $\hat{b}$-field has dimension L^2 and so has its one-form gauge parameter $\hat{\tilde{\xi}}$, since the hatted coordinates are dimensionless. We have that, schematically, $\hat{\tilde{\xi}} = R\tilde{\xi}$, which leads to the expected form for the b-field gauge transformations $\delta\hat{b} = \hat{\partial}\hat{\tilde{\xi}}$. This leads to an α' dependence in the relation between b-field gauge transformations and generalized coordinate transformations. In fact, defining dimension-free dual coordinates with standard periodicity $\hat{\tilde{x}} \equiv \frac{\tilde{x}}{R} = \frac{R}{\alpha'}\tilde{x}$, makes the coordinate transformation $\tilde{x}' = \tilde{x} - \tilde{\xi}$ turn into

$$\hat{\tilde{x}}' \;=\; \hat{\tilde{x}} - \frac{R}{\alpha'}\tilde{\xi} \;=\; \hat{\tilde{x}} - \frac{1}{\alpha'}\hat{\tilde{\xi}}\,. \tag{4.25}$$

Thus, working with dimensionless coordinates and metric and b-field of length dimension L^2 we have a relative factor of α' between the b-field gauge transformation and the generalized diffeomorphism. Dropping the hats on fields and coordinates and setting $\alpha' = 1$, the background field values are given by (4.22) and a b-field that is, in a particular gauge, linear in z,

$$b \;=\; b_{12} \;=\; -b_{21} \;=\; Hz\,, \tag{4.26}$$

while the periodicity condition of the double torus is $x^i \sim x^i + 2\pi$ and $\tilde{x}_i \sim \tilde{x}_i + 2\pi$. This is the convention we use from now on.

We close this section by mentioning that one can also view the above transformations as $O(2,2)$ monodromy transformations in the 1,2-directions. We first note that the b-field gauge transformation (4.17) acts on the Kähler parameter as

$$\rho' = \rho + 2\pi R_3 \bar{H} , \qquad (4.27)$$

while τ is invariant. With the above isomorphism this corresponds to the $O(2,2)$ matrix

$$h = \begin{pmatrix} 1 & 0 & 0 & 2\pi R_3 \bar{H} \\ 0 & 1 & -2\pi R_3 \bar{H} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (4.28)$$

In particular, it is straightforward to verify (4.27) and the invariance of τ.

4.3 Twisted 3-torus: f-flux

Now we consider a twisted 3-torus that is a field configuration related to the flat torus with H-flux discussed above by a T-duality in the 1-direction [36]. The metric is independent of x^α, $\alpha = 1, 2$, but depends on the coordinate z,

$$g = \begin{pmatrix} \frac{1}{R_1^2} & -\frac{Hz}{R_1^2} & 0 \\ -\frac{Hz}{R_1^2} & R_2^2 + \left(\frac{Hz}{R_1}\right)^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} . \qquad (4.29)$$

This is a three-dimensional so-called nilmanifold with no additional b-field.

We want to investigate whether this background is globally well-defined. As above we have to compare the metric at $z = 0$ and $z = 2\pi$. One finds after a quick computation for the complex structure τ of the 2-torus defined in (4.7)

$$\tau(2\pi) - \tau(0) = -2\pi H , \qquad (4.30)$$

while ρ is unchanged. This lack of periodicity can be compensated by an $O(2,2)$ transformation in the 1,2-directions, so that $\tau'(2\pi) = \tau(0)$, with the following matrix

$$h = \begin{pmatrix} 1 & 2\pi H & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2\pi H & 1 \end{pmatrix} , \qquad (4.31)$$

where we used again the explicit isomorphism (4.8). This matrix belongs to the $GL(2)$ subgroup. Thus, we can can reproduce this transformation by an accompanying general coordinate transformation in the (x^1, x^2) coordinates,

$$x^{1'} = x^1 ,$$
$$x^{2'} = x^2 - 2\pi H x^1 ,$$
$$\tilde{x}_1' = \tilde{x}_1 ,$$
$$\tilde{x}_2' = \tilde{x}_2 . \qquad (4.32)$$

Thus, the space is globally well-defined. Note that the $\tilde{x}$ coordinates are not transformed, in contrast to the naive $O(2,2)$ action of (4.31) on X^M. We could have realized the same field transformations with a (dual) transformation on $(\tilde{x}_1, \tilde{x}_2)$, leaving (x^1, x^2) invariant.[5]

4.4   3-torus with Q-flux: T-fold

Finally, we consider the situation obtained from the previous example by a T-duality transformation in the 2-direction. This gives the background

$$
g \;=\; f\begin{pmatrix} \frac{1}{R_1^2} & 0 & 0 \\ 0 & \frac{1}{R_2^2} & 0 \\ 0 & 0 & \frac{R_3^2}{f} \end{pmatrix}, \qquad b \;=\; f\begin{pmatrix} 0 & -\frac{Hz}{R_1^2 R_2^2} & 0 \\ \frac{Hz}{R_1^2 R_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.33}
$$

where

$$
f(z) \;=\; \left[1 + \left(\frac{Hz}{R_1 R_2}\right)^2\right]^{-1}. \tag{4.34}
$$

This configuration is known to be non-geometric. Indeed, as we will now discuss, when going around the z circle one cannot find a general coordinate or b-field gauge transformation which would make these fields globally well-defined. We can achieve this, however, by using a generalized coordinate transformation corresponding to a β-shift as the transition function between two patches on the z circle.

Let us compute the field values at $z = 0$ and $z = 2\pi$ for the Kähler parameter (4.7) of the 2-torus. We first compute with (4.33) that

$$
\rho(z) \;=\; \frac{1}{Hz - iR_1 R_2}. \tag{4.35}
$$

This implies

$$
\rho(2\pi) \;=\; \frac{1}{2\pi H - iR_1 R_2} \;=\; \frac{1}{2\pi H + \frac{1}{\rho(0)}} \;=\; \frac{\rho(0)}{1 + 2\pi H \rho(0)}. \tag{4.36}
$$

This (non-linear) transformation on ρ cannot be viewed as a gauge symmetry of supergravity. It can be viewed, however, as an $O(2,2)$ transformation in the 1,2-directions. Thus it can be compensated with the inverse transformation, given by the following matrix

$$
h \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2\pi H & 1 & 0 \\ -2\pi H & 0 & 0 & 1 \end{pmatrix}, \tag{4.37}
$$

as follows again with (4.8). Thus, in order to make this space globally well-defined one needs to resort to some notion as T-fold, where one allows for 'patchings with $O(d,d)$ transformations'. In standard supergravity, where $O(d,d)$ cannot be viewed as part of the gauge group, this is little more than words, but in DFT this idea can be given a concrete meaning. In fact, as reviewed above, $O(2,2)$ transformations can be viewed as generalized coordinate transformations in DFT.

---

[5]The square root of (4.31), acting both on the x and $\tilde{x}$ coordinates, also leads to the same field transformation but is illegal in general due to the periodicity implied by the torus topology.

Since (4.37) takes the form of a β transformation we infer from sec. 3.2(iii) that we need the generalized coordinate transformation (3.19) to reproduce the non-linear transformation (4.36) of ρ. One obtains

$$
\begin{aligned}
x^{1\prime} &= x^1 + \pi H \tilde{x}_2 \,, \\
x^{2\prime} &= x^2 - \pi H \tilde{x}_1 \,, \\
\tilde{x}'_1 &= \tilde{x}_1 \,, \\
\tilde{x}'_2 &= \tilde{x}_2 \,.
\end{aligned}
\tag{4.38}
$$

However, due to the factors of πH rather than $2\pi H \in \mathbb{Z}$, this transformation is not compatible with the torus identifications, but we can consider an alternative transformation, acting in the same way on fields, that rotates x^1 and x^2 as follows

$$
\begin{aligned}
x^{1\prime} &= x^1 + 2\pi H \tilde{x}_2 \,, \\
x^{2\prime} &= x^2 \,.
\end{aligned}
\tag{4.39}
$$

This transformation is well defined and implements the desired transformation of fields. It differs from (4.38) by a trivial gauge transformation of the type (2.13), with

$$
\chi = -\pi H \tilde{x}_1 \tilde{x}_2 \,.
\tag{4.40}
$$

This shows that in DFT this space is globally well-defined, given the generalized coordinate transformations mixing x and $\tilde{x}$ used for the patching after going around the circle in z direction. We finally note that the parameter H is now associated with a non-geometric Q-flux.

4.5 Genuine non-geometric backgrounds

After discussing the above chain of backgrounds obtained from the geometric H-flux background by T-duality transformations we now want to consider a background that is genuinely non-geometric in the sense that it is not T-dual to a geometric background. (Such backgrounds have also been discussed in [37].) As discussed in [38, 39] this background can be constructed as a truly asymmetric $\mathbb{Z}_2$ orbifold CFT. The corresponding background is a fibered torus with the following complex structure and Kähler parameters:

$$
\begin{aligned}
\tau(z) &= \frac{\tau_0 \cos(fz) + \sin(fz)}{\cos(fz) - \tau_0 \sin(fz)} \,, \qquad f \in \frac{1}{4} + \mathbb{Z} \,, \\
\rho(z) &= \frac{\rho_0 \cos(Hz) + \sin(Hz)}{\cos(Hz) - \rho_0 \sin(Hz)} \,, \qquad H \in \frac{1}{4} + \mathbb{Z} \,.
\end{aligned}
\tag{4.41}
$$

Here τ_0 and ρ_0 are arbitrary parameters of the background. In fact, this three-dimensional background does not solve the beta function equations for the underlying non-linear sigma model for arbitrary parameters τ_0 and ρ_0; it is not a conformal field theory. Only for $\tau_0 = \rho_0 = i$, the background becomes an CFT, namely the exactly solvable, freely acting asymmetric $\mathbb{Z}_2$ orbifold CFT. However, we may still consider these backgrounds as off-shell configurations for arbitrary parameters. One can show that this background is not T-dual to a geometric background. We

also mention that this background contains f, H and Q flux. Specifically one can identify the geometric flux with the f-parameter, whereas H, Q are generated by H.

In order to analyze the global structure we compare again the fields at $z = 0$ and $z = 2\pi$. We find an inversion of the complex structure as well as an inversion of the Kähler parameter of the fibre torus:

$$\tau(2\pi) \;=\; -\frac{1}{\tau(0)}\,, \qquad \rho(2\pi) \;=\; -\frac{1}{\rho(0)}\,. \tag{4.42}$$

The fixed point of this transformation, $\tau_0 = \rho_0 = i$, precisely agrees with the asymmetric orbifold point, mentioned already above. Note that at the fixed point this transformation (4.42) has trivial action on τ and on ρ. Nevertheless, it corresponds to an $O(2,2)$ monodromy transformation, which will still act non-trivially on the coordinates and the dual coordinates. The transformation (4.42) corresponds to the following $O(2,2)$ monodromy transformation:

$$h \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{4.43}$$

As discussed in sec. 3.2(iv), see (3.25), this particular $O(2,2)$ group element, which corresponds to an overall T-duality transformation, can be associated to a generalized coordinate transformation that is determined by the square root (3.26) of this matrix. However, since the matrix entries of (3.26) are half-integer valued this transformation is not compatible with the torus identifications assumed for the (doubled) coordinates. In order to remedy this we can give an alternative form of the generalized coordinate transformation that results from this one by a trivial gauge transformation, thus giving the same transformation of fields, however, in a way that is compatible with the torus identifications.

We now give the details of this 'trivial' generalized coordinate transformation. The original (illegal) transformation we write as

$$X'^M \;=\; A^M{}_N X^N\,, \tag{4.44}$$

where A is the constant matrix (3.26). The induced generalized coordinate transformation on fields is determined by the associated $\mathcal{F}$, which reads

$$\mathcal{F}(A)^M{}_N \;=\; \frac{1}{2}\left(\eta(A^{-1})^T\eta A + A\eta(A^{-1})^T\eta\right)^M{}_N\,. \tag{4.45}$$

We now act with a second, trivial transformation

$$X''^M \;=\; X'^M + \partial'^M\chi \;=\; A^M{}_N X^N + (A^{-1})_N{}^M \partial^N\chi\,, \tag{4.46}$$

where we used the chain rule in the second equation (and indices on A^{-1} are raised and lowered with η). Thus, in matrix notation we have

$$X'' \;=\; AX + \eta(A^{-1})^T\eta\,\vec{\partial}\chi \;=\; A\left(X + \vec{\partial}\chi\right), \tag{4.47}$$

where we used $A \in O(2,2)$, which implies $A = \eta(A^{-1})^T\eta$. We now give a function χ that leads to a particularly simple coordinate transformation that induces the required field transformation.

It reads[6]

$$\chi(x^1, x^2, \tilde{x}_1, \tilde{x}_2) = \frac{1}{2}\left(x^1 x^2 + \tilde{x}_1 \tilde{x}_2 - \tilde{x}_1 x^2 - x^1 \tilde{x}_2\right) - \frac{3}{2}\left(\tilde{x}_1 x^1 + \tilde{x}_2 x^2\right)$$
$$- \frac{1}{4}\left((x^1)^2 + (x^2)^2 + (\tilde{x}_1)^2 + (\tilde{x}_2)^2\right) . \tag{4.48}$$

Inserting (3.26) for A and this ansatz for χ in (4.47) it is straightforward to compute the resulting coordinate transformation $X \to X''$. Viewing this as a single coordinate transformation it is more convenient to write the new coordinates as X', and we finally get

$$\begin{aligned}
\tilde{x}_1' &= -\tilde{x}_2 , \\
\tilde{x}_2' &= x^1 , \\
x'^1 &= -x^2 , \\
x'^2 &= \tilde{x}_1 .
\end{aligned} \tag{4.49}$$

This corresponds to a constant coordinate transformation of the form (4.44), with matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{4.50}$$

One may easily check with (4.45) that the associated $\mathcal{F}$ is indeed of the required form (4.43). Since this transformation is integer-valued it is indeed compatible with the torus identifications. This suffices to show that in DFT the space is globally well-defined. It is curious to note that the above matrix has determinant -1. Thus, this coordinate transformation is orientation-reversing, and so realizing this fibered torus as a globally well-defined (doubled) space forces us to view it as some kind of generalized higher-dimensional Möbius strip. There are actually various different forms of valid coordinate transformations leading to the required field transformation, but we have checked that they have all determinant -1.

Let us finally mention that one may construct further non-geometric backgrounds that are even more exotic and that do not fit into the strongly constrained DFT known so far. For instance, recently a background has been constructed that contains a genuine R-flux and for which one accordingly expects only a non-local description, i.e., one which is not covered by T-folds. Specifically, this background can be constructed as an asymmetric $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold [39]. One feature of these backgrounds is that the expressions for the Kähler and complex structure moduli are inherently non-local in the sense that they depend simultaneously on the coordinate z and its dual $\tilde{z}$. As such these field configurations even violate the weak constraint, which requires $\partial_z \partial_{\tilde{z}}$ to annihilate all fields. The weak constraint is equivalent to the level-matching constraint for the massless string fields on the torus, but once we consider different backgrounds or include additional fields/states (say, in form of extra gauge fields) it is conceivable that there may be constraints that are compatible with these novel backgrounds. For the moment this remains as a very non-trivial open problem.

---

[6]Note that χ does not satisfy the weak constraint $\partial^M \partial_M \chi = 0$. While perhaps surprising, χ still gives a trivial gauge transformation because this only requires $\partial^M \chi \, \partial_M A = 0$, for any field A. This condition is satisfied because no field A depends on x^1, x^2 nor $\tilde{x}_1, \tilde{x}_2$.

5 Comments on the strong constraint

In this section we discuss the role of the strong constraint and the issues related to attempts to relax it. There are various reasons to believe that the strong constraint can and should be relaxed, ranging from string theory on a torus background to massive and gauged deformations of supergravity. We discuss the example of massive type IIA DFT, where the consistency of a weaker constraint with the gauge symmetries is simple to understand.

There have been a series of papers discussing the construction of gauged supergravity in lower dimensions by means of a generalized Scherk-Schwarz compactification of DFT [21–23].[7] Specifically, the Scherk-Schwarz ansatz allows for a dependence on the (doubled) internal coordinates and seems to lead naturally to gauged supergravity formulated with the embedding tensor technique [42,43]. It turns out that in order to obtain the most general lower-dimensional gauged supergravity theories a relaxation of the constraints, strong and weak, is required as one needs a certain restricted dependence on both the internal coordinates and their duals. In addition, the DFT action must be supplemented by a set of terms that would vanish if the constraints are enforced. The resulting lower-dimensional gauged supergravity is consistent, in particular gauge invariant, and in this sense the supplemented DFT action with Scherk-Schwarz ansatz is consistent with a relaxed constraint. In fact, [23] states that once the constraints required by closure of the gauge algebra are imposed, gauge invariance of the action can be established modulo the very same constraints. It would be useful to characterize explicitly and understand the scope of the proposed set of weaker constraints, in particular, because the weak constraint is a constraint in string theory. We hope that the simpler example of massive type IIA may be a guide in order to arrive at a conceptual understanding of generalized Scherk-Schwarz compactifications in DFT.

5.1 Implications of the strong constraint

The constraint (1.3) is interpreted in the strong sense that $\partial^M \partial_M$ annihilates all fields and gauge parameters, but also all products, so that for generic fields or parameters A and B we require

$$\partial^M \partial_M A \ = \ \partial^M \partial_M B \ = \ 0 \,, \qquad \partial^M A \, \partial_M B \ = \ 0 \,. \tag{5.1}$$

The first group of conditions is referred to as the weak form of the constraint. The first, together with the second, is referred to as the strong constraint.

As has been shown in [6], the strong constraint implies that, effectively, all fields depend only on half of the coordinates. They may depend only on x^i or only on $\tilde{x}_i$ or any combination obtained by an arbitrary $O(D, D)$ transformation. The subspaces corresponding to these restricted coordinates are also called totally null — as all tangent vectors are null with respect to the $O(D, D)$ invariant metric in (1.3). Therefore, we can state the strong constraint equivalently as follows:

Strong constraint: DFT fields only depend on the coordinates of a totally null subspace.

The conditions (5.1) are then a direct consequence of the strong constraint in this formulation.

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[7]We are grateful to D. Marques for a discussion and correspondence on this subject.

It is noteworthy that in explicit computations only the form (5.1) is ever required. The fact that (5.1) implies the above statement of the strong constraint is more nontrivial and we discuss the proof now. For this purpose, we first recall Witt's theorem on vector spaces:

Witt's Theorem: Let V be a finite dimensional vector space with a non-degenerate bilinear form. Any isometry between two subspaces of V can be extended to an isometry of V.

For our setup the vector space will be the $2D$-dimensional space $\mathbb{R}^{2D}$ of generalized momenta

$$P^M = \begin{pmatrix} p_i \\ w^i \end{pmatrix}, \tag{5.2}$$

and the non-degenerate bilinear form will be the $O(D,D)$ metric η_{MN}. Isometries of V are $O(D,D)$ transformations. Consider a field $A(x,\tilde{x})$ with a single Fourier mode

$$A(x,\tilde{x}) = A_P\, e^{ip_i x^i + iw^i \tilde{x}_i} = A_P\, e^{iP_K X^K}. \tag{5.3}$$

The constraint $\partial^M \partial_M = 0$ gives

$$\eta^{MN} P_M P_N = 0 \;\rightarrow\; P \cdot P = 0 \qquad P \text{ is null}. \tag{5.4}$$

If we have two fields $A = A_P e^{iP_K X^K}$ and $A' = A_{P'} e^{iP'_K X^K}$, both P and P' must be null, but the strong constraint $\partial^M A\, \partial_M A' = 0$ implies that, in addition, these two momenta must be orthogonal

$$P \cdot P' = 0. \tag{5.5}$$

It is thus the case that all momenta appearing in fields or gauge parameters span an N-dimensional isotropic subspace $\mathbb{S}^N$ of the momentum space $\mathbb{R}^{2D}$, namely, a subspace of null vectors. Indeed, since all momenta appearing on fields or gauge parameters must be null and any chosen pair must be orthogonal, any linear superposition of these momenta is null. Since isotropic subspaces of $\mathbb{R}^{2D}$ with metric η cannot have dimensionality larger than D, we must have $N \le D$.

Now consider the maximal isotropic subspace $\mathbb{E}^D$ of $\mathbb{R}^{2D}$ described by a basis of D vectors without winding

$$E_1 = \begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} e_2 \\ 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} e_3 \\ 0 \end{pmatrix}, \quad \ldots, \quad E_D = \begin{pmatrix} e_D \\ 0 \end{pmatrix}. \tag{5.6}$$

We argued above that the momenta that appear on the fields that satisfy the strong constraint span the isotropic space $\mathbb{S}^N$. Let V_i, with $i = 1, 2, \ldots N$ denote a basis for $\mathbb{S}^N$. Consider now the linear map m_N defined by the map of basis vectors

$$m_N: \; V_i \to E_i, \quad i = 1, 2, \ldots, N. \tag{5.7}$$

This map is clearly an isometry between $\mathbb{S}^N$ and $m_N(\mathbb{S}^N)$, both of which are subspaces of $\mathbb{R}^{2D}$. It follows by Witt's theorem that m_N can be extended to an $O(D,D)$ transformation of $\mathbb{R}^{2D}$. Therefore there is an $O(D,D)$ transformation that maps all relevant momenta to a subspace of vectors without winding. Fields without winding are fields that do not depend on $\tilde{x}$. This shows that there is an $O(D,D)$ transformation to a coordinate frame in which fields do not depend on the tilde coordinates.

5.2 Can the strong constraint be relaxed?

Let us now address the question: Can the strong constraint be relaxed? Asked in this generality the answer is undoubtedly yes. In fact, in closed string field theory on a torus background only the level-matching constraint is required, which is a weaker form of (5.1). For the massless fields (for which the number operators N and $\bar{N}$ both have eigenvalue equal to one) it reads

$$L_0 - \bar{L}_0 \ = \ -p_i w^i \ = \ 0 \,, \tag{5.8}$$

where p_i and w^i are the momentum and winding modes, respectively. Translating to coordinate space this constraint implies $\tilde{\partial}^i \partial_i = 0$ and thus $\partial^M \partial_M = 0$ when acting on the massless fields and the associated gauge parameters. In general, $L_0 - \bar{L}_0 = N - \bar{N} - p_i w^i = 0$, so massive fields have integrally quantized values of $\partial^M \partial_M$. Closed string field theory is in fact a fully consistent weakly constrained DFT, and so in the full string theory the doubled coordinates are undoubtedly physical and real. Of course, the full string theory is quite intricate and it is therefore of interest to focus just on the massless sector. For this sector, we can define the strong constraint by the requirement that the operator $\partial^M \partial_M$ that annihilates fields and gauge parameters, also annihilate all products of fields and/or gauge parameters.[8] Therefore, the more interesting question is this: Can the strong constraint be relaxed for the massless string fields only and/or to a finite order in α'? In the following we address the issues one encounters in relation to this question.

There are two main obstacles one encounters when trying to relax the strong constraint to the weak constraint.[9] First, one has to find an action and gauge transformations so that invariance of the action and closure of the gauge algebra require only the use of $\partial^M \partial_M A = 0$, not of $\partial^M A \partial_M B = 0$. This is a very non-trivial problem, as the latter condition is heavily used in most DFT computations. But the second obstacle is even more severe: one must make the symmetry variations consistent with the weak constraint. More precisely, if we impose $\tilde{\partial}^i \partial_i \Phi = 0$ for a generic DFT field Φ or a gauge parameter, consistency requires that the gauge variations, which read schematically

$$\delta_\xi \Phi \ = \ \xi \cdot \Phi \,, \tag{5.9}$$

should respect the constraint. Since we no longer have the strong constraint, the product of parameter and field in general does not satisfy the constraint, so that

$$\tilde{\partial}^i \partial_i (\delta_\xi \Phi) \ \neq \ 0 \,. \tag{5.10}$$

Thus the gauge variation is not compatible with the weak constraint. In order to remedy this we have to project out those Fourier modes that violate the constraint $p \cdot w = 0$. Denoting this projection by $[\,]$ we may try a new ansatz for the gauge transformations,

$$\delta_\xi \Phi \ = \ \left[\xi \cdot \Phi \right] \,. \tag{5.11}$$

Since we are projecting out unwanted Fourier modes, this modification of the gauge transformations introduces a non-locality into the theory. This projection and the associated non-locality

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[8]It is not clear if there is a useful definition of strongly constrained string field theory.
[9]One can try to relax both, as is the case in the works of [21–23].

are, however, present in closed string field theory as well, as this is the way it is consistent with the weaker level-matching constraint. After introducing the projectors, gauge invariance of the action and closure of the gauge algebra become a highly non-trivial issue and can only be established if the projector satisfies a sufficient number of algebraic identities, perhaps exhibiting a structure similar to the L^∞-algebras governing closed string field theory [13].[10]

The first obstacle of formulating gauge transformations whose gauge algebra closes and that leave an action invariant modulo only $\partial^M \partial_M A = 0$ has actually been solved in one particular case, the DFT for the Ramond-Ramond (RR) sector of type II. The second obstacle can be overcome by a partial weakening of the strong constraint. We discuss this now.

5.3   Massive type II: minor relaxation of the strong constraint

We start by recalling the basics of type II DFT as constructed in [20, 27]. The RR fields are encoded in a Majorana-Weyl spinor of the two-fold covering group $\mathrm{Spin}(10,10)$ of $SO(10,10)$, while the generalized metric is uplifted to an element $\mathbb{S}$ of $\mathrm{Spin}(10,10)$, satisfying $\mathbb{S} = \mathbb{S}^\dagger$. The Clifford algebra reads

$$\{\Gamma^M, \Gamma^N\} \;=\; 2\eta^{MN}\, \mathbf{1}\,, \tag{5.12}$$

so that due to the off-diagonal form of the $O(D,D)$ metric the gamma matrices can be identified with fermionic lowering and raising operators ψ_i and ψ^i, respectively,

$$\psi_i \;\equiv\; \frac{1}{\sqrt{2}}\,\Gamma_i\,, \quad \psi^i \;\equiv\; \frac{1}{\sqrt{2}}\,\Gamma^i\,, \tag{5.13}$$

with $(\psi_i)^\dagger = \psi^i$. We can thus introduce a Clifford vacuum $|0\rangle$, satisfying $\psi_i|0\rangle = 0$ for all i, and define the spinors by acting with the raising operator. The RR p-form potentials $C^{(p)}$ are then encoded in the spinor

$$\chi \;=\; \sum_{p=0}^{10} \frac{1}{p!}\, C_{i_1 \ldots i_p}\, \psi^{i_1} \ldots \psi^{i_p}|0\rangle\,. \tag{5.14}$$

The Dirac operator corresponding to the Clifford algebra (5.12),

$$\slashed{\partial} \;\equiv\; \frac{1}{\sqrt{2}}\,\Gamma^M \partial_M \;=\; \psi^i \partial_i + \psi_i \tilde{\partial}^i\,, \tag{5.15}$$

then acts on the spinor (5.14) as a natural $O(D,D)$ covariant extension of the exterior derivative. In fact, for $\tilde{\partial}^i = 0$ it acts on (5.14) by increasing the form degree by one and taking the totally antisymmetrized derivative, exactly as the exterior derivative on forms. Moreover, it generally squares to zero thanks to the first constraint in (5.1),

$$\slashed{\partial}^2 \;=\; \frac{1}{2}\Gamma^M \Gamma^N \partial_M \partial_N \;=\; \frac{1}{2}\eta^{MN}\partial_M \partial_N \;=\; 0\,, \tag{5.16}$$

where we used (5.12). Below we will need this relation only in the weak form, when $\slashed{\partial}^2$ acts on a single field or parameter.

---

[10]In collaboration with C. Hull two of us (OH and BZ) have obtained partial results along these lines, but they are as yet inconclusive.

The complete bosonic type II DFT action reads

$$S = \int d^{10}x\, d^{10}\tilde{x} \left(e^{-2d}\, \mathcal{R}(\mathcal{H}, d) + \frac{1}{4}(\partial\!\!\!/\chi)^\dagger\, \mathbb{S}\, \partial\!\!\!/\chi \right), \tag{5.17}$$

and is supplemented by the self-duality constraint

$$\partial\!\!\!/\chi = -\mathcal{K}\,\partial\!\!\!/\chi, \quad \mathcal{K} \equiv C^{-1}\mathbb{S}, \tag{5.18}$$

where C denotes the charge conjugation matrix of $\mathrm{Spin}(10,10)$. This theory is gauge invariant under the generalized diffeomorphisms parametrized by ξ^M, which act on the spinor χ as

$$\delta_\xi\chi = \xi^M\partial_M\chi + \frac{1}{2}\partial_M\xi_N\Gamma^M\Gamma^N\chi, \tag{5.19}$$

and under a new abelian gauge symmetry with a $\mathrm{Spin}(10,10)$ spinor parameter λ,

$$\delta_\lambda\chi = \partial\!\!\!/\lambda, \tag{5.20}$$

which reduces to the usual p-form gauge symmetry when $\tilde{\partial}^i = 0$. The invariance of $\partial\!\!\!/\chi$ under this symmetry (and thus the invariance of the action and self-duality constraint) is manifest thanks to $\partial\!\!\!/^2 = 0$.

As usual, the above action is only ξ^M gauge invariant if we impose the strong constraint, but as noted above $\partial\!\!\!/^2 = 0$ and thus λ invariance requires only the weaker constraint. It turns out that we can also reformulate the ξ^M gauge transformations of the RR fields so as to allow for a minor relaxation of this constraint. To this end we rewrite $\delta_\xi\chi$ as

$$\delta_\xi\chi = \xi^M\partial_M\chi - \frac{1}{2}\Gamma^M\Gamma^N\xi_N\partial_M\chi + \frac{1}{2}\Gamma^M\partial_M\left(\xi_N\Gamma^N\chi\right). \tag{5.21}$$

The last term in here takes the form of a field-dependent abelian gauge transformation with parameter $\lambda = \frac{1}{\sqrt{2}}\xi^N\Gamma_N\chi$, and so upon a field-dependent parameter redefinition this term can be eliminated. Using then the Clifford algebra in the remaining term above we arrive at

$$\delta_\xi\chi = \xi\!\!\!/\,\partial\!\!\!/\chi, \tag{5.22}$$

where we set $\xi\!\!\!/ = \frac{1}{\sqrt{2}}\Gamma^M\xi_M$. The claim is now that closure of the gauge algebra on χ and ξ^M and gauge invariance of the RR action require only the weaker constraint. More precisely, these computations require the conditions

$$\partial^M\partial_M A = 0, \qquad A = \left\{\chi, \lambda, \xi^M\right\}, \tag{5.23}$$

but no condition of the form $\partial^M A\,\partial_M B = 0$.

We thus have shown that for the RR sector the first of the obstacles discussed in the previous subsection has been solved. However, as stressed above in regard to the second obstacle, this is not yet sufficient to claim to have a weakly constrained theory. In fact, without the strong constraint the gauge transformations (5.22) are not consistent with the weak constraint, for the same reasons as in (5.10), but we can still relax the strong constraint somewhat using the observation that in contrast to (5.19) in (5.22) the field appears only under a derivative. Therefore, if we take the RR fields to be a sum of terms that depend arbitrarily on x^i, we can

also include terms that depend linearly on $\tilde{x}_i$ but have no x^i dependence. The gauge variation will then be $\tilde{x}$ independent and thus compatible with (5.23). We have therefore arrived at the following weakened version of the strong constraint:

Weakened strong constraint: The RR fields of DFT can only depend on the coordinates of a totally null subspace and at most linearly on the coordinates orthogonal to this space.

It turns out that thanks to this relaxation the type II DFT (5.17) now also encodes the massive type IIA theory due to Romans [40]. To see this we have to make an ansatz for the RR one-form that depends linearly on one of the $\tilde{x}_i$, say $\tilde{x}_1$,

$$C^{(1)}(x, \tilde{x}) \; = \; C_i(x)dx^i + m\tilde{x}_1 dx^1 \; , \tag{5.24}$$

where m is a mass parameter. Inserting this into the field strength F we obtain

$$F \; = \; \slashed{\partial}\chi \; = \; (\psi^i \partial_i + \psi_i \tilde{\partial}^i)\chi \; = \; F_{m=0} + \psi_i \tilde{\partial}^i (m\tilde{x}_1)\psi^1 |0\rangle \; . \tag{5.25}$$

Note that the $\slashed{\partial}$ operator acts non-trivially on the m-dependent part. In this last term the two operators ψ^i and ψ_i annihilate each other, leading to a non-vanishing contribution to the 'zero-form' field strength:

$$F^{(0)} \; = \; m \; . \tag{5.26}$$

Thus, the $\tilde{x}$-dependent part acts as a '(-1)-form' in the sense that acting with the (generalized) exterior derivative $\slashed{\partial}$ we obtain a zero-form. It has been noticed before in the literature that if one formally introduces (-1)-forms the formulation of massive supergravity simplifies [41]. Here we obtained a natural geometric interpretation of these exotic objects: (-1)-forms are 1-forms depending on the dual coordinates. Insertion of the ansatz (5.24) into type II DFT indeed precisely reproduces the massive type IIA theory [20].

We have thus seen that for the RR subsector of DFT the strong constraint can be relaxed somewhat so as to allow for a simultaneous dependence on coordinates and their duals, provided one of them enters only linearly. The resulting conventional spacetime theory then corresponds to a massive deformation.

6 Doubled α' geometry

As the generalized metric form of double field theory was developed it became clear that one could try to include higher-derivative terms in the action while preserving the continuous $O(D, D)$ symmetry of the theory as well as the gauge symmetries [7]. It was soon realized, however, that the expected higher derivative corrections, in particular, the Riemann-squared terms were difficult to include while preserving the symmetries of the theory [44].

This difficulty makes it instructive to discuss why we expect that the continuous $O(D, D)$ symmetry of the two-derivative DFT survives the inclusion of α' corrections from string theory. For this we will review an argument by Sen [45]. We then discuss and explain a few of the key results of the recent construction of α' corrected DFT presented in [19].

6.1 $O(d,d)$ survives α' corrections

We now consider the argument of [45], which explains why a string theory with d space-like coordinates described by d free scalar fields leads to a reduced string theory with $O(d,d)$ continuous symmetry. This is true including α' corrections. This result makes it plausible that a DFT with global $O(D,D)$ must exist upon inclusion of α' corrections.

For the standard matter CFT of free bosons the holomorphic and antiholomorphic sectors decouple. All correlators are invariant under simultaneous and independent rotations of the ∂X^i and $\bar{\partial} X^i$ currents, with $i = 1, 2, \ldots, d$. Sen's argument is couched in the language of string field theory and effective actions. It begins by stating that the string field encoding the fluctuations of the (internal) metric and b-field takes the form

$$(h_{ij} + b_{ij}) \, \alpha^i_{-1} \bar{\alpha}^j_{-1} c_1 \bar{c}_1 |0\rangle \, . \tag{6.1}$$

As a consequence of the correlators' invariance, the string field theory will have the exact symmetry under the $O(d) \times O(d)$ action

$$h + b \; \rightarrow \; \tilde{h} + \tilde{b} \; = \; S(h+b)R^T \, , \qquad S, R \in O(d) \, . \tag{6.2}$$

Consider now the effective field theory of the fields G_{ij}, B_{ij} as known with two derivatives. This theory uses a generalized metric $\mathcal{H}$ as in (1.2) with (g, b) replaced by (G, B), and has a symmetry

$$\mathcal{H} \; \rightarrow \; \tilde{\mathcal{H}} \; = \; \Omega \, \mathcal{H} \, \Omega^t \, , \tag{6.3}$$

where $\Omega \in O(d,d)$. There is an $O(d) \times O(d)$ subgroup described by the matrices

$$\Omega \; = \; \frac{1}{2} \begin{pmatrix} S+R & R-S \\ R-S & S+R \end{pmatrix} \, . \tag{6.4}$$

This subgroup leaves invariant the flat background $G_{ij} = \delta_{ij}$, $B_{ij} = 0$. Since we view the string field as fluctuations of G, B around the flat background, we have

$$\begin{aligned} G_{ij} &= \delta_{ij} + h_{ij} + \ldots \, , \\ B_{ij} &= b_{ij} + \ldots \, , \end{aligned} \tag{6.5}$$

where the dots indicate terms quadratic and higher order in the fluctuations. It is now possible to verify that the symmetry (6.3), (6.4), through the above expansion turns into the symmetry (6.2). Since this $O(d) \times O(d)$ of the string field theory is exact and holds to all orders of α' the symmetry should exist in the low-energy theory described with fields (G, B). This subgroup contains $2 \cdot \frac{1}{2} \cdot d(d-1) = d(d-1)$ generators.

We then note that any coordinate invariant theory with gravity and a dilaton (such as the reduced theory we are considering) has a $GL(d)$ symmetry arising from a $GL(d)$ transformation of the d coordinates: $x^i \rightarrow A^i{}_j x^j$. This symmetry should exist to all orders in α'. At the level of the G, B fields we have $G \rightarrow AGA^T$ and $B \rightarrow ABA^T$, while the dilaton is shifted as $\phi \rightarrow \phi + \ln(\det A)$. In terms of $O(d,d)$ these transformations arise from

$$\Omega \; = \; \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \, . \tag{6.6}$$

38

Not all of the $GL(d)$ generators are new: the diagonal subgroup $O(d)$ of $O(d) \times O(d)$ with $R = S$ is also a subgroup of $GL(d)$. Thus only $d^2 - \frac{1}{2}d(d-1) = \frac{1}{2}d(d+1)$ generators are new.

Finally, the low-energy theory theory must have, even with α' corrections, the Kalb-Ramond symmetries $\delta B_{ij} = \partial_i \epsilon_j - \partial_j \epsilon_i$ of the two-derivative theory. Taking $\epsilon_i = f_{ij} x^j$ with constant f_{ij}'s we get $\delta B_{ij} = f_{ji} - f_{ij} \equiv C_{ij}$, with C an antisymmetric constant matrix. This transformation is an $O(d, d)$ transformation with

$$\Omega = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}. \tag{6.7}$$

These so-called B-shifts amount to $\frac{1}{2}d(d-1)$ generators. With $O(d) \times O(d)$, the additional generators from $GL(d)$ and the B shifts we have identified all the generators of $O(d, d)$. Indeed the count works:

$$d(d-1) + \frac{1}{2}d(d+1) + \frac{1}{2}d(d-1) = 2d^2 - d = \frac{1}{2}(2d)(2d-1), \tag{6.8}$$

which is the number of generators of $O(d, d)$. This concludes the argument that the reduced theory of massless fields must have a global, continuous $O(d, d)$ symmetry even as α' corrections are included.

6.2 α'-geometry

The absence of a duality covariant generalized Riemann tensor that is fully determined in terms of the physical fields of DFT, see [44], made it clear that some deformation of the structures of the theory was needed in order to include the Riemann-squared terms that arise in the α'-expansion of the effective theory for the massless sector of closed strings. It was generally believed that the duality transformations of the fields would be modified while the gauge transformations, which include diffeomorphism and b-field transformations, would not be changed. Nevertheless, it was anticipated from string field theory that the opposite would be true: duality would remain manifest (thus un-corrected) while gauge symmetries would receive some kind of corrections.

Indeed, the formulation of [19] shows that the gauge structure of the theory is changed. Recall that gauge parameters $\Xi^M(X)$ contain components $(\tilde{\xi}_i, \xi^i)$ that depend both on $(x, \tilde{x})$. When we restrict ourselves to fields and gauge parameters that do not depend on $\tilde{x}$, $\tilde{\xi}_i$ becomes a one-form and ξ^i a vector.

In the two-derivative theory (no α' corrections) we had the following key structures

$$\langle \Xi_1 | \Xi_2 \rangle = \xi_1^M \xi_2^N \eta_{MN},$$

$$[\Xi_1, \Xi_2]_C^M = \xi_{[1}^N \partial_N \xi_{2]}^M - \frac{1}{2} \xi_1^K \overset{\leftrightarrow}{\partial}{}^M \xi_{2K}, \tag{6.9}$$

$$\widehat{\mathcal{L}}_\Xi V^M = \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M) V^P.$$

(In this section only we use the (anti)symmetrization convention $[ab] = ab - ba$, and $A \overset{\leftrightarrow}{\partial} B \equiv A \partial B - (\partial B) A$.) The first structure is the inner product, formed simply using the metric η. The inner product is a generalized scalar formed from two vectors. The second structure is the

C-bracket of two gauge parameters (vectors), giving a third. When restricted to parameters that do not depend on $\tilde{x}$, it reduces to the Courant bracket of generalized geometry. The third line defines the generalized Lie derivative of a generalized vector V along Ξ. The commutator of two such Lie derivatives, with two different gauge parameters, is a Lie derivative with respect to a parameter formed by taking the C-bracket of the two original parameters.

When we include α' corrections the above structures are modified. Interestingly, the modifications that were obtained are exact: they do not represent just first-order corrections to be supplemented by further, or infinitely many, terms to be determined. They are complete and self-consistent corrections. We do not write them with explicit factors of α' but rather the corrections are recognized by the higher number of derivatives; a factor of α' is associated with two derivatives. The new structures, denoted the same way as the ones above except for the Lie derivative, are given by

$$\langle \Xi_1 | \Xi_2 \rangle \;=\; \xi_1^M \xi_2^N \eta_{MN} - (\partial_N \xi_1^M)(\partial_M \xi_2^N)\,,$$

$$[\Xi_1, \Xi_2]_C^M \;=\; \xi_{[1}^N \partial_N \xi_{2]}^M - \frac{1}{2}\xi_1^K \overset{\leftrightarrow}{\partial}{}^M \xi_{2K} + \frac{1}{2}(\partial_K \xi_1^L)\overset{\leftrightarrow}{\partial}{}^M(\partial_L \xi_2^K)\,, \qquad (6.10)$$

$$\mathbf{L}_\Xi V^M \;=\; \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M)V^P - (\partial^M \partial_K \xi^L)\partial_L V^K\,.$$

Each structure has one additional term: the inner product has a two derivative term and both the bracket and the generalized Lie derivative now have a term with three derivatives. The commutator of two corrected Lie derivatives gives, exactly, a corrected Lie derivative with parameter given by the corrected C bracket. Moreover the Jacobiator of the corrected C bracket is still a trivial gauge parameter, as it was for the two-derivative theory and as needed for consistency. The correction to the generalized Lie derivative implies that the transformation law $V' = \mathcal{F}V$ with $\mathcal{F}$ given by (1.7) requires modification to order α'. We will not attempt here to find such modification.

Interestingly, the corrections in (6.10) do not vanish when we restrict to fields and parameters that do not depend on $\tilde{x}$. Therefore the above corrections imply corrections to the familiar structures of generalized geometry. Consider, for example, the inner product, which in the absence of $\tilde{x}$ dependence becomes

$$\langle \Xi_1 | \Xi_2 \rangle \;=\; \xi_1^i \tilde{\xi}_{2i} + \xi_2^i \tilde{\xi}_{1i} - \partial_i \xi_1^j\,\partial_j \xi_2^i\,. \qquad (6.11)$$

The last term is the higher-derivative correction. For the C bracket the vector part is not corrected, but the one-form part is

$$([\Xi_1, \Xi_2]_C)^i \;=\; \xi_{[1}^k \partial_k \xi_{2]}^i\,,$$

$$([\Xi_1, \Xi_2]_C)_i \;=\; \xi_{[1}^k \partial_k \tilde{\xi}_{2]i} + \tfrac{1}{2}\big(\xi_1^k \overset{\leftrightarrow}{\partial_i}\tilde{\xi}_{2k} + \tilde{\xi}_{1k}\overset{\leftrightarrow}{\partial_i}\xi_2^k\big) + \tfrac{1}{2}(\partial_k \xi_1^\ell)\overset{\leftrightarrow}{\partial_i}(\partial_\ell \xi_2^k)\,. \qquad (6.12)$$

The correction term is the last term on the right-hand side of the second line. Finally, for the Lie derivatives we have

$$(\mathbf{L}_\Xi V)^i \;=\; \xi^k \partial_k V^i - V^k \partial_k \xi^i$$

$$(\mathbf{L}_\Xi V)_i \;=\; \xi^k \partial_k V_i + \partial_i \xi^p\, V_p + (\partial_i \tilde{\xi}_p - \partial_p \tilde{\xi}_i)V^p - \partial_i \partial_k \xi^p \partial_p V^k\,. \qquad (6.13)$$

The correction is the last term on the second line and it only affects the Lie derivative of the one-form. Note, however, that this correction involves the vector V^k. Thus at order α' the one-form and the vector mix under generalized diffeomorphisms!

Given the gauge structure above, reference [19] looked for fields and gauge transformations. The key technical tool is a chiral CFT in the doubled space with a novel propagator and simplified OPE's, as the result of the strong constraint. The CFT has currents $Z^M(z) \equiv X'^M(z)$ and the dynamical fields are introduced through the weight-two operators

$$
\begin{aligned}
\mathcal{S} &\equiv \frac{1}{2}(Z^2 - \phi''), \\
\mathcal{T} &\equiv \tfrac{1}{2}\mathcal{M}^{MN}Z_M Z_N - \tfrac{1}{2}(\widehat{\mathcal{M}}^M Z_M)'.
\end{aligned}
\tag{6.14}
$$

Here we introduced the dilaton $\phi = -2d$, in order to comply with the conventions of [19], which enters in $\mathcal{S}$, while the *double metric* $\mathcal{M}$ enters in $\mathcal{T}$. The double metric is related but is not equal to the generalized metric $\mathcal{H}^{MN}$. While off-shell $\mathcal{H}$ squares to one, the former is unconstrained. The second term in $\mathcal{T}$, needed for consistency with gauge transformations, contains an auxiliary field $\widehat{\mathcal{M}}$. This field is determined in terms of the double metric and the dilaton by the condition $\mathrm{div}\,\mathcal{T} = 0$, where the divergence of an operator is defined through its OPE with $\mathcal{S}$. Thus $\mathcal{T}$ is a divergenceless operator.

Gauge transformations of any operator $\mathcal{O}$ are defined by the commutator $\delta_\Xi \mathcal{O} = \left[\int \Xi, \mathcal{O}\right]$, and are readily evaluated with the use of operator products. This was used in [19] to find the gauge transformations of $\mathcal{M}$ and ϕ

$$
\begin{aligned}
\delta_\Xi \mathcal{M}^{MN} &= \xi^P \partial_P \mathcal{M}^{MN} + (\partial^M \xi_P - \partial_P \xi^M)\mathcal{M}^{PN} + (\partial^N \xi_P - \partial_P \xi^N)\mathcal{M}^{NP} \\
&\quad - \tfrac{1}{2}\left[\partial^M \mathcal{M}^{PQ}\,\partial_P(\partial_Q \xi^N - \partial^N \xi_Q) + 2\partial_Q \mathcal{M}^{KM}\,\partial^N \partial_K \xi^Q + (M \leftrightarrow N)\right] \\
&\quad - \tfrac{1}{4}\partial_K \partial^{(M}\mathcal{M}^{PQ}\,\partial^{N)}\partial_P \partial_Q \xi^K,
\end{aligned}
\tag{6.15}
$$

$$
\delta_\Xi \phi = \xi \cdot \partial \phi + \partial \cdot \xi.
$$

The gauge transformation of the double metric $\mathcal{M}^{MN}$ receives α' and α'^2 corrections (second and third lines, respectively) since we recognize that the first line is simply the familiar generalized Lie derivative. The gauge transformation of the dilaton is unchanged.

For the dynamics, the key idea is that the equations of motion are the condition that $\mathcal{S}$ and $\mathcal{T}$ satisfy the Virasoro algebra. A gauge invariant action that implements this idea can be written and it takes the cubic form

$$
S = \int e^\phi \big(\langle \mathcal{T}|\mathcal{S}\rangle - \tfrac{1}{6}\langle \mathcal{T}|\mathcal{T} \star \mathcal{T}\rangle \big).
\tag{6.16}
$$

The definition of the various ingredients required here were given in [19]. For example, $\langle \cdot | \cdot \rangle$ above is a scalar inner-product between weight-two tensors and the $\star$-product of two weight-two tensors gives a divergenceless weight-two tensor as the answer. As it turns out, in terms of the double metric, the equation of motion takes the form $\mathcal{M}^2 = 1 + 2\mathcal{V}$, where $\mathcal{V}$ is quadratic in $\mathcal{M}$ and contains from two up to six derivatives. While the generalized metric squares to the identity, the double metric $\mathcal{M}$ squares to the identity up to higher derivatives terms. The replacement of $\mathcal{H}$ for the unconstrained $\mathcal{M}$ is a very significant departure from the two-derivative theory, forced by α' corrections.

7 Conclusions and Outlook

We have reviewed DFT with a focus on the geometrical aspects such as the notion of generalized manifold. We tried to convince the reader that the structures required by DFT inevitably require a generalization of the manifold structure. As the notion of a manifold is deeply ingrained in our intuition, this may appear to be a rather radical and speculative step, but we should stress that such a conclusion follows quite conservatively from what we know about string theory and, more specifically, string field theory. More precisely, this conclusion relies on the following inputs:

(i) Closed string field theory for a torus background

(ii) Background independence

(iii) Gauge transformations as some form of coordinate transformations

In fact, string field theory on a torus (input (i)) inevitably leads to doubled coordinates and gauge transformations involving doubled derivatives (explicitly known to cubic order). Then requiring a manifestly background independent formulation (input (ii)), thereby not requiring anymore a torus background, uniquely leads to the generalized Lie derivatives of DFT. Finally, if one wants to reproduce these generalized Lie derivatives as some sort of coordinate transformation $X \to X'$ (input (iii)), we are forced to adopt a notion of generalized coordinate transformations, thus requiring a generalized notion of manifold. Among the above assumptions, (iii) is perhaps the one that one could possibly imagine to abandon. Perhaps in the ultimate formulation of string theory the notion of manifold and coordinates will disappear altogether, but as long as we stick to coordinates, the emergence of a generalized manifold of the type discussed here seems inevitable.

The usefulness of DFT and its associated generalized geometric structures is evident in the striking economy of the corresponding formulations of the low-energy effective actions of string theory. The most conservative interpretation of DFT is to treat the winding coordinates as purely formal and to think of the doubled derivatives as $\partial_M = (0, \partial_i)$. In this case DFT may be viewed as the physical implementation of the 'generalized geometry' program of Hitchin-Gualtieri [15–17]. It is clear, however, that DFT goes beyond generalized geometry in at least two aspects: First, certain non-geometric backgrounds, such as those given in eq. (4.33) and (4.41), are not globally well-defined in standard supergravity (nor in the generalized geometry rewriting), but can be consistently patched together in DFT once we allow for generalized coordinate transformations that mix x and $\tilde{x}$ coordinates. In this way, DFT appears to provide a concrete framework for the inclusion of 'T-folds'. Second, the α'-corrections of string theory, formulated in DFT, lead to modifications of the gauge transformations and gauge algebra that are non-trivial even on the half-dimensional subspace relevant for generalized geometry [19]. In particular, Lie derivatives in the D-dimensional subspace receive α' corrections as do the defining structures of generalized geometry (like the Courant bracket).

Despite these first glimpses of a radically different geometry underlying string theory, it is clear that we are just beginning to understand these new structures and that the presently known DFT and its geometry are just first approximations to the full story. For instance, the

strongly constrained DFT allows only for rather mild deviations form standard geometry, as in the form of T-folds and some non-geometric backgrounds. Massive IIA gives a clear example of a possible relaxation of the strong constraint [20]. With the recent progress reported in [23] it is becoming clear that the most general gauged supergravities appearing in four dimensions may be obtained by a Scherk-Schwarz reduction of a suitably extended DFT. The weak and strong constraints are not assumed, instead, a set of weaker constraints arising from the closure of the gauge algebra is imposed. The recent reformulation of DFT given in [47] may be helpful to understand better this construction in that it puts DFT in a form already adapted to general Kaluza-Klein compactifications, thus representing the ideal starting point for a comparison of the lower-dimensional and higher-dimensional constraints. Finally, in regard to the (further generalized) α'-geometry we are just at the beginning, and it remains to be seen how the formulation of this geometry will enlighten our understanding of string theory in general.

Some further aspects about the covariant description of non-geometric fluxes in double field theory will be presented in [48], including their relation to gauged supergravity and T-folds, as well as a derivation of the fluxes associated to the backgrounds of section 4 from the generalized metric. Finally, let us make one more short comment on the appearance of non-commutativity and non-associativity in the presence of b-field gauge transformations and non-geometric backgrounds. Performing a world-sheet quantization procedure of the closed string coordinates $X^i(\tau, \sigma)$ in non-geometric target spaces, one loses commutativity and even associativity of the closed string coordinates (see refs. [38, 49–52]). It is tempting to speculate that this non-associativity of the closed string geometry, derived from the world-sheet theory, is related to the DFT non-associativity of generalized coordinate transformations. At the moment, however, we do not see a logical connection between these two non-associative structures, but this issue certainly deserves further studies.

We now mention a few other developments and possible generalization that we did not cover in more detail in the main text. DFT has been applied in various contexts, including the heterotic theory [53], massless and massive type II theories [20,27,54], and their supersymmetric extensions [55–57], and also leads to a compelling generalization of Riemannian geometry [14,44, 58–60], which in turn is closely related to (and an extension of) results in generalized geometry [15–18]. Another intriguing feature of DFT is that it makes manifest a 'double copy' property of supergravity, i.e., a certain factorization of gravity Feynman rules in terms of Yang-Mills-like Feynman rules, as was already pointed out in [3] and shown more explicitly in [61]. The ideas of DFT have also been generalized to various U-duality groups in order to describe truncations of $D = 11$ supergravity [62–70]. Until recently, however, they were restricted to rather severe truncations of $D = 11$ supergravity, but it has now been shown how to formulate complete gravity theories in fully U-duality covariant manner in [71,72], leading to an 'exceptional field theory' analogue of double field theory. Other developments and further results related to DFT have appeared in [73–77]. We finally note that short, early reviews can be found in [78]. More extensive and more recent reviews have appeared in [79].

We conclude by listing some outstanding questions:

- In DFT the b-field gauge transformations are treated geometrically as coordinate transformations on the doubled space. The known mathematical framework for b-fields is given

in the language of 'gerbes' and thus it would be useful to understand any relation between these approaches.

- How does the compactification on non-geometric backgrounds proceed in general? It should lead to gauged supergravity theories that so far had no higher-dimensional ancestor, as was recently discussed in [39] for the non-geometric form given in eq. (4.41).

- There should be a better understanding of the solution space for the weaker constraints used in the context of the extended DFT that yields gauged supergravities upon Scherk-Schwarz reduction. What kinds of doubled coordinate dependence do they allow? What is the relation to string theory, given that the weak constraint, which holds in string theory, is now relaxed?

- Since infinitesimal gauge transformations receive α' deformations, the same should happen for finite or large gauge transformations. Can we write a natural, closed-form expression for these corrected large gauge transformations?

- The α' deformations of the gauge transformations, bracket, and action may be generalized to other string theories. In particular, in the context of superstring theories, it may shed light on the subtle relation between duality symmetries and supersymmetry.

- In the 'exceptional field fheory' one may investigate the same generalizations and applications relevant to DFT – to supersymmetry, higher derivatives corrections, non-trivial backgrounds, etc.

Acknowledgments

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