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# CLT FOR SPECTRA OF SUBMATRICES OF WIGNER RANDOM MATRICES

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**ABSTRACT.** We prove a CLT for spectra of submatrices of real symmetric and Hermitian Wigner matrices. We show that if in the standard normalization the fourth moment of the off-diagonal entries is GOE/GUE-like then the limiting Gaussian process can be viewed as a collection of simply yet nontrivially correlated two-dimensional Gaussian Free Fields.

**Introduction.** Gaussian global fluctuations of eigenvalues of GUE, GOE, Wigner random matrices, and their generalizations is a well-studied subject, see e.g. Chapter 2 of [AGZ] and Chapter 9 of [BS] as well as references therein. One would usually concentrate on studying the spectrum of the full matrix, but it comes as no surprise that for large submatrices with a regular limiting behavior, the joint fluctuations would still be Gaussian. We prove this fact by a slight modification of the moment method presented in [AGZ].

It becomes more interesting when one looks at the limiting covariance structure. In what follows we assume that in the standard normalization the fourth moment of the off-diagonal entries of our matrices is the same as for GOE/GUE.

The first statement is that for such a (real symmetric or Hermitian) Wigner matrix, the joint fluctuations of spectra of nested submatrices formed by cutting out top left corners are described by the two-dimensional Gaussian Free Field (GFF), see e.g. [S] for definitions and basic properties of GFFs.

Although this result seems to be new, the appearance of the GFF is also not too surprising. Indeed, as was shown in [JN] and [OR], for GUE the eigenvalue ensemble of nested matrices arises as a limit of random surfaces, and for random surfaces the relevance of the GFF is widely anticipated, see [K], [BF] for rigorous results and further references. One might argue however that the GFF interpretation simplifies the description of the covariance in the one-matrix case, cf. Proposition 3 below.

The real novelty comes when one considers joint fluctuations for different nested sequences of submatrices. For each of the nested sequences the fluctuations are again described by the GFF. On the other hand, when different sequences have nontrivial and asymptotically regular intersections, these GFFs are correlated, and the exact form of the covariance kernel turns out to be simple. One could argue that it is as simple as one could hope for.

The resulting Gaussian process unites a large family of mutually correlated GFFs. Even for two GFFs the resulting Gaussian process seems to be new. An efficient description of the largest natural state space for this Gaussian process remains an open problem.

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It is natural to ask how universal the limiting process is. We believe that it also arises in the world of random surfaces, although it is not *a priori* clear how to vary the nested sequence there. The answer comes from representation theory — one views random surfaces as originating from restricting suitable representations to a maximal commutative subalgebra and then one varies that subalgebra. We will address these models in a later publication.

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**Wigner matrices.** Let  $\{Z_{ij}\}_{j>i\geq 1}$  and  $\{Y_i\}_{i\geq 1}$  be two families of independent identically distributed real-valued random variables with zero mean such that for any  $k \geq 1$

$$\max(\mathbb{E}|Z_{12}|^k, \mathbb{E}|Y_1|^k) < \infty.$$

Assume also that

$$\mathbb{E}Y_1^2 = 2, \quad \mathbb{E}Z_{12}^2 = 1, \quad \mathbb{E}Z_{12}^4 = 3.$$

Define a (real symmetric) *Wigner matrix*  $X$  by

$$X(i, j) = X(j, i) = \begin{cases} Z_{ij}, & i < j, \\ Y_i, & i = j. \end{cases}$$

An Hermitian variation of the same definition is as follows: Let  $\{Z_{ij}\}_{j>i\geq 1}$  now be complex-valued (i.i.d. mean zero) random variables with the same uniform bound on all moments. Assume that

$$\mathbb{E}Y_1^2 = 1, \quad \mathbb{E}|Z_{12}|^2 = 1, \quad \mathbb{E}|Z_{12}|^4 = 2.$$

Define an *Hermitian Wigner matrix*  $X$  by

$$X(i, j) = \overline{X(j, i)} = \begin{cases} Z_{ij}, & i < j, \\ Y_i, & i = j. \end{cases}$$

In the case when all the random variables  $Y_i, Z_{ij}$  (or  $Y_i, \Re Z_{ij}, \Im Z_{ij}$  in the Hermitian case) are Gaussian, the Wigner matrix is said to belong to the Gaussian Orthogonal Ensemble (GOE) in the real case, and Gaussian Unitary Ensemble (GUE) in the Hermitian case.

For any finite set  $B \subset \{1, 2, \dots\}$  we denote by  $X(B)$  the  $|B| \times |B|$  submatrix of the (real symmetric or Hermitian) Wigner matrix  $X$  formed by the intersections of the rows and columns of  $X$  marked by elements of  $B$ . Clearly, the distribution of  $X(B)$  depends only on  $|B|$ .

Traditionally one encodes the real symmetric and the Hermitian cases by a parameter  $\beta$  that takes value 1 for GOE and value 2 for GUE.

**The height function.** Let  $A = \{a_n\}_{n\geq 1}$  be an arbitrary sequence of pairwise distinct natural numbers. The *height function*  $H_A$  associated to  $A$  and a Wigner matrix  $X$  is a random integer-valued function on  $\mathbb{R} \times \mathbb{R}_{\geq 1}$  defined by

$$H_A(x, y) = \sqrt{\frac{\beta\pi}{2}} \left\{ \text{the number of eigenvalues of } X(\{a_1, \dots, a_{[y]}\}) \text{ that are } \geq x \right\}.$$

The convenience of the constant prefactor  $\sqrt{\beta\pi/2}$  will be evident shortly.

**Good families of sequences.** In what follows  $L > 0$  is a large parameter.

Let  $\{A_i\}_{i \in I}$  be a family of sequences of pairwise distinct natural numbers. Assume they all depend on  $L$ . Denote

$$A_i = \{a_{i,n}\}_{n \geq 1}, \quad A_{i,m} = \{a_{i,1}, \dots, a_{i,m}\}, \quad i \in I, \quad m \in \mathbb{N}.$$

We say that  $\{A_i\}_{i \in I}$  is a *good family* if for any  $i, j \in I$  and  $x, y \in \mathbb{R}_{>0}$  there exists a limit

$$\alpha(i, x; j, y) = \lim_{L \rightarrow \infty} \frac{|A_{i,[xL]} \cap A_{j,[yL]}|}{L}.$$

Here is an example of a good family:  $I = \{1, 2, 3, 4\}$  and

$$a_{1,n} = n, \quad a_{2,n} = 2n, \quad a_{3,n} = 2n + 1, \quad a_{4,n} = \begin{cases} n + L, & n \leq L, \\ n - L, & L < n \leq 2L, \\ n, & n > 2L. \end{cases}$$

Note, however, that the index set  $I$  does not have to be finite.

**Correlated Gaussian Free Fields.** Let  $\{A_i\}_{i \in I}$  be a good family of sequences as above. Take a family of copies of the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$  indexed by  $I$  and consider their union

$$\mathbb{H}(I) = \bigcup_{i \in I} \mathbb{H}_i.$$

Introduce a function  $C : \mathbb{H}(I) \times \mathbb{H}(I) \rightarrow \mathbb{R} \cup \{-\infty\}$  via

$$C_{ij}(z, w) = \frac{1}{2\pi} \ln \left| \frac{\alpha(i, |z|^2; j, |w|^2) - zw}{\alpha(i, |z|^2; j, |w|^2) - z\bar{w}} \right|, \quad i, j \in I, \quad z \in \mathbb{H}_i, \quad w \in \mathbb{H}_j,$$

where  $\alpha(\cdot)$  is as above. Note that for  $i = j$

$$C_{ii}(z, w) = \frac{1}{2\pi} \ln \left| \frac{\min(|z|^2, |w|^2) - zw}{\min(|z|^2, |w|^2) - z\bar{w}} \right| = -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \bar{w}} \right|$$

is the Green function for the Laplace operator on  $\mathbb{H}$  with Dirichlet boundary conditions.

**Proposition 1.** *For any good family of sequences as above, there exists a generalized Gaussian process on  $\mathbb{H}(I)$  with the covariance kernel  $C(z, w)$  as above. More exactly, for any finite family of test functions  $f_m(z) \in C_0(\mathbb{H}_{i_m})$  and  $i_1, \dots, i_M \in I$ , the covariance matrix*

$$\text{cov}(f_k, f_l) = \int_{\mathbb{H}} \int_{\mathbb{H}} f_k(z) f_l(w) C_{i_k i_l}(z, w) dz d\bar{z} dw d\bar{w}, \quad k, l = 1, \dots, M,$$

*is positive-definite.*

Denote the resulting generalized Gaussian process by  $\mathcal{G}_{\{A_i\}_{i \in I}}$ .

The proof of Proposition 1 will be given later.

**Complex structure.** Let  $A$  be a sequence of pairwise distinct integers. The height function  $H_A$  is naturally defined on  $\mathbb{R} \times \mathbb{R}_{\geq 1}$ . Having the large parameter  $L$ , we would like to scale  $(x, y) \mapsto (L^{-\frac{1}{2}}x, L^{-1}y)$ , which lands us in  $\mathbb{R} \times \mathbb{R}_{>0}$ .

Wigner's semicircle law implies that with  $L \gg 1$ ,  $x \sim L^{\frac{1}{2}}$ ,  $y \sim L$ , after rescaling with overwhelming probability the eigenvalues (or, equivalently, the places of growth of the height function in  $x$ -direction) are concentrated in the domain

$$\{(x, y) \in \mathbb{R} \times \mathbb{R}_{>0} \mid -2\sqrt{y} \leq x \leq 2\sqrt{y}\}.$$

Let us identify the interior of this domain with  $\mathbb{H}$  via the map

$$\Omega : (x, y) \mapsto \frac{x}{2} + i\sqrt{y - \left(\frac{x}{2}\right)^2}.$$

Its inverse has the form

$$\Omega^{-1}(z) = (x(z), y(z)) = (2\Re(z), |z|^2).$$

Note that this map sends the boundary of the domain to the real line.

Thanks to  $\Omega$  we can now speak of the height function  $H_A$  as being defined on  $\mathbb{H}$ ; we will use the notation

$$H_A^\Omega(z) = H_A(L^{\frac{1}{2}}x(z), Ly(z)), \quad z \in \mathbb{H}.$$

Note that we have incorporated rescaling in this definition.

**Main result.** Let  $X$  be a (real symmetric or Hermitian) Wigner matrix. Let  $\{A_i\}_{i \in I}$  be a good family of sequences. We argue that the collection of the centralized random height functions

$$H_{A_i}^\Omega(z_i) - \mathbb{E}H_{A_i}^\Omega(z_i), \quad i \in I, \quad z_i \in \mathbb{H}_i,$$

viewed as distributions, converges as  $L \rightarrow \infty$  to the generalized Gaussian process  $\mathcal{G}_{\{A_i\}_{i \in I}}$ .

One needs to verify the convergence on a suitable set of test functions. The exact statement that we prove is the following.

**Theorem 2.** *Pick  $i \in I$ ,  $y > 0$ , and  $k \in \mathbb{Z}_{\geq 0}$ . Define a moment of the random height function by*

$$M_{i,y,k} = \int_{-\infty}^{+\infty} x^k (H_{A_i}(L^{\frac{1}{2}}x, Ly) - \mathbb{E}H_{A_i}(L^{\frac{1}{2}}x, Ly)) dx.$$

*Then as  $L \rightarrow \infty$ , these moments converge, in the sense of finite dimensional distributions, to the moments of  $\mathcal{G}_{\{A_i\}_{i \in I}}$  defined as*

$$\mathcal{M}_{i,y,k} = \int_{z \in \mathbb{H}_i, |z|^2=y} (x(z))^k \mathcal{G}_{\{A_i\}_{i \in I}}(z) \frac{dx(z)}{dz} dz.$$

**Moments as traces.** Let us rescale the variable  $x = L^{-\frac{1}{2}}u$  in the definition of  $M_{i,y,k}$  and then integrate by parts. Since the derivative of the height function  $H_{A_i}(u, [Ly])$  in  $u$  is

$$\frac{d}{du} H_{A_i}(u, [Ly]) = -\sqrt{\frac{\beta\pi}{2}} \sum_{s=1}^{[Ly]} \delta(u - \lambda_s),$$

where  $\{\lambda_s\}_{1 \leq s \leq [Ly]}$  are the eigenvalues of  $X(A_{i,[Ly]})$ , we obtain

$$\begin{aligned} M_{i,y,k} &= L^{-\frac{k+1}{2}} \sqrt{\frac{\beta\pi}{2}} \left( \sum_{s=1}^{[Ly]} \frac{\lambda_s^{k+1}}{k+1} - \mathbb{E} \sum_{s=1}^{[Ly]} \frac{\lambda_s^{k+1}}{k+1} \right) \\ &= \frac{L^{-\frac{k+1}{2}}}{k+1} \sqrt{\frac{\beta\pi}{2}} \left( \text{Tr}(X(A_{i,[Ly]})^{k+1}) - \mathbb{E} \text{Tr}(X(A_{i,[Ly]})^{k+1}) \right). \end{aligned}$$

We can now reformulate the statement of Theorem 2 as follows.

**Theorem 2'.** *Let  $X$  be a Wigner matrix. Let  $k_1, \dots, k_m \geq 1$  be integers, and let  $B_1, \dots, B_m$  be subsets of  $\mathbb{N}$  dependent on the large parameter  $L$  such that there exists limits*

$$b_p = \lim_{L \rightarrow \infty} \frac{|B_p|}{L} > 0, \quad c_{pq} = \lim_{L \rightarrow \infty} \frac{|B_p \cap B_q|}{L}, \quad p, q = 1, \dots, m.$$

*Then the  $m$ -dimensional random vector*

$$(1) \quad \left( L^{-\frac{k_p}{2}} \left( \text{Tr}(X(B_p)^{k_p}) - \mathbb{E} \text{Tr}(X(B_p)^{k_p}) \right) \right)_{p=1}^m$$

*converges (in distribution and with all moments) to the zero mean  $m$ -dimensional Gaussian random variable  $(\xi_p)_{p=1}^m$  with the covariance*

$$\begin{aligned} (2) \quad & \mathbb{E} \xi_p \xi_q \\ &= \frac{2k_p k_q}{\beta\pi} \oint_{|z|^2=b_p} \oint_{|w|^2=b_q} (x(z))^{k_p-1} (x(w))^{k_q-1} \frac{1}{2\pi} \ln \left| \frac{c_{pq} - zw}{c_{pq} - z\bar{w}} \right| \frac{dx(z)}{dz} \frac{dx(w)}{dw} dz dw. \end{aligned}$$

$|z|^2=b_p \quad |w|^2=b_q$   
 $\Im z > 0 \quad \Im w > 0$

**Proof of Theorem 2'.** The argument closely follows that given in Section 2.1.7 of [AGZ] in the case of one set  $B_j \equiv B$ . One proves the convergence of moments, which is sufficient to also claim the convergence in distribution for Gaussian limits.

Any joint moment of the coordinates of (1) is written as a finite combination of contributions corresponding to suitably defined graphs that are in their turn associated to words. The only difference of the multi-set case with the one-set case is that one needs to keep track of the *alphabets* these words are built from: A word corresponding to coordinate number  $p$  of (1) would have to be built from the alphabet that coincides with the set  $B_p$ . Equivalently, the corresponding graphs will have their vertices labeled by elements of  $B_p$ .

Since all sizes  $|B_p|$  have order  $L$ , and  $|B_1 \cup \dots \cup B_m| = O(L)$ , the estimate showing that all contributions not coming from matchings are negligible (Lemma 2.1.34 in [AGZ]) carries over without difficulty. It only remains to compute the covariance.

For real symmetric Wigner matrices in the one-set case the limits of the variances of the coordinates of (1) are given by (2.1.44) in [AGZ]. It reads (with  $k = k_p$  for a  $p$  between 1 and  $m$ )

$$(3) \quad 2k^2 C_{\frac{k-1}{2}}^2 + k^2 C_{\frac{k}{2}}^2 + \sum_{r=3}^{\infty} \frac{2k^2}{r} \left( \sum_{\substack{k_i \geq 0 \\ 2 \sum_{i=1}^r k_i = k-r}} \prod_{i=1}^r C_{k_i} \right)^2,$$

where  $\{C_k\}_{k \geq 1}$  are the Catalan numbers, and we assume  $C_a = 0$  unless  $a \in \{0, 1, 2, \dots\}$ . The Catalan number  $C_k$  counts the number of rooted planar trees with  $k$  edges, and different terms of (3) have the following interpretation (see [AGZ] for detailed explanations):

- The first term comes from two trees with  $(k-1)/2$  edges each that hang from a common vertex; the factor  $k^2$  originates from choices of certain starting points on each tree united with the common vertex, and the extra 2 is actually  $\mathbb{E}Y_1^2$ .
- The second term comes from two trees with  $k/2$  edges each that are glued along one edge. There are  $k/2$  choices of this edge for each of the trees, there is an additional  $2 = \mathbb{E}Z_{12}^4 - 1$ , and another additional 2 responsible of the choice of the orientation of the gluing.
- The third term comes from two graphs each of which is a cycle of length  $r$  with pendant trees hanging off each of the vertices of the cycle; the total number of edges in the extra trees being  $(k-r)/2$  (this must be an integer). As for the first term, there is an extra  $k^2 = k \cdot k$  coming from the choice of the starting points and also an extra 2 for the choice of the gluing orientation along the cycle.

For each of the three terms the total number of vertices in the resulting graph is equal to  $k$ , and if one labels each vertex with a letter from an alphabet of cardinality  $|B|$  this would yield a factor of

$$|B|(|B| - 1) \cdots (|B| - k + 1) = |B|^k + O(|B|^{k-1}).$$

Normalization by  $|B|^k$  yields (3).

In the general case, in order to evaluate the covariance

$$(4) \quad L^{-\frac{k_p+k_q}{2}} \mathbb{E} \left[ (\text{Tr}(X(B_p)^{k_p}) - \mathbb{E} \text{Tr}(X(B_p)^{k_p})) (\text{Tr}(X(B_q)^{k_q}) - \mathbb{E} \text{Tr}(X(B_q)^{k_q})) \right]$$

in the limit, we need to employ the same graph counting, except for the two graphs being glued now correspond to different values  $k_p$  and  $k_q$  of  $k$ , and their vertices are marked by letters of different alphabets  $B_p$  and  $B_q$ .

- The first term gives  $2k_p k_q C_{\frac{k_p-1}{2}} C_{\frac{k_q-1}{2}}$  for the graph counting, and an extra

$$|B_p \cap B_q| \cdot (|B_p| - 1)(|B_p| - 2) \cdots (|B_p| - \frac{k_p+1}{2}) \cdot (|B_q| - 1)(|B_q| - 2) \cdots (|B_q| - \frac{k_q+1}{2})$$

for the vertex labeling (the factor  $|B_p \cap B_q|$  comes from the only common vertex).

Normalized by  $L^{-\frac{k_p+k_q}{2}}$  this yields

$$2k_p k_q C_{\frac{k_p-1}{2}} C_{\frac{k_q-1}{2}} c_{pq} b_p^{\frac{k_p-1}{2}} b_q^{\frac{k_q-1}{2}}.$$

• The second term has  $k_p k_q C_{\frac{k_p}{2}} C_{\frac{k_q}{2}}$  from the graph counting and  $c_{pq}^2 b_p^{\frac{k_p}{2}-1} b_q^{\frac{k_q}{2}-1}$  from the label counting; a total of

$$k_p k_q C_{\frac{k_p}{2}} C_{\frac{k_q}{2}} c_{pq}^2 b_p^{\frac{k_p}{2}-1} b_q^{\frac{k_q}{2}-1}$$

• For the third term in the same way we obtain

$$\sum_{r=3}^{\infty} \frac{2k_p k_q}{r} \left( \sum_{\substack{s_i \geq 0 \\ 2 \sum_{i=1}^r s_i = k_p - r}} \prod_{i=1}^r C_{s_i} \right) \left( \sum_{\substack{t_i \geq 0 \\ 2 \sum_{i=1}^r t_i = k_q - r}} \prod_{i=1}^r C_{t_i} \right) c_{pq}^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}}$$

Thus, the asymptotic value of the covariance (4) is

$$\begin{aligned} & 2k_p k_q C_{\frac{k_p-1}{2}} C_{\frac{k_q-1}{2}} c_{pq} b_p^{\frac{k_p-1}{2}} b_q^{\frac{k_q-1}{2}} + k_p k_q C_{\frac{k_p}{2}} C_{\frac{k_q}{2}} c_{pq}^2 b_p^{\frac{k_p}{2}-1} b_q^{\frac{k_q}{2}-1} \\ & + \sum_{r=3}^{\infty} \frac{2k_p k_q}{r} \left( \sum_{\substack{s_i \geq 0 \\ 2 \sum_{i=1}^r s_i = k_p - r}} \prod_{i=1}^r C_{s_i} \right) \left( \sum_{\substack{t_i \geq 0 \\ 2 \sum_{i=1}^r t_i = k_q - r}} \prod_{i=1}^r C_{t_i} \right) c_{pq}^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}} \end{aligned}$$

We now use the fact that for any  $S = 0, 1, 2, \dots$

$$\sum_{\substack{s_i \geq 0 \\ \sum_{i=1}^r s_i = S}} \prod_{i=1}^r C_{s_i} = \binom{2S+r}{S} \frac{r}{2S+r},$$

see (5.70) in [GKP]. This allows us to rewrite the asymptotic covariance in terms of binomial coefficients:

$$\begin{aligned} & 2 \binom{k_p}{(k_p-1)/2} \binom{k_q}{(k_q-1)/2} c_{pq} b_p^{\frac{k_p-1}{2}} b_q^{\frac{k_q-1}{2}} \\ & + 4 \binom{k_p}{k_p/2-1} \binom{k_q}{k_q/2-1} c_{pq}^2 b_p^{\frac{k_p-2}{2}} b_q^{\frac{k_q-2}{2}} \\ & + \sum_{r=3}^{\infty} 2r \binom{k_p}{(k_p-r)/2} \binom{k_q}{(k_q-r)/2} c_{pq}^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}} \\ & = \sum_{r=1}^{\infty} 2r \binom{k_p}{(k_p-r)/2} \binom{k_q}{(k_q-r)/2} c_{pq}^r b_p^{\frac{k_p-r}{2}} b_q^{\frac{k_q-r}{2}} \end{aligned}$$

Using the binomial theorem, we can write this expression as a double contour integral

$$(5) \quad \frac{2}{(2\pi i)^2} \iint_{\text{const}_1 = |z| < |w| = \text{const}_2} \left( z + \frac{b_p}{z} \right)^{k_p} \left( w + \frac{b_q}{w} \right)^{k_q} \frac{c_{pq}}{b_p} \frac{dz dw}{\left( \frac{c_{pq}}{b_p} z - w \right)^2}.$$



Consider the right-hand side of (2) and assume that  $|z|^2 = b_p < b_q = |w|^2$ . Observe that

$$\begin{aligned} 2 \ln \left| \frac{c_{pq} - zw}{c_{pq} - z\bar{w}} \right| &= -2 \ln \left| \frac{\frac{c_{pq}}{b_p} z - w}{\frac{c_{pq}}{b_p} \bar{z} - \bar{w}} \right| \\ &= -\ln \left( \frac{c_{pq}}{b_p} z - w \right) + \ln \left( \frac{c_{pq}}{b_p} z - \bar{w} \right) + \ln \left( \frac{c_{pq}}{b_p} \bar{z} - w \right) - \ln \left( \frac{c_{pq}}{b_p} \bar{z} - \bar{w} \right). \end{aligned}$$

This allows us to rewrite the right-hand side of (2) as a double contour integral over complete circles in the form

$$-\frac{k_p k_q}{2\beta\pi^2} \oint_{|z|^2=b_p} \oint_{|w|^2=b_q} (x(z))^{k_p-1} (x(w))^{k_q-1} \ln \left( \frac{c_{pq}}{b_p} z - w \right) \frac{dx(z)}{dz} \frac{dx(w)}{dw} dz dw.$$

Recalling that  $\beta = 1$  and noting that

$$k_p (x(z))^{k_p-1} \frac{dx(z)}{dz} = \frac{d(x(z))^{k_p}}{dz}, \quad k_q (x(w))^{k_q-1} \frac{dx(w)}{dw} = \frac{d(x(w))^{k_q}}{dw},$$

we integrate by parts in  $z$  and  $w$  and recover (5). The proof for  $b_p = b_q$  is obtained by continuity of both sides, and to see that the needed identity holds for  $b_p > b_q$  it suffices to observe that both sides are symmetric in  $p$  and  $q$ .

The argument in the case of Hermitian Wigner matrices is exactly the same, except in the combinatorial part for the first term the factor 2 is missing due to the change in  $\mathbb{E}Y_1^2$ , in the second term 2 is missing due to the change in  $\mathbb{E}|Z_{12}|^4$ , and in the third term 2 is missing because there is no choice in the orientation of two  $r$ -cycles that are being glued together.  $\square$

**Proof of Proposition 1.** We need to show that for any complex numbers  $\{u_k\}_{k=1}^M$

$$\sum_{k,l=1}^M u_k \bar{u}_l \int_{\mathbb{H}} \int_{\mathbb{H}} f_k(z) f_l(w) C_{i_k i_l}(z, w) dz d\bar{z} dw d\bar{w} \geq 0.$$

We can approximate the integration over the two-dimensional domains by finite sums of one-dimensional integrals over semi-circles of the form  $|z| = \text{const}$ . On each semi-circle we further uniformly approximate the (continuous) integrand by a polynomial in  $\Re(z)$ . Finally, for the polynomials the nonnegativity follows from Theorem 2'.  $\square$

**Chebyshev polynomials.** One way to describe the limiting covariance structure in the one-matrix case is to show that traces of the Chebyshev polynomials of the matrix are asymptotically independent, see [J]. A similar effect takes place for submatrices as well.

For  $n = 0, 1, 2, \dots$  let  $T_n(x)$  be the  $n$ th degree Chebyshev polynomial of the first kind:

$$T_n(x) = \cos(n \arccos x), \quad T_n(\cos(x)) = \cos(nx).$$

For any  $a > 0$ , let  $T_n^a(x) = T_n(\frac{x}{a})$  be the rescaled version of  $T_n$ .

**Proposition 3.** *In the assumptions of Theorem 2', for any  $p, q = 1, \dots, m$*

$$\lim_{L \rightarrow \infty} \mathbb{E} \left[ \left( \text{Tr}(T_{k_p}^{2\sqrt{b_p L^{k_p}}}(X(B_p))) - \mathbb{E} \text{Tr}(T_{k_p}^{2\sqrt{b_p L^{k_p}}}(X(B_p))) \right) \right. \\ \left. \times \left( \text{Tr}(T_{k_q}^{2\sqrt{b_q L^{k_q}}}(X(B_q))) - \mathbb{E} \text{Tr}(T_{k_q}^{2\sqrt{b_q L^{k_q}}}(X(B_q))) \right) \right] = \delta_{k_p k_q} \frac{k_p}{2\beta} \left( \frac{c_{pq}}{\sqrt{b_p b_q}} \right)^{k_p}.$$

*Proof.* Using (5) and assuming  $b_p < b_q$  we obtain

$$\begin{aligned} & \lim_{L \rightarrow \infty} \mathbb{E} \left[ \left( \text{Tr}(T_{k_p}^{2\sqrt{b_p L^{k_p}}}(X(B_p))) - \mathbb{E} \text{Tr}(T_{k_p}^{2\sqrt{b_p L^{k_p}}}(X(B_p))) \right) \right. \\ & \quad \left. \times \left( \text{Tr}(T_{k_q}^{2\sqrt{b_q L^{k_q}}}(X(B_q))) - \mathbb{E} \text{Tr}(T_{k_q}^{2\sqrt{b_q L^{k_q}}}(X(B_q))) \right) \right] \\ &= \frac{2}{\beta(2\pi i)^2} \iint_{b_p=|z| < |w|=b_q} T_{k_p}(\cos(\arg(z))) T_{k_q}(\cos(\arg(w))) \frac{c_{pq}}{b_p} \frac{dzdw}{(\frac{c_{pq}}{b_p}z - w)^2} \\ &= \frac{1}{2\beta(2\pi i)^2} \iint_{b_p=|z| < |w|=b_q} \left( \left( \frac{z}{\sqrt{b_p}} \right)^{k_p} + \left( \frac{\sqrt{b_p}}{z} \right)^{k_p} \right) \left( \left( \frac{w}{\sqrt{b_q}} \right)^{k_q} + \left( \frac{\sqrt{b_q}}{w} \right)^{k_q} \right) \\ & \quad \times \frac{c_{pq}}{b_p} \frac{dzdw}{(\frac{c_{pq}}{b_p}z - w)^2}. \end{aligned}$$

Writing  $(\frac{c_{pq}}{b_p}z - w)^{-2}$  as a series in  $z/w$  we arrive at the result. Continuity and symmetry of both sides of the limiting relation removes the assumption  $b_p < b_q$ .  $\square$

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