Magnetic Field Diffusion Through
the Alcator A Liner

R.S. Granetz

Francis Bitter National Magnet Laboratory
and
Plasma Fusion Center

Massachusetts Institute of Technology
Cambridge, Massachusetts   02139

Introduction

In order to measure poloidal magnetic field fluctuations, Alcator A
is fitted with sets of discrete pickup coils located just outside the
stainless steel vacuum vessel. Although the bellows sections of the
liner are only 0.5 mm thick, the pickup coils are positioned on a
smooth steel sleeve, 1/16 " thick, located between two bellows sections.
For future experiments involving feedback stabilization of magnetic
island instabilities, it is necessary to know the amplitude and phase
of these magnetic perturbations inside the stainless steel liner
relative to that measured by the coils outside the liner.

Theoretical

The diffusion equation for the magnetic field is obtained from
two of Maxwell's equations:

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \qquad (1)$$

$$\nabla \times \vec{E} = \frac{-1}{c} \frac{\partial \vec{B}}{\partial t} \qquad (2)$$

We make the following assumptions:

$$\left\{ \begin{array}{c} \vec{H}(t) \\ \vec{E}(t) \end{array} \right\} = \left\{ \begin{array}{c} \vec{H} \\ \vec{E} \end{array} \right\} e^{-i\omega t} \qquad (3)$$

$$\frac{1}{c} \frac{\partial \vec{D}}{\partial t} = 0 \qquad (4)$$

$$\vec{J} = \sigma \vec{E} \qquad \text{(Ohm's law)} \qquad (5)$$

and $\quad \vec{B} = \mu \vec{H} \quad$ in stainless steel. $\qquad (6)$

Equations (1) and (2) become:

$$\vec{H} = \frac{-ic}{\mu\omega} \nabla \times \vec{E} \qquad (7)$$

$$\vec{E} = \frac{c}{4\pi\sigma} \nabla \times \vec{H} \qquad (8)$$

Substituting (8) into (7) gives:

$$\vec{H} = \frac{-ic^2}{4\pi\mu\omega\sigma}\left[\nabla(\underbrace{\nabla\cdot\vec{A}}_{0}) - \nabla^2\vec{H}\right] \tag{9}$$

$$\nabla^2\vec{H} + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right)\vec{H} = 0 \tag{10}$$

When dealing with audio frequency magnetic fields diffusing into the surface of a good conductor, $H_\perp \simeq 0$. $(H_\perp/H_\parallel \simeq \sqrt{\frac{\mu\omega}{4\pi\sigma}} \sim \delta/\lambda)$. In plane geometry, the vector Helmholtz equation separates completely with the result:

$$\vec{H}(x) = \vec{H}_{\parallel_0} e^{-x/\delta} e^{ix/\delta} \tag{11}$$

where

$$\delta = \left[\frac{2c^2}{4\pi\mu\omega\sigma}\right]^{1/2} = \text{skin depth} \tag{12}$$

and x = distance into stainless steel.

Since in the case of interest, the stainless steel liner is quite thin compared to its cross section radius, we expect this solution to be valid. However, a bit more precise result can be obtained by solving the Helmholtz equation is cylindrical geometry. Unfortunately, it does not separate completely in these coordinates[2]:

$$\nabla^2 H_r - \frac{1}{r^2}H_r - \frac{2}{r^2}\frac{\partial H_\theta}{\partial\theta} + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right)H_r = 0$$

$$\nabla^2 H_\theta - \frac{1}{r^2}H_\theta + \frac{2}{r^2}\frac{\partial H_r}{\partial\theta} + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right)H_\theta = 0 \tag{13}$$

$$\nabla^2 H_z + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right)H_z = 0$$

Since the pickup coils only measure $\dot{H}_\theta$, and $H_r = H_\perp \simeq 0$ anyway, we are only interested in the second equation, with the H_r term dropped.

$$\frac{\partial^2 H_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_\theta}{\partial\theta^2} + \frac{\partial^2 H_\theta}{\partial z^2} - \frac{1}{r^2}H_\theta + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right) = 0 \tag{14}$$

subject to the periodicity conditions:

$$H_\theta(r,\theta,z) = H_\theta\left(r,\theta\pm\frac{2\pi}{m},z\right) = H_\theta\left(r,\theta,z\pm\frac{2\pi R}{n}\right) \tag{15}$$

where m and n are the standard MHD mode numbers characterizing the
magnetic islands[3,4]. Another boundary condition is:

$$\lim_{\omega \to \infty} H_\theta(r, \theta, z) = 0 \tag{16}$$

Equation (14) can be solved by assuming that:

$$H_\theta(r, \theta, z) = H_\theta(r) \, \Theta(\theta) \, Z(z) \tag{17}$$

and using the method of separation of variables. We get (see appendix):

$$H_\theta(r, \theta, z) = H_\theta(r) \, e^{\pm im\theta \pm in\frac{z}{R}} \tag{18}$$

where $H_\theta(r)$ satisfies the equation:

$$r^2 \frac{d^2 H_\theta}{dr^2} + r \frac{dH_\theta}{dr} + (k^2 r^2 - p^2) H_\theta = 0 \tag{19}$$

and

$$k^2 = \frac{-n^2}{R^2} + i \frac{4\pi \mu \omega \sigma}{c^2} = \frac{-n^2}{R^2} + i \frac{2}{\delta^2} \tag{20}$$

$$p^2 = 1 + m^2 \tag{21}$$

This is the simple Bessel's equation, and the complete solution is
normally written[5]:

$$H_\theta(r, \theta, z) = \left[c_1 J_p(kr) + c_2 J_{-p}(kr) \right] e^{\pm im\theta \pm in\frac{z}{R}} \tag{22}$$

where C_1, C_2, and $J_{\pm p}$ are complex because k is complex. Unfortunately,
$J_{\pm p}(Z)$ blows up for $|Z| \to \infty$, which corresponds to $\omega \to \infty$. This doesn't
mean that solution (22) is theoretically incorrect, because we can
adjust C_1 and C_2 to satisfy condition (16). But the sum becomes com-
putationally impossible to handle.

Instead we have to use another pair of independent solutions[6]:

$$H_\theta(r, \theta, z) = \left[c_1 I_p(ikr) + c_2 K_p(ikr) \right] e^{\pm im\theta \pm in\frac{z}{R}} \tag{23}$$

As $\omega \to \infty$, I_p blows up, while K_p vanishes. Thus $C_1 = 0$ and the entire
solution inside the stainless steel is:

$$H_\theta(r, \theta, z) = H_{\theta_c} K_p(ikr) \, e^{\pm im\theta \pm in\frac{z}{R}} \tag{24}$$

Since we are only interested in low m and n numbers, we can
further simplify equation(24). In the audio frequencies we are
concerned with, $0.3 < \delta < 0.6$ cm. Thus

$$\frac{2}{\delta^2} >>> \frac{n^2}{R^2} \qquad \text{(R=54 cm)}$$

Therefore $k \simeq i^{1/2} \frac{\sqrt{2}}{\delta}$ (exact for n = 0) and the solution is
virtually independent of toroidal mode number n (for low n). The
function $K_p(i^{3/2}\sqrt{2}\frac{r}{\delta})$ can be written in terms of ker and kei functions[7]:

$$K_p\left(i^{3/2}\sqrt{2}\frac{r}{\delta}\right) \propto ker_p\left(\sqrt{2}\frac{r}{\delta}\right) - i\,kei_p\left(\sqrt{2}\frac{r}{\delta}\right) \qquad (25)$$

This form can also be written in terms of special magnitude and phase
functions: $ker_p(\sqrt{2}\frac{r}{\delta}) - i\,kei_p(\sqrt{2}\frac{r}{\delta}) = N_p(\sqrt{2}\frac{r}{\delta})e^{-i\phi_p(\sqrt{2}\frac{r}{\delta})}$ \qquad (26)

Since the liner has $12.60 < r < 12.75$ cm, the argument $\sqrt{2}\frac{r}{\delta} \sim 45$,
and the asymptotic expansions of N_p and ϕ_p can be used:

$$N_p\left(\sqrt{2}\frac{r}{\delta}\right) \propto \sqrt{\frac{\delta}{r}}\,e^{-r/\delta}$$

$$\phi_p\left(\sqrt{2}\frac{r}{\delta}\right) = \frac{-r}{\delta} + const. \qquad (27)$$

Notice that the asymptotic expansions are independent of the poloidal
mode number, m (for low m).

The final result is:

$$H_\theta(r,\theta,z) = H_{\theta_c}\sqrt{\frac{\delta}{r}}\,e^{-r/\delta}\,e^{ir/\delta}\,e^{\pm im\theta \pm in\frac{z}{R}} \qquad (28)$$

Notice that the only difference between this result and the flat
plane is the factor $\sqrt{\frac{\delta}{r}}$. Since the liner is thin, the range of r is
small and the extra factor hardly changes anyway.

Putting in the numbers gives the following attenuation and phase
shifts for H_θ through the stainless steel liner (for low m and n mode):

ν (kHz)	Attenuation	Phase shift
5	0.775	14.2°
10	0.700	20.1°
15	0.647	24.6°
20	0.605	28.5°

Experiment

The poloidal magnetic diffusion was actually measured using a spare section (60°) of the Alcator A bellows. Two identical magnetic pick-up coils similar to those on the machine were attached at a single location on the liner, one on the inside surface and one on the outside surface. A power amplifier was used to drive several amps (A.C.) through wires running inside the liner, parallel to its axis. (Thus the toroidal mode number, n, was zero.) By using the proper number of wires, fields with poloidal mode numbers, m, of 0,1, and 2 were generated. The results are tabulated here:

	ν (kHz)	Attenuation	Phase shift
	5	0.74 ± 0.05	15 ± 5°
m=0	10	0.63	16°
	15	0.54	17°
	20	0.50	20°
	5	0.58 ± 0.03	34° ± 3°
m=1	10	0.40	47°
	15	0.29	50°
	20	0.24	52°

ν (kHz)	Attenuation	Phase shift
5	0.59 ± 0.03	$32° \pm 3°$
10	0.42	49°
15	0.32	57°
20	0.26	63°

m=2

Conclusion

Except for the lower frequency m=0 measurements, the theoretical results do not match the measured values well at all. It has been suggested[8] that errors might be due to the fact that the experimental bellows was of finite length, although if it wasn't (such as in a real tokamak, where the bellows connects with itself), the measured m=0 values should more closely match the m=1 and m=2 cases. This would mean the theory would not match any measured values at all. Possibly the discrepancy is due to the ripples in the bellows.

There are several other important differences between the experiment done here and the actual Alcator tokamak. First of all, these measurements were done at room temperature, whereas Alcator operates near liquid nitrogen temperature. The conductivity of its stainless steel liner doubles under these cryogenic conditions, and therefore the magnetic field attenuation should be greater.

Secondly, the Alcator vacuum vessel is closely surrounded by a thick copper shell, which should compress the poloidal field against the outside of the bellows, thus effectively, the magnetic field will not be attenuated as much. This effect was measured at room temperature by postioning a thick piece of copper just over the outside pickup coil. An increase in the external B-field of 24 ± 1 % was seen for m=0 (10-20 khz) and

of $60 \overset{=}{\pm} 5$ % for m=2 (20 khz). No measurable increase was seen on the inside coil, and the phase was independent of the copper. At cryogenic temperatures, the conductivity of copper goes up by a factor of 7, so if anything, the compression outside the bellows will be greater in Alcator.

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Substitute equation (17) into (14) and divide by HΘZ to get:

$$\frac{H''}{H} + \frac{1}{r}\frac{H'}{H} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} - \frac{1}{r^2} + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right) = 0 \tag{A1}$$

Rearrange to isolate Z(z):

$$\frac{H''}{H} + \frac{1}{r}\frac{H'}{H} + \frac{1}{r^2}\frac{\Theta''}{\Theta} - \frac{1}{r^2} + i\left(\frac{4\pi\mu\omega\sigma}{c^2}\right) = -\frac{Z''}{Z} = \text{constant} \tag{A2}$$

Thus $\quad Z(z) \propto e^{\pm i\alpha z} \quad$ and $\tag{A3}$

in order to satisfy the periodicity condition on z (equation 15),

$$\alpha = \frac{n}{R} \tag{A4}$$

Substitute this back into (A2), multiply by r^2, and isolate Θ(θ):

$$r^2\frac{H''}{H} + r\frac{H'}{H} + \left(\frac{-n^2}{R^2} + i\frac{4\pi\mu\omega\sigma}{c^2}\right)r^2 - 1 = -\frac{\Theta''}{\Theta} = \text{constant} \tag{A5}$$

Thus $\quad \Theta(\theta) \propto e^{\pm i\alpha\theta} \tag{A6}$

and in order to satisfy the periodicity condition on θ (equation 15),

$$\alpha = m \tag{A7}$$

Substitute this back into (A5), multiply by H, and rearrange to get:

$$r^2 H'' + r H' + \left[\left(\frac{-n^2}{R^2} + i\frac{4\pi\mu\omega\sigma}{c^2}\right)r^2 - (1+m^2)\right]H = 0 \tag{A8}$$