PSFC/JA-03-13

Drift-Ordered Fluid Equations for Field-Aligned Modes in Low-β Collisional Plasma with Equilibrium Pressure Pedestals

Simakov, A.N., Catto, P.J

June 2003

Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139 USA

This work was supported by the U.S. Department of Energy, Grant No. DE-FG02-91ER-54109.

Submitted for publication to *Physics of Plasmas*

Drift-Ordered Fluid Equations for Field-Aligned Modes in Low-β Collisional Plasma with Equilibrium Pressure Pedestals

Andrei N. Simakov and Peter J. Catto

Massachusetts Institute of Technology, Plasma Science and Fusion Center,

167 Albany Street, Cambridge, Massachusetts 02139

(June 26, 2003)

Abstract

Starting from the complete short mean-free path fluid equations describing magnetized plasmas, assuming that plasma pressure is small compared to magnetic pressure, considering field-aligned plasma fluctuations, and adopting an ordering in which the plasma species flow velocities are much smaller than the ion thermal speed, a system of non-linear equations for plasma density, electron and ion temperatures, parallel ion flow velocity, parallel current, electrostatic potential, perturbed parallel electromagnetic potential, and perturbed magnetic field is derived. The equations obtained allow sharp equilibrium radial gradients of plasma quantities, and are shown to contain the neoclassical (Pfirsch-Schlüter) results for plasma current, parallel ion flow velocity (with the correct temperature gradient terms), and parallel gradients of equilibrium electron and ion temperatures. Special care is taken to insure the divergence-free character of perturbed magnetic field and total plasma current, as well as local particle number and electron and ion energy conservation.

I. Introduction

Properly modelling and understanding turbulent transport in magnetically confined plasmas is one of the important challenges facing magnetic fusion. To confine a plasma for fusion applications normally the most virulent ideal magnetohydrodynamic instabilities must be avoided so that the turbulent evolution of the plasma is on the diamagnetic frequency time scale with ion flow velocities well below the ion thermal speed. The highly non-linear nature of the plasma evolution is then typically modelled by numerically solving a reduced system of equations. At the edge of a tokamak plasma the perpendicular wavelengths and scale lengths must be allowed to be comparable and far shorter than the parallel wavelengths and scale lengths. The lack of spatial and temporal scale separation between the turbulent correlation and transport evolution times and lengths has resulted in the adoption of short mean-free path fluid descriptions as one means of obtaining a tractable description of edge plasmas.

The short mean-free path description of magnetized plasma as originally formulated by Braginskii[1] assumes an ordering in which the ion mean flow is on the order of the ion thermal speed. Mikhailovskii and Tsypin[2] realized that this ordering is not the one of most interest in many practical situations in which the flow is well below the ion thermal speed and on the order of the ion heat flux divided by the pressure. In their ordering the ion flow velocity is allowed to be on the order of the diamagnetic drift velocity – the case of interest for most fusion devices in general, and the edge of many tokamaks in particular. Indeed, most short mean-free path treatments of turbulence in magnetized plasmas must use some version of the Mikhailovskii and Tsypin results to properly treat the temperature gradient terms in the gyroviscosity and parallel ion viscosity. As in all short mean-free path treatments, the ion gyroradius is assumed small compared to all perpendicular lengths and the mean-free path is assumed small compared to all parallel lengths.

Starting from the Mikhailovskii and Tsypin[2] description with some corrections[3] a reduced system of non-linear low-beta equations is derived for fluctuations with parallel variations on the scale of the connection length and perpendicular variations on the scale of the pedestal (the region of sharp gradients just inside the separatrix of a tokamak) width. The resulting system of reduced equations describe the evolution of the plasma density, electron and ion temperatures, parallel ion flow velocity, and vorticity (or, equivalently, electrostatic potential), when supplemented by the parallel Ohm's and Ampere's laws. They are constructed to locally conserve number and electron and ion energy, and insure that the magnetic field[4] and plasma current remain divergence free. In addition to describing plasma turbulence, these reduced equations also allow standard neoclassical Pfirsch-Schlüter results[5] for the current, ion heat flux and ion flow velocity to be recovered.

Existing reduced non-linear collisional numerical descriptions[6–9] only approximately preserve conservation properties, neglect or retain an incomplete expression for the parallel ion viscosity, or even invoke a Braginskii-like high flow closure. The effects of these and other approximations discussed in more detail in Sec. XIII, are hard to quantify since the codes are so time consuming to run that the impact of an approximation is hard to determine. However, the use of a Braginskii closure and/or a neglect or incomplete treatment of parallel ion viscosity means that the proper Pfirsch-Schlüter parallel ion flows will not be retained. This shortcoming is potentially serious since flow shear often plays an important role in the evolution and suppression of turbulence. The consequences of departures from number, energy and/or charge conservation are more difficult to judge, but since the effect is to introduce spatially and temporally varying sources or sinks of particles, energy, and/or charge, it is possible that such errors will become serious when turbulence is modelled on transport time scales.

The paper is organized in the following way. In Sec. II we discuss in detail the orderings and assumptions employed and in Sec. III the starting (corrected Mikhailovskii and Tsypin) equations are briefly summarized. Section IV derives expressions for the electric field and perturbed divergence-free magnetic field in terms of electrostatic potential and the parallel component of the perturbed electromagnetic potential, with the latter related to the parallel plasma current. Sections V through X present derivations of the reduced equations for plasma density, parallel current, electron pressure and temperature, ion parallel flow velocity, ion energy and temperature, and vorticity (the equation for the electrostatic potential), respectively. Section XI demonstrates that the reduced equations locally conserve the total energy of plasma plus the electromagnetic field. In Sec. XII we show that these equations allow the recovery of the standard neoclassical Pfirsch-Schlüter results for parallel current, ion parallel flow velocity, and parallel gradients of electron and ion temperatures. Our results are discussed and conclusions drawn in Sec. XIII. Finally, we use Appendix A to rewrite $\boldsymbol{\nabla} \times (\boldsymbol{B}/B^2)$ with $\boldsymbol{B}$ the total magnetic field in a convenient form, Appendix B to derive all the necessary expressions related to the parallel species viscous stress tensor, Appendix C to do the same but for the ion gyroviscous stress tensor, and Appendix D to derive the leading order expression for the ion "polarization" velocity and for the divergence of its contribution to the total plasma current.

II. Orderings

We assume that plasma equilibrium possesses two independent length scales: a slow scale, L_s, associated with gradients and curvature of the equilibrium magnetic field, $\boldsymbol{B}_0$, that is on the order of the tokamak major radius, R_0; and a fast scale, L_f, associated with sharp radial gradients of plasma pressure and density. The scale L_f is assumed to be much smaller than the tokamak minor radius, a, which, in turn, can be on the order of R_0:

$$\epsilon \equiv \frac{L_f}{L_s} \ll 1. \tag{1}$$

In addition, we concentrate on field-aligned fluctuations and assume

$$k_\parallel \sim L_s^{-1}, \qquad k_\perp \sim L_f^{-1}, \tag{2}$$

where $k_\parallel$ and $k_\perp$ are the components of the fluctuation wave vector parallel and perpendicular to the magnetic field, respectively.

We consider low-β,

$$\beta \equiv \frac{8\pi\,(p_i + p_e)}{B^2} \ll 1, \tag{3}$$

magnetized, and collision-dominated plasmas,

$$\delta_j \equiv \frac{\rho_j}{L_f} \sim k_\perp \rho_j \ll 1, \qquad \Delta_j \equiv \frac{\lambda_j}{L_s} \sim k_\parallel \lambda_j \ll 1, \tag{4}$$

where p_i and p_e are the ion and electron pressure, respectively, $\rho_j \equiv v_{Tj}/|\Omega_j|$ is the gyroradius, and $\lambda_j \equiv v_{Tj}/\nu_j$ is mean-free path with $v_{Tj} \equiv \sqrt{2T_j/m_j}$ the thermal speed, $\Omega_j \equiv Z_j eB/m_j c$ the gyrofrequency, and ν_j the characteristic collision frequency[1] for species j ($j = e, i$):

$$\nu_e \equiv \frac{4\sqrt{2\pi}\Lambda Z_i^2 e^4 n_i}{3\sqrt{m_e}\, T_e^{3/2}}, \qquad \nu_i \equiv \frac{4\sqrt{\pi}\Lambda Z_i^4 e^4 n_i}{3\sqrt{m_i}\, T_i^{3/2}}. \tag{5}$$

As usual, T_j, n_j, m_j, $Z_j e$, Λ, c and B denote temperature, density, mass and charge of species j, Coulomb logarithm, the speed of light, and the magnitude of the magnetic field, respectively.

We shall treat the small parameters Δ_j and δ_i on equal footing while deriving our fluid equations for plasma turbulence:

$$\Delta_j \sim \delta_i. \tag{6}$$

In addition to the leading order terms, we also retain in our equations the small terms necessary to recover the standard Pfirsch-Schlüter neoclassical expressions[5] for the plasma current, radial ion heat flux, and parallel ion flow velocity, which are only valid for $\delta_i \ll \Delta_i$. We resist the urge to retain the more complicated viscosity expressions required for our $\Delta_i \sim \delta_i$ ordering since the more general collisional neoclassical expressions for fluxes and flows are not available in the literature.

We also assume that

$$|\Omega_j| \gg \nu_j \gg \omega \sim \delta_i^2 \Omega_i, \tag{7}$$

where ω is the characteristic frequency of interest, and that

$$\frac{V_j}{v_{Ti}} \ll 1, \tag{8}$$

where V_j is the flow velocity of species j. We shall see in Sec. VIII that when turbulence is present $V_{\parallel i} \sim (\epsilon/\delta_i)\, v_{Ti}$. It is therefore necessary to assume that $\epsilon \ll \delta_i$ to be consistent with a drift ordering of V_j. It follows from inequality (7) that $\epsilon \gg \delta_i \Delta_i$, so we find

$$\delta_i \gg \epsilon \gg \delta_i \Delta_i. \tag{9}$$

Notice that the Braginskii fluid equations[1] consider the MHD ordering, $V_j \sim v_{Ti}$, so important terms are missing in the expressions for the viscous stress tensor. Indeed, the use of standard Braginskii expressions for the species viscous stress tensors leads to the appearance of large spurious terms that are cancelled by heat flow terms in the drift-ordered equations of Mikhailovskii and Tsypin.[2]

Finally, we allow the amplitude of fluctuations in plasma quantities, such as density, ion and electron temperatures and flow velocities, to be on the order of corresponding unperturbed quantities. At the same time, we shall see that fluctuations of magnetic field, $\boldsymbol{B}_1$, are $\beta \ll 1$ times smaller than the unperturbed field $\boldsymbol{B}_0$.

III. Starting Equations

We begin by briefly summarizing the exact moments of the kinetic equation for plasma species j, namely the continuity equation,

$$\frac{\partial n_j}{\partial t} + \boldsymbol{\nabla} \cdot (n_j \boldsymbol{V}_j) = S_j^n; \tag{10}$$

the momentum conservation equation,

$$\frac{\partial}{\partial t}(m_j n_j \boldsymbol{V}_j) + \boldsymbol{\nabla} p_j + \boldsymbol{\nabla} \cdot \left(\overleftrightarrow{\pi}_j + m_j n_j \boldsymbol{V}_j \boldsymbol{V}_j\right) = Z_j e n_j \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{V}_j \times \boldsymbol{B}\right) + \boldsymbol{R}_j + \boldsymbol{S}_j^M; \tag{11}$$

and the energy conservation equation,

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_j + \frac{1}{2}m_j n_j V_j^2\right) + \boldsymbol{\nabla} \cdot \boldsymbol{Q}_j = W_j + \boldsymbol{V}_j \cdot (\boldsymbol{R}_j + Z_j e n_j \boldsymbol{E}) + S_j^E, \tag{12}$$

where S_j^n, $\boldsymbol{S}_j^M$ and S_j^E are density, momentum and energy sources for species j, respectively.

The viscous stress tensor, $\overleftrightarrow{\pi}_j$, is given by[2,3]

$$\overleftrightarrow{\pi}_j = \overleftrightarrow{\pi}_{cj} + \overleftrightarrow{\pi}_{gj} + \overleftrightarrow{\pi}_{\perp j}, \tag{13}$$

where the "parallel" viscous stress tensor is

$$\overleftrightarrow{\pi}_{cj} = \left(\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}} - \frac{1}{3}\overleftrightarrow{I}\right)\pi_{cj} \tag{14}$$

with

$$\begin{gathered}
\pi_{cj} = -\eta_j\left(3\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}} - \overleftrightarrow{I}\right) : \left(\overleftrightarrow{\alpha}_j - \xi_j\,\overleftrightarrow{\gamma}_j\right) + \zeta_j, \quad \overleftrightarrow{\alpha}_j = \boldsymbol{\nabla}\boldsymbol{V}_j + \frac{2}{5p_j}\boldsymbol{\nabla}\boldsymbol{q}_j, \\
\overleftrightarrow{\gamma}_j = \frac{2}{5p_j}\left[\boldsymbol{q}_j\left(\boldsymbol{\nabla}\ln p_j - \boldsymbol{F}_j\right) - \left(2\boldsymbol{q}_j - \boldsymbol{q}_j^*\right)\boldsymbol{\nabla}\ln T_j - \boldsymbol{\nabla}\left(\boldsymbol{q}_j - \boldsymbol{q}_j^*\right)\right], \quad \boldsymbol{F}_i = 0, \\
\boldsymbol{F}_e = \frac{\boldsymbol{R}_e}{p_e}, \quad \boldsymbol{q}_i^* = -\frac{4}{15}\boldsymbol{q}_{\parallel i}, \quad \boldsymbol{q}_e^* = 1.58\,\frac{p_e}{m_e\nu_e}\boldsymbol{\nabla}_\parallel T_e + 0.08p_e\left(\boldsymbol{V}_{\parallel e} - \boldsymbol{V}_{\parallel i}\right), \\
\eta_i = 0.96\frac{p_i}{\nu_i}, \quad \eta_e = 0.73\frac{p_e}{\nu_e}, \quad \zeta_i = \frac{3}{2p_iv_{Ti}^2}\left(0.20q_{\parallel i}^2 - 0.085q_i^2\right), \\
\zeta_e = \frac{3}{2p_ev_{Te}^2}\left[0.109\left(q_{\parallel e}^2 - \frac{1}{3}q_e^2\right) + 0.092\boldsymbol{q}_e\cdot\boldsymbol{q}_e^* + 0.069q_e^{*2}\right],
\end{gathered} \tag{15}$$

$\boldsymbol{R}_e$ the electron-ion friction force defined below, $\xi_i = 0.61$, $\xi_e = 0.98$, $\overleftrightarrow{I}$ the unit tensor, and $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$. The so-called "gyroviscous" part of the stress tensor is

$$\overleftrightarrow{\pi}_{gj} = \frac{p_j}{4\Omega_j}\left\{\hat{\boldsymbol{b}}\times\left[\overleftrightarrow{\alpha}_j + \overleftrightarrow{\alpha}_j^{\mathrm{T}}\right]\cdot\left(\overleftrightarrow{I} + 3\,\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}}\right) - \left(\overleftrightarrow{I} + 3\,\hat{\boldsymbol{b}}\,\hat{\boldsymbol{b}}\right)\cdot\left[\overleftrightarrow{\alpha}_j + \overleftrightarrow{\alpha}_j^{\mathrm{T}}\right]\times\hat{\boldsymbol{b}}\right\}, \tag{16}$$

where a superscript "T" is used to denote a transpose of a dyad. The "perpendicular" part of the stress tensor is a factor of $\nu_j/|\Omega_j|$ smaller than the "gyroviscous" part and negligible for our considerations.

Expressions (14) - (16) reduce to the corresponding Braginskii expressions[1] in the limit $\boldsymbol{q}_j, \boldsymbol{q}_j^* \to 0$. Equations (14) and (15) coincide with the Mikhailovskii and Tsypin[2] expressions for the ions in the limit $\zeta_i \to 0$, and expression (16) is identical to theirs. The electron viscous stress tensor, $\overleftrightarrow{\pi}_e$, was not evaluated by Mikhailovskii and Tsypin, so for completeness we have evaluated it and the ζ_i and ζ_e modifications as explained in Ref. [3].

The energy flux is given by the expression

$$\boldsymbol{Q}_j = \left(\frac{5}{2}n_jT_j + \frac{1}{2}m_jn_jV_j^2\right)\boldsymbol{V}_j + \overleftrightarrow{\pi}_j\cdot\boldsymbol{V}_j + \boldsymbol{q}_j \tag{17}$$

with[1]

$$\boldsymbol{q}_i = -\kappa_{\parallel i}\boldsymbol{\nabla}_\parallel T_i + \frac{5p_i}{2m_i\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_i - \kappa_{\perp i}\boldsymbol{\nabla}_\perp T_i, \tag{18}$$

$$\boldsymbol{q}_e = -\kappa_{\parallel e}\boldsymbol{\nabla}_\parallel T_e + \frac{5p_e}{2m_e\Omega_e}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_e - \kappa_{\perp e}\boldsymbol{\nabla}_\perp T_e - 0.71\frac{T_e\boldsymbol{J}_\parallel}{e} + \frac{3\nu_e}{2\Omega_e}\frac{T_e\hat{\boldsymbol{b}}\times\boldsymbol{J}}{e},$$

and

$$\kappa_{\parallel i} = \frac{125 p_i}{32 m_i \nu_i}, \qquad \kappa_{\perp i} = 2\frac{p_i \nu_i}{m_i \Omega_i^2}, \qquad \kappa_{\parallel e} = 3.16\frac{p_e}{m_e \nu_e}, \qquad \kappa_{\perp e} = 4.66\frac{p_e \nu_e}{m_e \Omega_e^2} \tag{19}$$

the usual Braginskii heat-conduction coefficients.

Finally, the electron-ion friction forces are[1]

$$\boldsymbol{R}_e = -\boldsymbol{R}_i = e n_e \left(\frac{\boldsymbol{J}_\parallel}{\sigma_\parallel} + \frac{\boldsymbol{J}_\perp}{\sigma_\perp} \right) - 0.71 n_e \boldsymbol{\nabla}_\parallel T_e + \frac{3\nu_e}{2\Omega_e} n_e \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_e \tag{20}$$

with $\sigma_\perp = 0.51\sigma_\parallel = (e^2 n_e / m_e \nu_e)$, while the energy exchange terms are

$$W_i = \frac{3 m_e n_e \nu_e}{m_i} (T_e - T_i), \qquad W_e = -W_i - \boldsymbol{R}_e \cdot (\boldsymbol{V}_e - \boldsymbol{V}_i). \tag{21}$$

In what follows we assume for simplicity that $Z_i = 1 = -Z_e$, so that quasineutrality requires $n_i = n_e \equiv n$ and $S_i^n = S_e^n \equiv S^n$.

IV. Electromagnetic Fields

In this section we derive equations for evolution of electromagnetic fields. We assume that the equilibrium magnetic field varies in space on the slow scale only, while the perturbed magnetic field can vary on both the fast and slow scales. In addition, we assume that both perturbed and unperturbed plasma quantities, such as plasma pressure, vary on both slow and fast scales.

Then, following the assumptions of Sec. II and the procedure of Ref. [10], we write the magnetic field, $\boldsymbol{B}$, vector potential, $\boldsymbol{A}$, and plasma pressure, p, in the form

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_0(\boldsymbol{x}_s) + \boldsymbol{B}_1(\boldsymbol{x}_f, \boldsymbol{x}_s, t), \\ \boldsymbol{A} &= \boldsymbol{A}_0(\boldsymbol{x}_s, t) + \boldsymbol{A}_1(\boldsymbol{x}_f, \boldsymbol{x}_s, t), \\ p &= p_0(\boldsymbol{x}_f, \boldsymbol{x}_s) + p_1(\boldsymbol{x}_f, \boldsymbol{x}_s, t). \end{aligned} \tag{22}$$

Here, we use subscripts "0" and "1" to indicate equilibrium and perturbed quantities, respectively, denote the fast and slow spatial variation by $\boldsymbol{x}_f$ and $\boldsymbol{x}_s$, and assume $B_1 \sim \epsilon B_0$, $A_1 \sim \epsilon^2 A_0$, but $p_1 \sim p_0$. Derivatives must now be taken with respect to both slow and fast variables,

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_f + \boldsymbol{\nabla}_s, \tag{23}$$

where $\boldsymbol{\nabla}_f \equiv \partial/\partial \boldsymbol{x}_f \sim L_f^{-1}$ and $\boldsymbol{\nabla}_s \equiv \partial/\partial \boldsymbol{x}_s \sim L_s^{-1}$. Since $k_\parallel$ is assumed to be small, we have to require

$$\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}_f = 0. \tag{24}$$

Using Eqs. (22) and (23) and $\boldsymbol{\nabla}_s \cdot \boldsymbol{B}_0 = 0$ to rewrite $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ we obtain

$$0 = \boldsymbol{\nabla}_s \cdot \boldsymbol{B}_0 + \boldsymbol{\nabla}_f \cdot \boldsymbol{B}_1 + \boldsymbol{\nabla}_s \cdot \boldsymbol{B}_1 = \boldsymbol{\nabla}_f \cdot \boldsymbol{B}_1 + \mathrm{O}\left(\epsilon\right), \tag{25}$$

which implies

$$\boldsymbol{B}_1 = \boldsymbol{\nabla}_f A_{\parallel 1} \times \hat{\boldsymbol{b}}_0 + B_{\parallel 1} \hat{\boldsymbol{b}}_0 + \mathrm{O}\left(\epsilon\right), \tag{26}$$

where $A_{\parallel 1} \equiv \boldsymbol{A}_1 \cdot \hat{\boldsymbol{b}}_0$ and $B_{\parallel 1} \equiv \boldsymbol{B}_1 \cdot \hat{\boldsymbol{b}}_0$ are to be determined, and $\hat{\boldsymbol{b}}_0 \equiv \boldsymbol{B}_0 / B_0$.

Employing Ampere's law next we find

$$\frac{4\pi}{c}\boldsymbol{J} = \frac{4\pi}{c}\boldsymbol{J}_0 + \boldsymbol{\nabla}_f \times \boldsymbol{B}_1 + \mathrm{O}\left(\epsilon\right) = \frac{4\pi}{c}\boldsymbol{J}_0 + \boldsymbol{\nabla}_f B_{\parallel 1} \times \hat{\boldsymbol{b}}_0 - \hat{\boldsymbol{b}}_0 \nabla_f^2 A_{\parallel 1} + \mathrm{O}\left(\epsilon\right), \tag{27}$$

so that

$$\nabla_f^2 A_{\parallel 1} = \frac{4\pi}{c}\left(\boldsymbol{J}_0 - \boldsymbol{J}\right) \cdot \hat{\boldsymbol{b}}_0 + \mathrm{O}\left(\epsilon\right), \tag{28}$$

where $\boldsymbol{J} = e n_i \left(\boldsymbol{V}_i - \boldsymbol{V}_e\right)$ and $\boldsymbol{J}_0 = c \boldsymbol{\nabla} \times \boldsymbol{B}_0 / 4\pi$.

Writing

$$\boldsymbol{\kappa} \equiv \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \hat{\boldsymbol{b}} = \hat{\boldsymbol{b}}_0 \cdot \boldsymbol{\nabla}_s \hat{\boldsymbol{b}}_0 + \mathrm{O}\left(\epsilon\right) = \hat{\boldsymbol{b}}_0 \cdot \boldsymbol{\nabla} \hat{\boldsymbol{b}}_0 + \mathrm{O}\left(\epsilon\right), \tag{29}$$

we see that magnetic field curvature is unperturbed to leading order. Using Ampere's law in the leading order equilibrium pressure balance equation we find

$$\frac{4\pi}{c}\boldsymbol{J}_0 \times \boldsymbol{B}_0 = \left(\boldsymbol{\nabla}_s \times \boldsymbol{B}_0\right) \times \boldsymbol{B}_0 = \tag{30}$$
$$B_0^2 \boldsymbol{\kappa} - \frac{1}{2}\left(\overleftrightarrow{\boldsymbol{I}} - \hat{\boldsymbol{b}}_0 \hat{\boldsymbol{b}}_0\right) \cdot \boldsymbol{\nabla}_s B_0^2 + \mathrm{O}\left(\epsilon\right) = 4\pi\, \boldsymbol{\nabla}_f p_0 + \mathrm{O}\left(\epsilon\right),$$

so that using $\boldsymbol{\nabla}_f p_0 \sim p_0 / L_f$ and $\boldsymbol{\kappa} \sim \boldsymbol{\nabla}_s \sim 1/L_s$ gives

$$\epsilon \sim \beta. \tag{31}$$

Consequently, sharp equilibrium pressure gradients are only consistent with smooth magnetic field profiles when β is small.

Employing the perturbed pressure balance equation,

$$-\frac{B_0}{4\pi}\boldsymbol{\nabla}_f B_{\parallel 1} \approx \boldsymbol{\nabla}_f p_1 + \mathrm{O}(\epsilon)\,, \tag{32}$$

we obtain

$$\frac{p_1}{p_0} = -\frac{2}{\beta_0}\frac{B_{\parallel 1}}{B_0} + \mathrm{O}(\epsilon)\,, \tag{33}$$

so that pressure perturbations are $\beta_0^{-1} \equiv B_0^2/8\pi p_0 \gg 1$ times larger than parallel magnetic field perturbations. Therefore, in view of Eq. (31), the ordering of perturbations in Eq. (22) is justified.

Notice, that Eq. (26) evaluates the perturbed magnetic field $\boldsymbol{B}_1$ to leading order only, so that $\boldsymbol{\nabla}\cdot\boldsymbol{B}_1$ is in general not zero. To remove this deficiency for axisymmetric tokamaks we follow Kruger[4] and employ flux or "straight field-line" coordinates[10] ψ, ζ, θ with ψ the poloidal magnetic flux and ζ and θ the toroidal and poloidal angles, respectively. Then, both the equilibrium axisymmetric magnetic field and the perturbed magnetic field can be written in manifestly divergence-free form,[10]

$$\boldsymbol{B}_0 = (\boldsymbol{\nabla}\zeta - q\boldsymbol{\nabla}\theta)\times\boldsymbol{\nabla}\psi \tag{34}$$

and

$$\boldsymbol{B}_1 = \boldsymbol{\nabla}G\times\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}H\times\boldsymbol{\nabla}\theta, \tag{35}$$

with $q = q(\psi) = \boldsymbol{B}_0\cdot\boldsymbol{\nabla}\zeta/\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta$ the equilibrium safety factor, and the functions G and H to be determined from the equations

$$\frac{\partial G}{\partial\psi} = \frac{p-p_0}{B_0^2} + \frac{I}{B_0}\frac{\partial A_{\parallel 1}}{\partial\psi} + \frac{q\,(\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta)}{IB_0}\frac{\partial A_{\parallel 1}}{\partial\zeta}, \qquad H = \frac{qR^2B_0A_{\parallel 1}}{I} - qG, \tag{36}$$

which follow from Eq. (33) to leading order. Here, R is the cylindrical radial coordinate, $I = R^2\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\zeta$, and $A_{\parallel 1}$ should be determined from Eq. (28) to leading order, which we take to be

$$\nabla^2 A_{\parallel 1} = \frac{4\pi}{c}\left(\hat{\boldsymbol{b}}_0\cdot\boldsymbol{J}_0 - \hat{\boldsymbol{b}}\cdot\boldsymbol{J}\right). \tag{37}$$

The above method of obtaining a divergence-free perturbed magnetic field can be generalized for different axisymmetric as well as non-axisymmetric coordinates.[4]

Next, we obtain expressions for parallel and perpendicular electric fields using the general expression

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{38}$$

where ϕ is the electrostatic potential. Neglecting the induced electric field $\boldsymbol{E}_{\text{ind}} \equiv -c^{-1}\partial \boldsymbol{A}_0/\partial t$, recalling that $B_{\parallel 1}\hat{\boldsymbol{b}}_0 \approx \boldsymbol{\nabla}_f \times \boldsymbol{A}_{\perp 1}$, and employing Eqs. (33) and (7) to estimate

$$A_{\perp 1} \sim \beta B_0 L_f, \qquad \left|\frac{1}{c}\frac{\partial \boldsymbol{A}_{\perp 1}}{\partial t}\right| \sim \frac{1}{c}\,\delta_i^2 \Omega_i \beta B_0 L_f, \tag{39}$$

we find

$$\frac{|c^{-1}\partial \boldsymbol{A}_{\perp 1}/\partial t|}{|\boldsymbol{\nabla}_\perp \phi|} \sim \beta \ll 1, \tag{40}$$

where the estimates $|\boldsymbol{\nabla}_\perp \phi| \sim (T_e/eL_f)$ and $T_e \sim T_i$ were used. Consequently, the perpendicular electric field is electrostatic to leading order,

$$\boldsymbol{E}_\perp = -\boldsymbol{\nabla}_\perp \phi, \tag{41}$$

where $\mathrm{O}(\beta)$ corrections are neglected. Using Eq. (37) to estimate $A_{\parallel 1}$ in terms of $J_\parallel$ and noticing that $\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$ results in $J_\parallel \sim J_\perp$ we find

$$A_{\parallel 1} \sim \beta B_0 L_f, \qquad \frac{|c^{-1}\partial A_{\parallel 1}/\partial t|}{|\nabla_\parallel \phi|} \sim \frac{\beta}{\epsilon}. \tag{42}$$

In view of Eq. (31) both electrostatic and inductive parts of the parallel electric field must be retained, so that

$$E_\parallel = -\nabla_\parallel \phi - \frac{1}{c}\frac{\partial A_{\parallel 1}}{\partial t}, \tag{43}$$

where $\mathrm{O}(\epsilon)$ corrections are ignored. Here and elsewhere, $\nabla_\parallel \equiv \hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}$ refers to a gradient in a direction of the *total* magnetic field (22).

V. Plasma Density Equation

In this section we derive an equation describing the evolution of the plasma density. We begin by obtaining an expression for perpendicular flow velocity $\boldsymbol{V}_{\perp j}$ of species

j. It follows from the perpendicular component of momentum conservation equation (11) that

$$\boldsymbol{V}_{\perp j} = \boldsymbol{V}_{E\times B} + \boldsymbol{V}_{dj} + \boldsymbol{V}_{pj}, \tag{44}$$

where the $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic velocities are given by

$$\boldsymbol{V}_{E\times B} \equiv c\frac{\boldsymbol{E}\times\hat{\boldsymbol{b}}}{B} = c\,\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\phi}{B} + \mathrm{O}\left(\epsilon,\beta\right), \quad \boldsymbol{V}_{dj} \equiv c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}p_j}{Z_j e n_j B}. \tag{45}$$

The polarization velocity given by

$$\boldsymbol{V}_{pj} \equiv \frac{1}{\Omega_j}\hat{\boldsymbol{b}}\times\left(\frac{\partial\boldsymbol{V}_j}{\partial t} + \boldsymbol{V}_j\cdot\boldsymbol{\nabla}\boldsymbol{V}_j + \frac{\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_j + m_j\boldsymbol{V}_j S^n - \boldsymbol{S}_j^M - \boldsymbol{R}_j}{m_j n}\right) \tag{46}$$

is a small correction to the $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic velocities (of order δ_i^2 for ions and $\sqrt{m_e/m_i}$ times smaller for electrons), which can normally be neglected for electrons, but should be retained for ions to allow evaluation of the radial electric field (Sec. X). The right-hand side of Eq. (46) will be substantially simplified when we derive the vorticity equation.

Using expression (44) with $j\to e$ in the electron continuity equation (10), defining

$$\boldsymbol{V}_E \equiv c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\phi}{B}, \tag{47}$$

and retaining in $\boldsymbol{V}_{pe}$ only the contribution from the electron momentum source $\boldsymbol{S}_e^M$, which can become important when radio frequency waves are launched into the plasma,

$$\boldsymbol{V}_{pe} \equiv \frac{c}{enB}\hat{\boldsymbol{b}}\times\boldsymbol{S}_e^M, \tag{48}$$

we obtain a conservative form of the plasma density evolution equation,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\left[n\left(\boldsymbol{V}_E + \boldsymbol{V}_{de} + \boldsymbol{V}_{\parallel e} + \boldsymbol{V}_{pe}\right)\right] = S^n. \tag{49}$$

It is convenient in some cases to use the *exact* identities

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{V}_E &= c\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)\cdot\boldsymbol{\nabla}\phi, \\
\boldsymbol{\nabla}\cdot\left(n\boldsymbol{V}_{dj}\right) &= \frac{c}{Z_j e}\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)\cdot\boldsymbol{\nabla}p_j, \\
\boldsymbol{\nabla}\cdot\boldsymbol{V}_{\parallel j} &= B\nabla_\parallel\left(\frac{V_{\parallel j}}{B}\right),
\end{aligned} \tag{50}$$

where $\nabla_\parallel = \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}$, to rewrite Eq. (49) as

$$\frac{\partial n}{\partial t} + \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel e}\right) \cdot \boldsymbol{\nabla} n = \frac{c}{e}\left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{b}}}{B}\right) \cdot \left(\boldsymbol{\nabla} p_e - en\boldsymbol{\nabla}\phi\right) \tag{51}$$
$$-nB\nabla_\parallel \left(\frac{V_{\parallel e}}{B}\right) - \boldsymbol{\nabla} \cdot \left(n\boldsymbol{V}_{pe}\right) + S^n.$$

Since no further approximations are required to obtain (51) it has the same conservation property as (49). Notice in particular that the approximation $\boldsymbol{\nabla} \times (\hat{\boldsymbol{b}}/B) \to (2/B)\, \hat{\boldsymbol{b}} \times \boldsymbol{\kappa}$ is not used (see Appendix A).

VI. Electron Parallel Momentum Equation (Ohm's Law)

Using the continuity equation (10) to rewrite the momentum conservation equation (11) for electrons, dotting the resulting expression by $\hat{\boldsymbol{b}}$, and dividing by $m_e n$, we arrive at

$$\frac{\partial V_{\parallel e}}{\partial t} + \boldsymbol{V}_e \cdot \boldsymbol{\nabla} V_{\parallel e} - \left(\frac{\partial \hat{\boldsymbol{b}}}{\partial t} + \boldsymbol{V}_e \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}}\right) \cdot \boldsymbol{V}_e = \tag{52}$$
$$-\frac{e}{m_e}E_\parallel + \frac{R_{\parallel e} - \nabla_\parallel p_e}{m_e n} - \frac{\hat{\boldsymbol{b}} \cdot \left(\boldsymbol{\nabla} \cdot \overleftrightarrow{\pi}_e\right)}{m_e n} + \frac{S^M_{\parallel e} - m_e V_{\parallel e} S^n}{m_e n}.$$

Neglecting inertial (small by m_e/m_i) and viscous [small by $\sqrt{m_e/m_i}\,(\epsilon\Delta_i/\delta_i)$] terms and employing Eq. (20) to find

$$\frac{R_{\parallel e} - \nabla_\parallel p_e}{m_e n} = 0.51\nu_e \left(V_{\parallel i} - V_{\parallel e}\right) - \frac{1}{m_e n}\left(1.71 n\nabla_\parallel T_e + T_e \nabla_\parallel n\right), \tag{53}$$

we obtain the required Ohm's law for the parallel current,

$$\frac{enJ_\parallel}{\sigma_\parallel} = 1.71 n \nabla_\parallel T_e + T_e \nabla_\parallel n + enE_\parallel - S^M_{\parallel e}, \tag{54}$$

where $E_\parallel$ is given by Eq. (43).

VII. Electron Energy Equation

To simplify the electron energy conservation equation, Eq. (12) with $j \to e$, we first notice that contributions from the electron kinetic energy under the time derivative and from the electron kinetic energy and viscosity in the expression for $\boldsymbol{Q}_e$ are small and can be neglected. Neglecting inertial and viscous terms in the electron momentum conservation equation, Eq. (11) with $j \to e$, we next find

$$\boldsymbol{V}_e \cdot (\boldsymbol{R}_e - en\boldsymbol{E}) \approx \boldsymbol{V}_e \cdot \left(\boldsymbol{\nabla} p_e - \boldsymbol{S}_e^M\right) = \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel e}\right) \cdot \left(\boldsymbol{\nabla} p_e - \boldsymbol{S}_e^M\right). \tag{55}$$

In addition, we use expressions (18) – (21) for $\boldsymbol{q}_e$ and W_e, and notice that

$$\frac{5}{2}\, p_e \boldsymbol{V}_{de} + \frac{5p_e}{2m_e\Omega_e}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_e = \frac{5}{2m_e\Omega_e}\,\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\left(p_e T_e\right). \tag{56}$$

Then we employ $\boldsymbol{J}_\perp \approx c\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\left(p_e + p_i\right)/B$ to obtain

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \left(\kappa_{\perp e}\boldsymbol{\nabla}_\perp T_e - \frac{3\nu_e T_e}{2\Omega_e e}\hat{\boldsymbol{b}} \times \boldsymbol{J}\right) + \frac{\boldsymbol{J}}{ne} \cdot \left(\frac{ne}{\sigma_\perp}\boldsymbol{J}_\perp + \frac{3\nu_e n}{2\,\Omega_e}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_e\right) \approx \\
&\boldsymbol{\nabla} \cdot \left\{\frac{\nu_e p_e^2}{m_e n\Omega_e^2}\left[4.66\boldsymbol{\nabla}_\perp \ln T_e - \frac{3}{2}\left(1 + \frac{T_i}{T_e}\right)\boldsymbol{\nabla}_\perp \ln\left(p_e + p_i\right)\right]\right\} \\
&+\frac{\nu_e}{m_e n\Omega_e^2}\left\{\left[\nabla_\perp\left(p_i + p_e\right)\right]^2 - \frac{3}{2}n\boldsymbol{\nabla}_\perp T_e \cdot \boldsymbol{\nabla}_\perp\left(p_e + p_i\right)\right\},
\end{aligned} \tag{57}$$

and observe that the right-hand side of Eq. (57) is approximately $\left(\delta_i/\Delta_i\right)\sqrt{m_e/m_i}$ times smaller than divergence of the right-hand side of Eq. (56) and can therefore be neglected. As a result, we find a conservative form of the electron energy equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{3}{2}\,p_e\right) + \boldsymbol{\nabla} \cdot \left[\frac{5}{2}\,p_e\left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel e} + \boldsymbol{V}_{pe}\right) + \frac{5}{2m_e\Omega_e}\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\left(p_e T_e\right) - \kappa_{\parallel e}\boldsymbol{\nabla}_\parallel T_e\right. \\
&\left.-\frac{0.71 T_e}{e}\boldsymbol{J}_\parallel\right] = -W_i + \frac{J_\parallel^2}{\sigma_\parallel} - \frac{0.71\nabla_\parallel T_e}{e}\,J_\parallel + \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel e}\right) \cdot \left(\boldsymbol{\nabla} p_e - \boldsymbol{S}_e^M\right) + S_e^E.
\end{aligned} \tag{58}$$

Using definitions (45), (47), and (48) of $\boldsymbol{V}_{de}$, $\boldsymbol{V}_E$, and $\boldsymbol{V}_{pe}$, respectively, and the plasma density evolution equation (49), without further approximations Eq. (58) can be rewritten as the equation for T_e:

$$\begin{aligned}
&\frac{3n}{2}\left[\frac{\partial T_e}{\partial t} + \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel e} + \boldsymbol{V}_{pe}\right) \cdot \boldsymbol{\nabla} T_e\right] = \frac{cT_e}{e}\left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{b}}}{B}\right) \cdot \left(\boldsymbol{\nabla} p_e - en\boldsymbol{\nabla}\phi + \frac{5}{2}\,n\boldsymbol{\nabla} T_e\right) \\
&+B\nabla_\parallel\left(\frac{\kappa_{\parallel e}\nabla_\parallel T_e}{B}\right) - p_e B\nabla_\parallel\left(\frac{V_{\parallel e}}{B}\right) + \frac{0.71}{e}\,T_e B\nabla_\parallel\left(\frac{J_\parallel}{B}\right) + \frac{J_\parallel^2}{\sigma_\parallel} \\
&-p_e\boldsymbol{\nabla} \cdot \boldsymbol{V}_{pe} - W_i + S_e^E - \boldsymbol{S}_e^M \cdot \left(\boldsymbol{V}_E + \boldsymbol{V}_{de} + \boldsymbol{V}_{\parallel e}\right) - \frac{3}{2}\,T_e S^n.
\end{aligned} \tag{59}$$

Equation (59) retains the exact conservation properties of Eq. (58).

VIII. Ion Parallel Momentum Equation

Adding the momentum conservation equations (11) for electrons and ions and dropping electron inertial and viscous terms as small by factors of m_e/m_i and $\sqrt{m_e/m_i}$, respectively, compared to their ion counterparts we obtain

$$\begin{gathered}\frac{\partial}{\partial t}\left(m_i n \boldsymbol{V}_i\right)+\boldsymbol{\nabla}\cdot\left(m_i n \boldsymbol{V}_i \boldsymbol{V}_i\right)+\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_i= \\ -\boldsymbol{\nabla}\left(p_i+p_e\right)+\frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}+\boldsymbol{S}_i^M+\boldsymbol{S}_e^M .\end{gathered} \tag{60}$$

Dotting Eq. (60) by $\hat{\boldsymbol{b}}$ and neglecting $m_i n \boldsymbol{V}_i \cdot \partial\hat{\boldsymbol{b}}/\partial t$ and $m_i n \boldsymbol{V}_i \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}}\cdot\boldsymbol{V}_i$ as small compared to $\partial\left(m_i n V_{\|i}\right)/\partial t$ and $\boldsymbol{\nabla}\cdot\left(m_i n V_{\|i}\boldsymbol{V}_i\right)$, respectively, by a factor of ϵ we find

$$\begin{gathered}\frac{\partial}{\partial t}\left(m_i n V_{\|i}\right)+\boldsymbol{\nabla}\cdot\left[m_i n V_{\|i}\left(\boldsymbol{V}_E+\boldsymbol{V}_{di}+\boldsymbol{V}_{\|i}+\boldsymbol{V}_{pi}\right)\right]= \\ -\nabla_{\|}\left(p_i+p_e\right)-\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_i\right)+S_{\|i}^M+S_{\|e}^M,\end{gathered} \tag{61}$$

independent of the precise form of $\boldsymbol{V}_{pi}$.

Using Eq. (13) and neglecting the ion perpendicular viscous stress tensor contributions we next write

$$\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_i\right)\approx\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_{ci}\right)+\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_{gi}\right),$$

where the first term on the right-hand side can be conveniently evaluated by using Eq. (B2) from Appendix B,

$$\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_{ci}\right)=\frac{2}{3}B^{3/2}\nabla_{\|}\left(\frac{\pi_{ci}}{B^{3/2}}\right). \tag{62}$$

The term $\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_{gi}\right)$ is given in a turbulent plasma to leading order by Eq. (C11) from Appendix C,

$$\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot \overleftrightarrow{\pi}_{gi}\right)\approx\boldsymbol{\nabla}\cdot\left(\frac{p_i}{\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}V_{\|i}\right). \tag{63}$$

Using results (62) and (63) in Eq. (61) and noticing that

$$\boldsymbol{\nabla}\cdot\left(m_i n V_{\|i}\boldsymbol{V}_{di}+\frac{p_i}{\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}V_{\|i}\right)=\boldsymbol{\nabla}\cdot\left[\frac{1}{\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_i V_{\|i}\right)\right] \tag{64}$$

yields

$$\frac{\partial}{\partial t}\left(m_i n V_{\|i}\right)+\boldsymbol{\nabla}\cdot\left[m_i n V_{\|i}\left(\boldsymbol{V}_E+\boldsymbol{V}_{\|i}+\boldsymbol{V}_{pi}\right)+\frac{1}{\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_i V_{\|i}\right)\right]= \tag{65}$$

$$-\nabla_{\|}\left(p_i+p_e\right)-\frac{2}{3}B^{3/2}\,\nabla_{\|}\left(\frac{\pi_{ci}}{B^{3/2}}\right)+S^M_{\|i}+S^M_{\|e},$$

where π_{ci} is given by Eq. (B8) with $j\to i$. Notice that we have still not had to specify a precise form for $\boldsymbol{V}_{pi}$.

In a turbulent plasma the largest terms in Eq. (65) are $\boldsymbol{\nabla}\cdot\left(m_i n V_{\|i}\boldsymbol{V}_E\right)$ and $\nabla_{\|}\left(p_i+p_e\right)$. Balancing them gives $V_{\|i}\sim\left(\epsilon/\delta_i\right)v_{Ti}\gg V_{\perp i}\sim\delta_i v_{Ti}$, which is consistent with inequalities (9). The smaller parallel viscosity term was retained in Eq. (65) to recover the correct Pfirsch-Schlüter parallel ion flow when $\Delta_i\gg\delta_i$ (see Sec. XII). To treat equilibria with $\Delta_i\sim\delta_i$ correctly one would also have to retain terms such as $\boldsymbol{V}_i\cdot\boldsymbol{\nabla}\hat{\boldsymbol{b}}\cdot\boldsymbol{V}_i$, $\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{\perp i}\right)$, as well as higher order corrections to $\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{gi}\right)$ in Eq. (65). The retention of such terms is beyond the scope of all turbulence and Pfirsch-Schlüter investigations at present.

IX. Ion Energy Equation

According to our orderings the ion thermal energy is large compared to ion kinetic energy. Moreover, as follows from the discussion at the end of Sec. VIII, $V_{\|i}\gg V_{\perp i}$, so that "parallel" ion kinetic energy is large compared to "perpendicular" ion kinetic energy. We are therefore justified in retaining in our reduced ion energy equation the effects of the former while neglecting those associated with the latter. This simplification allows us to decouple the vorticity equation from ion energy conservation.

Starting from the ion energy conservation equation, Eq. (12) with $j\to i$, we then drop terms $m_i n\left(\boldsymbol{V}_{\perp i}\right)^2/2$ under the time derivative and $m_i n\left(\boldsymbol{V}_{\perp i}\right)^2\boldsymbol{V}_i/2$ and $\overleftrightarrow{\pi}_i\cdot\boldsymbol{V}_{\perp i}$ under the divergence. Neglecting the ion perpendicular viscosity we next write

$$\overleftrightarrow{\pi}_i\cdot\boldsymbol{V}_{\|i}\approx\left(\overleftrightarrow{\pi}_{ci}+\overleftrightarrow{\pi}_{gi}\right)\cdot\boldsymbol{V}_{\|i}, \tag{66}$$

where the first term in the right-hand side can be rewritten using definition (14) as

$$\overleftrightarrow{\pi}_{ci}\cdot\boldsymbol{V}_{\|i}=\frac{2}{3}\pi_{ci}\boldsymbol{V}_{\|i}$$

with π_{ci} given by Eq. (B8), whereas the second term in the right-hand side can be shown using expressions (C2) – (C3) and inequality $\epsilon \gg \delta_i \Delta_i$ [see Eq. (9)] to be given to leading order by

$$\overleftrightarrow{\pi}_g \cdot \boldsymbol{V}_i \approx \frac{p_i}{2\Omega_i} \hat{\boldsymbol{b}} \times V_{\parallel i}^2. \tag{67}$$

Again using the electron momentum conservation equation with inertial and viscous terms neglected, we find

$$\boldsymbol{V}_i \cdot (\boldsymbol{R}_i + en\boldsymbol{E}) \approx \boldsymbol{V}_i \cdot \left(\boldsymbol{S}_e^M - \boldsymbol{\nabla} p_e + \frac{1}{c} \boldsymbol{J} \times \boldsymbol{B} \right), \tag{68}$$

where it follows from plasma momentum conservation equation (60) that

$$\boldsymbol{J} = \boldsymbol{J}_\parallel + \frac{c}{B} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} (p_i + p_e) + en (\boldsymbol{V}_{pi} - \boldsymbol{V}_{pe}). \tag{69}$$

Employing the local ambipolarity condition,

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0, \tag{70}$$

to write

$$\boldsymbol{\nabla} \cdot \left(\frac{c}{4\pi} \boldsymbol{B} \times \boldsymbol{\nabla} \phi \right) = J_\parallel \nabla_\parallel \phi - \boldsymbol{V}_E \cdot \boldsymbol{\nabla} (p_i + p_e) + en (\boldsymbol{V}_{pi} - \boldsymbol{V}_{pe}) \cdot \boldsymbol{\nabla} \phi, \tag{71}$$

we find after some straightforward algebra

$$\boldsymbol{V}_i \cdot (\boldsymbol{R}_i + en\boldsymbol{E}) \approx J_\parallel \nabla_\parallel \phi - \boldsymbol{\nabla} \cdot \left(\frac{c}{4\pi} \boldsymbol{B} \times \boldsymbol{\nabla} \phi \right) + \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel i} \right) \cdot \left(\boldsymbol{S}_e^M - \boldsymbol{\nabla} p_e \right). \tag{72}$$

Employing Eq. (56) with $e \to i$ and pulling everything together we obtain a conservative form of the reduced ion energy equation:

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{3}{2} p_i + \frac{1}{2} m_i n V_{\parallel i}^2 \right) + \boldsymbol{\nabla} \cdot \left[\frac{5}{2} p_i \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel i} + \boldsymbol{V}_{pi} \right) + \frac{5}{2 m_i \Omega_i} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} (p_i T_i) \right. \\ &\left. + \frac{1}{2} m_i n V_{\parallel i}^2 \left(\boldsymbol{V}_E + \boldsymbol{V}_{di} + \boldsymbol{V}_{\parallel i} + \boldsymbol{V}_{pi} \right) + \frac{2}{3} \pi_{ci} \boldsymbol{V}_{\parallel i} + \frac{p_i}{2\Omega_i} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} V_{\parallel i}^2 - \kappa_{\parallel i} \boldsymbol{\nabla}_\parallel T_i - \kappa_{\perp i} \boldsymbol{\nabla}_\perp T_i \right] \\ &= W_i + J_\parallel \nabla_\parallel \phi - \boldsymbol{\nabla} \cdot \left(\frac{c}{4\pi} \boldsymbol{B} \times \boldsymbol{\nabla} \phi \right) + \left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel i} \right) \cdot \left(\boldsymbol{S}_e^M - \boldsymbol{\nabla} p_e \right) + S_i^E. \end{aligned} \tag{73}$$

Using the ion parallel momentum equation (65), the density evolution equation (49), and the ambipolarity condition (70), without any further approximations we can cast Eq. (73) in the form of an equation for T_i:

$$\frac{3n}{2}\left[\frac{\partial T_i}{\partial t}+\left(\boldsymbol{V}_E+\boldsymbol{V}_{\|i}+\boldsymbol{V}_{pi}\right)\cdot\boldsymbol{\nabla}T_i\right]=-\frac{cT_i}{e}\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)\cdot\left(\boldsymbol{\nabla}p_i+en\boldsymbol{\nabla}\phi+\frac{5}{2}n\boldsymbol{\nabla}T_i\right)$$
$$+B\nabla_\|\left(\frac{\kappa_{\|i}\nabla_\|T_i}{B}\right)-p_iB\nabla_\|\left(\frac{V_{\|i}}{B}\right)+\boldsymbol{\nabla}\cdot\left(\kappa_{\perp i}\boldsymbol{\nabla}_\perp T_i-p_i\boldsymbol{V}_{pi}-\frac{c}{4\pi}\boldsymbol{B}\times\boldsymbol{\nabla}\phi\right)$$
$$-\frac{2\pi_{ci}}{3\sqrt{B}}\nabla_\|\left(\sqrt{B}V_{\|i}\right)+\boldsymbol{V}_E\cdot\left[\boldsymbol{S}_e^M-\boldsymbol{\nabla}\left(p_i+p_e\right)\right]+J_\|\nabla_\|\phi+W_i$$
$$+S_i^E-S_{\|i}^MV_{\|i}+\left(\frac{m_iV_{\|i}^2}{2}-\frac{3}{2}T_i\right)S^n. \tag{74}$$

Equation (74) for T_i retains the exact conservation properties of Eq. (73), and we still have not had to specify a precise form for $\boldsymbol{V}_{pi}$.

X. Vorticity Equation

To derive an equation for radial electric field we substitute expression (69) for plasma current $\boldsymbol{J}$ into the ambipolarity condition (70) to obtain an equation for the vorticity,

$$\varpi\equiv\boldsymbol{\nabla}\cdot\left[\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi+\boldsymbol{\nabla}_\perp p_i\right)\right]. \tag{75}$$

Our results so far are independent of the form we adopt for $\boldsymbol{V}_{pi}$, but to obtain an equation for the vorticity we must adopt a sensible form for $\boldsymbol{\nabla}\cdot(n\boldsymbol{V}_{pi})$. The details are given in Appendix D. Using expression (D5) for $\boldsymbol{\nabla}\cdot(en\boldsymbol{V}_{pi})$, $\boldsymbol{\nabla}\cdot\boldsymbol{J}$ becomes

$$\frac{\partial\varpi}{\partial t}=B\nabla_\|\left(\frac{J_\|}{B}\right)+c\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)\cdot\boldsymbol{\nabla}\left(p_i+p_e\right)$$
$$\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}_\perp\left(\frac{c}{2B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}p_i\right)+\frac{\varpi}{2}\boldsymbol{V}_E+\left(\frac{enc}{2B\Omega_i}\nabla_\perp^2\phi\right)\left(\boldsymbol{V}_E+\boldsymbol{V}_{di}\right)\right] \tag{76}$$
$$+\boldsymbol{\nabla}\cdot\left[\left(\frac{ec}{2B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}n\right)\boldsymbol{\nabla}_\perp\phi+c\frac{\hat{\boldsymbol{b}}}{B}\times\left(\boldsymbol{\kappa}\,\pi_{ci}-\frac{1}{3}\boldsymbol{\nabla}\pi_{ci}\right)-c\frac{\hat{\boldsymbol{b}}}{B}\times\left(\boldsymbol{S}_i^M+\boldsymbol{S}_e^M\right)\right].$$

Notice, that the first two terms in the right-hand side of the vorticity equation (76) are large compared to the rest of the terms both when turbulence is present and in equilibrium. Therefore these two terms should always nearly cancel to leading order.

The main advantage of vorticity equation (76) over other versions existing in the literature is that it is exact to order δ_i^2 and it is written in a conservative form, thereby insuring a divergence-free current.

XI. Conservation Properties of Reduced Equations

It has already been mentioned that the fluid equations derived in previous sections insure the divergence-free nature of the total magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$ and total plasma current $\boldsymbol{J}$, as well as conserve the total number of particles. Next, we show that they are constructed in such a way as to conserve the total energy of the system.

Adding the electron and ion energy conservation equations (58) and (73), using parallel Ohm's law (54), and noticing that according to Eq. (37)

$$\frac{J_\parallel}{c}\frac{\partial A_{\parallel 1}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\left(\boldsymbol{\nabla} A_{\parallel 1}\right)^2}{8\pi} + \frac{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\times\left(A_{\parallel 1}\hat{\boldsymbol{b}}_0\right)}{4\pi}\right] + \boldsymbol{\nabla}\cdot\left[\frac{c}{4\pi}\left(-\frac{1}{c}\frac{\partial A_{\parallel 1}}{\partial t}\right)\boldsymbol{\nabla} A_{\parallel 1}\right],$$

where $\left(-c^{-1}\partial A_{\parallel 1}/\partial t\right)\boldsymbol{\nabla} A_{\parallel 1} \approx \left(-c^{-1}\partial \boldsymbol{A}_\parallel/\partial t\right)\times\boldsymbol{B}_1$, we find that the total energy of our system can change only due to the energy sources S_i^E and S_e^E,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\frac{3}{2}\left(p_i + p_e\right) + \frac{m_i n V_{\parallel i}^2}{2} + \frac{\left(\boldsymbol{\nabla} A_{\parallel 1}\right)^2}{8\pi} + \frac{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\times\left(A_{\parallel 1}\hat{\boldsymbol{b}}_0\right)}{4\pi}\right\}\\
&+\boldsymbol{\nabla}\cdot\left\{\frac{5}{2}p_e\left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel e} + \boldsymbol{V}_{pe}\right) + \frac{5}{2}p_i\left(\boldsymbol{V}_E + \boldsymbol{V}_{\parallel i} + \boldsymbol{V}_{pi}\right) + \frac{5}{2m_i\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_iT_i - p_eT_e\right)\right.\\
&\quad+\frac{1}{2}m_i n V_{\parallel i}^2\left(\boldsymbol{V}_E + \boldsymbol{V}_{di} + \boldsymbol{V}_{\parallel i} + \boldsymbol{V}_{pi}\right) + \frac{2}{3}\pi_{ci}\boldsymbol{V}_{\parallel i} + \frac{p_i}{2\Omega_i}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla} V_{\parallel i}^2 + \frac{c}{4\pi}\boldsymbol{B}\times\boldsymbol{\nabla}\phi \qquad (77)\\
&\left.+\frac{c}{4\pi}\left(-\frac{1}{c}\frac{\partial A_{\parallel 1}}{\partial t}\right)\boldsymbol{\nabla} A_{\parallel 1} - \kappa_{\parallel i}\boldsymbol{\nabla}_\parallel T_i - \kappa_{\perp i}\boldsymbol{\nabla}_\perp T_i - \kappa_{\parallel e}\boldsymbol{\nabla}_\parallel T_e - \frac{0.71 T_e}{e}\boldsymbol{J}_\parallel\right\} = \left(S_i^E + S_e^E\right).
\end{aligned}$$

XII. Recovering Pfirsch-Schlüter Results

In this section we use our equations in the absence of sources and sinks to demonstrate that they contain the standard Pfirsch-Schlüter results[5,11] for a plasma in an axisymmetric tokamak. We write the equilibrium magnetic field in the form

$$\boldsymbol{B} = I\left(\psi\right)\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi \qquad (78)$$

and define the flux-surface average as $\langle \cdots \rangle_\theta \equiv (1/V') \oint [(\cdots)\, \mathrm{d}\theta / \boldsymbol{B} \cdot \boldsymbol{\nabla} \theta]$ with $V' \equiv \oint [\mathrm{d}\theta / \boldsymbol{B} \cdot \boldsymbol{\nabla} \theta]$. To obtain standard Pfirsch-Schlüter results we must also assume

$$\Delta_i \gg \delta_i, \tag{79}$$

so that plasma density n, electron and ion temperatures T_e and T_i, and electrostatic potential ϕ are functions of only the poloidal magnetic flux to leading order. As a result

$$\frac{\nabla_\parallel \ln T_i}{\nabla_\parallel \ln B} \ll 1, \tag{80}$$

and the poloidal variation of B is responsible for the Pfirsch-Schlüter flows. It can be shown *a posteriori* that the left-hand side of Eq. (80) is of order δ_i/Δ_i.

Employing the plasma density evolution equation (51) for the orderings (79) and (80), we notice that the first two terms on the right-hand side are dominant. We use Eq. (A13) and the leading order terms in Eq. (70) to rewrite $\boldsymbol{\nabla} p_e$ in terms of $\boldsymbol{\nabla} p_i$ and $V_{\parallel e}$ in terms of $V_{\parallel i}$. As a result, observing that for $\boldsymbol{\kappa} \approx 4\pi \boldsymbol{\nabla} \left(p_i + p_e\right)/B^2 + \boldsymbol{\nabla}_\perp \ln B$,

$$\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \psi \approx -I \nabla_\parallel \ln B, \tag{81}$$

we find

$$\nabla_\parallel \left[\frac{V_{\parallel i}}{B} + \frac{cIT_i}{eB} \left(\frac{1}{p_i} \frac{\partial p_i}{\partial \psi} + \frac{e}{T_i} \frac{\partial \phi}{\partial \psi} \right) \right] = 0$$

or

$$V_{\parallel i} = K_i\left(\psi\right) B - \frac{cIT_i}{eB} \left(\frac{1}{p_i} \frac{\partial p_i}{\partial \psi} + \frac{e}{T_i} \frac{\partial \phi}{\partial \psi} \right), \tag{82}$$

where the flux function $K_i\left(\psi\right)$ is to be determined.

Next, we consider the ion temperature evolution equation (74). For the Pfirsch-Schlüter orderings, the first three terms in the right-hand side are dominant. Using Eqs. (A13), (81) and (82), and recalling that $\kappa_{\parallel i} = (125 p_i / 32 m_i \nu_i)$ we find

$$\nabla_\parallel T_i = \frac{16 cI m_i \nu_i}{25 eB} \frac{\partial T_i}{\partial \psi} + \tilde{K}_i\left(\psi\right) B. \tag{83}$$

The unknown flux function $\tilde{K}_i\left(\psi\right)$ is obtained by multiplying Eq. (83) by B and flux-surface averaging, resulting in the final expression

$$\nabla_\parallel T_i = \frac{16 cI m_i \nu_i}{25 eB} \left(1 - \frac{B^2}{\langle B^2 \rangle_\theta} \right) \frac{\partial T_i}{\partial \psi}. \tag{84}$$

In a similar fashion the vorticity equation (76) gives an expression for the Pfirsch-Schlüter current:

$$J_\parallel = \frac{cI}{B}\left(\frac{B^2}{\langle B^2\rangle_\theta} - 1\right)\frac{\partial\left(p_i + p_e\right)}{\partial\psi} + \frac{B\left\langle BJ_\parallel\right\rangle_\theta}{\langle B^2\rangle_\theta}, \tag{85}$$

where the parallel Ohm's law [Eq. (54)] can be used to find $\left\langle BJ_\parallel\right\rangle_\theta = \sigma_\parallel\left\langle BE_\parallel\right\rangle_\theta$. Notice that bootstrap current corrections to $J_\parallel$ are neglected as small compared to Pfirsch-Schlüter current.

In the electron temperature evolution equation (59) the term $B\nabla_\parallel\left(\kappa_{\parallel e}\nabla_\parallel T_e/B\right)$ is a factor of $\sqrt{m_i/m_e}$ larger, than the other terms, which are in turn comparable to $B\nabla_\parallel\left(\kappa_{\parallel i}\nabla_\parallel T_i/B\right)$. As a result, we expect

$$\nabla_\parallel T_e \sim \sqrt{m_e/m_i}\,\nabla_\parallel T_i. \tag{86}$$

Although small, $\nabla_\parallel T_e$ can be evaluated exactly using our fluid equations. Details of such calculation can be found for example in Ref. [12].

To completely determine $V_{\parallel i}$ we use the ion parallel momentum equation (65) to evaluate the flux function $K_i(\psi)$. Multiplying this equation by B and flux-surface averaging we annihilate the large pressure gradient term and obtain to leading order

$$\left\langle\pi_{ci}\nabla_\parallel B\right\rangle_\theta \approx 0, \tag{87}$$

where π_{ci} is given by Eq. (B8) with $j \to i$. Using expressions (82) and (84) gives

$$\left\langle\frac{2\,\eta_i}{\sqrt{B}}\nabla_\parallel\left(\sqrt{B}V_{\parallel i}\right)\nabla_\parallel B\right\rangle_\theta \approx \tag{88}$$
$$\eta_i\left[3K_i(\psi)\left\langle\left(\nabla_\parallel B\right)^2\right\rangle_\theta + \frac{cIT_i}{e}\left(\frac{1}{p_i}\frac{\partial p_i}{\partial\psi} + \frac{e}{T_i}\frac{\partial\phi}{\partial\psi}\right)\left\langle\left(\nabla_\parallel \ln B\right)^2\right\rangle_\theta\right]$$

and

$$\left\langle\frac{4\,\eta_i}{5p_i\sqrt{B}}\nabla_\parallel\left[\sqrt{B}\left(q_{\parallel i} + \xi_i q_{\parallel i} - \xi_i q_i^*\right)\right]\nabla_\parallel B\right\rangle_\theta \approx \tag{89}$$
$$1.78\eta_i\frac{cI}{e}\frac{\partial T_i}{\partial\psi}\left[\left\langle\left(\nabla_\parallel \ln B\right)^2\right\rangle_\theta + 3\frac{\left\langle\left(\nabla_\parallel B\right)^2\right\rangle_\theta}{\langle B^2\rangle_\theta}\right].$$

It follows from Eqs. (65) and (84) that

$$\frac{\nabla_\parallel \ln\left(p_e + p_i\right)}{\nabla_\parallel \ln T_i} \sim \Delta_i^2 \ll 1$$

so using Eq. (86) we obtain

$$\nabla_\parallel p_i \approx \frac{p_e}{T_i + Z_i T_e} \nabla_\parallel T_i. \tag{90}$$

Consequently, Eqs. (82), (84) and (90) yield

$$\left\langle \frac{4\xi_i\,\eta_i}{5p_i} \left[\left(2q_{\parallel i} - q_i^*\right) \nabla_\parallel \ln T_i - q_{\parallel i}\left(\nabla_\parallel \ln p_i - F_{\parallel i}\right)\right] \nabla_\parallel B \right\rangle_\theta \tag{91}$$
$$\propto \left\langle \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla} B}{B^3}\left(1 - \frac{B^2}{\left\langle B^2\right\rangle_\theta}\right)^2 \right\rangle_\theta = 0.$$

Similarly,

$$\left\langle \eta_i\, \zeta_i\, \nabla_\parallel B \right\rangle_\theta = 0. \tag{92}$$

Using Eq. (81) we can also write

$$\left\langle \eta_i \boldsymbol{\kappa}\cdot\left(\boldsymbol{V}_E + \boldsymbol{V}_{di} + \left(1+\xi_i\right) c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla} T_i}{eB}\right)\nabla_\parallel B \right\rangle_\theta \approx \tag{93}$$
$$\eta_i \frac{cIT_i}{e}\left(\frac{1}{p_i}\frac{\partial p_i}{\partial \psi} + \frac{e}{T_i}\frac{\partial \phi}{\partial \psi} + 1.61\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}\right)\left\langle \left(\nabla_\parallel \ln B\right)^2 \right\rangle_\theta.$$

Neglecting the last term on the right-hand side of Eq. (B8) as small, and using Eqs. (88), (89), (91) – (93) in constraint (87), we obtain

$$K_i\left(\psi\right) \approx -\frac{cIT_i}{eB^2}\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}\left(1.78\frac{B^2}{\left\langle B^2\right\rangle_\theta} + 0.057\frac{B^2\left\langle\left(\nabla_\parallel \ln B\right)^2\right\rangle_\theta}{\left\langle\left(\nabla_\parallel B\right)^2\right\rangle_\theta}\right). \tag{94}$$

As a result, the standard Pfirsch-Schlüter expression[5] for the ion parallel flow,

$$V_{\parallel i}^{\rm PS} = -\frac{cIT_i}{eB}\left[\frac{1}{p_i}\frac{\partial p_i}{\partial \psi} + \frac{e}{T_i}\frac{\partial \phi}{\partial \psi} + \frac{1}{T_i}\frac{\partial T_i}{\partial \psi}\left(1.8\frac{B^2}{\left\langle B^2\right\rangle_\theta} + 0.054\frac{B^2\left\langle\left(\nabla_\parallel \ln B\right)^2\right\rangle_\theta}{\left\langle\left(\nabla_\parallel B\right)^2\right\rangle_\theta}\right)\right], \tag{95}$$

is recovered, but with numerical coefficients 1.78 and 0.057 instead of 1.8 and 0.054, respectively.

Finally, we remark that Eq. (76) could in principle be used to determine equilibrium radial electric field if contributions from ion gyroviscosity (and perhaps ion perpendicular viscosity) were retained to a high enough order both in this equation and in the parallel momentum equation. To see this we flux-surface average the ambipolarity condition $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ and use Eqs. (78) and (60) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\psi} V' \left\langle \frac{I}{B} \hat{\boldsymbol{b}} \cdot \left[\boldsymbol{\nabla} \cdot (m_i n \boldsymbol{V}_i \boldsymbol{V}_i) + \boldsymbol{\nabla} (p_i + p_e) + \boldsymbol{\nabla} \cdot \overleftrightarrow{\pi}_i \right] \right\rangle_\theta \tag{96}$$
$$- \frac{\mathrm{d}}{\mathrm{d}\psi} V' \left\langle R^2 \boldsymbol{\nabla} \zeta \cdot \left[\boldsymbol{\nabla} \cdot (m_i n \boldsymbol{V}_i \boldsymbol{V}_i) + \boldsymbol{\nabla} \cdot \overleftrightarrow{\pi}_i \right] \right\rangle_\theta = 0.$$

The first term in the left-hand side of Eq. (96) is obviously zero due to parallel momentum conservation. In addition, using the fact that $\overleftrightarrow{\pi}_i$ is a symmetric tensor, the second term in the left-hand side of Eq. (96) can be cast in the form

$$- \frac{\mathrm{d}^2}{\mathrm{d}\psi^2} V' \left\langle R^2 \boldsymbol{\nabla} \zeta \cdot \left(\overleftrightarrow{\pi}_{gi} + \overleftrightarrow{\pi}_{\perp i} \right) \cdot \boldsymbol{\nabla} \psi \right\rangle_\theta , \tag{97}$$

which is used to evaluate the equilibrium radial electric field in Ref. [5].

To derive accurate turbulent equations only the leading order contributions to the divergence of the ion gyroviscous stress tensor are retained in our equations (the gyroviscous cancellation). Moreover, the effects of the ion perpendicular viscosity are neglected. As a result, we cannot obtain any credible estimate of the equilibrium or Pfirsch-Schlüter radial electric field. As in all previous treatments, it is assumed small compared to the turbulence-generated radial electric field. To help insure that this is the case our $\boldsymbol{V}_{pi}$ is constructed such that the largest terms in $\boldsymbol{J}$ automatically satisfy $\langle \boldsymbol{J} \cdot \boldsymbol{\nabla} \psi \rangle_\theta = 0$ in the axisymmetric and steady state limit.

XIII. Discussion and Conclusions

We consider a low-β, collisional, magnetized plasma with the equilibrium magnetic field $\boldsymbol{B}_0$, current $\boldsymbol{J}_0 = (c/4\pi) \boldsymbol{\nabla} \times \boldsymbol{B}_0$, and pressure p_0 satisfying $\boldsymbol{J}_0 \times \boldsymbol{B}_0 = c \boldsymbol{\nabla} p_0$. The non-linear behavior of such a plasma in the self-consistently evolving magnetic field in the presence of diamagnetic drift frequency field-aligned modes is described by Eqs. (49) [or equivalently (51)] for the plasma density, (65) for ion parallel flow velocity, (54) for parallel current, Eqs. (58) and (73) for electron and ion energy [or equivalently (59) and (74) for electron and ion temperatures], (76) for plasma vorticity (or equivalently the electrostatic potential), (37) for the parallel component of perturbed electromagnetic potential, (35) and (36) for perturbed magnetic field, and Eqs. (41) and (43) for perpendicular and parallel electric fields, respectively. In these equations, velocities $\boldsymbol{V}_E$, $\boldsymbol{V}_{dj}$, $\boldsymbol{V}_{pe}$, and $\boldsymbol{V}_{pi}$ are given by Eqs. (47), (45), (48), and (D6), respectively. In addition, $\boldsymbol{V}_{\parallel e} = \left(V_{\parallel i} - J_{\parallel}/en\right)\hat{\boldsymbol{b}}$, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$, $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, $\boldsymbol{\kappa} \equiv \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}}$, $\boldsymbol{\nabla}_{\parallel} = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}$, $\boldsymbol{\nabla}_{\perp} = -\hat{\boldsymbol{b}} \times \left(\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\right)$, π_{ci} is given by Eq. (B8) with $j \to i$, and ϖ is given by Eq. (75).

Our system of equations is written in such a way that electric charge, particle number and total energy density are conserved locally, and a divergence-free nature of magnetic field is preserved. Any further approximations to these equations must be done with great care to avoid destroying these properties. Sharp radial gradients of equilibrium plasma quantities (plasma edge pedestals) are permitted by the formulation. In addition, our equations are shown to include the standard neoclassical (Pfirsch-Schlüter) results for parallel current, parallel ion flow velocity with the correct temperature gradient terms (these temperature gradients terms cannot be obtained from the Braginskii equations), and parallel gradients of electron and ion temperatures. However, our equations in their present form do not allow recovery of the neoclassical equilibrium radial electric field since the expression for ion gyroviscous stress tensor is not retained with sufficient accuracy in the parallel ion momentum and vorticity equations. The same statement pertains to the equations in Refs. [6–9] and is the price that must be paid because the gyroviscous cancellation is only approximate. The conventional believe is that the radial electric field is due to plasma turbulence and not due to plasma equilibrium; however, the resulting field can be somewhat sensitive to the details of the approximations employed.

Finally, comparing our system of equations with those of Refs. [6–9] we note the following important differences. First, we employ a self-consistent expression for the ion parallel viscous stress tensor, $\overleftrightarrow{\pi}_{ci}$, whereas Refs. [6,8] use Braginskii expressions, Ref. [7] neglects it altogether, and Ref. [9] uses some neoclassical expression with the temperature gradients set to zero. In particular, our expression for $\overleftrightarrow{\pi}_{ci}$ allows the neoclassical Pfirsch-Schlüter expression for $V_{\|i}$ to be recovered, while all other treatments do not. Second, the ion gyroviscous stress tensor is used in Refs. [6,7,9] in its Braginskii high-flow form. This feature may influence the value of the radial electric field produced by those equations. In general, the use of the Braginskii viscosity leads to large spurious terms in the expressions for the parallel and gyroviscosity. Third, Refs. [6–8] appear to neglect the term $\boldsymbol{J}_0 \cdot \hat{\boldsymbol{b}}_0$ is Eq. (37), which does not seem to be a good approximation since this term is comparable to $J_\|$. Fourth, Refs. [7,8] use the approximation $\boldsymbol{\nabla} \times \left(\hat{\boldsymbol{b}}/B\right) \rightarrow 2\hat{\boldsymbol{b}} \times \boldsymbol{\kappa}/B$ in their final equations, which is inconsistent with Eqs. (A1) and (A11). Fifth, while Refs. [6,7,9] acknowledge the possibilities of problems caused by forms of equations not conserving energy, particle number density, divergence-free current and so on, Refs. [6–9] ultimately employ final sets of equations which do not satisfy some or all of these conservation properties.

XIV. Acknowledgement

The authors are grateful to Dr. X. Q. Xu for kindly providing them with his notes on the ion gyroviscous stress tensor, and to Drs. J. J. Ramos and X. Q. Xu for a number of enlightening discussions. They are also grateful to Dr. S. E. Kruger for bringing his procedure as summarized by Eqs. (35) and (36) to their attention and providing a copy of his Ph.D. thesis.

This work was supported by U.S. Department of Energy Grant No. DE-FG02-91ER-54109 at the Massachusetts Institute of Technology.

Appendix A. Alternative Expression for $\boldsymbol{\nabla} \times \left(\hat{\boldsymbol{b}}/B\right)$

In this appendix we obtain an alternative expression for $\boldsymbol{\nabla} \times \left(\hat{\boldsymbol{b}}/B\right)$. We start by noting that

$$\left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{b}}}{B}\right) = \frac{2}{B}\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} + \frac{4\pi\left(\boldsymbol{J}_{\parallel} - \boldsymbol{J}_{\perp}\right)}{cB^2} \tag{A1}$$

follows from

$$\left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{b}}}{B}\right) = \frac{\boldsymbol{\nabla} \times \hat{\boldsymbol{b}}}{B} + \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{B^2} \tag{A2}$$

and

$$\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} = -\hat{\boldsymbol{b}} \times \left[\hat{\boldsymbol{b}} \times \left(\boldsymbol{\nabla} \times \hat{\boldsymbol{b}}\right)\right] = \boldsymbol{\nabla} \times \hat{\boldsymbol{b}} - \hat{\boldsymbol{b}}\left(\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \times \hat{\boldsymbol{b}}\right) \tag{A3}$$

with

$$\boldsymbol{\nabla} \times \hat{\boldsymbol{b}} = \frac{\boldsymbol{\nabla} \times \boldsymbol{B}}{B} + \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{B} = \frac{4\pi \boldsymbol{J}}{cB} + \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} B}{B}, \tag{A4}$$

and

$$\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \times \hat{\boldsymbol{b}} = \frac{4\pi J_{\parallel}}{cB}. \tag{A5}$$

Employing the following leading order approximation for perpendicular current,

$$\boldsymbol{J}_{\perp} \approx \frac{c\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\left(p_i + p_e\right)}{B}, \tag{A6}$$

we find

$$\frac{4\pi}{cB^2}\boldsymbol{J}_{\perp} \cdot \boldsymbol{\nabla} f \sim \frac{\beta k_{\perp}^2 f}{B}, \tag{A7}$$

where f can stand for plasma density, temperature or electrostatic potential. Noticing that $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ results in $\boldsymbol{J}_{\parallel} \sim \boldsymbol{J}_{\perp}$ we obtain in a similar fashion

$$\frac{4\pi}{cB^2}\boldsymbol{J}_{\parallel} \cdot \boldsymbol{\nabla} f \sim \frac{\beta k_{\perp} k_{\parallel} f}{B} \sim \epsilon \frac{\beta k_{\perp}^2 f}{B}. \tag{A8}$$

Since

$$\frac{1}{B}\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} f \sim \frac{k_{\perp} f}{B L_s}, \tag{A9}$$

we find

$$\frac{\left(4\pi/cB^2\right)\boldsymbol{J}_{\perp} \cdot \boldsymbol{\nabla} f}{\left(2/B\right)\hat{\boldsymbol{b}} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} f} \sim \beta k_{\perp} L_s \sim \frac{\beta}{\epsilon}. \tag{A10}$$

Then, it follows from Eq. (31) that

$$\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)=\frac{2}{B}\,\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}-\frac{4\pi\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_i+p_e\right)}{B^3}+\mathrm{O}\left(\beta\right). \tag{A11}$$

Notice, that in the absence of fluctuations estimate (A7) changes to

$$\frac{4\pi}{cB^2}\boldsymbol{J}_\perp\cdot\boldsymbol{\nabla}f\sim\frac{\beta f}{BL_sL_f}, \tag{A12}$$

and expression (A11) simplifies further to become

$$\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)=\frac{2}{B}\,\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}+\mathrm{O}\left(\beta\right). \tag{A13}$$

Appendix B. Parallel Viscosity

In this appendix we evaluate terms necessary for our fluid equations, which involve the parallel viscous stress tensor of species j. Recalling Eq. (14) we find

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{cj}=\left[\hat{\boldsymbol{b}}\left(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}}\right)+\boldsymbol{\kappa}\right]\pi_{cj}+\hat{\boldsymbol{b}}\nabla_\parallel\pi_{cj}-\frac{1}{3}\boldsymbol{\nabla}\pi_{cj}, \tag{B1}$$

so that

$$\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{cj}\right)=\left(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}}\right)\pi_{cj}+\frac{2}{3}\,\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\pi_{cj}=\frac{2}{3}\,B^{3/2}\nabla_\parallel\left(\frac{\pi_{cj}}{B^{3/2}}\right). \tag{B2}$$

Using Eq. (B1) we also obtain

$$\boldsymbol{\nabla}\cdot\left[\frac{\hat{\boldsymbol{b}}}{B}\times\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{cj}\right)\right]=\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\kappa}}{B}\,\pi_{cj}\right)-\frac{1}{3}\left(\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{b}}}{B}\right)\cdot\boldsymbol{\nabla}\pi_{ci}, \tag{B3}$$

which is required to evaluate $\boldsymbol{\nabla}\cdot(en\boldsymbol{V}_{pi})$.

Next, it is necessary to evaluate π_{cj}. Noticing that

$$\left(3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}-\overleftrightarrow{I}\right):\boldsymbol{\nabla}\boldsymbol{V}_j=2\nabla_\parallel V_{\parallel j}-3\boldsymbol{\kappa}\cdot\boldsymbol{V}_{\perp j}-\boldsymbol{\nabla}\cdot\boldsymbol{V}_{\perp j}-V_{\parallel j}\left(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}}\right), \tag{B4}$$

employing expression (44) for $\boldsymbol{V}_{\perp j}$ with $\boldsymbol{V}_{pj}$ dropped, and using Eq. (A11) we find

$$\left(3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}-\overleftrightarrow{I}\right):\boldsymbol{\nabla}\boldsymbol{V}_j\approx\frac{2}{\sqrt{B}}\nabla_\parallel\left(\sqrt{B}V_{\parallel j}\right)-\boldsymbol{\kappa}\cdot\left(\boldsymbol{V}_E+\boldsymbol{V}_{dj}\right) \tag{B5}$$
$$-c\frac{\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}n\times\boldsymbol{\nabla}T_j}{Z_jenB}+\frac{4\pi c}{B^3}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_e+p_i\right)\cdot\left(\boldsymbol{\nabla}\phi+\frac{\boldsymbol{\nabla}p_j}{Z_jen}\right).$$

Employing expression (18) for $\boldsymbol{q}_j$ and following the same procedure we obtain

$$\frac{2}{5p_j}\left(3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}-\overleftrightarrow{I}\right):\boldsymbol{\nabla}\boldsymbol{q}_j\approx\frac{4}{5p_j\sqrt{B}}\nabla_\parallel\left(\sqrt{B}q_{\parallel j}\right)-\boldsymbol{\kappa}\cdot\left(c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_j}{Z_jeB}\right)$$
$$+c\frac{\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}n\times\boldsymbol{\nabla}T_j}{Z_jenB}+\frac{4\pi c}{Z_jeB^3}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_e+p_i\right)\cdot\boldsymbol{\nabla}T_j, \tag{B6}$$

where the last term in Eq. (18) for $\boldsymbol{q}_e$ was neglected as small by the factor of $(\nu_e/|\Omega_e|)\ll 1$ compared to electron diamagnetic heat flux. Notice that the large $\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}n\times\boldsymbol{\nabla}T_j$ terms cancel when π_{cj} is formed. This does not happen when the Braginskii expression for $\overleftrightarrow{\pi}_{cj}$ is employed. Therefore, the use of the Braginskii form of $\overleftrightarrow{\pi}_{cj}$ leads to a large spurious term.

Similarly, we find

$$\left(3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}-\overleftrightarrow{I}\right):\overleftrightarrow{\gamma}_j\approx-\frac{4}{5p_j\sqrt{B}}\nabla_\parallel\left[\sqrt{B}\left(q_{\parallel j}-q_j^*\right)\right]+\boldsymbol{\kappa}\cdot\left(c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_j}{Z_jeB}\right) \tag{B7}$$
$$-\frac{4}{5p_j}\left[\left(2q_{\parallel j}-q_j^*\right)\nabla_\parallel\ln T_j-q_{\parallel j}\left(\nabla_\parallel\ln p_j-F_{\parallel j}\right)\right]-\frac{4\pi c}{Z_jeB^3}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_e+p_i\right)\cdot\boldsymbol{\nabla}T_j.$$

Substituting results (B5) - (B7) into Eq. (15) we finally obtain

$$\pi_{cj}\approx\zeta_j+\eta_j\Bigg\{\boldsymbol{\kappa}\cdot\left[\boldsymbol{V}_E+\boldsymbol{V}_{dj}+\left(1+\xi_j\right)c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_j}{Z_jeB}\right]$$
$$-\frac{2}{\sqrt{B}}\nabla_\parallel\left(\sqrt{B}V_{\parallel j}\right)-\frac{4}{5p_j\sqrt{B}}\nabla_\parallel\left[\sqrt{B}\left(q_{\parallel j}+\xi_jq_{\parallel j}-\xi_jq_j^*\right)\right] \tag{B8}$$
$$-\frac{4\xi_j}{5p_j}\left[\left(2q_{\parallel j}-q_j^*\right)\nabla_\parallel\ln T_j-q_{\parallel j}\left(\nabla_\parallel\ln p_j-F_{\parallel j}\right)\right]$$
$$-\frac{4\pi c}{B^3}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\left(p_e+p_i\right)\cdot\left[\boldsymbol{\nabla}\phi+\frac{\boldsymbol{\nabla}p_j}{Z_jen}+\left(1+\xi_j\right)\frac{\boldsymbol{\nabla}T_j}{Z_je}\right]\Bigg\}.$$

Appendix C. Leading Order Ion Gyroviscosity

In this appendix we evaluate the leading order expressions for $\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{gi}$ and $\hat{\boldsymbol{b}}\cdot\left(\boldsymbol{\nabla}\cdot\overleftrightarrow{\pi}_{gi}\right)$. Using the tensor identity

$$\hat{\boldsymbol{b}}\times\left[\boldsymbol{\nabla}\boldsymbol{a}+\left(\boldsymbol{\nabla}\boldsymbol{a}\right)^{\mathrm{T}}\right]\cdot\left(\overleftrightarrow{I}-\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)-\left(\overleftrightarrow{I}-\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)\cdot\left[\boldsymbol{\nabla}\boldsymbol{a}+\left(\boldsymbol{\nabla}\boldsymbol{a}\right)^{\mathrm{T}}\right]\times\hat{\boldsymbol{b}}= \tag{C1}$$
$$4\,\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\boldsymbol{a}\cdot\left(\overleftrightarrow{I}-\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)-2\left(\overleftrightarrow{I}\times\hat{\boldsymbol{b}}\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{a}-\nabla_\parallel\boldsymbol{a}\cdot\hat{\boldsymbol{b}}\right)-2\left(\overleftrightarrow{I}-\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\times\boldsymbol{a},$$

which is valid for an arbitrary vector $\boldsymbol{a}$ and an arbitrary unit vector $\hat{\boldsymbol{b}}$, we can rewrite expression (16) for $\overleftrightarrow{\pi}_{gi}$ in the form

$$\overleftrightarrow{\pi}_{gi} = \overleftrightarrow{\pi}_{gi}^{V} + \overleftrightarrow{\pi}_{gi}^{q} \tag{C2}$$

with

$$\begin{aligned} \overleftrightarrow{\pi}_{gi}^{V} \equiv \frac{p_i}{\Omega_i}\Big\{ & \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \boldsymbol{V}_i - \left(\nabla_\parallel \boldsymbol{V}_i \times \hat{\boldsymbol{b}}\right) \hat{\boldsymbol{b}} + \hat{\boldsymbol{b}} \left(\hat{\boldsymbol{b}} \times \boldsymbol{\nabla} \boldsymbol{V}_i \cdot \hat{\boldsymbol{b}} - \nabla_\parallel \boldsymbol{V}_i \times \hat{\boldsymbol{b}}\right) \\ & -\frac{1}{2}\left(\overleftrightarrow{I} \times \hat{\boldsymbol{b}}\right)\left(\boldsymbol{\nabla} \cdot \boldsymbol{V}_i - \nabla_\parallel \boldsymbol{V}_i \cdot \hat{\boldsymbol{b}}\right) - \frac{1}{2}\left(\overleftrightarrow{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right) \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \times \boldsymbol{V}_i \Big\} \end{aligned} \tag{C3}$$

and $\overleftrightarrow{\pi}_{gi}^{q}$ given by Eq. (C3) but with the substitutions $p_i \to 2/5$ and $\boldsymbol{V}_i \to \boldsymbol{q}_i$.

To derive the lowest order gyroviscous contributions, we need only consider the case of a straight, homogeneous, time-independent magnetic field, and we can neglect parallel derivatives ($\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \to 0$) to obtain from Eqs. (C2) and (C3)

$$\begin{aligned} \boldsymbol{\nabla} \cdot \overleftrightarrow{\pi}_{gi} \approx & -m_i n \boldsymbol{V}_{di} \cdot \boldsymbol{\nabla} \boldsymbol{V}_i + \hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\left(\frac{p_i}{2\Omega_i} \boldsymbol{\nabla} \cdot \boldsymbol{V}_{\perp i} + \frac{1}{5\Omega_i} \boldsymbol{\nabla} \cdot \boldsymbol{q}_{\perp i}\right) \\ & + \boldsymbol{\nabla}_\perp \left[\frac{p_i}{2\Omega_i} \boldsymbol{\nabla} \cdot \left(\hat{\boldsymbol{b}} \times \boldsymbol{V}_i\right) + \frac{1}{5\Omega_i} \boldsymbol{\nabla} \cdot \left(\hat{\boldsymbol{b}} \times \boldsymbol{q}_i\right)\right]. \end{aligned} \tag{C4}$$

Since

$$\boldsymbol{\nabla} \cdot \boldsymbol{V}_{\perp i} \approx \boldsymbol{\nabla} \cdot \boldsymbol{V}_{di} \approx -\boldsymbol{V}_{di} \cdot \boldsymbol{\nabla} \ln n \tag{C5}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{q}_{\perp i} \approx \boldsymbol{\nabla} \cdot \left(\frac{5 c p_i}{2eB} \hat{\boldsymbol{b}} \times \boldsymbol{\nabla} T_i\right) \approx -\frac{5}{2} n \boldsymbol{V}_{di} \cdot \boldsymbol{\nabla} T_i, \tag{C6}$$

it is easy to see that

$$\frac{p_i}{2\Omega_i} \boldsymbol{\nabla} \cdot \boldsymbol{V}_{\perp i} + \frac{1}{5\Omega_i} \boldsymbol{\nabla} \cdot \boldsymbol{q}_{\perp i} \approx 0. \tag{C7}$$

Again, large $\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} n \times \boldsymbol{\nabla} T_i$ terms cancel because the diamagnetic heat and particle flows are comparable. Similarly,

$$\begin{aligned} & \frac{p_i}{2\Omega_i} \boldsymbol{\nabla} \cdot \left(\hat{\boldsymbol{b}} \times \boldsymbol{V}_i\right) + \frac{1}{5\Omega_i} \boldsymbol{\nabla} \cdot \left(\hat{\boldsymbol{b}} \times \boldsymbol{q}_i\right) \approx \\ & -\frac{1}{2 m_i \Omega_i^2}\left[p_i \boldsymbol{\nabla}_\perp \cdot \left(e \boldsymbol{\nabla}_\perp \phi + \frac{\boldsymbol{\nabla}_\perp p_i}{n}\right) + \boldsymbol{\nabla}_\perp \cdot \left(p_i \boldsymbol{\nabla}_\perp T_i\right)\right]. \end{aligned} \tag{C8}$$

As a result,

$$\nabla\cdot\overleftrightarrow{\pi}_{gi}\approx -m_i n\boldsymbol{V}_{di}\cdot\nabla\boldsymbol{V}_i \tag{C9}$$
$$-\nabla_\perp\left\{\frac{1}{2m_i\Omega_i^2}\left[p_i\nabla_\perp\cdot\left(e\nabla_\perp\phi+\frac{\nabla_\perp p_i}{n}\right)+\nabla_\perp\cdot\left(p_i\nabla_\perp T_i\right)\right]\right\},$$

and

$$\hat{\boldsymbol{b}}\cdot\left(\nabla\cdot\overleftrightarrow{\pi}_{gi}\right)\approx -m_i n\boldsymbol{V}_{di}\cdot\nabla V_{\parallel i}. \tag{C10}$$

Notice, that expression (C10) is equivalent, because of our straight and homogeneous magnetic field approximation, to the following convenient expression:

$$\hat{\boldsymbol{b}}\cdot\left(\nabla\cdot\overleftrightarrow{\pi}_{gi}\right)\approx\nabla\cdot\left(\frac{p_i}{\Omega_i}\hat{\boldsymbol{b}}\times\nabla V_{\parallel i}\right). \tag{C11}$$

Appendix D. Ion "polarization" velocity $\boldsymbol{V}_{pi}$

In this appendix we derive the leading order expression for the ion "polarization" velocity, $\boldsymbol{V}_{pi}$, defined by Eq. (46), and then we evaluate $\nabla\cdot(en\boldsymbol{V}_{pi})$. First, we notice that $\boldsymbol{V}_{pi}$ enters the ion parallel momentum and energy (and temperature) equations only as a small correction to $\boldsymbol{V}_E$ and so can be given in these equations by an approximate expression, which need not even retain all the leading order terms. It is important, however, to keep all the leading order terms in $\nabla\cdot(en\boldsymbol{V}_{pi})$ in the vorticity equation in a divergence-free manner.

First, we observe that the electron-ion friction forces in $\boldsymbol{V}_{pi}$ and $\boldsymbol{V}_{pe}$ cancel exactly and cannot lead to a perpendicular current, so $\boldsymbol{R}_i$ must be dropped in the expression for $\boldsymbol{V}_{pi}$ (since it was ignored in $\boldsymbol{V}_{pe}$). Then we use the leading order expression (C9) for $\nabla\cdot\overleftrightarrow{\pi}_{gi}$ in the definition (46) to find

$$\boldsymbol{V}_{pi}\approx\frac{\hat{\boldsymbol{b}}}{\Omega_i}\times\left[\frac{\partial\boldsymbol{V}_i}{\partial t}+\boldsymbol{V}_E\cdot\nabla\boldsymbol{V}_i+\frac{\nabla\cdot\overleftrightarrow{\pi}_{ci}+m_i\boldsymbol{V}_iS^n-\boldsymbol{S}_i^M}{m_i n}\right] \tag{D1}$$
$$-\frac{c}{enB}\hat{\boldsymbol{b}}\times\nabla\left\{\frac{1}{2m_i\Omega_i^2}\left[p_i\nabla_\perp\cdot\left(e\nabla_\perp\phi+\frac{\nabla_\perp p_i}{n}\right)+\nabla_\perp\cdot\left(p_i\nabla_\perp T_i\right)\right]\right\},$$

where we neglect $\boldsymbol{V}_{\parallel i}\cdot\nabla\boldsymbol{V}_i$ as an order $(\epsilon/\delta_i)^2\ll 1$ correction to $\boldsymbol{V}_E\cdot\nabla\boldsymbol{V}_i$.

Next, we obtain the leading order expression for $\boldsymbol{\nabla}\cdot(en\boldsymbol{V}_{pi})$. Noticing that the second term in Eq. (D1) gives zero to leading order and using Eq. (B3) to evaluate the contribution due to the ion parallel viscosity we write

$$\boldsymbol{\nabla}\cdot(en\boldsymbol{V}_{pi}) \approx \boldsymbol{\nabla}\cdot\left[en\frac{\hat{\boldsymbol{b}}}{\Omega_i}\times\left(\frac{\partial}{\partial t}+\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\right)(\boldsymbol{V}_E+\boldsymbol{V}_{di})\right]-\boldsymbol{\nabla}\cdot\left(c\frac{\hat{\boldsymbol{b}}}{B}\times\boldsymbol{S}_i^M\right)$$
$$+\boldsymbol{\nabla}\cdot\left[c\,\frac{\hat{\boldsymbol{b}}}{B}\times\left(\boldsymbol{\kappa}\,\pi_{ci}-\frac{1}{3}\boldsymbol{\nabla}\pi_{ci}\right)\right]-\boldsymbol{\nabla}\cdot\left[\frac{S^n}{n}\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi+\boldsymbol{\nabla}_\perp p_i\right)\right], \qquad \text{(D2)}$$

where we neglect $\partial\boldsymbol{V}_{\|i}/\partial t+\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\boldsymbol{V}_{\|i}$ terms as small by a factor of $(\epsilon/\delta_i)^2$.

Employing the approximation of a straight, homogeneous, time-independent magnetic field and zero parallel gradients and noticing that under such an approximation $\boldsymbol{\nabla}\cdot(n\boldsymbol{V}_i)\approx\boldsymbol{V}_E\cdot\boldsymbol{\nabla}n$, we obtain to leading order

$$en\frac{\hat{\boldsymbol{b}}}{\Omega_i}\times\left(\frac{\partial}{\partial t}+\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\right)(\boldsymbol{V}_E+\boldsymbol{V}_{di})-\frac{S^n}{n}\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi+\boldsymbol{\nabla}_\perp p_i\right)\approx \qquad \text{(D3)}$$
$$-\left(\frac{\partial}{\partial t}+\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\right)\left[\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi+\boldsymbol{\nabla}_\perp p_i\right)\right].$$

Using the vector identity

$$(\boldsymbol{a}\cdot\boldsymbol{\nabla})\,\boldsymbol{c}=\frac{1}{2}[\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{c})+\boldsymbol{a}(\boldsymbol{\nabla}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{\nabla}\cdot\boldsymbol{a})$$
$$-\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{c})-\boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{c})-\boldsymbol{c}\times(\boldsymbol{\nabla}\times\boldsymbol{a})],$$

which is valid for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{c}$, the convective term on the right-hand side of Eq. (D3) can be rewritten as

$$\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\left[\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi+\boldsymbol{\nabla}_\perp p_i\right)\right]\approx\frac{1}{2}\left\{\boldsymbol{\nabla}_\perp\left(\frac{c}{B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}p_i\right)+\left(\frac{ec}{B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}n\right)\boldsymbol{\nabla}_\perp\phi\right.$$
$$\left.+\varpi\boldsymbol{V}_E+\left(\frac{enc}{B\Omega_i}\nabla_\perp^2\phi\right)(\boldsymbol{V}_E+\boldsymbol{V}_{di})-\boldsymbol{\nabla}\times\left[\boldsymbol{V}_E\times\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi+\boldsymbol{\nabla}_\perp p_i\right)\right]\right\}, \qquad \text{(D4)}$$

where the plasma vorticity ϖ is given by Eq. (75).

Using results (D3) and (D4) in Eq. (D2) we obtain the convenient form

$$\boldsymbol{\nabla}\cdot(en\boldsymbol{V}_{pi})\approx-\frac{\partial\varpi}{\partial t}+\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}_\perp\left(\frac{c}{2B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}p_i\right)+\left(\frac{ec}{2B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}n\right)\boldsymbol{\nabla}_\perp\phi\right]$$
$$+\boldsymbol{\nabla}\cdot\left[\frac{\varpi}{2}\boldsymbol{V}_E+\left(\frac{enc}{2B\Omega_i}\nabla_\perp^2\phi\right)(\boldsymbol{V}_E+\boldsymbol{V}_{di})\right]-\boldsymbol{\nabla}\cdot\left(c\frac{\hat{\boldsymbol{b}}}{B}\times\boldsymbol{S}_i^M\right) \qquad \text{(D5)}$$
$$+\boldsymbol{\nabla}\cdot\left[c\,\frac{\hat{\boldsymbol{b}}}{B}\times\left(\boldsymbol{\kappa}\,\pi_{ci}-\frac{1}{3}\boldsymbol{\nabla}\pi_{ci}\right)\right],$$

used in Eq. (76).

Consequently, in the ion parallel momentum and energy (and temperature) equations we can use the following approximate expression for $\boldsymbol{V}_{pi}$, which retains only terms necessary to obtain the leading order expression (D5) for $\boldsymbol{\nabla}\cdot(en\boldsymbol{V}_{pi})$:

$$
\begin{aligned}
en\boldsymbol{V}_{pi} \equiv -\frac{\partial}{\partial t}\left[\frac{c}{B\Omega_i}\left(en\boldsymbol{\nabla}_\perp\phi + \boldsymbol{\nabla}_\perp p_i\right)\right] + \boldsymbol{\nabla}_\perp\left(\frac{c}{2B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}p_i\right) \\
+\frac{\varpi}{2}\boldsymbol{V}_E + \left(\frac{enc}{2B\Omega_i}\nabla_\perp^2\phi\right)\left(\boldsymbol{V}_E + \boldsymbol{V}_{di}\right) + \left(\frac{ec}{2B\Omega_i}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}n\right)\boldsymbol{\nabla}_\perp\phi \\
+c\,\frac{\hat{\boldsymbol{b}}}{B}\times\left(\boldsymbol{\kappa}\,\pi_{ci} - \frac{1}{3}\boldsymbol{\nabla}\pi_{ci}\right) - c\,\frac{\hat{\boldsymbol{b}}}{B}\times\boldsymbol{S}_i^M.
\end{aligned}
\tag{D6}
$$

Notice, that the first term on the right-hand side of Eq. (D6) is not in general perpendicular to $\boldsymbol{B}$. The spurious parallel component of $\boldsymbol{V}_{pi}$ is small and not of concern.

References

[1]S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), vol. 1, p. 205.

[2]A. B. Mikhailovskii and V. S. Tsypin, Beitr. Plasmaphys. **24**, 335 (1984) and references therein.

[3]P. J. Catto and A. N. Simakov, *A drift ordered short mean-free path description for magnetized plasma with strong spatial anisotropy*, to be published.

[4]S. E. Kruger, *Generalized Reduced Magnetohydrodynamic Equations*, Ph. D. thesis, the University of Wisconsin-Madison, 1999.

[5]R. D. Hazeltine, Phys. Fluids **17**, 961 (1974).

[6]M. Yagi and W. Horton, Phys. Plasmas **1**, 2135 (1994).

[7]A. Zeiler, J. F. Drake, and B. Rogers, Phys. Plasmas **4**, 2134 (1997).

[8]X. Q. Xu, R. H. Cohen, T. D. Rognlien, and J. R. Myra, Phys. Plasmas **7**, 1951 (2000).

[9]L. E. Sugiyama and W. Park, Phys. Plasmas **7**, 4644 (2000).

[10]R. D. Hazeltine, J. D. Meiss, *Plasma Confinement*, in *Frontiers in Physics*, edited by David Pines (Addison-Wesley, Redwood City CA, 1992) v. 86.

[11]F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976); S. P. Hirshman and D. J. Sigmar, Nucl. Fusion **21**, 1079 (1981).

[12]P. J. Catto, P. Helander, J. W. Connor, and R. D. Hazeltine, Phys. Plasmas **5**, 3961 (1998).