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Evaluation of the Neoclassical Radial Electric Field in a Collisional Tokamak

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Abstract

The neoclassical electric field in a tokamak is determined by the conservation of toroidal angular momentum. In the steady state in the absence of momentum sources and sinks it is explicitly evaluated by the condition that radial flux of toroidal angular momentum vanishes. For a collisional or Pfirsch-Schlüter short mean-free path ordering with sub-sonic plasma flows we find that there are two limiting cases of interest. The first is the simpler case of a strongly up-down asymmetric tokamak (for example, just inside the separatrix of a single-null-divertor configuration) for which the lowest order gyro-viscosity does not vanish and must be balanced by the leading order collisional viscosity in order to determine the radial electric field. The second case is the more complicated case of an up-down symmetric tokamak for which the gyro-viscosity must be evaluated to higher order and again balanced by the lowest order collisional viscosity to determine the radial electric field.

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I. INTRODUCTION

In tokamaks the pressure times the ion flow velocity is normally of the same order as the diamagnetic and parallel ion heat flows so that a drift ordering is most appropriate. The high flow ordering of Braginskii[1] is not valid since it ignores heat flow corrections to the pressure anisotropy, the gyro-viscosity and the collisional perpendicular viscosity. Mikhailovskii and Tsypin[2] realized this shortcoming of the Braginskii treatment and employed a more appropriate drift ordering in an attempt to retain these important heat flow modifications to the pressure tensor. However, Catto and Simakov[3] showed that the truncated polynomial technique they used together with their neglect of modifications to the pressure anisotropy due to the non-linear nature of the ion-ion collision operator resulted in an erroneous expression for the collisional perpendicular viscosity because the ion distribution function was not retained to second order in the ion gyro-radius and mean-free path expansions. As a result, the full expressions for the viscosity of a collisional plasma in the drift ordering have only recently become available. These expressions are obtained by treating the gyro-radius and mean-free path expansion parameters

$$\delta = \frac{\rho}{L_\perp} \quad \text{and} \quad \Delta = \frac{\lambda}{L_\parallel} \tag{1}$$

on equal footing and so are valid in the Pfirsch-Schlüter regime of tokamak transport which assumes $\delta \ll \Delta$. Here, the parallel scale length is denoted by $L_\parallel$ and can be comparable or even much larger than the perpendicular scale length $L_\perp$. In addition, $\rho = v_i/\Omega$ is the ion gyro-radius and $\lambda = v_i/\nu$ is the Coulomb mean-free path, with $v_i = \sqrt{2T_i/M}$ the ion thermal speed, ν the ion-ion collision frequency,[1] and $\Omega = eB/Mc$ the ion gyro-frequency, with B the magnitude of the magnetic field, c the speed of light, and e, T_i and M the ion charge (the case of singly-charged ions is considered for simplicity), temperature and mass. The drift ordering assumes the mean ion flow velocity $\boldsymbol{V}$ to be on the order of the diamagnetic drift velocity, which is on the order of the ion diamagnetic and collisional parallel heat fluxes $\boldsymbol{q}$ divided by the ion pressure $p_i = nT_i$, with n the ion density. As a result, Ref. [[3]] assumes

$$\frac{|\boldsymbol{V}|}{v_i} \sim \frac{|\boldsymbol{q}|}{p_i v_i} \sim \delta \sim \Delta \tag{2}$$

with $|\boldsymbol{V}| \sim V_\| \sim |\boldsymbol{V}_\perp|$. Using the standard Pfirsch-Schlüter regime expansion $\delta \ll \Delta$ we have recently shown[4] that the Hazeltine's[5] expression for the parallel ion flow can be obtained from a constraint on the pressure anisotropy - a result that we will employ later in the lowest order form of the ion flow velocity.

The focus of the work herein is on the evaluation of the radial electric field in a tokamak in the Pfirsch-Schlüter regime of neoclassical transport. To perform this evaluation we begin with the complete expressions for the gyro-viscosity and the order (ν/Ω) smaller perpendicular collisional viscosity as obtained in Ref. [[3]]. The results we obtain do not agree with the pioneering results found by Hazeltine[5] for the following three reasons: (i) his expression for the radial flux of toroidal angular momentum is incomplete[6]; (ii) he assumed that both the ion pressure and electrostatic potential separately had no poloidal variation rather than requiring that they need only satisfy parallel ion momentum conservation constraint; and (iii) he solved a drift kinetic equation[7] that can be shown to be missing some of the second order in the δ expansion terms needed to obtain the full gyro-viscosity by his method of evaluation. As a result of these assumptions his treatment did not require him to consider the perpendicular collisional viscosity, which is the only place the electrostatic potential appears when these three deficiencies are removed.

Our evaluation requires that we retain both the gyro-viscosity and the collisional perpendicular viscosity to determine the radial electric field. We demonstrate that when the full gyro-viscosity is evaluated the final expression to lowest and next order in the Pfirsch-Schlüter expansion does not contain the electrostatic potential. As a result, the radial electric field only enters in the collisional viscosity which also contains other classical collisional viscous terms that are smaller by $(B_p/B)^2$ than the higher order neoclassical terms from the gyro-viscosity, where B_p is the poloidal magnetic field and B is the magnitude of the total magnetic field. Of course, for an isothermal plasma the viscosity must simplify to an expression that vanishes for a radial Maxwell-Boltzmann ion response. In the presence of ion temperature variation obtaining the electric field in terms of density and temperature gradients is far more complicated. The simplest case is then for a strongly up-down asymmetric tokamak, for example, just inside the separatrix in single-null-divertor geometry. In this case only the lowest order gyro-viscosity needs to be retained and it is proportional to the radial ion temperature gradient. Part of the up-down asymmetric contribution to the gyro-viscosity was found by Hinton and Wong[8] in their large flow ordering $|\boldsymbol{V}| \sim v_i$. Consequently, to determine the radial electric field we then only need the portion of the lowest order collisional perpendicular viscosity that depends on the radial gradients of the electrostatic potential and the density. The coefficient of the gyro-viscous temperature gradient term is formally Ω/ν larger than the coefficient of the collisional perpendicular viscosity so that the shear in the electric field, and thereby the toroidal ion flow, can be surprisingly large for strongly up-down asymmetric tokamaks. A more complicated situation arises for an up-down symmetric tokamak in which the lowest order gyro-viscous contribution vanishes and gyro-viscosity has to be evaluated to higher order in the (δ/Δ) expansion. In addition, the lowest order collisional perpendicular viscosity must be completely evaluated to determine which terms are small by $(B_p/B)^2$ and can be neglected. In this case the gyro-viscosity and collisional perpendicular viscosity are the same order in collisionality - no (ν/Ω) factor appears.

The up-down asymmetry in the gyro-viscosity arises in part because the lowest order ion flow velocity is of the form $\boldsymbol{V} = \omega R^2 \boldsymbol{\nabla}\zeta + u\boldsymbol{B}$, where ω and u are flux functions to lowest order, R is the cylindrical radial coordinate, ζ is the toroidal angle, and $\boldsymbol{B}$ the magnetic field, so that the poloidal variation of the flow on the asymmetric flux surfaces gives rise to an up-down asymmetric radial flux of toroidal angular momentum. A similar term arises because the heat flux is of the same form to the lowest order so that the heat flux modifications to the ion viscosity result in a terms of the same order. The behavior is somewhat similar to the radial Pfirsch-Schlüter ion heat flux which is driven by the up-down asymmetric poloidal variation of the ion temperature in both up-down symmetric and asymmetric tokamaks. However, unlike the heat flux case, where the parallel temperature variation can be related to the radial temperature gradient to bring in the collision frequency, gyro-viscous momentum transport does not depend explicitly on collisionality because u is independent of collision frequency - only its form is determined by collisions. As a result, the up-down asymmetric contribution to the gyro-viscosity appears to be formally Ω/ν larger than customary large flow estimates for the viscosity.[8–11]

In the next section, we discuss the normally accepted evaluation of the steady state electric field in the Pfirsch-Schlüter regime[5] and note its shortcomings. Section III discusses the orderings and assumptions appropriate for the Pfirsch- Schlüter regime and Sec. IV employs them to evaluate the ion particle and heat fluxes to the requisite order. These fluxes are required to evaluate the gyro-viscosity in Sec. V and the collisional perpendicular viscosity in Sec. VI. In Sec. VII we present the complete results for determining the radial electric field in a tokamak and discuss their implications.

II. BACKGROUND

Before presenting our evaluation of the radial flux of toroidal ion angular momentum,

$$\Pi \equiv \left\langle R^2 \boldsymbol{\nabla}\zeta \cdot \left(M \int \mathrm{d}^3v\, \tilde{f} \boldsymbol{v}\boldsymbol{v} \right) \cdot \boldsymbol{\nabla}\psi \right\rangle_\theta , \tag{3}$$

we outline what has historically been the procedure and point out its limitations. We employ the tokamak coordinates ψ, θ, and ζ which are the poloidal magnetic flux and the poloidal and toroidal angles, respectively. We also define the flux-surface average as $\langle \ldots \rangle_\theta \equiv (1/V') \oint [\mathrm{d}\theta\,(\ldots)/\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta]$, with $V' \equiv \oint [\mathrm{d}\theta/\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta]$. In this equation $\tilde{f} \equiv f - \bar{f}$ is the gyro-phase dependent portion of the ion distribution function, $\bar{f} \equiv \langle f \rangle_\varphi$ is the gyro-phase independent portion of the ion distribution function, and the gyro-average is defined as $\langle \ldots \rangle_\varphi \equiv (1/2\pi) \oint \mathrm{d}\varphi\,(\ldots)$, with φ the gyro-angle.

Hazeltine[5] is considered to be the first to develop a procedure for the evaluation of the radial electric field in the Pfirsch-Sclüter regime. To do so he employed the expression for $\tilde{f}$ in terms of $\bar{f}$ that he used to derive his drift kinetic equation,[7] namely,

$$\begin{aligned} \tilde{f} = \boldsymbol{v} \cdot \left\{ \frac{\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}|_{\varepsilon,\mu,\varphi} \bar{f}}{\Omega} - \boldsymbol{v}_E \left(\frac{\partial \bar{f}}{\partial \varepsilon} + \frac{1}{B}\frac{\partial \bar{f}}{\partial \mu} \right) - \boldsymbol{v}_M \frac{1}{B}\frac{\partial \bar{f}}{\partial \mu} \right\} \\ - \left(\boldsymbol{v}_\perp \boldsymbol{v} \times \hat{\boldsymbol{b}} + \boldsymbol{v} \times \hat{\boldsymbol{b}}\, \boldsymbol{v}_\perp \right) : \boldsymbol{\nabla}\hat{\boldsymbol{b}}\ \frac{v_\parallel}{4\Omega B}\frac{\partial \bar{f}}{\partial \mu}, \end{aligned} \tag{4}$$

in Eq. (3) to find the result

$$\Pi \rightarrow \left\langle \frac{MI}{B} \int \mathrm{d}^3v\, v_\parallel [(\boldsymbol{v}_E + \boldsymbol{v}_M)\cdot\boldsymbol{\nabla}\psi] \bar{f} \right\rangle_\theta . \tag{5}$$

In the preceding equations $\boldsymbol{\nabla}|_{\varepsilon,\mu,\varphi}$ is the gradient with respect to spatial coordinates taken at fixed $\varepsilon \equiv v^2/2$, $\mu \equiv v_\perp^2/2B$, and φ, $\boldsymbol{v}_E \equiv c\boldsymbol{E}\times\hat{\boldsymbol{b}}/B$ is the $\boldsymbol{E}\times\boldsymbol{B}$ velocity, the magnetic drifts are given by $\boldsymbol{v}_M \equiv \hat{\boldsymbol{b}} \times [\mu \boldsymbol{\nabla} B + v_\parallel^2(\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}}) + v_\parallel \partial\hat{\boldsymbol{b}}/\partial t]/\Omega$, $\boldsymbol{E}$ is the electric field, $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$, $\boldsymbol{B}$ is taken in the form

$$\boldsymbol{B} = I(\psi)\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\psi, \tag{6}$$

and "parallel" and "perpendicular" refer to directions along and across $\boldsymbol{B}$. To complete his evaluation Hazeltine then used his drift-kinetic equation[7] to solve for $\bar{f}$ for a collisional plasma, inserted the solution into Eq. (5) and after assuming no poloidal variation in the electrostatic potential and ion pressure he arrived at the equation for the Pfirsch-Schlüter radial electric field by setting $\Pi = 0$.

The solution procedure employed in Ref. [5] seems to have been adopted as the standard Pfirsch-Schlüter procedure for determining the radial electric field[12] even though it has three subtle flaws, which invalidate both the procedure and the final answer. These are briefly described in the remainder of this section to make it clear why we cannot use the same procedure and why the corrected results derived herein are of a completely different form.

The first difficulty is that the expression (4) is incomplete. Indeed, if the small parameters δ and Δ are introduced and used to expand the ion distribution function, $\bar{f} = \bar{f}_0 + \bar{f}_1 + \bar{f}_2 + \cdots$, $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 + \cdots$, it can be shown that $\tilde{f}_1$ does not contribute to the integral on the left-hand side of Eq. (3) and so the second order portion of $\tilde{f}$, $\tilde{f}_2$, is required. If the exact kinetic equation for $\tilde{f}$ is solved order by order, $\tilde{f}_2$ is found to contain contributions from both $\bar{f}_1$ and $\tilde{f}_1$. However, $\tilde{f}_2$ as given by expression (4) only contains contributions due to $\bar{f}_1$ and neglects those due to $\tilde{f}_1$. Therefore, expression (4) is only accurate to the first order in $\delta \sim \Delta$. A more detailed proof of this statement is presented in the Appendix A.

The second shortcoming is that even if expression (4) for $\tilde{f}_2$ were accurate to the order required, substituting it into Eq. (3) would result not in constraint (5) but instead would give[6]

$$\Pi \to \left\langle \frac{MI}{B} \int \mathrm{d}^3v\, \bar{f} v_\parallel [(\boldsymbol{v}_E + \boldsymbol{v}_M) \cdot \boldsymbol{\nabla}\psi] \right\rangle_\theta + \left\langle \frac{M\boldsymbol{\nabla} \cdot (\hat{\boldsymbol{b}}\, |\boldsymbol{\nabla}\psi|^2)}{2\Omega} \int \mathrm{d}^3v\, \bar{f} v_\parallel \mu \right\rangle_\theta . \quad (7)$$

The contribution from the second term turns out to be large for up-down asymmetric tokamaks.

The final flaw in the procedure is the assumption that $\boldsymbol{B} \cdot \boldsymbol{\nabla} p_i$ and $\boldsymbol{B} \cdot \boldsymbol{\nabla}\phi$ are equal to zero separately rather than allowing them to satisfy the leading order parallel ion momentum conservation equation,

$$\boldsymbol{B} \cdot (\boldsymbol{\nabla} p_i + en\boldsymbol{\nabla}\phi) = 0, \quad (8)$$

which along with the total pressure $p_i + p_e$ and electron temperature T_e being flux functions gives[4,13]

$$-en\boldsymbol{B} \cdot \boldsymbol{\nabla}\phi = \boldsymbol{B} \cdot \boldsymbol{\nabla} p_i \approx \frac{p_e}{T_i + T_e} \boldsymbol{B} \cdot \boldsymbol{\nabla} T_i. \quad (9)$$

Here and elsewhere $p_e = nT_e$ is the electron pressure. To write parallel gradients in terms of the radial ion temperature gradient the usual lowest order Pfirsch-Schlüter relation between the parallel and radial temperature gradients[5] is employed

$$\boldsymbol{B} \cdot \boldsymbol{\nabla} T_i \approx \frac{16I\nu B}{25\Omega} \left(1 - \frac{B^2}{\langle B^2 \rangle_\theta}\right) \frac{\partial T_i}{\partial \psi}. \quad (10)$$

In subsequent sections we will present a procedure free from these flaws and use it to evaluate the radial flux of toroidal angular momentum. To do so we will have to consider the collisional perpendicular viscosity as well as the ion gyro-viscosity, and will make liberal use of the other Pfirsch-Schlüter results from the seminal work of Hazeltine.[5] We begin by giving a brief discussion of orderings and assumptions in the next section.

III. ORDERINGS AND ASSUMPTIONS

For our evaluation of the radial flux of toroidal ion angular momentum we adopt the standard neoclassical collisional transport orderings for the Pfirsch-Schlüter regime (see, for example, Refs. [7,14,15]). In particular, we employ both small ion gyro-radius and short mean-free path expansions as noted in Eq. (1), where the parallel scale length $L_\parallel$ is normally taken to be of order the connection length qR_0, with q the tokamak safety factor and R_0 the tokamak major radius.

As in all Pfirsch-Schlüter treatments we also assume that n, T_e, T_i and the electrostatic potential, ϕ, are functions of ψ only in leading order, so that the dominant poloidal variation is due to poloidal variation of the magnetic field. In particular, we assume

$$\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\ln T_i}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\ln B} \ll 1. \tag{11}$$

It can be shown *a posteriori* that the left-hand side of Eq. (11) is of order (δ_p/Δ), where $\delta_p \equiv \rho_p/w$, with $\rho_p \equiv (B/B_p)(v_i/\Omega)$ the poloidal ion gyro-radius, and $w \sim L_\perp$ is the characteristic macroscopic radial length scale. Then, Eq. (11) requires

$$\delta \lesssim \delta_p \ll \Delta \ll 1. \tag{12}$$

The time scale of interest is assumed to be that associated with the Pfirsch-Schlüter ion radial heat transport, namely

$$\frac{\partial}{\partial t} \sim \frac{\chi}{w^2} \sim q^2\nu\delta^2, \tag{13}$$

where[16] $\chi \sim q^2\nu\rho^2$ is the ion thermal diffusivity. To keep the orderings as general as possible we assume the aspect ratio $\epsilon \equiv r/R_0$, with r the minor radius of a flux surface, to be of order unity, $\epsilon \sim 1$, and $B \sim B_p$, so that $\rho_p \sim \rho$.

Assuming the characteristic time scale for the variation of the vector potential, $\boldsymbol{A}$, to be much longer than the heat transport time scale, the electric field, $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - c^{-1}\partial\boldsymbol{A}/\partial t$, is electrostatic to the order we require:

$$\frac{c^{-1}|\partial\boldsymbol{A}/\partial t|}{|\boldsymbol{\nabla}\phi|} \ll \frac{\epsilon^2 q\delta}{\Delta} \sim \frac{\delta}{\Delta} \ll 1, \tag{14}$$

where we estimate $A \sim B_p r$, $e\phi \sim T_e$.

Using the preceding assumptions and orderings we can evaluate the ion particle and heat flows to the requisite order for general tokamak geometry.

IV. ION PARTICLE AND HEAT FLOWS

To evaluate the radial flux of toroidal ion angular momentum from the expression for the ion stress tensor it is necessary to know the ion particle and heat flows to the lowest and next order in the Pfirsch-Schlüter expansion in $(\delta/\Delta) \ll 1$. In this section we derive these flows in forms convenient to carry out the evaluation of the gyro- and collisional viscosities in the next two sections.

As usual, to the order required the ion flow velocity is given by the sum of the parallel, $\boldsymbol{E}\times\boldsymbol{B}$ and diamagnetic velocities,

$$\boldsymbol{V} = V_\parallel\hat{\boldsymbol{b}} + c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}\phi}{B} + c\frac{\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}p_i}{enB}, \tag{15}$$

where $V_\parallel$ remains to be determined. Substituting this sum into the continuity equation,

$$\boldsymbol{\nabla}\cdot(n\boldsymbol{V}) = 0, \tag{16}$$

and taking into account the parallel component of the leading order ion momentum conservation equation, Eq. (8), we obtain

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\left[\frac{nV_\parallel}{B} + \frac{cIn}{B^2}\left(\frac{\partial\phi}{\partial\psi} + \frac{1}{en}\frac{\partial p_i}{\partial\psi}\right)\right] = 0, \tag{17}$$

so that

$$V_{\parallel} = K(\psi)\frac{B}{n} - \frac{cI}{B}\left(\frac{\partial\phi}{\partial\psi} + \frac{1}{en}\frac{\partial p_i}{\partial\psi}\right). \tag{18}$$

The flux function $K(\psi)$ is determined by the parallel momentum constraint on the pressure anisotropy and is given to the lowest order by[4,5]

$$K(\psi) \approx -\frac{cI}{e}\left\langle n\frac{\partial T_i}{\partial\psi}\right\rangle_\theta \left(1.78\frac{1}{\langle B^2\rangle_\theta} + 0.057\frac{\left\langle(\nabla_\parallel \ln B)^2\right\rangle_\theta}{\left\langle(\nabla_\parallel B)^2\right\rangle_\theta}\right). \tag{19}$$

Substituting expression (18) into Eq. (15) we arrive at

$$\boldsymbol{V} = \frac{K(\psi)}{n}\boldsymbol{B} - c\left(\frac{\partial\phi}{\partial\psi} + \frac{1}{en}\frac{\partial p}{\partial\psi}\right)R^2\boldsymbol{\nabla}\zeta, \tag{20}$$

where we note from Eq. (20) that $\boldsymbol{V}\cdot\boldsymbol{\nabla}\psi = 0$, so that there is no radial ion particle flux to the order we require. Since n, p_i, and ϕ are flux functions to the leading order the lowest order form of Eq. (20) can be written as

$$\boldsymbol{V} = \frac{K(\psi)}{\langle n\rangle_\theta}\boldsymbol{B} - c\left(\frac{\mathrm{d}\langle\phi\rangle_\theta}{\mathrm{d}\psi} + \frac{1}{e\langle n\rangle_\theta}\frac{\mathrm{d}\langle p_i\rangle_\theta}{\mathrm{d}\psi}\right)R^2\boldsymbol{\nabla}\zeta \equiv u(\psi)\boldsymbol{B} + \omega(\psi)R^2\boldsymbol{\nabla}\zeta. \tag{21}$$

The ion heat flux $\boldsymbol{q}$ is given to the order required by the sum of the parallel and diamagnetic heat fluxes,

$$\boldsymbol{q} = q_\parallel\hat{\boldsymbol{b}} + \frac{5cp_i}{2eB}\hat{\boldsymbol{b}}\times\boldsymbol{\nabla}T_i, \tag{22}$$

where the Pfirsch-Schlüter form of $q_\parallel$ must be determined to the leading two orders, and in accordance with Eq. (22)

$$\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi = -\frac{5cIp_i}{2eB^2}\boldsymbol{B}\cdot\boldsymbol{\nabla}T_i \tag{23}$$

with $\boldsymbol{B}\cdot\boldsymbol{\nabla}T_i$ given by Eq. (10). To the order required the ion energy conservation equation may be written as

$$\boldsymbol{\nabla}\cdot\left[\boldsymbol{q} + \left(e\phi + \frac{5}{2}T_i\right)n\boldsymbol{V}\right] = \langle\boldsymbol{\nabla}\cdot\boldsymbol{q}\rangle_\theta\,, \tag{24}$$

where the right-hand side does not vanish because, unlike the ion particle flux, the ion heat flux has a radial component - the Pfirsch-Schlüter radial ion heat flux. Again using Eq. (8) we may rewrite the preceding conveniently as

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\frac{q_\parallel}{B}+\frac{5cIp_i}{2eB^2}\frac{\partial T_i}{\partial\psi}+\frac{5}{2}K(\psi)T_i\right)=F(\psi)\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)+S-\langle S\rangle_\theta\,, \tag{25}$$

where

$$S\equiv-(\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta)\frac{\partial}{\partial\psi}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\right), \tag{26}$$

and we define the flux function $F(\psi)$ by the equation

$$K(\psi)\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}p_i}{n}\approx\frac{16cIMK(\psi)\langle\nu\rangle_\theta\langle T_e\rangle_\theta}{25e(\langle T_i\rangle_\theta+\langle T_e\rangle_\theta)}\frac{\mathrm{d}\langle T_i\rangle_\theta}{\mathrm{d}\psi}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)\equiv F(\psi)\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right). \tag{27}$$

Solving for $q_\parallel$ we find

$$q_\parallel=B\left[L(\psi)-\frac{5}{2}K(\psi)T_i-\frac{5cIp_i}{2eB^2}\frac{\partial T_i}{\partial\psi}\right] \tag{28}$$
$$+B\left[F(\psi)\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)+\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}(S-\langle S\rangle_\theta)\right],$$

where the unknown flux function $L(\psi)$ can be obtained from the parallel heat flux constraint,[1] $\left\langle q_\parallel B\right\rangle_\theta=0$:

$$L(\psi)=\frac{5}{2}K(\psi)\frac{\langle B^2T_i\rangle_\theta}{\langle B^2\rangle_\theta}+\frac{5cI}{2e\langle B^2\rangle_\theta}\left\langle p_i\frac{\partial T_i}{\partial\psi}\right\rangle_\theta \tag{29}$$
$$-\frac{1}{\langle B^2\rangle_\theta}\left[F(\psi)\left\langle B^2\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)\right\rangle_\theta+\left\langle B^2\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}(S-\langle S\rangle_\theta)\right\rangle_\theta\right].$$

Since n, p_i, and ϕ are flux functions in the leading order we can use Eqs. (28) and (29) to write the leading order form of Eq. (22) as

$$\boldsymbol{q}=\frac{5c}{2e}\langle p_i\rangle_\theta\frac{\partial\langle T_i\rangle_\theta}{\partial\psi}\left(\frac{I}{\langle B^2\rangle_\theta}\boldsymbol{B}-R^2\boldsymbol{\nabla}\zeta\right)\equiv\frac{5}{2}[g(\psi)\boldsymbol{B}+s(\psi)R^2\boldsymbol{\nabla}\zeta]. \tag{30}$$

The preceding expressions for the flows are to be used to evaluate the radial flux of toroidal angular momentum, which we write as

$$\left\langle R^2\boldsymbol{\nabla}\zeta\cdot(Mn\boldsymbol{V}\boldsymbol{V}+\overleftrightarrow{\pi})\cdot\boldsymbol{\nabla}\psi\right\rangle_\theta=0. \tag{31}$$

It follows from Eq. (20) that the $\boldsymbol{VV}$ term does not contribute to constraints (31) to the order we require.

The ion viscous stress tensor is given by the sum of the so-called *parallel*, *gyro-*, and *perpendicular* viscous stress tensors[1]: $\overleftrightarrow{\pi}=\overleftrightarrow{\pi}_{\parallel}+\overleftrightarrow{\pi}_{g}+\overleftrightarrow{\pi}_{\perp}$. The parallel viscous stress tensor is a diagonal matrix, $\overleftrightarrow{\pi}_{\parallel}=(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}-\overleftrightarrow{I}/3)\pi_{\parallel}$, and therefore does not contribute to constraint (31).

The gyro-viscous stress tensor is conveniently written in terms of the ion particle and heat flows as[2,3]

$$\overleftrightarrow{\pi}_{g}=\frac{p_i}{4\Omega}\left\{\hat{\boldsymbol{b}}\times\left[\overleftrightarrow{\alpha}+\overleftrightarrow{\alpha}^{\mathrm{T}}\right]\cdot\left(\overleftrightarrow{I}+3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)-\left(\overleftrightarrow{I}+3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right)\cdot\left[\overleftrightarrow{\alpha}+\overleftrightarrow{\alpha}^{\mathrm{T}}\right]\times\hat{\boldsymbol{b}}\right\}, \tag{32}$$

where $\overleftrightarrow{\alpha}\equiv\boldsymbol{\nabla V}+2\boldsymbol{\nabla q}/(5p_i)$ and $\overleftrightarrow{\alpha}^{\mathrm{T}}$ denotes a transpose of $\overleftrightarrow{\alpha}$.

In terms of the same flows the perpendicular viscous stress tensor is given by[3]

$$\begin{aligned}\overleftrightarrow{\pi}_{\perp}= & -\frac{3\nu}{10\Omega^2}\left[\overleftrightarrow{W}+\overleftrightarrow{W}_1+3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\cdot\left(\overleftrightarrow{W}+\overleftrightarrow{W}_1\right)+3\left(\overleftrightarrow{W}+\overleftrightarrow{W}_1\right)\cdot\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\right] \\ & -\frac{9M\nu}{200p_iT_i\Omega}\left[\hat{\boldsymbol{b}}\times\boldsymbol{q}\left(\boldsymbol{q}+\frac{31}{15}\boldsymbol{q}_{\parallel}\right)+\left(\boldsymbol{q}+\frac{31}{15}\boldsymbol{q}_{\parallel}\right)\hat{\boldsymbol{b}}\times\boldsymbol{q}\right],\end{aligned} \tag{33}$$

with

$$\overleftrightarrow{W}=p_i\left[\overleftrightarrow{\alpha}+\overleftrightarrow{\alpha}^{\mathrm{T}}-\frac{2}{3}(\overleftrightarrow{I}:\overleftrightarrow{\alpha})\overleftrightarrow{I}\right]$$

and

$$\overleftrightarrow{W}_1=\frac{3}{10}\left[-\boldsymbol{\nabla}\left(\boldsymbol{q}+\frac{1}{10}\boldsymbol{q}_{\parallel}\right)+\left(\boldsymbol{q}-\frac{1}{6}\boldsymbol{q}_{\parallel}\right)\boldsymbol{\nabla}\ln p_i-\left(\frac{3}{4}\boldsymbol{q}-\frac{13}{120}\boldsymbol{q}_{\parallel}\right)\boldsymbol{\nabla}\ln T_i\right] \\ +\text{Transpose},$$

where $\overleftrightarrow{I}:\overleftrightarrow{\alpha}$ is the trace of the tensor $\overleftrightarrow{\alpha}$.

In the section that follows we insert the flows into the gyro-viscosity to evaluate its contribution to the radial flux of toroidal angular momentum through the first two orders. Then, the section after gives the evaluation of the collisional perpendicular viscosity to the lowest order.

V. GYRO-VISCOUS CONTRIBUTION

To evaluate the gyro-viscous contribution to the radial flux of toroidal angular momentum we begin by casting it into a more convenient form by substituting expression (32) for $\overleftrightarrow{\pi}_g$ into Eq. (31) and using the equality

$$\overleftrightarrow{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} = \frac{\boldsymbol{\nabla}\psi\boldsymbol{\nabla}\psi + (\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\psi)(\hat{\boldsymbol{b}} \times \boldsymbol{\nabla}\psi)}{|\boldsymbol{\nabla}\psi|^2}.$$

The expression obtained,

$$R^2\boldsymbol{\nabla}\zeta\cdot \overleftrightarrow{\pi}_{gi} \cdot\boldsymbol{\nabla}\psi = \frac{B}{2\Omega}\Bigg[\left(\frac{I^2}{B^2} - R^2\right)\left(p_i\boldsymbol{\nabla}\cdot\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\cdot\boldsymbol{q}\right) \tag{34}$$
$$+ \left(R^2 - \frac{3I^2}{B^2}\right)\hat{\boldsymbol{b}}\cdot\left(p_i\boldsymbol{\nabla}\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\boldsymbol{q}\right)\cdot\hat{\boldsymbol{b}} + 2R^2\boldsymbol{\nabla}\zeta\cdot\left(p_i\boldsymbol{\nabla}\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\boldsymbol{q}\right)\cdot R^2\boldsymbol{\nabla}\zeta\Bigg],$$

can be evaluated term by term.

Using continuity (16) and parallel momentum (8), as well as Eqs. (20), (27) to rewrite the ion energy equation (24) as

$$p_i\boldsymbol{\nabla}\cdot\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\cdot\boldsymbol{q} = \frac{2}{5}\left\langle\boldsymbol{\nabla}\cdot\boldsymbol{q}\right\rangle_\theta - \frac{3}{5}F(\psi)\left(1 - \frac{B^2}{\langle B^2\rangle_\theta}\right), \tag{35}$$

we find for the contribution from the first term

$$\left\langle\left(\frac{I^2}{B^2} - R^2\right)\left(p_i\boldsymbol{\nabla}\cdot\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\cdot\boldsymbol{q}\right)\right\rangle_\theta = \tag{36}$$
$$\frac{2}{5}\left\langle\frac{I^2}{B^2} - R^2\right\rangle_\theta\langle\boldsymbol{\nabla}\cdot\boldsymbol{q}\rangle_\theta - \frac{3}{5}F(\psi)\left\langle\left(\frac{I^2}{B^2} - R^2\right)\left(1 - \frac{B^2}{\langle B^2\rangle_\theta}\right)\right\rangle_\theta.$$

To evaluate the third term in Eq. (34) we first rewrite it by noting

$$\left\langle 2R^2\boldsymbol{\nabla}\zeta\cdot\left(p_i\boldsymbol{\nabla}\boldsymbol{V} + \frac{2}{5}\boldsymbol{\nabla}\boldsymbol{q}\right)\cdot R^2\boldsymbol{\nabla}\zeta\right\rangle_\theta = \left\langle\left(p_i\boldsymbol{V} + \frac{2}{5}\boldsymbol{q}\right)\cdot\boldsymbol{\nabla}R^2\right\rangle_\theta =$$
$$\left\langle\boldsymbol{\nabla}\cdot\left[R^2\left(p_i\boldsymbol{V} + \frac{2}{5}\boldsymbol{q}\right)\right]\right\rangle_\theta - \left\langle R^2\boldsymbol{\nabla}\cdot\left(p_i\boldsymbol{V} + \frac{2}{5}\boldsymbol{q}\right)\right\rangle_\theta.$$

Recalling that for any vector $\boldsymbol{a}$

$$\langle\boldsymbol{\nabla}\cdot\boldsymbol{a}\rangle_\theta = \frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}\left(V'\langle\boldsymbol{a}\cdot\boldsymbol{\nabla}\psi\rangle_\theta\right), \tag{37}$$

that to the order required $\boldsymbol{V} \cdot \boldsymbol{\nabla}\psi = 0$, and using Eq. (24) and then Eqs. (8) and (27), we find the desired form

$$\left\langle 2R^2\boldsymbol{\nabla}\zeta \cdot \left(p_i \boldsymbol{\nabla V} + \frac{2}{5}\boldsymbol{\nabla q}\right) \cdot R^2\boldsymbol{\nabla}\zeta \right\rangle_\theta = \qquad (38)$$
$$\frac{2}{5}\left[\left\langle \boldsymbol{\nabla} \cdot \left(R^2\boldsymbol{q}\right)\right\rangle_\theta - \left\langle R^2\right\rangle_\theta \left\langle \boldsymbol{\nabla} \cdot \boldsymbol{q}\right\rangle_\theta - F(\psi)\left\langle R^2\left(1 - \frac{B^2}{\langle B^2\rangle_\theta}\right)\right\rangle_\theta\right].$$

The evaluation of the remaining term in Eq. (34) is more involved. We begin by observing that

$$\hat{\boldsymbol{b}} \cdot \left(p_i \boldsymbol{\nabla V} + \frac{2}{5}\boldsymbol{\nabla q}\right) \cdot \hat{\boldsymbol{b}} = p_i \nabla_\| V_\| + \frac{2}{5}\nabla_\| q_\| - \boldsymbol{\kappa} \cdot \left(p_i \boldsymbol{V}_\perp + \frac{2}{5}\boldsymbol{q}_\perp\right), \qquad (39)$$

where $\boldsymbol{\kappa} \equiv \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}}$ is the magnetic field line curvature. Using Eqs. (18) and (28), respectively, we can write

$$p_i \nabla_\| V_\| = p_i K(\psi) \nabla_\| \left(\frac{B}{n}\right) + \frac{cIp_i}{B^2}\left(\frac{\partial \phi}{\partial \psi}\nabla_\| B + \frac{1}{en}\frac{\partial p_i}{\partial \psi}\right) \qquad (40)$$
$$- \frac{cIp_i}{B}\nabla_\| \left(\frac{\partial \phi}{\partial \psi} + \frac{1}{en}\frac{\partial p_i}{\partial \psi}\right)$$

and

$$\frac{2}{5}\nabla_\| q_\| = \frac{2}{5}L(\psi)\nabla_\| B - K(\psi)\nabla_\|(BT_i) + \frac{cIp_i}{eB^2}\frac{\partial T_i}{\partial \psi}\nabla_\| B - \frac{cI}{eB}\nabla_\| \left(p_i \frac{\partial T_i}{\partial \psi}\right)$$
$$+ \frac{2}{5}F(\psi)\nabla_\| B \int \frac{\mathrm{d}\theta}{\boldsymbol{B} \cdot \boldsymbol{\nabla}\theta}\left(1 - \frac{B^2}{\langle B^2\rangle_\theta}\right) + \frac{2}{5}F(\psi)\left(1 - \frac{B^2}{\langle B^2\rangle_\theta}\right) \qquad (41)$$
$$+ \frac{2}{5}\nabla_\| B \int \frac{\mathrm{d}\theta}{\boldsymbol{B} \cdot \boldsymbol{\nabla}\theta}(S - \langle S\rangle_\theta) + \frac{2}{5}(S - \langle S\rangle_\theta).$$

Employing Eqs. (8), (15), (22) and (23) together with expressions

$$\boldsymbol{\kappa} \times \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\psi \approx I\nabla_\| \ln B, \quad \boldsymbol{\kappa} \times \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\theta = -\frac{I\boldsymbol{B} \cdot \boldsymbol{\nabla}\theta}{2B^3}\frac{\partial}{\partial \psi}[B^2(1 + \beta)], \qquad (42)$$

where $\beta \equiv 8\pi(p_i + p_e)/B^2 \equiv 8\pi p/B^2$, we obtain

$$\boldsymbol{\kappa} \cdot \left(p_i \boldsymbol{V}_\perp + \frac{2}{5}\boldsymbol{q}_\perp\right) = \frac{cIp_i}{B^2}\left(\frac{\partial \phi}{\partial \psi} + \frac{1}{en}\frac{\partial p_i}{\partial \psi} + \frac{1}{e}\frac{\partial T_i}{\partial \psi}\right)\nabla_\| B \qquad (43)$$
$$+ \frac{1}{5B^2}(\boldsymbol{q} \cdot \boldsymbol{\nabla}\psi)\frac{\partial}{\partial \psi}[B^2(1 + \beta)].$$

Employing the equality

$$\frac{cI}{B}\left[p_i\nabla_\parallel\left(\frac{\partial\phi}{\partial\psi}+\frac{1}{en}\frac{\partial p_i}{\partial\psi}\right)+\frac{1}{e}\nabla_\parallel\left(p_i\frac{\partial T_i}{\partial\psi}\right)\right]-\frac{2}{5}S=\frac{2}{5}(\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi)\frac{\partial}{\partial\psi}\left[\ln\left(\frac{I}{B^2}\right)\right] \quad (44)$$

obtained by making use of Eq. (23) and

$$ep_i\nabla_\parallel\left(\frac{\partial\phi}{\partial\psi}+\frac{1}{en}\frac{\partial p_i}{\partial\psi}\right)=\left(\frac{\partial T_i}{\partial\psi}\right)\nabla_\parallel p_i-\left(\frac{\partial p_i}{\partial\psi}\right)\nabla_\parallel T_i$$

and then using results (40), (41), (43) and $\langle S\rangle_\theta=-\langle\boldsymbol{\nabla}\cdot\boldsymbol{q}\rangle_\theta$, we obtain

$$\begin{aligned}\hat{\boldsymbol{b}}\cdot\left(p_i\boldsymbol{\nabla V}+\frac{2}{5}\boldsymbol{\nabla q}\right)\cdot\hat{\boldsymbol{b}}&=\frac{2}{5}L(\psi)\nabla_\parallel B-\frac{3}{5}F(\psi)\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)+\frac{2}{5}\langle\boldsymbol{\nabla}\cdot\boldsymbol{q}\rangle_\theta\\&+\frac{2}{5}\nabla_\parallel B\left[F(\psi)\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)+\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}(S-\langle S\rangle_\theta)\right]\\&-\frac{2}{5}\left\{\frac{\partial}{\partial\psi}\left[\ln\left(\frac{I}{B^2}\right)\right]+\frac{1}{2B^2}\frac{\partial}{\partial\psi}[B^2(1+\beta)]\right\}(\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi).\end{aligned} \quad (45)$$

Observing that

$$\left\langle\frac{\nabla_\parallel B}{B^2}\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}Q\right\rangle_\theta=\left\langle\frac{Q}{2B^2}\right\rangle_\theta \quad (46)$$

we obtain the desired form for the last remaining (second) term in Eq. (34):

$$\begin{aligned}&\left\langle\left(R^2-\frac{3I^2}{B^2}\right)\hat{\boldsymbol{b}}\cdot\left(p_i\boldsymbol{\nabla V}+\frac{2}{5}\boldsymbol{\nabla q}\right)\cdot\hat{\boldsymbol{b}}\right\rangle_\theta=\frac{2}{5}L(\psi)\left\langle R^2\nabla_\parallel B\right\rangle_\theta\\&-\frac{3}{5}F(\psi)\left\langle\left(R^2-\frac{2I^2}{B^2}\right)\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)\right\rangle_\theta+\frac{2}{5}\left\langle R^2-\frac{9I^2}{2B^2}\right\rangle_\theta\langle\boldsymbol{\nabla}\cdot\boldsymbol{q}\rangle_\theta\\&+\frac{3I^2}{5}\left[\left\langle\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{q}}{B^2}\right)\right\rangle_\theta+\left\langle\frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi}{B^4}\frac{\partial B^2}{\partial\psi}\right\rangle_\theta\right]\\&+\frac{2}{5}\left\langle R^2\nabla_\parallel B\left[F(\psi)\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)+\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}(S-\langle S\rangle_\theta)\right]\right\rangle_\theta\\&-\frac{2}{5}\left\langle\left(R^2-\frac{3I^2}{B^2}\right)\left\{\frac{\partial}{\partial\psi}\left[\ln\left(\frac{I}{B^2}\right)\right]+\frac{1}{2B^2}\frac{\partial}{\partial\psi}[B^2(1+\beta)]\right\}(\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi)\right\rangle_\theta.\end{aligned} \quad (47)$$

Combining the intermediate results (36), (38) and (47) we arrive at the full contribution from the gyro-viscous stress tensor:

$$
\begin{aligned}
\left\langle R^2 \boldsymbol{\nabla}\zeta \cdot \overleftrightarrow{\pi}_g \cdot \boldsymbol{\nabla}\psi \right\rangle_\theta &= \frac{B}{5\Omega} L(\psi) \left\langle R^2 \nabla_\| B \right\rangle_\theta \\
&- \frac{B}{5\Omega} F(\psi) \left\langle \left(R^2 - \frac{3I^2}{2B^2} \right) \left(1 - \frac{B^2}{\langle B^2 \rangle_\theta} \right) \right\rangle_\theta - \frac{B}{5\Omega} \left\langle R^2 + \frac{7I^2}{2B^2} \right\rangle_\theta \langle \boldsymbol{\nabla} \cdot \boldsymbol{q} \rangle_\theta \\
&+ \frac{3I^2 B}{10\Omega} \left[\left\langle \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{q}}{B^2} \right) \right\rangle_\theta + \left\langle \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla}\psi}{B^4} \frac{\partial B^2}{\partial \psi} \right\rangle_\theta \right] + \frac{B}{5\Omega} \left\langle \boldsymbol{\nabla} \cdot (R^2 \boldsymbol{q}) \right\rangle_\theta \\
&+ \frac{B}{5\Omega} \left\langle R^2 \nabla_\| B \left[F(\psi) \int \frac{\mathrm{d}\theta}{\boldsymbol{B} \cdot \boldsymbol{\nabla}\theta} \left(1 - \frac{B^2}{\langle B^2 \rangle_\theta} \right) + \int \frac{\mathrm{d}\theta}{\boldsymbol{B} \cdot \boldsymbol{\nabla}\theta} (S - \langle S \rangle_\theta) \right] \right\rangle_\theta \\
&- \frac{B}{5\Omega} \left\langle \left(R^2 - \frac{3I^2}{B^2} \right) \left\{ \frac{\partial}{\partial \psi} \left[\ln \left(\frac{I}{B^2} \right) \right] + \frac{1}{2B^2} \frac{\partial}{\partial \psi} [B^2 (1 + \beta)] \right\} (\boldsymbol{q} \cdot \boldsymbol{\nabla}\psi) \right\rangle_\theta .
\end{aligned}
\tag{48}
$$

This result is neoclassical in nature because it depends on relation (10) between the poloidal and radial variation of ion temperature.

The formally largest contribution to the right-hand side of this equation comes from the first term and upon inserting the lowest order expression for $L(\psi)$ it is equal to

$$
\left\langle R^2 \boldsymbol{\nabla}\zeta \cdot \overleftrightarrow{\pi}_g \cdot \boldsymbol{\nabla}\psi \right\rangle_\theta \rightarrow \frac{B}{2\Omega} \left(K(\psi) \langle T_i \rangle_\theta + \frac{cI}{e \langle B^2 \rangle_\theta} \langle p_i \rangle_\theta \frac{\partial \langle T_i \rangle_\theta}{\partial \psi} \right) \left\langle R^2 \nabla_\| B \right\rangle_\theta . \tag{49}
$$

Notice that this contribution vanishes for an up-down symmetric tokamak equilibria, in which case all the other terms in Eq. (48) must be retained. Only for a strongly up-down asymmetric tokamak will the lowest order expression be adequate. Strong asymmetry is expected to occur just inside the separatrix of a single-null-divertor configuration.

It is important to notice in the preceding results that the electrostatic potential does not enter. Consequently, the radial electric field cannot be determined by setting the right-hand side of Eq. (48) or Eq. (49) equal to zero. In fact, the radial electric field first enters in the collisional perpendicular viscosity contribution to the radial flux of toroidal angular momentum which is evaluated in the next section.

VI. PERPENDICULAR VISCOSITY CONTRIBUTION

The radial electric field enters the radial flux of ion toroidal angular momentum only through the collisional perpendicular ion viscosity which we need only evaluate to lowest order. To begin we note that we can employ Eq. (33) to write the contribution from the perpendicular viscosity to constraint (31) as follows:

$$\begin{aligned} R^2\boldsymbol{\nabla}\zeta\cdot \overleftrightarrow{\pi}_\perp \cdot\boldsymbol{\nabla}\psi = \\ -\frac{3\nu}{10\Omega^2}\left[R^2\boldsymbol{\nabla}\zeta\cdot \overleftrightarrow{W}_* \cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot \overleftrightarrow{W}_* \cdot R^2\boldsymbol{\nabla}\zeta + \frac{3I}{B}\left(\hat{\boldsymbol{b}}\cdot \overleftrightarrow{W}_* \cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot \overleftrightarrow{W}_* \cdot\hat{\boldsymbol{b}}\right)\right] \\ -\frac{9M\nu}{200p_iT_i\Omega}R^2\boldsymbol{\nabla}\zeta\cdot\left[\hat{\boldsymbol{b}}\times\boldsymbol{q}\left(\boldsymbol{q}+\frac{31}{15}\boldsymbol{q}_\parallel\right)+\left(\boldsymbol{q}+\frac{31}{15}\boldsymbol{q}_\parallel\right)\hat{\boldsymbol{b}}\times\boldsymbol{q}\right]\cdot\boldsymbol{\nabla}\psi, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \overleftrightarrow{W}_* \equiv p_i\boldsymbol{\nabla}\boldsymbol{V} + \frac{1}{10}\boldsymbol{\nabla}\boldsymbol{q} - \frac{3}{100}\boldsymbol{\nabla}\boldsymbol{q}_\parallel \\ +\frac{3}{10}\boldsymbol{q}\left(\boldsymbol{\nabla}\ln p_i - \frac{3}{4}\boldsymbol{\nabla}\ln T_i\right) - \frac{1}{20}\boldsymbol{q}_\parallel\left(\boldsymbol{\nabla}\ln p_i - \frac{13}{20}\boldsymbol{\nabla}\ln T_i\right). \end{aligned}$$

Due to smallness of the perpendicular viscosity it is sufficient to employ the lowest order expressions (21) and (30) for $\boldsymbol{V}$ and $\boldsymbol{q}$, respectively. Then,

$$\boldsymbol{\nabla}\boldsymbol{V} = R^2\boldsymbol{\nabla}\omega\boldsymbol{\nabla}\zeta + R\,\omega(\boldsymbol{\nabla}R\boldsymbol{\nabla}\zeta - \boldsymbol{\nabla}\zeta\boldsymbol{\nabla}R) + \boldsymbol{\nabla}u\boldsymbol{B} + u\boldsymbol{\nabla}\boldsymbol{B}. \tag{51}$$

A similar expression can be written for $\boldsymbol{\nabla}\boldsymbol{q}$. Noticing that

$$0 = \boldsymbol{\nabla}\zeta\times(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\psi) = (\boldsymbol{\nabla}\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\zeta - \boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\psi)$$

gives

$$R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{B}\cdot\boldsymbol{\nabla}\psi = -R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\psi\cdot\boldsymbol{B} = -\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\psi\cdot R^2\boldsymbol{\nabla}\zeta = \boldsymbol{B}\cdot\boldsymbol{\nabla}(R^2\boldsymbol{\nabla}\zeta)\cdot\boldsymbol{\nabla}\psi,$$

and employing expression (51) we find

$$R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot R^2\boldsymbol{\nabla}\zeta = R^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\omega + \frac{uI}{R^2}\right). \tag{52}$$

Observing that the leading order total (electron + ion) momentum equation gives

$$B\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\psi = \boldsymbol{\nabla}\psi\cdot\left(\boldsymbol{\nabla}B + \frac{4\pi\boldsymbol{\nabla}p}{B}\right),$$

and using Eq. (51) we also obtain

$$\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot\hat{\boldsymbol{b}} = \boldsymbol{\nabla}\psi\cdot\left\{\frac{I}{B}\boldsymbol{\nabla}\omega + B\boldsymbol{\nabla}u + \frac{u}{B}\boldsymbol{\nabla}\left[B^2\left(1+\frac{\beta}{2}\right)\right]\right\}. \quad (53)$$

Combining results (52) and (53) we obtain the contribution from the $\boldsymbol{\nabla}\boldsymbol{V}$ terms in $\overleftrightarrow{W}_*$ to Eq. (50) as

$$\begin{aligned} R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot R^2\boldsymbol{\nabla}\zeta + \frac{3I}{B}\left(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\boldsymbol{V}\cdot\hat{\boldsymbol{b}}\right) &= \quad (54) \\ \boldsymbol{\nabla}\psi\cdot\left\{\left(R^2 + \frac{3I^2}{B^2}\right)\boldsymbol{\nabla}\omega + 4I\boldsymbol{\nabla}u + uR^2\boldsymbol{\nabla}\left(\frac{I}{R^2}\right) + \frac{3uI}{B^2}\boldsymbol{\nabla}\left[B^2\left(1+\frac{\beta}{2}\right)\right]\right\}. \end{aligned}$$

An analogous result for $(2/5)\boldsymbol{\nabla}\boldsymbol{q}$ instead of $\boldsymbol{\nabla}\boldsymbol{V}$ can be obtained from Eq. (54) by substituting $s(\psi)$ and $g(\psi)$ for $\omega(\psi)$ and $u(\psi)$, respectively.

To evaluate the $\boldsymbol{\nabla}\boldsymbol{q}_\parallel$ contribution from $\overleftrightarrow{W}_*$ we notice that to the lowest order

$$\boldsymbol{q}_\parallel = \frac{5}{2}\left[g(\psi) + \frac{s(\psi)I(\psi)}{B^2}\right]\boldsymbol{B},$$

where

$$\boldsymbol{\nabla}\left(\frac{sI}{B^2}\boldsymbol{B}\right) = \frac{\boldsymbol{\nabla}(sI)\hat{\boldsymbol{b}}}{B} + sI\left(\frac{\boldsymbol{\nabla}\hat{\boldsymbol{b}}}{B} - \frac{\boldsymbol{\nabla}B\,\hat{\boldsymbol{b}}}{B^2}\right), \quad (55)$$

gives

$$R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\left(\frac{sI}{B^2}\boldsymbol{B}\right)\cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{sI}{B^2}\boldsymbol{B}\right)\cdot R^2\boldsymbol{\nabla}\zeta = R^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{sI^2}{R^2B^2}\right)$$

and

$$\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\left(\frac{sI}{B^2}\boldsymbol{B}\right)\cdot\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\left(\frac{sI}{B^2}\boldsymbol{B}\right)\cdot\hat{\boldsymbol{b}} = \boldsymbol{\nabla}\psi\cdot\left[\frac{\boldsymbol{\nabla}(sI)}{B} + \frac{sI}{2B^3}\boldsymbol{\nabla}\left(B^2\beta\right)\right].$$

Using the preceding results and employing Eqs. (52) and (53) to evaluate the contribution from the $g(\psi)\boldsymbol{B}$ portion of $\boldsymbol{q}_\parallel$ we obtain

$$\frac{2}{5}\left[R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\boldsymbol{q}_\parallel\cdot\boldsymbol{\nabla}\psi+\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\boldsymbol{q}_\parallel\cdot R^2\boldsymbol{\nabla}\zeta+\frac{3I}{B}\left(\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\boldsymbol{q}_\parallel\cdot\boldsymbol{\nabla}\psi+\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\boldsymbol{q}_\parallel\cdot\hat{\boldsymbol{b}}\right)\right]= \tag{56}$$
$$\boldsymbol{\nabla}\psi\cdot R^2\boldsymbol{\nabla}\left(\frac{sI^2}{R^2B^2}+\frac{gI}{R^2}\right)+\boldsymbol{\nabla}\psi\cdot\frac{3I}{B^2}\left[\boldsymbol{\nabla}(sI+gB^2)+\left(\frac{sI}{2B^2}+\frac{g}{2}\right)\boldsymbol{\nabla}\left(B^2\beta\right)\right].$$

The contributions to Eq. (50) from the remaining terms in $\overleftrightarrow{W}_*$ are easily evaluated, as is the remaining term in Eq. (50). Combining these results with (54) (and the analogous result for $\boldsymbol{\nabla}\boldsymbol{q}$) and (56), we arrive at the desired expression for the contribution from the perpendicular viscosity:

$$\begin{aligned}
&R^2\boldsymbol{\nabla}\zeta\cdot\overleftrightarrow{\pi}_\perp\cdot\boldsymbol{\nabla}\psi=\\
&-\frac{3\nu}{10\Omega^2}\left(R^2+\frac{3I^2}{B^2}\right)\left(p_i\boldsymbol{\nabla}\omega+\frac{1}{4}\boldsymbol{\nabla}s\right)\cdot\boldsymbol{\nabla}\psi\\
&-\frac{6\nu}{5\Omega^2}I\left[p_i\boldsymbol{\nabla}u+\frac{7}{40}\boldsymbol{\nabla}g-\frac{3}{40}\boldsymbol{\nabla}\left(\frac{sI}{B^2}\right)\right]\cdot\boldsymbol{\nabla}\psi\\
&-\frac{3\nu}{10\Omega^2}\left(p_iu+\frac{7}{40}g-\frac{3}{40}\frac{sI}{B^2}\right)\left\{R^2\boldsymbol{\nabla}\left(\frac{I}{R^2}\right)+\frac{3I}{B^2}\boldsymbol{\nabla}\left[B^2\left(1+\frac{\beta}{2}\right)\right]\right\}\cdot\boldsymbol{\nabla}\psi\\
&-\frac{3\nu}{10\Omega^2}\left[\frac{5}{2}gI+s\left(\frac{3}{4}R^2+\frac{7}{4}\frac{I^2}{B^2}\right)\right]\boldsymbol{\nabla}\ln p_i\cdot\boldsymbol{\nabla}\psi\\
&+\frac{3\nu}{10\Omega^2}\left[\frac{24}{5}gI+s\left(\frac{3}{2}R^2+\frac{33}{10}\frac{I^2}{B^2}\right)\right]\boldsymbol{\nabla}\ln T_i\cdot\boldsymbol{\nabla}\psi.
\end{aligned}\tag{57}$$

We have not had to employ relation (10) so these terms may be viewed as classical.

It is useful to note that the ion temperature and pressure gradient terms and magnetic gradient terms contained in expression (57) are classical and therefore at least q^2 times smaller than the neoclassical terms from Eq. (48). Consequently, it is normally (i.e. when $q^2\gg1$) safe to neglect these terms in Eq. (57) and keep only the electrostatic potential gradient and ion pressure gradient terms that try to drive the system towards a radial Maxwell-Boltzmann relation:

$$R^2\boldsymbol{\nabla}\zeta\cdot\overleftrightarrow{\pi}_\perp\cdot\boldsymbol{\nabla}\psi\rightarrow-\frac{3\nu}{10\Omega^2}\left(R^2+\frac{3I^2}{B^2}\right)p_i\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega. \tag{58}$$

VII. RESULTS

The prime purpose of the work outlined herein is to complete the description of Pfirsch-Schlüter transport in general tokamak geometry by properly evaluating the electric field and therefore the toroidal ion flow. The key results for our evaluation of the radial electric field in a tokamak can be summarized by giving two limiting cases. The first is the simpler strongly up-down asymmetric case that corresponds to the situation expected just inside the separatrix of a single-null-divertor configuration. The radial electric field in this situation is found by setting the sum of Eqs. (49) and (58) equal to zero. With $K(\psi)$ from Eq. (19) inserted this result may be written as follows:

$$\begin{gathered}\left\langle \left(R^2 - \frac{I^2}{B^2}\right)\left(R^2 + \frac{3I^2}{B^2}\right)\right\rangle_\theta \frac{\mathrm{d}\omega}{\mathrm{d}\psi}\bigg|_a = \\ -\frac{5I}{3M\nu\left\langle B^2\right\rangle_\theta}\frac{\partial T_i}{\partial\psi}\left(0.78 + 0.057\frac{\left\langle B^2\right\rangle_\theta\left\langle(\nabla_\parallel \ln B)^2\right\rangle_\theta}{\left\langle(\nabla_\parallel B)^2\right\rangle_\theta}\right)\left\langle R^2\nabla_\parallel B\right\rangle_\theta \approx \\ -\frac{1.3I}{M\nu\left\langle B^2\right\rangle_\theta}\frac{\partial T_i}{\partial\psi}\left\langle R^2\nabla_\parallel B\right\rangle_\theta ,\end{gathered} \tag{59}$$

where the last form simply ignores the small term with the 0.057 coefficient.

The up-down symmetric case is more awkward since the radial electric field should be found by setting the sum of Eq. (48) with $\left\langle R^2\nabla_\parallel B\right\rangle_\theta = 0$ and Eq. (58) to zero. The resulting expression is given in terms of the ion heat flow by the following equation:

$$\begin{gathered}\frac{3\nu p_i B}{2\Omega}\left\langle \left(R^2 - \frac{I^2}{B^2}\right)\left(R^2 + \frac{3I^2}{B^2}\right)\right\rangle_\theta \frac{\mathrm{d}\omega}{\mathrm{d}\psi}\bigg|_s = \\ F(\psi)\left\langle R^2\nabla_\parallel B\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right) - \left(R^2 - \frac{3I^2}{2B^2}\right)\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right)\right\rangle_\theta + \\ \frac{\mathrm{d}}{\mathrm{d}\psi}\left\langle\left(R^2+\frac{3I^2}{2B^2}\right)(\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi)\right\rangle_\theta - \left\langle R^2+\frac{7I^2}{2B^2}\right\rangle_\theta\frac{\mathrm{d}\left\langle\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi\right\rangle_\theta}{\mathrm{d}\psi} - \\ \frac{\mathrm{d}}{\mathrm{d}\psi}\left\langle R^2\nabla_\parallel B\int\frac{\mathrm{d}\theta}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta}\left[(\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi) - \langle\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi\rangle_\theta\right]\right\rangle_\theta ,\end{gathered} \tag{60}$$

where

$$\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi = -\frac{8I^2\nu p_i}{5M\Omega^2}\frac{\partial T_i}{\partial\psi}\left(1-\frac{B^2}{\langle B^2\rangle_\theta}\right).$$

and neglecting the term with 0.057 coefficient in $K(\psi)$

$$F(\psi)\approx 0.71\frac{\langle T_e\rangle_\theta}{\langle T_e\rangle_\theta+\langle T_i\rangle_\theta}\frac{\mathrm{d}\ln\langle T_i\rangle_\theta}{\mathrm{d}\psi}\frac{\langle\boldsymbol{q}\cdot\boldsymbol{\nabla}\psi\rangle_\theta}{\langle B^2\rangle_\theta\langle B^{-2}\rangle_\theta-1}.$$

This expression was obtained assuming that radial variation of ion temperature is much faster than that of magnetic field and geometric quantities, as expected, for example, in the pedestal region just inside the separatrix. The indefinite integrals in the preceding expression are written in a secular-free form so that only the poloidal variation of the integrand matters. For example, for a large aspect ratio concentric circular flux surface model with $B = B_0R_0/R$ only the $F(\psi)$ term contributes and we find after some algebra

$$\left.\frac{r}{\Omega_0}\frac{\mathrm{d}\omega}{\mathrm{d}r}\right|_s \approx -0.19\,q^3\rho_0^2\frac{T_e}{T_e+T_i}\left(\frac{\mathrm{d}\ln T_i}{\mathrm{d}r}\right)^2, \tag{61}$$

where $\Omega_0 = eB_0/Mc$ and $\rho_0 = v_i/\Omega_0$. Notice that $\mathrm{d}\omega/\mathrm{d}r < 0$ so ω decreases away from the magnetic axis.

The up-down symmetric expression written in terms of the radial ion heat flow can be seen to be formally ν/Ω times smaller than the asymmetric result if $\left\langle R^2\nabla_\parallel B\right\rangle_\theta \sim RB$. However, more generally the right-hand sides of both of these expressions must be added together to determine the radial electric field. Moreover, if the radial variation of the magnetic field is comparable to that of the ion temperature then the remaining terms in Eq. (48) must be retained. In addition, if the safety factor q is of order unity then the full collisional perpendicular viscosity (57) must be used.

Expressions (59) and (60) relate the shear in the rotation frequency or the departure from rigid rotation to the radial ion temperature gradient. Or equivalently, recalling Eq. (21), they relate the shear in the electric field to the radial ion temperature gradient. In the strongly asymmetric case the strength of this shear can be controlled by the geometric details of how the single-null geometry affects $\left\langle R^2\nabla_\parallel B\right\rangle_\theta$. In the symmetric case the situation is more complex. Details of the magnetic geometry still enter, but apparently in a much less sensitive way.

Of course, knowing $\mathrm{d}\omega/\mathrm{d}\psi$ an integration can be performed to determine ω provided it is known on some flux surface inside the separatrix within the radial domain of validity of a collisional treatment. For example, for a collisional pedestal the ω at the separatrix might be most convenient.

It is worth remarking that none of our results in the presence of ion temperature variation are equivalent to assuming that the toroidal flow vanishes. Strictly speaking, the assumption of a vanishing toroidal flow is not consistent with the neoclassical form (21) of the ion flow for a tokamak with radial temperature variation since it requires $\omega R^2 + uI = 0$ and the poloidal angle or R dependence of these two terms differ. As a result, after a few ion-ion collision times, an assumption that the toroidal flow vanishes is only possible if there is no ion temperature variation and the plasma is known to be non-rotating at some flux surface.

Before closing, we discuss estimates of the momentum relaxation time τ_m based on the conservation of total toroidal angular momentum equation,

$$\left\langle \frac{\partial}{\partial t}\left[Mn(\omega R^2 + uI)\right]\right\rangle_\theta + \frac{1}{V'}\frac{\mathrm{d}}{\mathrm{d}\psi}V'\left\langle R^2\boldsymbol{\nabla}\zeta\cdot \overset{\leftrightarrow}{\pi}\cdot\boldsymbol{\nabla}\psi\right\rangle_\theta = \left\langle \frac{1}{c}\boldsymbol{J}\cdot\boldsymbol{\nabla}\psi\right\rangle_\theta, \qquad (62)$$

with $\boldsymbol{J}$ the total plasma current. The term on the right-hand side is normally neglected since it can be shown to be small by the square of the Alfvén speed over the speed of light. To evaluate τ_m we assume that ω and $\partial T_i/\partial\psi$ both vanish at $t = 0^-$ (i.e. plasma is in radial Maxwell-Boltzmann equilibrium and stationary) and then at

$t = 0^+$ we impose a $\partial T_i/\partial\psi$ not equal to zero. Then,

$$\tau_{\rm m} \sim \left. \frac{MnuI}{\mathrm{d}\langle R^2\boldsymbol{\nabla}\zeta\cdot \overleftrightarrow{\pi}_g \cdot\boldsymbol{\nabla}\psi\rangle_\theta/\mathrm{d}\psi} \right|_{t=0^+}, \tag{63}$$

where the subscript "eq" is used to indicate that the equilibrium value of the quantity is employed. Not surprisingly, for an up-down symmetric tokamak we find from Eqs. (19), (21) and (48) that $(uI)|_{t=0^+} \sim (R_0^2B/M\Omega)\partial T_i/\partial\psi$, $\langle R^2\boldsymbol{\nabla}\zeta\cdot \overleftrightarrow{\pi}_g \cdot\boldsymbol{\nabla}\psi\rangle_\theta|_{t=0^+} \sim (R_0^3B^3\nu p_i/M\omega^3B_pw)\partial T_i/\partial\psi$, and the usual Pfirsch-Schlüter result is recovered [recall Eq. (13)]

$$\tau_{\rm ms} \sim \frac{w^2}{q^2\nu\rho^2} \tag{64}$$

giving momentum relaxation to be q^2 faster than classical. However, for the up-down asymmetric case Eq. (49) predicts $\langle R^2\boldsymbol{\nabla}\zeta\cdot \overleftrightarrow{\pi}_g \cdot\boldsymbol{\nabla}\psi\rangle_\theta \sim (BR_0p_i/M\Omega^2)\partial T_i/\partial\psi \left\langle R^2\nabla_\parallel B\right\rangle_\theta$, which gives the different result

$$\tau_{\rm ma} \sim \frac{w^2}{q^2\nu\rho^2}\left[q\left(\frac{\delta}{\Delta}\right)\frac{R_0B_p}{\left\langle R^2\nabla_\parallel B\right\rangle_\theta}\right]. \tag{65}$$

Notice that

$$\frac{\tau_{\rm ma}}{\tau_{\rm ms}} \sim q\left(\frac{\delta}{\Delta}\right)\frac{R_0B_p}{\left\langle R^2\nabla_\parallel B\right\rangle_\theta} \tag{66}$$

and recall that the Pfirsch-Schlüter expansion procedure requires $(\delta/\Delta) \ll 1$. As a result, if the up-down asymmetry just inside the separatrix is strong, then it is possible for momentum to relax faster than according to the usual Pfirsch-Schlüter estimate.

To see that the asymmetry can be quite severe, we may use the definition of the flux surface average to integrate by parts and obtain the alternate form for $\left\langle R^2\nabla_\parallel B\right\rangle_\theta$:

$$\left\langle R^2\nabla_\parallel B\right\rangle_\theta = -\frac{1}{2}\left\langle(\boldsymbol{B}\cdot\boldsymbol{\nabla}R^2)\ln B^2\right\rangle_\theta = -\frac{1}{2V'}\oint \mathrm{d}R^2\ln\left(\frac{I^2}{R^2}+B_p^2\right). \tag{67}$$

Noticing that this result will vanish if $B_p = 0$ or if B_p is up-down symmetric, we obtain the rough estimate $\oint \mathrm{d}R^2\ln(I^2/R^2+B_p^2) \sim rR_0[(B_p^{\rm up})^2 - (B_p^{\rm down})^2]/B^2$, where

$B_p^{\rm up}$ and $B_p^{\rm down}$ are typical values of B_p evaluated at the same R at the top and bottom of the tokamak. Using $V' \sim r/B_p$ and assuming $(B_p^{\rm up})^2 - (B_p^{\rm down})^2 \sim B_p^2$ we then find $\left\langle R^2 \nabla_\parallel B \right\rangle_\theta \sim (\epsilon/q)^2 R_0 B_p$. Since we are near the separatrix we can take $\epsilon \sim 1$ and obtain

$$\frac{\tau_{\rm ma}}{\tau_{\rm ms}} \sim q^3 \left(\frac{\delta}{\Delta}\right). \tag{68}$$

Only when $q^3(\delta/\Delta) \ll 1$ in the pedestal region of a single-null configuration it is possible to get momentum relaxation at faster than the Pfirsch-Schlüter rate for a collisional edge. Normally, this restriction is hard to satisfy. Therefore, even though up-down asymmetry can increase the momentum relaxation rate, it is unlikely to be competitive with anomalous relaxation.[17] However, the neoclassical effects evaluated herein might be expected to contribute to the ion flow and flow shear in a tokamak and thereby influence the turbulence level.

To make our estimate of the momentum relaxation time for the up-down symmetric case more precise we give the lowest order momentum conservation equation for the large aspect ratio concentric circular flux surface model:

$$\left\langle \frac{\partial}{\partial t}\left[n\left(\omega + \frac{uI}{R_0^2}\right)\right]\right\rangle_\theta = \frac{\rm d}{{\rm d}r}\left\{ \frac{6\nu p_i}{5M\Omega_0^2}\left[\frac{{\rm d}\omega}{{\rm d}r} + 0.19\frac{q^3\rho_0^2\Omega_0}{r}\frac{T_e}{T_e+T_i}\left(\frac{{\rm d}\ln T_i}{{\rm d}r}\right)^2\right]\right\}. \tag{69}$$

Again, assuming that ω is negligible at $t = 0$ and inserting $u = K/n$ from Eq. (21) we find the up-down symmetric momentum relaxation time to be

$$\tau_{\rm ms} \approx 7.8\left(\frac{T_e+T_i}{T_e}\right)\frac{w^2}{q^2\nu\rho_0^2}. \tag{70}$$

The large coefficient indicates that up-down symmetric momentum relaxation is slower than our rough estimate predicts so asymmetric momentum relaxation may play a role more easily than estimate (68) implies. For a collisional edge plasma with $n = 10^{14}$ cm^{-3}, $T = 100$ eV, $R = 100$ cm, $B = 5$ T, and $q = 3$, and an edge scale length of $w = 1$ cm, $\tau_{\rm ms} \sim 50$ msec. We also remark that if we estimate the relaxation time in the absence of temperature gradients then we find the classical momentum relaxation time, $\tau_{\rm mc} = (5w^2/3\nu\rho_0^2)$, which is often comparable to $\tau_{\rm ms}$.

Finally, we remark that even though toroidal momentum relaxation is anomalous, once this transient phase is complete the resulting steady state may be near or similar to the one found here. Indeed, the steady state solutions we find make definite predictions about the electric field and toroidal flow inside the separatrix and these can be compared to experiments with collisional edges to gain insight into whether the steady state has neoclassical as well as anomalous features.

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Appendix A. $\tilde{f}_2$ EVALUATION FOR DRIFT KINETICS

In this appendix we show that Hazeltine's $\tilde{f}_2$ as given by Eq. (4) is incomplete to second order in the gyro-radius expansion. We begin by writing the exact kinetic equation in terms of the velocity variables ε, μ and φ and then subtracting off its gyro-phase average to obtain the following general equation for $\tilde{f}$:

$$\tilde{f} = \frac{1}{\Omega}\int \mathrm{d}\varphi \left\{ \boldsymbol{v}_\perp \cdot \boldsymbol{\nabla}|_{\varepsilon,\mu,\varphi}\bar{f} + (\dot{\varepsilon} - \langle\dot{\varepsilon}\rangle_\varphi)\frac{\partial \bar{f}}{\partial \varepsilon} + (\dot{\mu} - \langle\dot{\mu}\rangle_\varphi)\frac{\partial \bar{f}}{\partial \mu} \right\}$$
$$+\frac{1}{\Omega}\int \mathrm{d}\varphi \left\{ \frac{\partial \tilde{f}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}|_{\varepsilon,\mu,\varphi}\tilde{f} - \left\langle \boldsymbol{v}\cdot\boldsymbol{\nabla}|_{\varepsilon,\mu,\varphi}\tilde{f}\right\rangle_\varphi + \dot{\varepsilon}\frac{\partial \tilde{f}}{\partial \epsilon} - \left\langle \dot{\varepsilon}\frac{\partial \tilde{f}}{\partial \epsilon}\right\rangle_\varphi \right. \tag{A1}$$
$$\left. +\dot{\mu}\frac{\partial \tilde{f}}{\partial \mu} - \left\langle \dot{\mu}\frac{\partial \tilde{f}}{\partial \mu}\right\rangle_\varphi + (\dot{\varphi}+\Omega)\frac{\partial \tilde{f}}{\partial \varphi} - \left\langle (\dot{\varphi}+\Omega)\frac{\partial \tilde{f}}{\partial \varphi}\right\rangle_\varphi + \langle C(f)\rangle_\varphi - C(f) \right\},$$

where $\dot{Q} \equiv \partial Q/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla}Q + (e/M)(\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_v Q$ for an arbitrary quantity Q and C is the ion collision operator. Hazeltine obtains Eq. (4) from Eq. (A1) by *dropping* the second integral term on the right-hand side. Accordingly, we denote this portion of $\tilde{f}$ as $\tilde{f}^{\mathrm{H}}$. The remaining portion of $\tilde{f}$ we denote by $\tilde{f}^{\mathrm{NH}}$, where it is clear that $\tilde{f}^{\mathrm{NH}}$ should be obtained iteratively via the gyroradius expansion.

In order to explicitly evaluate $\tilde{f}_2^{\mathrm{H}}$ we have to substitute $\bar{f}_1$ into Eq. (4), where for a collisional plasma[5,13]

$$\bar{f}_1 = \frac{M}{T_i}\left\{ V_\parallel - \frac{2q_\parallel}{5p_i}\left[L_1^{(3/2)}\left(x^2\right) - \frac{4}{15}L_2^{(3/2)}\left(x^2\right)\right]\right\} v_\parallel f_M(v), \tag{A2}$$

where $f_M(v) \equiv n(M/2\pi T_i)^{3/2}\exp(-Mv^2/2T_i)$ is a Maxwellian and $L_i^{(j+1/2)}\left(x^2\right)$, $i, j = 0, 1, 2, \ldots$, are generalized Laguerre polynomials, with $x^2 \equiv Mv^2/2T_i$. Carrying out the evaluation we find

$$\tilde{f}_2^{\mathrm{H}} = \frac{1}{\Omega}\left[\left(\boldsymbol{v}_\parallel + \frac{1}{4}\boldsymbol{v}_\perp\right)\boldsymbol{v}\times\hat{\boldsymbol{b}} + \boldsymbol{v}\times\hat{\boldsymbol{b}}\left(\boldsymbol{v}_\parallel + \frac{1}{4}\boldsymbol{v}_\perp\right)\right] : \tag{A3}$$
$$\left\{ \boldsymbol{\nabla}\left(\frac{\bar{f}_1}{v_\parallel}\hat{\boldsymbol{b}}\right) - \frac{eM\boldsymbol{E}}{T_i^2}\left[\boldsymbol{V}_\parallel - \frac{2\boldsymbol{q}_\parallel}{5p_i}\left(L_1^{(5/2)}(x^2) - \frac{4}{15}L_2^{(5/2)}(x^2)\right)\right] f_M(v)\right\}.$$

We have neglected the $\partial\hat{\boldsymbol{b}}/\partial t$ term in $\boldsymbol{v}_M$ because we treat time variation as being on the diamagnetic drift frequency time scale here and elsewhere in our evaluation rather than on the scale of a transit time.

Next, we evaluate $\tilde{f}_2^{\mathrm{NH}}$. Inserting $\bar{f} = f_M(v)$ into Eq. (4) we obtain

$$\tilde{f}_1 = \frac{M}{T_i}\left[\boldsymbol{V}_\perp - \frac{2\boldsymbol{q}_\perp}{5p_i}L_1^{(3/2)}(x^2)\right]\cdot \boldsymbol{v} f_M(v). \tag{A4}$$

By inserting $\tilde{f}_1$ into Eq. (A1) we can show that

$$\tilde{f}_2^{\mathrm{NH}} = \frac{1}{\Omega}\left[\left(\boldsymbol{v}_\parallel + \frac{1}{4}\boldsymbol{v}_\perp\right)\boldsymbol{v}\times\hat{\boldsymbol{b}} + \boldsymbol{v}\times\hat{\boldsymbol{b}}\left(\boldsymbol{v}_\parallel + \frac{1}{4}\boldsymbol{v}_\perp\right)\right] : \tag{A5}$$
$$\left\{\boldsymbol{\nabla}\left[\frac{M}{T_i}\left(\boldsymbol{V}_\perp - \frac{2\boldsymbol{q}_\perp}{5p_i}L_1^{(3/2)}(x^2)\right) f_M(v)\right] - \frac{eM\boldsymbol{E}}{T_i^2}\left[\boldsymbol{V}_\perp - \frac{2\boldsymbol{q}_\perp}{5p_i}L_1^{(5/2)}(x^2)\right] f_M(v)\right\}.$$

Combining the results (A3) and (A5) we obtain $\tilde{f}_2 = \tilde{f}_2^{\mathrm{H}} + \tilde{f}_2^{\mathrm{NH}}$. To demonstrate we recover the same expression for the second order gyro-phase dependent piece of the ion distribution function as obtained in Ref. [3] [see Eq. (23)] we rewrite our result for $\tilde{f}$ in terms of $\boldsymbol{w} \equiv \boldsymbol{v} - \boldsymbol{V}$ variables. We use the leading order ion momentum equation $\boldsymbol{\nabla} p_i - en\boldsymbol{E} \approx \Omega M n \boldsymbol{V}\times\hat{\boldsymbol{b}}$ to simplify, and complete the identification by taking into account that when rewritten in terms of $\boldsymbol{w}$ variables $f_M(v)$ and $f_1 = \bar{f}_1 + \tilde{f}_1$ can be seen to contain the remaining second order gyro-phase dependent terms.

The preceding demonstrates explicitly that Eq. (4) for $\tilde{f}$ does not recover the correct answer for collisional plasma. The same defect is present for collisionless plasmas and has important implications for momentum transport and flow shear. In particular, an incorrect answer is obtained if, instead of using a moment approach, Eq. (4) is used to directly evaluate gyro-viscous stress tensor. More importantly, since Eq. (4) for $\tilde{f}$ was used to obtain a drift-kinetic equation for $\bar{f}$ in Ref. [7], this drift-kinetic equation is only correct to first order in the gyroradius expansion. It is missing important information in second order, which means that both the pressure anisotropy or parallel viscous stress tensor (the evaluation of which requires $\bar{f}_2$) and the perpendicular viscous stress tensor (which can be evaluated using f_2 by a moment approach) are obtained incorrectly using the Hazeltine drift-kinetic equation.[7]

It is instructive to see which portion of the ion gyro-viscous stress tensor and consequently which part of the leading order portion of the gyro-viscous stress tensor contribution to the constraint (31) is recovered when expression (4) for $\tilde{f}$ is employed. Using Eq. (A3) for $\tilde{f}_2^{\rm H}$ we obtain $\overleftrightarrow{\pi}_g^{\rm H}$ to be of the form of Eq. (32) with $\overleftrightarrow{\alpha}$ replaced by $\overleftrightarrow{\alpha}^{\rm H} \equiv \boldsymbol{\nabla V}_\parallel + (2/5p_i)\boldsymbol{\nabla q}_\parallel$. Similarly, using Eq. (A5) for $\tilde{f}_2^{\rm NH}$ we find $\overleftrightarrow{\pi}_g^{\rm NH}$ to be given by Eq. (32) with $\overleftrightarrow{\alpha}$ replaced by $\overleftrightarrow{\alpha}^{\rm NH} \equiv \boldsymbol{\nabla V}_\perp + (2/5p_i)\boldsymbol{\nabla q}_\perp$. Clearly, $\overleftrightarrow{\pi}_g = \overleftrightarrow{\pi}_g^{\rm H} + \overleftrightarrow{\pi}_g^{\rm NH}$.

Using the leading order expressions (21) and (30) for $\boldsymbol{V}$ and $\boldsymbol{q}$, respectively, we can show that the leading order contribution to constraint (31) from $\overleftrightarrow{\pi}_g^{\rm H}$ is

$$\left\langle R^2 \boldsymbol{\nabla}\zeta \cdot \overleftrightarrow{\pi}_g^{\rm H} \cdot \boldsymbol{\nabla}\psi \right\rangle_\theta \to \frac{B}{2\Omega} \left\langle R^2 \nabla_\parallel B \right\rangle_\theta (p_i u + g) - \frac{3B}{4\Omega} \left\langle R^2 B \nabla_\parallel \left(\frac{I}{B^2} \right) \right\rangle_\theta (p_i \omega + s). \tag{A6}$$

In addition,

$$\left\langle R^2 \boldsymbol{\nabla}\zeta \cdot \overleftrightarrow{\pi}_g^{\rm NH} \cdot \boldsymbol{\nabla}\psi \right\rangle_\theta \to \frac{3B}{4\Omega} \left\langle R^2 B \nabla_\parallel \left(\frac{I}{B^2} \right) \right\rangle_\theta (p_i \omega + s). \tag{A7}$$

The sum of results (A6) and (A7) is consistent with Eq. (48) since in the leading order $p_i u + g = (2/5)L$.

As a further check we also note that we can recover (A6) by using expression (A2) for $\bar{f}_1$ in constraint (7) to find that the leading order portion of the right-hand side is equal to

$$\left(\frac{B}{2\Omega} \right) \left\langle \frac{1}{B} \nabla_\parallel \left(\frac{R^2 B^2 - I^2}{B} \right) \left(p_i V_\parallel + \frac{2}{5} q_\parallel \right) \right\rangle_\theta \to$$
$$\left(\frac{B}{2\Omega} \right) \left\langle R^2 \nabla_\parallel B \right\rangle_\theta (p_i u + g) - \frac{3B}{4\Omega} \left\langle R^2 B \nabla_\parallel \left(\frac{I}{B^2} \right) \right\rangle_\theta (p_i \omega + s),$$

while constraint (5) gives zero in the leading order.

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