PSFC/JA-05-9

General Expression of the Gyroviscous Force

J.J. Ramos

June, 2005.

Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge MA 02139, U.S.A.

This work was supported by the U.S. Department of Energy under Grant No. DE-FG05-91ER-54109.

General Expression of the Gyroviscous Force

J.J. Ramos

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge MA, U.S.A.

Abstract

Assuming only small gyromotion periods and Larmor radii compared to any other time and length scales, and retaining the lowest significant order in $\delta = \rho_i/L \ll 1$, the general expression of the ion gyroviscous stress tensor is presented. This expression covers both the "fast dynamics" (or "magnetohydrodynamic") ordering, where the time derivative and ion gyroviscous stress are first order in δ relative to the ion gyrofrequency and scalar pressure respectively, and the "slow dynamics" (or "drift") ordering, where the time derivative and ion gyroviscous stress are respectively second order in δ. This general stress tensor applies to arbitrary collisionality and does not require the distribution function to be close to a Maxwellian. Its exact divergence (gyroviscous force) is written in closed vector form, allowing for arbitrary magnetic geometry, parallel gradients and flow velocities. Considering in particular the contribution from the velocity gradient (rate of strain) term, the final form of the momentum conservation equation after the "gyroviscous cancellation" and the "effective renormalization of the perpendicular pressure by the parallel vorticity" is precisely established.

I. Introduction.

The inclusion of finite ion Larmor radius (FLR) effects in the fluid moment equations, is a fundamental part of the so-called "extended magnetohydrodynamic" (extended-MHD) or "multi-fluid" description of magnetized plasmas. Extended-MHD theories are currently the subject of very active research, since they are recognized to be necessary to explain many important phenomena such as the sawtooth, neoclassical-tearing and edge-localized modes in tokamaks, the stability of field-reversed-configurations (FRC), or the magnetic reconnection processes in general. The main FLR effect in the ion momentum conservation equation is the gyroviscous force. This term, in its most elementary form which takes into account only a simplified contribution from the velocity gradient (rate of strain) tensor, has long been known to allow the diamagnetic stabilization of single-fluid modes[1−4]. However, realistic theoretical analyses and numerical simulations that could live up to the expectation of a reliable predictive capability, need a more accurate treatment of the gyroviscosity. There are two aspects to this. First, the appropriate form of the stress tensor should be used according to the plasma regime under consideration, bearing in mind that the simplest and most popular form that involves just the velocity gradient tensor[2−6] applies only to high collisionality and fast (MHD-like) time evolution with sonic flows. Second, an accurate evaluation of the divergence of the stress tensor should be carried out, allowing for realistic magnetic geometry, finite parallel gradients, and compressible flow velocities with comparable parallel and perpendicular components.

Proper expressions of the gyroviscous stress tensor Π^{gyr}, applicable to different collisionality regimes, are available in the literature[6−10]. These have been derived for either the "fast dynamics" ordering characterized by $u \sim v_{thi}$, $\partial/\partial t \sim \delta\Omega_{ci}$ and $\Pi^{gyr} \sim \delta p$, or the "slow dynamics" ordering characterized by $u \sim \delta v_{thi}$, $\partial/\partial t \sim \delta^2\Omega_{ci}$ and $\Pi^{gyr} \sim \delta^2 p$ (here $\delta = \rho_i/L \ll 1$ is the ratio between the ion gyroradius and other length scales, Ω_{ci} and v_{thi} are the ion gyrofrequency and thermal speed, u is the macroscopic flow velocity and p is the scalar pressure). Braginskii's[6] form applies to high collisionality and fast dynamics, Mikhailowskii-Tsypin's[7] applies to high collisionality and slow dynamics, and Macmahon's[8] applies to collisionless or arbitrary collisionality regimes and fast dynamics. The Simakov-Catto[9] result was derived for slow dynamics without explicit assumptions on the collisionality, but requiring that the distribution function would still be a Maxwellian in lowest order. The results of Ref. 10 are completely general (within the lowest significant order in the small-δ asymptotic expansions), do not require the distribution function to be close to a Maxwellian and contain all the above as special limits. As far as the implementation of these results is concerned, only Braginskii's expression has so far been included or is in the process of being included in the state of the art numerical simulation codes[11–14].

With regard the evaluation of the divergence of the gyroviscous stress tensor (the gyroviscous force), only approximate results have been reported[2–5,15–22] and implemented numerically[11–13], even when consideration was limited to the simplest Braginskii form. Routinely made approximations include constant magnetic field, neglect of parallel derivatives, incompressible or mostly perpendicular flow, weak anisotropy, low beta or electrostatic limits. The purpose of this work is to provide the exact expression of the gyroviscous force, in coordinate-free vector form, without invoking any of those subsidiary assumptions and based on the general stress tensor derived in Ref. 10. An explicit gyroviscous force is not necessary in a numerical scheme that uses the weak form of the discretized equations such as the one adopted by the NIMROD code[14]. In this case, only the scalar products with a set of basis functions are used and, following partial integration, only the stress tensor (not its divergence) is needed explicitly. However, besides its theoretical interest, the availability of an expression of the force will always be useful to enforce possible cancellations and to provide the possibility of other numerical schemes.

II. The general gyroviscous stress.

The gyroviscous stress is defined as the traceless and perpendicular (i.e. $\Pi^{gyr}_{ii} = \Pi^{gyr}_{ij} b_i b_j = 0$) part of the stress tensor in the fluid rest frame that does not depend explicitly on the collision frequencies. The fluid rest frame stress tensor can be uniquely split into its Chew-Goldberger-Low (CGL) part and its traceless perpendicular part:

$$m \int d^3\mathbf{v}\ (v_i - u_i)(v_j - u_j)\ f(\mathbf{v}, \mathbf{x}, t)\ =\ p_\perp \delta_{ij}\ +\ (p_\parallel - p_\perp) b_i b_j\ +\ \hat{P}_{ij}\ , \tag{1}$$

where $\hat{P}_{ii} = \hat{P}_{ij} b_i b_j = 0$. The tensor $\hat{P}_{ij}$ can in turn be uniquely split into parts that do and do not depend explicitly on the collision frequencies, and this specifies the gyroviscous stress:

$$\hat{P}_{ij} \;=\; \Pi^{gyr}_{ij} \;+\; \Pi^{coll}_{\perp ij} \;. \tag{2}$$

In Eq.(1), $f(\mathbf{v}, \mathbf{x}, t)$ is the distribution function, $\mathbf{u}(\mathbf{x}, t)$ is the macroscopic flow velocity,

$$\int d^3\mathbf{v}\; v_i\; f(\mathbf{v}, \mathbf{x}, t) \;=\; n\; u_i \;, \tag{3}$$

with $n(\mathbf{x}, t)$ the particle density,

$$\int d^3\mathbf{v}\; f(\mathbf{v}, \mathbf{x}, t) \;=\; n \;, \tag{4}$$

$p_\parallel(\mathbf{x}, t)$ and $p_\perp(\mathbf{x}, t)$ are the parallel and perpendicular pressures, and $\mathbf{b}(\mathbf{x}, t) = \mathbf{B}/B$ is the magnetic unit vector. It is also useful to introduce the mean scalar pressure $p = (p_\parallel + 2p_\perp)/3$. All the analysis in this paper refers to the ion variables, so the ion species index is dropped throughout. A completely similar analysis could be carried out for the electrons, but electron gyroviscosity and other electron Larmor radius effects are usually neglected due to the small electron mass.

Analogously, the third rank stress-flux tensor can be written as:

$$m \int d^3\mathbf{v}\; (v_i - u_i)(v_j - u_j)(v_k - u_k) f \;=\; q_{T\parallel} \delta_{[ij} b_{k]} \;+\; (2q_{B\parallel} - 3q_{T\parallel}) b_i b_j b_k \;+\; \Theta^{gyr}_{ijk} \;+\; \Theta^{coll}_{\perp ijk} \;, \tag{5}$$

where the CGL variables $q_{T\parallel}$ and $q_{B\parallel}$ are the parallel fluxes of perpendicular heat and parallel heat respectively, and $\Theta^{gyr}_{iij} b_j \;=\; \Theta^{gyr}_{ijk} b_i b_j b_k \;=\; \Theta^{coll}_{\perp iij} b_j \;=\; \Theta^{coll}_{\perp ijk} b_i b_j b_k \;=\; 0$. In our notation, the square brackets around indices represent the minimal sum over permutations of uncontracted indices needed to yield completely symmetric tensors.

Considering the $v_i v_j$ moment of the kinetic equation for $f(\mathbf{v}, \mathbf{x}, t)$, it follows that Π^{gyr}_{ij} can always be expressed[10,23,24] as

$$\Pi^{gyr}_{ij} \;=\; \frac{1}{4} \epsilon_{[ikl} b_k K^{gyr}_{lm} \Big(\delta_{mj]} + 3 b_m b_{j]} \Big) \;, \tag{6}$$

and the general form of the tensor K^{gyr}_{ij} is given in Ref. 10. Within the lowest significant order in the fundamental expansion parameter δ, but keeping enough terms to cover both the fast dynamics and slow dynamics orderings with a single formula, it is:

$$K_{ij}^{gyr} \;=\; \frac{m}{eB}\left[\,p_\perp \frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial (q_{T\parallel} b_{j]})}{\partial x_{[i}} \;+\; b_{[i}\,c_{j]} \;+\; \frac{\partial \Theta_{ijk}^{gyr}}{\partial x_k}\,\right], \tag{7}$$

where

$$\mathbf{c} \;=\; (2q_{B\parallel} - 3q_{T\parallel})\kappa \;+\; \left(\frac{p_\parallel - p_\perp}{B}\right)\left\{2(\mathbf{B}\cdot\nabla)\mathbf{u} - \nabla\times\left[\frac{1}{en}\nabla p_\perp + \frac{1}{en}(\mathbf{B}\cdot\nabla)\left(\frac{p_\parallel - p_\perp}{B}\mathbf{b}\right)\right]\right\} \tag{8}$$

and $\kappa = (\mathbf{b}\cdot\nabla)\mathbf{b}$ is the magnetic curvature.

The collision-independent perpendicular stress-flux tensor Θ_{ijk}^{gyr} is a quantity of order $\delta p v_{th}$, which is needed only in the slow dynamics ordering where $u = O(\delta v_{th})$ and $K_{ij}^{gyr} = O(\delta^2 p)$. For this case, and within the required accuracy of $O(\delta p v_{th})$, the result of Ref. 10 can be written as:

$$\Theta_{ijk}^{gyr} \;=\; 2b_{[i}b_j q_{B\perp k]}^{gyr} \;+\; \frac{1}{2}\Big(\delta_{[ij} - b_{[i}b_j\Big) q_{T\perp k]}^{gyr} \;+\; \frac{\alpha}{2}\epsilon_{[ilm}b_j b_l\left(\frac{\partial b_n}{\partial x_m} + \frac{\partial b_m}{\partial x_n}\right)\Big(\delta_{nk]} - b_n b_{k]}\Big) \tag{9}$$

with

$$\mathbf{q}_{B\perp}^{gyr} \;=\; \frac{1}{eB}\mathbf{b}\times\left[\frac{1}{2}p_\perp\nabla\left(\frac{p_\parallel}{n}\right) + \frac{p_\parallel(p_\parallel - p_\perp)}{n}\kappa + \frac{1}{5}\nabla(\tilde{r}_\perp^{(0)} + \tilde{r}_\Delta^{(0)}) + (\tilde{r}_\parallel^{(0)} - \tilde{r}_\perp^{(0)} - \tilde{r}_\Delta^{(0)})\kappa\right], \tag{10}$$

$$\mathbf{q}_{T\perp}^{gyr} \;=\; \frac{1}{eB}\mathbf{b}\times\left[2p_\perp\nabla\left(\frac{p_\perp}{n}\right) + \frac{1}{5}\nabla(4\tilde{r}_\perp^{(0)} - \tilde{r}_\Delta^{(0)}) + \tilde{r}_\Delta^{(0)}\kappa\right], \tag{11}$$

and

$$\alpha \;=\; \frac{1}{eB}\left[\frac{p_\perp(p_\parallel - p_\perp)}{2n} + \tilde{r}_\Delta^{(0)}\right]. \tag{12}$$

Here, $\mathbf{q}_{B\perp}^{gyr}$ and $\mathbf{q}_{T\perp}^{gyr}$ are the collision-independent parts of the perpendicular fluxes of parallel heat and perpendicular heat respectively. The scalars $\tilde{r}_\parallel^{(0)}$, $\tilde{r}_\perp^{(0)}$ and $\tilde{r}_\Delta^{(0)}$, whose precise definition is given in Appendix A, are three independent components of the fourth rank fluid moment, evaluated on the difference between the actual zeroth-order distribution function and a two-temperature Maxwellian.

The divergence of the collision-independent perpendicular stress-flux tensor $\partial\Theta_{ijk}^{gyr}/\partial x_k$ was not evaluated explicitly in Ref. 10 in the most general, strongly anisotropic case. The details of this calculation are now given in Appendix B. We note that terms proportional to δ_{ij} and $b_i b_j$ in K_{ij}^{gyr} do not contribute to Π_{ij}^{gyr}. Thus, bringing the result of Eqs.(57,58) to (7) and dropping the δ_{ij} and $b_i b_j$ terms, we get the final expression:

$$K_{ij}^{gyr} \;=\; \frac{m}{eB}\left\{\, p_\perp \frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial}{\partial x_{[i}}\left[\left(q_{T\parallel} - \frac{\alpha j_\parallel}{B}\right) b_{j]} + \frac{1}{2} q^{gyr}_{T\perp j]}\right] \;+\; b_{[i}\,(c_{j]} + d_{j]}) \;+\right.$$

$$\left. +\; \kappa_{[i}\; g_{\perp j]} \;+\; \epsilon_{[ilm}\left[\left(\nabla\cdot(\alpha\mathbf{b})\; b_l + \alpha\kappa_l\right)\frac{\partial b_{j]}}{\partial x_m} + \alpha b_l \frac{\partial \kappa_{j]}}{\partial x_m}\right]\,\right\}, \tag{13}$$

where $j_\parallel = \mathbf{b}\cdot(\nabla\times\mathbf{B})$ is the parallel current,

$$\mathbf{g}_\perp = 2\mathbf{q}^{gyr}_{B\perp} - \frac{1}{2}\mathbf{q}^{gyr}_{T\perp} = \frac{1}{eB}\mathbf{b}\times\left[p_\perp\nabla\left(\frac{p_\parallel - p_\perp}{n}\right) + \frac{2p_\parallel(p_\parallel - p_\perp)}{n}\kappa + \frac{1}{2}\nabla\tilde{r}^{(0)}_\Delta + \left(2\tilde{r}^{(0)}_\parallel - 2\tilde{r}^{(0)}_\perp - \frac{5}{2}\tilde{r}^{(0)}_\Delta\right)\kappa\right] \tag{14}$$

and

$$\mathbf{d} \;=\; \frac{3\alpha j_\parallel}{B}\,\kappa \;+\; \nabla\times\Big[\mathbf{g}_\perp\times\mathbf{b} - \alpha(\nabla\cdot\mathbf{b})\mathbf{b}\Big] \;+\; 2\Big\{\Big[\mathbf{g}_\perp + \nabla\times(\alpha\mathbf{b})\Big]\cdot\nabla\Big\}\mathbf{b}\;. \tag{15}$$

This general formula (6,13) for the gyroviscous stress takes into account all the details of the magnetic geometry, and is valid for strongly anisotropic and far from Maxwellian distribution functions. If the distribution function were Maxwellian or just isotropic in lowest order, then $(p_\parallel - p_\perp)$ would vanish in lowest order as would $(\tilde{r}^{(0)}_\parallel - \tilde{r}^{(0)}_\perp)$ and $\tilde{r}^{(0)}_\Delta$. In this particular case, the lowest significant order expression (13) for K_{ij}^{gyr} would lack the α, $\mathbf{g}_\perp$ and $\mathbf{d}$ terms.

III. Special limits.

The formerly known gyroviscosity tensors, which apply to different more specific regimes, can be recovered as special limits of our general expression. In a high collisionality regime, the lowest-order distribution function is Maxwellian, therefore $(p_\parallel - p_\perp) \ll p$, $q \ll p v_{th}$ and $\tilde{r}^{(0)}_\parallel = \tilde{r}^{(0)}_\perp = \tilde{r}^{(0)}_\Delta = 0$. It also follows that, at high collisionality, $(2q_{B\parallel} - 3q_{T\parallel}) \ll \delta p v_{th}$ and $|2\mathbf{q}^{gyr}_{B\perp} - \frac{1}{2}\mathbf{q}^{gyr}_{T\perp}| \ll \delta p v_{th}$. If besides one considers fast dynamics with sonic flows, $u \sim v_{th}$, the gyroviscous stress is $\Pi^{gyr} \sim \delta p$. Within this first-order accuracy, Eq.(13) reduces then to the Braginskii form[6]:

$$K^{gyr}_{ij} \;=\; \frac{mp_\perp}{eB}\,\frac{\partial u_{j]}}{\partial x_{[i}}\;, \tag{16}$$

in which case one can take $p_\perp = p$.

Considering high collisionality but slow dynamics with diamagnetic flows, $u \sim \delta v_{th}$, the leading order gyroviscous stress is $\Pi^{gyr} \sim \delta^2 p$. If we keep this second-order accuracy using the above high collisionality simplifications, Eq.(13) reduces to the Mikhailowskii-Tsypin form[7,23]:

$$K^{gyr}_{ij} \;=\; \frac{m}{eB}\left[\,p_\perp\frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial}{\partial x_{[i}}\left(q_{T\parallel}b_{j]} + \frac{1}{2}q^{gyr}_{T\perp j]}\right)\right]\;, \tag{17}$$

in which case $p_\perp = p$ and $q_{T\parallel}b_j + \frac{1}{2}q^{gyr}_{T\perp j} = \frac{2}{5}[(q_{T\parallel} + q_{B\parallel})b_j + q^{gyr}_{T\perp j} + q^{gyr}_{B\perp j}]$.

Without any assumptions on the collisionality so that the distribution function is allowed to be far from Maxwellian and highly anisotropic, but considering the fast dynamics ordering so that only $O(\delta p)$ accuracy needs to be retained, Eq.(13) reduces to:

$$K^{gyr}_{ij} \;=\; \frac{m}{eB}\left\{\,p_\perp\frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial(q_{T\parallel}b_{j]})}{\partial x_{[i}} \;+\; b_{[i}\left[(2q_{B\parallel} - 3q_{T\parallel})\kappa_{j]} + 2(p_\parallel - p_\perp)b_k\frac{\partial u_{j]}}{\partial x_k}\right]\right\}\;, \tag{18}$$

in agreement with Macmahon's result[8].

Finally, we may consider the slow dynamics ordering without any explicit reference to the collisionality regime, but assuming that the lowest-order distribution function would still be Maxwellian or at least isotropic. At low collisionality, this is guaranteed only under some special circumstances such as equilibria with closed magnetic surfaces. In this case, as discussed in the preceding section, the α, $\mathbf{g}_\perp$ and $\mathbf{d}$ terms drop from Eq.(13). Also, the term proportional to $(p_\parallel - p_\perp)$ in the vector $\mathbf{c}$ (8) becomes negligible within the leading order accuracy $\Pi^{gyr} \sim \delta^2 p$. Thus, we get[9,10]:

$$K^{gyr}_{ij} \;=\; \frac{m}{eB}\left[\,p_\perp\frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial}{\partial x_{[i}}\left(q_{T\parallel}b_{j]} + \frac{1}{2}q^{gyro}_{T\perp j]}\right) \;+\; b_{[i}\;(2q_{B\parallel} - 3q_{T\parallel})\kappa_{j]}\right]\;. \tag{19}$$

Here $p_\perp$ can be taken equal to p, and the reduced expression for $\mathbf{q}_{T\perp}^{gyr}$ follows from the corresponding limit of Eq.(11). If one assumes a Maxwellian lowest-order distribution function[9], this is

$$\mathbf{q}_{T\perp}^{gyr} \;=\; \frac{2p}{eB}\mathbf{b}\times\nabla\left(\frac{p}{n}\right) \,, \tag{20}$$

and if one assumes an isotropic but not necessarily Maxwellian lowest-order distribution function[10] with $\tilde{r}_\parallel^{(0)} = \tilde{r}_\perp^{(0)} = \tilde{r}^{(0)} \neq 0$, it is

$$\mathbf{q}_{T\perp}^{gyr} \;=\; \frac{2}{eB}\mathbf{b}\times\left[p\nabla\left(\frac{p}{n}\right)+\frac{2}{5}\nabla\tilde{r}^{(0)}\right] . \tag{21}$$

IV. Explicit gyroviscous force and momentum conservation equation.

The divergence of the gyroviscous stress tensor contributes the gyroviscous force term to the momentum conservation equation. In order to obtain an explicit representation of the gyroviscous force vector, it is convenient to split the stress tensor in five terms according to the five terms in the r.h.s. of Eq.(13):

$$\Pi_{ij}^{gyr} \;=\; \sum_{N=1}^{5}\Pi_{ij}^{gyrN} \;=\; \frac{1}{4}\epsilon_{[ikl}b_k\left(\sum_{N=1}^{5}K_{lm}^{gyrN}\right)\left(\delta_{mj]}+3b_m b_{j]}\right) \,, \tag{22}$$

with

$$K_{ij}^{gyr1} \;=\; \frac{mp_\perp}{eB}\,\frac{\partial u_{j]}}{\partial x_{[i}} \,, \tag{23}$$

$$K_{ij}^{gyr2} \;=\; \frac{m}{eB}\,\frac{\partial}{\partial x_{[i}}\left[\left(q_{T\parallel}-\frac{\alpha j_\parallel}{B}\right)b_{j]}+\frac{1}{2}q_{T\perp j]}^{gyr}\right] \,, \tag{24}$$

$$K_{ij}^{gyr3} = \frac{m}{eB}\;\; b_{[i}\;(c_{j]}+d_{j]}) \,, \tag{25}$$

$$K_{ij}^{gyr4} = \frac{m}{eB}\;\;\kappa_{[i}\;g_{\perp j]} \,, \tag{26}$$

and

$$K_{ij}^{gyr5} \;=\; \frac{m}{eB}\;\epsilon_{[ilm}\left[\left(\nabla\cdot(\alpha\mathbf{b})\;b_l+\alpha\kappa_l\right)\frac{\partial b_{j]}}{\partial x_m}+\alpha b_l\frac{\partial\kappa_{j]}}{\partial x_m}\right] . \tag{27}$$

The first term, driven by the velocity gradient (or rate of strain) tensor, is the one most commonly considered[2−6,11−18]. However, only approximate calculations of the corresponding force vector have been reported, as far as this author is aware. The exact result in coordinate-free form, whose derivation is detailed in Appendix C, is:

$$\nabla \cdot \Pi^{gyr1} \;=\; -\, m\, n\, (\mathbf{u}_* \cdot \nabla)\mathbf{u} \;-\; \nabla\chi \;-$$

$$-\,\nabla \times \left\{ \frac{mp_\perp}{eB} \left[(\mathbf{b}\cdot\nabla)\mathbf{u} \;+\; \frac{1}{2}\left(\nabla\cdot\mathbf{u} - 3\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}\Big]\right)\,\mathbf{b}\,\right]\right\} +$$

$$+\,(\mathbf{B}\cdot\nabla)\left\{ \frac{mp_\perp}{eB^2}\,\mathbf{b}\times\Big[3(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega\Big] \;+\; \frac{\chi}{B}\,\mathbf{b}\right\} . \tag{28}$$

Here, $\omega = \nabla \times \mathbf{u}$ is the vorticity with the scalar χ proportional to its parallel component,

$$\chi = \frac{mp_\perp}{2eB}\,\mathbf{b}\cdot\omega\ , \tag{29}$$

and $\mathbf{u}_*$ is the magnetization velocity:

$$\mathbf{u}_* = -\frac{1}{en}\nabla\times\left(\frac{p_\perp}{B}\mathbf{b}\right) . \tag{30}$$

The second term includes the contribution from the gradients of the heat fluxes. Since it has the same form as the first one, the corresponding piece of the gyroviscous force can be obtained by direct substitution:

$$\nabla\cdot\Pi^{gyr2} \;=\; \nabla\cdot\Pi^{gyr1}\Big[p_\perp \to 1\ ;\ \mathbf{u}\to(q_{T\parallel} - \alpha j_\parallel/B)\mathbf{b} + \mathbf{q}^{gyr}_{T\perp}/2\Big] . \tag{31}$$

The Π^{gyr3} and Π^{gyr4} terms are in the form of symmetrized tensor products of vectors (diadic forms):

$$\Pi^{gyr3} \;=\; \frac{m}{eB}\,\Big\{[\mathbf{b}\times(\mathbf{c}+\mathbf{d})]\ \mathbf{b} \;+\; \mathbf{b}\ [\mathbf{b}\times(\mathbf{c}+\mathbf{d})]\Big\} \tag{32}$$

and

$$\Pi^{gyr4} \;=\; \frac{m}{4eB}\,\Big[(\mathbf{b}\times\kappa)\;\mathbf{g}_\perp \;+\; \mathbf{g}_\perp\,(\mathbf{b}\times\kappa) \;+\; (\mathbf{b}\times\mathbf{g}_\perp)\;\kappa \;+\; \kappa\;(\mathbf{b}\times\mathbf{g}_\perp)\Big]\;. \tag{33}$$

Therefore the evaluation of their divergence is straightforward and, using standard vector identities, we can write:

$$\nabla\cdot\Pi^{gyr3} \;=\; \nabla\times\left\{\mathbf{B}\times\Big[\frac{m}{eB^2}(\mathbf{c}+\mathbf{d})\Big]\right\} \;+\; (\mathbf{B}\cdot\nabla)\Big[\frac{2m}{eB^2}(\mathbf{c}+\mathbf{d})\Big] \tag{34}$$

and

$$\nabla\cdot\Pi^{gyr4} \;=\; \frac{m}{2eB}\left\{\Big[(\nabla\cdot\kappa)\mathbf{b}-(\mathbf{b}\cdot\nabla)\kappa\Big]\times\mathbf{g}_\perp \;+\; \Big[\nabla\cdot(\mathbf{b}\times\kappa)\Big]\;\mathbf{g}_\perp\right\}+$$

$$+\left\{\Big[\nabla\cdot\Big(\frac{m\;\mathbf{g}_\perp}{2eB}\Big)-\frac{m\;\mathbf{g}_\perp\cdot\kappa}{2eB}\Big]\;\mathbf{b}\;-\;(\mathbf{b}\cdot\nabla)\Big(\frac{m\;\mathbf{g}_\perp}{2eB}\Big)\right\}\times\kappa\;+\;\Big[\nabla\cdot\Big(\frac{m\;\mathbf{b}\times\mathbf{g}_\perp}{2eB}\Big)\Big]\;\kappa\;+$$

$$+\left\{\frac{m\;\mathbf{g}_\perp\cdot(\nabla\times\kappa)}{2eB}\;-\;\frac{m\;j_\parallel\;\mathbf{g}_\perp\cdot\kappa}{2eB^2}\;+\;\kappa\cdot\Big[\nabla\times\Big(\frac{m\;\mathbf{g}_\perp}{2eB}\Big)\Big]\right\}\;\mathbf{b}\;. \tag{35}$$

Like Π^{gyr4}, the Π^{gyr5} term needs to be retained only in the slow dynamics ordering and then only if the lowest-order distribution function is anisotropic. Its divergence is evaluated following procedures similar to those used in the Appendices B and C. Without elaborating on the details, the result is:

$$\nabla\cdot\Pi^{gyr5} \;=\; -\nabla\left[\frac{m\alpha}{2eB}(\nabla\cdot\kappa)+\nabla\cdot\Big(\frac{m\nabla\cdot(\alpha\mathbf{b})}{2eB}\;\mathbf{b}\Big)\right]\;+$$

$$+\;\nabla\times\left\{\mathbf{b}\times\left(\frac{m\alpha}{4eB}\;\xi_\perp+\frac{m\nabla\cdot(\alpha\mathbf{b})}{2eB}\;\kappa\right)\;+\;\left[\frac{m\alpha}{2eB}\nabla\cdot\Big(\frac{j_\parallel}{B}\;\mathbf{b}\Big)+\frac{m\nabla\cdot(\alpha\mathbf{b})\;j_\parallel}{2eB^2}\right]\mathbf{b}\right\}\;+$$

$$+\;(\mathbf{B}\cdot\nabla)\left\{\frac{m\alpha}{2eB^2}\Big[\xi_\perp+(\nabla\cdot\kappa)\;\mathbf{b}\Big]+\frac{m\nabla\cdot(\alpha\mathbf{b})}{2eB^2}\;\kappa+\frac{1}{B}\nabla\Big[\frac{m\nabla\cdot(\alpha\mathbf{b})}{2eB}\Big]\right\}\;+$$

$$+\;\mathbf{b}\times\left\{\left(\Big[\nabla\times\Big(\frac{m\nabla\cdot(\alpha\mathbf{b})}{2eB}\;\mathbf{b}\Big)\Big]\cdot\nabla\right)\mathbf{b}\;+\;\left(\Big[\nabla\times\Big(\frac{m\alpha}{eB}\;\mathbf{b}\Big)\Big]\cdot\nabla\right)\kappa\right\}\;+$$

$$+\;\frac{m\alpha}{eB}\Big[\nabla\cdot\Big(\frac{j_\parallel}{B}\;\mathbf{b}\Big)\Big]\;\mathbf{b}\times\kappa\;-\;\Big[(\mathbf{b}\times\kappa)\cdot\nabla\Big]\Big(\frac{m\alpha}{eB}\;\mathbf{b}\times\kappa\Big)\;-\;\frac{m\nabla\cdot(\alpha\mathbf{b})}{2eB}\;\eta\;-\;\frac{m\alpha}{eB}\;\zeta\;, \tag{36}$$

where

$$\xi_\perp \;=\; 5(\kappa\cdot\nabla)\mathbf{b} \;-\; (\nabla\cdot\mathbf{b})\,\kappa \;+\; \mathbf{b}\times\Big\{3\Big[(\mathbf{b}\times\kappa)\cdot\nabla\Big]\mathbf{b} - \frac{5j_\parallel}{B}\,\kappa\Big\}\;, \tag{37}$$

and η, ζ are the vectors with components

$$\eta_i \;=\; \epsilon_{ikl}\;\epsilon_{jmn}\;b_j\;\frac{\partial b_k}{\partial x_m}\,\frac{\partial b_l}{\partial x_n}\;, \tag{38}$$

$$\zeta_i \;=\; \epsilon_{ikl}\;\epsilon_{jmn}\;b_j\;\frac{\partial b_k}{\partial x_m}\,\frac{\partial \kappa_l}{\partial x_n}\;. \tag{39}$$

In the momentum conservation equation, parts of $\nabla\cdot\Pi^{gyr1}$ (28) can be combined with the divergence of the Reynolds stress and the CGL stress. Collecting all the terms, the final form of the complete momentum conservation equation is

$$\begin{aligned} & mn\frac{\partial\mathbf{u}}{\partial t} \;+\; mn\Big[(\mathbf{u}-\mathbf{u}_*)\cdot\nabla\Big]\mathbf{u} \;+\; \nabla(p_\perp-\chi) \;+\; (\mathbf{B}\cdot\nabla)\left(\frac{p_\parallel-p_\perp+\chi}{B}\,\mathbf{b}\right) \;- \\ & -\;\nabla\times\left\{\frac{mp_\perp}{eB}\;\left[(\mathbf{b}\cdot\nabla)\mathbf{u} \;+\; \frac{1}{2}\Big(\nabla\cdot\mathbf{u}-3\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}\Big]\Big)\;\mathbf{b}\;\right]\right\} \;+ \\ & +\;(\mathbf{B}\cdot\nabla)\left\{\frac{mp_\perp}{eB^2}\;\mathbf{b}\times\Big[3(\mathbf{b}\cdot\nabla)\mathbf{u}+\mathbf{b}\times\omega\Big]\right\} \;+ \\ & +\;\nabla\cdot\left(\sum_{N=2}^{5}\Pi^{gyrN}+\Pi_\perp^{coll}\right) \;-\; en\;(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \;-\; \mathbf{F}^{coll} \;=\; 0\;, \end{aligned} \tag{40}$$

where $\mathbf{E}$ stands for the electric field and $\mathbf{F}^{coll}$ for the collisional friction force. Thus, the magnetization velocity $\mathbf{u}_*$ substracts from the total flow velocity $\mathbf{u}$ in the convective derivative operator $\mathbf{u}\cdot\nabla$. This property is famously known as the "gyroviscous cancellation"[3,4,15–19]. However, contrary to widespread lore, the cancelled part of $\mathbf{u}$ is the magnetization velocity $\mathbf{u}_*$, not the diamagnetic drift velocity $\mathbf{u}_d = \mathbf{b} \times \nabla p_\perp/(enB)$. Only for a constant magnetic field is $\mathbf{u}_* = \mathbf{u}_d$. Also, we observe that the parallel vorticity term χ acts as an effective renormalization of the perpendicular pressure:

$$p_\perp \to p_\perp - \chi = p_\perp \Big(1 - \frac{m}{2eB}\, \mathbf{b} \cdot \omega\Big) \ , \tag{41}$$

leaving the parallel pressure unaffected. The other two terms (a curl and the parallel derivative of a perpendicular vector) which still stem from the Braginskii piece $\nabla \cdot \Pi^{gyr1}$, along with the remaining $\nabla \cdot (\sum_{N=2}^{5} \Pi^{gyrN})$ piece (31,34-36), complete the gyroviscous force contribution to the momentum equation.

Acknowledgements.

The author thanks S. Jardin and the Center for Extended MHD Modeling (CEMM) for their interest and support, and for suggesting the coordinate-free evaluation of the gyroviscous force. Useful discussions with P. Catto are also acknowledged. This work was sponsored by the U.S. Department of Energy under Grant No. DE-FG05-91ER-54109 at the Massachusetts Institute of Technology.

Appendix A: The zero-Larmor-radius fourth rank fluid moment.

In its lowest order, the collision-independent perpendicular stress-flux tensor $\Theta^{gyr}_{ijk} = O(\delta p v_{th})$ (9-12) involves[10] the divergence of the zero-Larmor-radius fourth rank moment:

$$\bar{N}^{(0)}_{ijkl} \;=\; m^2 \int d^3\mathbf{v}\ (v_i - u_i)(v_j - u_j)(v_k - u_k)(v_l - u_l)\ f^{(0)}. \tag{42}$$

Here, $f^{(0)} = f^{(0)}(m|\mathbf{v}-\mathbf{u}|^2/2, \lambda, \mathbf{x}, t)$ is the zero-Larmor-radius distribution function which depends on the velocity space coordinates through the fluid-rest-frame energy, $m|\mathbf{v}-\mathbf{u}|^2/2$, and the pitch angle, $\sin\lambda = (\mathbf{v}-\mathbf{u})\cdot\mathbf{b}/|\mathbf{v}-\mathbf{u}|$, but is independent of the gyrophase. Then, writing

$$\bar{N}^{(0)}_{ijkl} \;=\; \frac{1}{n}\,\Big[p_\perp \delta_{[ij} + (p_\| - p_\perp) b_{[i} b_j\Big]\,\Big[p_\perp \delta_{kl]} + (p_\| - p_\perp) b_k b_{l]}\Big] \;+\; \tilde{N}^{(0)}_{ijkl}\ , \tag{43}$$

one gets

$$\tilde{N}^{(0)}_{ijkl} \;=\; \frac{1}{5}\Big(2\tilde{r}^{(0)}_\perp - \frac{1}{2}\tilde{r}^{(0)}_\Delta\Big)\delta_{[ij}\delta_{kl]} \;+\; \frac{1}{2}\tilde{r}^{(0)}_\Delta \delta_{[ij} b_k b_{l]} \;+\; \Big(2\tilde{r}^{(0)}_\| - 2\tilde{r}^{(0)}_\perp - \frac{7}{2}\tilde{r}^{(0)}_\Delta\Big) b_i b_j b_k b_l \tag{44}$$

where

$$\tilde{r}^{(0)}_\perp \;=\; \frac{m^2}{4}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \cos^2\lambda\ (f^{(0)} - f_{2M}), \tag{45}$$

$$\tilde{r}^{(0)}_\| \;=\; \frac{m^2}{2}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \sin^2\lambda\ (f^{(0)} - f_{2M}), \tag{46}$$

$$\tilde{r}^{(0)}_\Delta \;=\; \frac{m^2}{4}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \cos^2\lambda\ (5\sin^2\lambda - 1)(f^{(0)} - f_{2M}) \tag{47}$$

and f_{2M} is the two-temperature Maxwellian:

$$f_{2M}(m|\mathbf{v}-\mathbf{u}|^2/2, \lambda, \mathbf{x}, t) \;=\; \left(\frac{m}{2\pi}\right)^{3/2} \frac{n^{5/2}}{p_\perp\, p_\|^{1/2}}\ \ \exp\!\left[-\frac{m\ n\ |\mathbf{v}-\mathbf{u}|^2}{2}\left(\frac{\cos^2\lambda}{p_\perp} \;+\; \frac{\sin^2\lambda}{p_\|}\right)\right]. \tag{48}$$

If the zeroth-order distribution function $f^{(0)}$ were isotropic (not necessarily Maxwellian), i.e. independent of λ with $p_\| = p_\perp$, one would have $\tilde{r}^{(0)}_\| = \tilde{r}^{(0)}_\perp$ and $\tilde{r}^{(0)}_\Delta = 0$. With a Maxwellian zeroth-order distribution function, $\tilde{r}^{(0)}_\|$, $\tilde{r}^{(0)}_\perp$ and $\tilde{r}^{(0)}_\Delta$ would vanish.

Appendix B: Divergence of the anisotropic stress-flux tensor.

This Appendix will outline the evaluation of the divergence of the strongly anisotropic stress-flux tensor[10] in the slow dynamics ordering (9-12):

$$\Theta_{ijk}^{gyr} \;=\; \frac{1}{2}\delta_{[ij}q_{T\perp k]}^{gyr} \;+\; b_{[i}b_j g_{\perp k]} \;+\; \frac{\alpha}{2}\epsilon_{[ilm}b_j b_l\left(\frac{\partial b_n}{\partial x_m}+\frac{\partial b_m}{\partial x_n}\right)\Big(\delta_{nk]}-b_n b_{k]}\Big) \;, \tag{49}$$

where the notation $\mathbf{g}_\perp = 2\mathbf{q}_{B\perp}^{gyr} - \frac{1}{2}\mathbf{q}_{T\perp}^{gyr}$ is used.

For any vector $\mathbf{A}$ with curl $\mathbf{C} = \nabla\times\mathbf{A}$, we have the identity

$$\frac{\partial A_j}{\partial x_i} \;=\; \frac{\partial A_i}{\partial x_j} \;+\; \epsilon_{ijk}C_k \;, \tag{50}$$

and for the magnetic unit vector $\mathbf{b}$,

$$\nabla\times\mathbf{b} \;=\; \mathbf{b}\times\kappa \;+\; \frac{j_\|}{B}\mathbf{b} \;. \tag{51}$$

Therefore we can write

$$\frac{\partial b_n}{\partial x_m}+\frac{\partial b_m}{\partial x_n} \;=\; 2\frac{\partial b_n}{\partial x_m} \;+\; b_n\kappa_m \;-\; b_m\kappa_n \;+\; \frac{j_\|}{B}\epsilon_{nmp}b_p \;, \tag{52}$$

hence

$$\epsilon_{ilm}b_l\left(\frac{\partial b_n}{\partial x_m}+\frac{\partial b_m}{\partial x_n}\right)\Big(\delta_{nk}-b_n b_k\Big) \;=\; 2\epsilon_{ilm}b_l\frac{\partial b_k}{\partial x_m} \;+\; \frac{j_\|}{B}\Big(b_i b_k - \delta_{ik}\Big) \tag{53}$$

and

$$\Theta_{ijk}^{gyr} \;=\; \delta_{[ij}\left(\frac{1}{2}q_{T\perp k]}^{gyro} - \frac{\alpha j_\|}{B}b_{k]}\right) \;+\; b_{[i}b_j\left(g_{\perp k]} + \frac{\alpha j_\|}{B}b_{k]}\right) \;+\; \alpha\epsilon_{[ilm}b_j b_l\frac{\partial b_{k]}}{\partial x_m} \;. \tag{54}$$

The next step is to carry out an integration by parts in the last term, which yields

$$\Theta_{ijk}^{gyr} \;=\; \delta_{[ij}\left(\frac{1}{2}q_{T\perp k]}^{gyro} - \frac{\alpha j_\|}{B}b_{k]}\right) \;+\; b_{[i}b_j h_{k]} \;+\; \epsilon_{[ilm}\frac{\partial(\alpha b_l b_j b_{k]})}{\partial x_m} \;, \tag{55}$$

where $\mathbf{h} = \mathbf{g}_\perp + (\alpha j_\|/B)\mathbf{b} + \nabla\times(\alpha\mathbf{b})$.

The divergence of this last expression can be readily evaluated:

$$\frac{\partial \Theta^{gyr}_{ijk}}{\partial x_k} \;=\; \frac{\partial}{\partial x_k}\left(\frac{1}{2}q^{gyr}_{T\perp k} - \frac{\alpha j_\parallel}{B} b_k\right)\,\delta_{ij} \;+\; \frac{\partial}{\partial x_{[i}}\left(\frac{1}{2}q^{gyr}_{T\perp j]} - \frac{\alpha j_\parallel}{B} b_{j]}\right) \;+\; \frac{\partial h_k}{\partial x_k}\; b_i b_j \;+\; \kappa_{[i}\; h_{j]} \;+$$

$$+\; b_{[i}\;\left(b_k \frac{\partial h_{j]}}{\partial x_k} + h_k \frac{\partial b_{j]}}{\partial x_k} + \frac{\partial b_k}{\partial x_k}\; h_{j]}\right) \;+\; \epsilon_{[ilm}\frac{\partial}{\partial x_m}\left[\frac{\partial(\alpha b_k)}{\partial x_k} b_l b_{j]} + \alpha \kappa_l b_{j]} + \alpha b_l \kappa_{j]}\right] \;. \tag{56}$$

Finally, expanding the last $\partial/\partial x_m$ derivative, using standard vector identities and collecting like terms, we get:

$$\frac{\partial \Theta^{gyr}_{ijk}}{\partial x_k} \;=\; \nabla\cdot\left(\frac{1}{2}\mathbf{q}^{gyr}_{T\perp} - \frac{\alpha j_\parallel}{B}\mathbf{b}\right)\,\delta_{ij} \;+\; \frac{\partial}{\partial x_{[i}}\left(\frac{1}{2}q^{gyr}_{T\perp j]} - \frac{\alpha j_\parallel}{B} b_{j]}\right) \;+\; 3\nabla\cdot\mathbf{h}\; b_i b_j \;+$$

$$+\; \kappa_{[i}\; g_{\perp j]} \;+\; b_{[i}\; d_{j]} \;+\; \epsilon_{[ilm}\left[\Big(\nabla\cdot(\alpha\mathbf{b})\; b_l + \alpha\kappa_l\Big)\frac{\partial b_{j]}}{\partial x_m} + \alpha b_l \frac{\partial \kappa_{j]}}{\partial x_m}\right] \;, \tag{57}$$

with

$$\mathbf{d} \;=\; \frac{3\alpha j_\parallel}{B}\;\kappa \;+\; \nabla\times\Big[\mathbf{g}_\perp\times\mathbf{b} - \alpha(\nabla\cdot\mathbf{b})\mathbf{b}\Big] \;+\; 2\Big\{\Big[\mathbf{g}_\perp + \nabla\times(\alpha\mathbf{b})\Big]\cdot\nabla\Big\}\mathbf{b} \;. \tag{58}$$

Appendix C: Divergence of the velocity-gradient-driven gyroviscosity tensor.

In this Appendix we shall evaluate the divergence of the velocity-gradient-driven part[6] of the gyroviscous stress tensor (22,23):

$$\Pi_{ij}^{gyr1} \;=\; \frac{mp_\perp}{4eB}\epsilon_{[ikl}b_k\left(\frac{\partial u_m}{\partial x_l}+\frac{\partial u_l}{\partial x_m}\right)\left(\delta_{mj]}+3b_m b_{j]}\right)\;. \tag{59}$$

Following the procedure of Appendix B, we apply the identity (50) to the vector $\mathbf{u}$ with $\omega = \nabla\times\mathbf{u}$, to get:

$$\epsilon_{ikl}b_k\left(\frac{\partial u_m}{\partial x_l}+\frac{\partial u_l}{\partial x_m}\right)\left(\delta_{mj}+3b_m b_j\right) \;=\; \epsilon_{ikl}b_k\left[\left(2\frac{\partial u_j}{\partial x_l}+\epsilon_{jln}\omega_n\right) \;+\; 3\left(2\frac{\partial u_l}{\partial x_m}+\epsilon_{lmn}\omega_n\right)b_m b_j\right] \;=$$

$$=\; 2\epsilon_{ikl}b_k\frac{\partial u_j}{\partial x_l} \;-\; b_k\omega_k\delta_{ij} \;+\; \left(6\epsilon_{ikl}b_k b_m\frac{\partial u_l}{\partial x_m}-2\omega_i+3b_k\omega_k b_i\right)b_j\;. \tag{60}$$

Therefore we can write:

$$\Pi_{ij}^{gyr1} \;=\; \frac{mp_\perp}{2eB}\;\epsilon_{[ikl}b_k\frac{\partial u_{j]}}{\partial x_l} \;-\; \chi\;\delta_{ij} \;+\; B_{[i}\;a_{j]}\;, \tag{61}$$

where

$$\chi = \frac{mp_\perp}{2eB}\;\mathbf{b}\cdot\omega \tag{62}$$

and

$$\mathbf{a} \;=\; \frac{mp_\perp}{2eB^2}\;\mathbf{b}\times\left[3(\mathbf{b}\cdot\nabla)\mathbf{u}+\mathbf{b}\times\omega\right] \;+\; \frac{\chi}{2B}\;\mathbf{b}\;. \tag{63}$$

After an integration by parts, Eq.(61) becomes

$$\Pi_{ij}^{gyr1} \;=\; \frac{\partial}{\partial x_l}\left(\frac{mp_\perp}{2eB}\;\epsilon_{[ikl}b_k u_{j]}\right) \;-\; \frac{mn}{2}\;u_{[i}\;u_{*j]} \;-\; \chi\;\delta_{ij} \;+\; B_{[i}\;a_{j]}\;, \tag{64}$$

with

$$\mathbf{u}_* = -\frac{1}{en}\nabla\times\left(\frac{p_\perp}{B}\mathbf{b}\right)\;. \tag{65}$$

We can now evaluate easily the divergence of the latter expression (64):

$$\frac{\partial \Pi^{gyr1}_{ij}}{\partial x_j} \;=\; \epsilon_{ikl} \frac{\partial}{\partial x_l} \left[u_j \frac{\partial}{\partial x_j} \left(\frac{m p_\perp}{2eB}\; b_k \right) + \frac{\partial u_j}{\partial x_j}\; \frac{m p_\perp}{2eB}\; b_k \right] \;-\; \frac{m}{2}\; \frac{\partial (n u_{[i}\; u_{*j]})}{\partial x_j} \;-\; \frac{\partial \chi}{\partial x_i} \;+\; \frac{\partial B_{[i}\; a_{j]})}{\partial x_j}\;, \quad (66)$$

or using vector notation, taking into account $\nabla \cdot (n\mathbf{u}_*) = 0$, $\nabla \cdot \mathbf{B} = 0$ and standard vector identities,

$$\nabla \cdot \Pi^{gyr1} \;=\; -\nabla \times \left[(\mathbf{u} \cdot \nabla) \left(\frac{m p_\perp}{2eB}\; \mathbf{b} \right) +\; (\nabla \cdot \mathbf{u})\; \frac{m p_\perp}{2eB}\; \mathbf{b} \right] -$$

$$-\frac{m}{2}\; \left[\nabla \times (n\mathbf{u}_* \times \mathbf{u}) + 2n(\mathbf{u}_* \cdot \nabla)\mathbf{u} \right] \;-\; \nabla \chi \;+\; \nabla \times (\mathbf{B} \times \mathbf{a}) \;+\; 2(\mathbf{B} \cdot \nabla)\mathbf{a}\;. \quad (67)$$

Finally, we collect terms and use some further vector identities to arrive at the result:

$$\nabla \cdot \Pi^{gyr1} \;=\; -\; \nabla \times \left\{ \frac{m p_\perp}{eB}\; \left[(\mathbf{b} \cdot \nabla)\mathbf{u} \;+\; \frac{1}{2} \left(\nabla \cdot \mathbf{u} - 3\mathbf{b} \cdot \left[(\mathbf{b} \cdot \nabla)\mathbf{u} \right] \right)\; \mathbf{b}\; \right] \right\} -$$

$$-\; m\; n\; (\mathbf{u}_* \cdot \nabla)\mathbf{u} \;-\; \nabla \chi \;+\; 2(\mathbf{B} \cdot \nabla)\mathbf{a}\;. \quad (68)$$

References.

[1] M.N. Rosenbluth, N.A. Krall and N. Rostoker, Nucl. Fusion Suppl. Pt.1, 143 (1962).

[2] K.V. Roberts and J.B. Taylor, Phys. Rev. Lett. **8**, 197 (1962).

[3] B. Coppi, Phys. Fluids **7**, 1501 (1964).

[4] M.N. Rosenbluth and A. Simon, Phys. Fluids **8**, 1300 (1965).

[5] A.M. Kaufman, La théorie des gaz neutres et ionizés (Hermann et Cie., Paris, 1960).

[6] S.I. Braginskii in Reviews of Plasma Physics Vol.1, M.A. Leontovich ed. (Consultants Bureau, New York, 1965) p.205.

[7] A.B. Mikhailovskii and V.S. Tsypin, Plasma Phys. **13**, 785 (1971).

[8] A. Macmahon, Phys. Fluids **8**, 1840 (1965).

[9] A.N. Simakov and P.J. Catto, Phys. Plasmas **12**, 012105 (2005).

[10] J.J. Ramos, Phys. Plasmas **12**, 052102 (2005).

[11] A. Zeiler, J.F. Drake and B. Rogers, Phys. Plasmas **4**, 2134 (1997).

[12] X.Q. Xu and R.H. Cohen, Contrib. Plasma Phys. **36**, 158 (1998).

[13] L.E. Sugiyama and W. Park, Phys. Plasmas **7**, 4644 (2000).

[14] C.R. Sovinec, A.H. Glasser, T.A. Gianakon, D.C. Barnes, R.A. Nebel, S.E. Kruger, D.D. Schnack, S.J. Plimpton, A. Tarditi, M.S. Chu and the NIMROD team, J. Comput. Phys. **195**, 355 (2004).

[15] T.K. Chu, B. Coppi, H.W. Hendel and F.W. Perkins, Phys. Fluids **12**, 203 (1969).

[16] F.L. Hinton and C.W. Horton, Phys. Fluids **14**, 116 (1971).

[17] R.D. Hazeltine and J.D. Meiss, Phys. Rep. **121**, 1 (1985).

[18] Z. Chang and J.D. Callen, Phys. Fluids B **4**, 1167 (1992).

[19] R.D. Hazeltine and J.D. Meiss, Plasma Confinement (Addison Wesley, Redwood City, CA, 1992).

[20] A.I. Smolyakov, Can. J. Phys. **76**, 321 (1998).

[21] E.V. Belova, Phys. Plasmas **8**, 3936 (2001).

[22] A.N. Simakov and P.J. Catto, Phys. Plasmas **10**, 4744 (2003).

[23] A.B. Mikhailovskii and V.S. Tsypin, Beitr. Plasmaphys., **24**, 335 (1984).

[24] C.T. Hsu, R.D. Hazeltine and P.J. Morrison, Phys. Fluids **29**, 1480 (1986).