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Neutral beam injection (NBI) is known to significantly affect radial transport in a tokamak plasma. However, the effect on collisional transport, accounting for strong toroidal plasma rotation, which is typical in the presence of NBI, had not been addressed. Furthermore, recent observations have shown poloidal velocities, in the presence of NBI, significantly in excess of the standard neoclassical value. Motivated by this, the additional collisional radial bulk ion fluxes of particles, heat and toroidal angular momentum, and the poloidal velocity, driven by fast ions from NBI have been evaluated for a low-collisionality, pure plasma. The effects of toroidal acceleration caused by the beam are seen to dominate at large aspect ratio. Higher order velocity space structure of the fast ion distribution function can be significant at tight aspect ratio. The driven poloidal velocity depends strongly on system parameters, becoming larger at higher beam density and lower beam energy.

I. INTRODUCTION

It is recognized that there is a significant response of a tokamak plasma to the injection of neutral particle beams. The effect on collisional, radial transport, of the thermal bulk species in a tokamak plasma, due to the fast ions resulting from neutral beam injection (NBI), has been studied by a variety of authors. Connor and Cordey(1) determined the cross-field particle flux in the presence of NBI, whilst, more recently, Helander and Akers(2) extended the work on electron transport, evaluating the neoclassical bulk electron particle and heat fluxes, for both high and low-collisionality electrons. Hinton and Kim(3) considered the effect on bulk ion heat transport, as well as the bulk ion poloidal flow, whilst Wang(4) determined the momentum and heat friction forces between the fast ions and thermal bulk plasma species, using the linearised Fokker-Planck collision operator, primarily in order to evaluate the bootstrap current in the presence of fast ions. Essentially, the effects of NBI are due to the additional friction experienced by the bulk plasma with the beam ions. This alters the flows of the species, thereby affecting transport both directly and indirectly, as the friction between the bulk species also drives transport(5).

Hinton and Kim(3) considered a quasi-steady-state situation, in which the toroidal torques were closely balanced, so there was no toroidal acceleration of the plasma. As they noted, to ensure momentum conservation, this required an unspecified turbulent diffusion of momentum to balance the toroidal momentum input from the beam. They found that the friction with fast ions produces a contribution to the bulk ion heat flux, which may be comparable to the neoclassical flux driven by the temperature gradient and whose radial direction depends on the system. Motivated by their work, we have considered the situation in which turbulence is assumed to be suppressed, which may be the case, for example, in the region of an internal transport barrier(6). Toroidal acceleration of the bulk plasma must therefore occur to ensure momentum conservation.

Hinton and Kim(3) took the plasma to consist of two distinct components, a thermal bulk plasma component and a fast ion component, with their associated electrons. They used the formulation presented by Hinton and Wong(7) to establish a drift kinetic equation for the bulk ion distribution function, containing a stronger, that is lower order, source term than Hinton and Wong retained, but restricted to the case of small toroidal rotation speed of the bulk plasma. The source represented the effect of the neutral beam and was explicitly taken to be a Fokker-Planck collision operator describing collisions between the fast and bulk ions. An approximation to this collision operator was obtained by expanding in the small ratio of the bulk to fast ion velocity. Hinton and Kim(3) retained only the term representing momentum transfer between the two species, but neglecting its energy dependence, which was the simplest term of the expansion with the correct parity to contribute to the heat transport. They were thus able to use the Hirshman-Sigmar moment method(5) to calculate approximate expressions for the bulk poloidal velocity and bulk ion heat flux, in the limit of large aspect ratio.

We follow the approach of Hinton and Kim(3), but allow for arbitrary toroidal rotation speed of the bulk plasma. We also retain the time dependent terms in the drift kinetic equation for the bulk ion distribution function, which are required to ensure particle, energy and angular momentum conservation in the presence of only collisional radial transport. The mechanism by which the transfer of toroidal momentum between fast ions and thermal ions takes place has been discussed by many authors, including Hinton and Rosenbluth in Ref. [8]. As was done by Hinton and Kim(3), we represent the neutral beam injection via a collision operator, expanded in the small ratio of the bulk to fast ion velocity, but also include a source term which represents the fast ions which have slowed down and joined the thermal plasma. All low order terms of the expansion are retained here, thus an alternative method to that used by Hinton and Kim(3) is required to determine the radial transport. Hinton and Wong(7) obtained flux-friction relations via which the neoclassical radial fluxes of particles, heat and toroidal angular momentum could be evaluated. We derive the modifications to these relations due to the presence of NBI, specifically the additional direct friction and the inclusion of time dependent terms, which vanish in the case of a weaker source, then evaluate the radial fluxes.

We find that both the time dependent effects and the terms of the collision operator neglected by Hinton and Kim play a significant role. Specifically, the total heat flux produced by the drive retained by Hinton and Kim, at large aspect ratio, is now small and a return current present in the accelerating system provides the dominant drive. Contributions from the higher order pitch angle structure of the fast ion distribution function can be significant at tighter aspect ratio.

Finally, we note that recent observations on DIII-D(9) and JET(10) indicate that the poloidal flow velocity of plasma species is larger than that predicted by standard neoclassical theory. It is typically the case that NBI is the dominant form of heating during these discharges. In their work, Hinton and Kim(3) found that the bulk ion poloidal rotation driven by friction with the fast ions is comparable in magnitude to the standard neoclassical value and again the relative directions depend on the system. We have also evaluated the bulk ion poloidal velocity and compared it to that expected without NBI. Again, we find that the dominant effect driving the poloidal velocity at large aspect ratio is now a return current. The magnitude of the driven velocity depends strongly on the parameters of the system, for example, the ratio of the energies of the fast and bulk ion components in the plasma.

The paper is organized as follows. In Section II, the drift kinetic equation for the bulk ions in the presence of a strong source is obtained by expansion of the Fokker-Planck equation, following Ref. [7], and expressions for the radial fluxes are given. In Sec. III, the form in which neutral beam injection is included as a source term in the drift kinetic equation is motivated, as is the adopted form of the bulk ion collision operator. The conservation laws are discussed in Sec. IV and the time dependencies of the bulk ion parameters determined. Then, in Sec. V, the solution of the drift kinetic equation is obtained and the radial fluxes evaluated. The limits of these results in the case of subsonic toroidal rotation of the bulk plasma are presented. The poloidal velocity of the bulk ions is evaluated in Sec. VI and, finally, the results are discussed in Sec. VII.

II. FOKKER-PLANCK EQUATION WITH STRONG SOURCE TERM

Hinton and Wong[7] and Catto, Bernstein and Tessarotto[11] analyzed neoclassical bulk ion transport in a pure plasma, consisting of bulk ions and electrons, and rotating with arbitrary toroidal velocity. They used a systematic expansion of the Fokker-Planck equation,

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \nabla f_i + \frac{e}{m_i} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_i}{\partial \mathbf{v}} = C_i + S_i. \tag{1}$$

satisified by the bulk ion distribution function, f_i. Other symbols have their usual meaning and throughout a subscript i implies a property of the singly charged bulk ions. The basic expansion parameter, δ, is taken to be the ratio of the ion gyroradius, ρ_i, to the typical macroscopic scale length, L, and a subscript indicates the order of expansion. Whilst Hinton and Wong finally evaluated the radial transport in the presence of a weak source, S_i, taken to be second order in δ at most, the initial formulation was presented for an arbitrary source. We have therefore used their work as a basis to determine the radial transport in the presence of a strong source term, that is S_i taken to be first order. In this section, we will summarize the results of the expansion, then present the modifications to the drift kinetic equation and expressions for the radial fluxes arising from retaining a strong source term. The following sections will be concerned with using this formulation to determine the effects on transport, in a low-collisionality pure plasma, in the case when the source term represents neutral beam injection.

A. Expansion of the Fokker-Planck Equation

The time derivative is formally expanded, corresponding to evolution of the plasma on well-separated timescales: $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1} + \ \ldots$, where the terms are of order $v_{Ti}/L, \delta v_{Ti}/L \ \ldots$, and v_{Ti} is the bulk ion thermal velocity. We assume that beam modification of the bulk ion distribution function will enter as an order δ correction to the ion Maxwellian, which is shown below to be the zeroth order distribution function. A more precise statement will be given near the end of Sec. VI. The bulk ion collision frequency, ν_i, is initially taken to be of the order of the bulk ion transit frequency, v_{Ti}/L, and small compared with the bulk ion gyrofrequency $\Omega_i = eB/m_i$, such that $\nu_i \sim \delta\Omega_i$. The general axisymmetric form for the magnetic field is assumed: $\mathbf{B} = I\nabla\phi + \nabla\phi \times \nabla\Psi$, where ϕ is the toroidal angle, $I = RB_\phi$, R is the major radius, B_ϕ is the toroidal component of the magnetic field and Ψ is the poloidal flux function, so $|\nabla\Psi| = RB_p$. The poloidal field satisfies $B_p^2 = B^2 - I^2/R^2$.

Hinton and Wong take the leading order term in the expansion of the electric field, $\mathbf{E} = \mathbf{E}_{-1} + \mathbf{E}_0 + \mathbf{E}_1 + \ \ldots$, to be large. The electromagnetic terms in the Fokker-Planck equation therefore approximately balance, allowing for a bulk flow velocity which is comparable to the ion thermal speed. It is assumed that no changes in the magnetic field occur on the timescales of interest, so the electric field is given by an electrostatic potential, up to order zero. Hinton and Wong then transform Eq. (1) to a frame moving with the local rotation velocity, $\mathbf{u}_0$, which is found to be purely toroidal throughout the timescales we consider:

$$\mathbf{u}_0 = \omega\left(\Psi, t\right) R\hat{e}_\phi. \tag{2}$$

Here $\hat{e}_\phi$ is a unit vector in the toroidal direction and the angular rotation frequency, ω, is related to the potential via: $\omega = -\partial\Phi_{-1}/\partial\Psi$. Φ_{-1} is seen to be constant around a flux surface and as in Ref. [13], we include the flux surface average of Φ_0 in Φ_{-1}.

The zeroth order terms in the expansion are found to describe the relaxation of the lowest-order distribution function to a Maxwellian on the fast, zeroth order, timescale,

$$f_{i0} = n_{i0}\pi^{-3/2} v_{Ti}^{-3} e^{-\left(v'^2/v_{Ti}^2\right)}. \tag{3}$$

Here $\mathbf{v}' = \mathbf{v} - \omega R\hat{e}_\phi$ is the velocity in the rotating frame, $v_{Ti}^2 = 2T_i/m_i$ is the square of the bulk ion thermal velocity, where the bulk ion temperature, T_i, is a flux surface function. The zeroth order density, $n_{i0} = \int d^3v f_{i0}$, varies around a magnetic flux surface in the following way:

$$n_{i0} = N_i\left(\Psi\right) \exp\left(-\frac{e}{T_i}\tilde{\Phi}_0 + \frac{m_i\omega^2 R^2}{2T_i}\right), \tag{4}$$

where $N_i\left(\Psi\right)$ is an arbitrary flux function and $\tilde{\Phi}_0$, the poloidally varying part of Φ_0, is determined by requiring charge neutrality. As in Ref. [7], we wish to consider only effects occuring on slower timescales. Thus we assume that equilibrium has been achieved on the t_0 timescale and f_{i0} may subsequently be taken to be given by Eq. (3).

The form of the gyrophase-dependent piece of the first order correction to the bulk ion distribution function, $\tilde{f}_{i1}$, is determined from the zeroth order terms of the expansion of the Fokker-Planck equation and thus the expression obtained in Ref. [7] (Eq. (85)) is still valid in the presence of a strong source. The drift kinetic equation for $\bar{f}_{i1} = f_{i1} - \tilde{f}_{i1}$, the gyroaverage of f_{i1}, derives from the first order terms of the expansion of Eq. (1):

$$\Omega_i \frac{\partial f_{i2}}{\partial \zeta} = \frac{\partial f_{i0}}{\partial t_1} + \Lambda\left(f_{i1}\right) + \left(\frac{e}{m_i}\mathbf{E}_1 - \frac{\partial \mathbf{u}_0}{\partial t_1}\right) \cdot \frac{\partial f_{i0}}{\partial \mathbf{v}'} - C_{ii}^l\left(f_{i1}\right) - S_{i1}. \tag{5}$$

As in both Refs. [7] and [3], we take the effects of electron-ion collisions to be higher order in δ, for both thermal and fast ions, due to the small electron-ion mass ratio, so here C_{ii}^l is the linearized bulk ion self-collision operator. S_{i1} is the first order source term and as in Ref. [7] we have introduced cylindrical velocity coordinates, $v_\parallel$, $v_\perp$ and the gyroangle, ζ, in the rotating frame, such that $\mathbf{v}' = v_\parallel \mathbf{b} + v_\perp\left(\hat{e}_1 \cos\zeta + \hat{e}_2 \sin\zeta\right)$. The subscript parallel refers to the magnetic field direction and $\mathbf{b} = \mathbf{B}/B$, $\hat{e}_1$ and $\hat{e}_2$ are unit orthogonal vectors. The linear operator Λ, is not directly affected by the inclusion of a first order source and is defined by:

$$\Lambda\left(f\right) = \left(\mathbf{v}' + \mathbf{u}_0\right) \cdot \nabla' f - \left(\frac{e}{m_i}\nabla\Phi_0 + \frac{\partial \mathbf{u}_0}{\partial t_0} + \left(\mathbf{v}' + \mathbf{u}_0\right) \cdot \nabla\mathbf{u}_0\right) \cdot \frac{\partial f}{\partial \mathbf{v}'}$$

where, with the spatial gradient ∇ taken at constant $v_\parallel, v_\perp$ and ζ:

$$\nabla' f = \nabla f + \left(\nabla\mathbf{b}\right) \cdot \mathbf{v}_\perp \left(\frac{\partial f}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial f}{\partial v_\perp}\right) + \left[\left(\nabla\hat{e}_2\right)\cos\zeta - \left(\nabla\hat{e}_1\right)\sin\zeta\right] \cdot \frac{\mathbf{v}'}{v_\perp}\frac{\partial f}{\partial \zeta}$$

Note the appearance of terms in Eq. (5) describing the variation, on the t_1 timescale, of the parameters of the bulk ion Maxwellian established on the t_0 timescale, which ensure the conservation of particles, energy and angular momentum injected by the source. These variations will be determined explicitly in Sec. IV, for the case when the source term represents strong NBI.

We now gyroaverage Eq. (5) and following Ref. [7], substitute the expression for $\tilde{f}_{i1}$, remembering that the fast thermalisation on the t_0 timescale is assumed to be complete. As Hinton and Wong note, the resulting equation is simplified by the choice of $\mathbf{x}, t, \mu$ and H as independent variables, where the latter are, respectively, the magnetic moment and energy of a bulk ion in the rotating frame:

$$\mu = m_i v_\perp^2 / 2B \tag{6}$$

and

$$H = \frac{m_i}{2}\left(v_\parallel^2 + v_\perp^2\right) + e\tilde{\Phi}_0 - \frac{m_i \omega^2 R^2}{2}. \tag{7}$$

Thus the linearized drift kinetic equation for $\bar{f}_{i1}$, retaining the effects of a first order source term, is:

$$\frac{\partial f_{i0}}{\partial t_1} + v_\parallel \mathbf{b}\cdot\nabla \bar{f}_{i1} + \frac{e}{T} v_\parallel \mathbf{b}\cdot\nabla\Phi_1 f_{i0} + \frac{m_i}{T} v_\parallel \mathbf{b}\cdot\frac{\partial \mathbf{u}_0}{\partial t_1} f_{i0} + v_\parallel f_{i0}\sum_{j=1}^{3} A_j(\Psi)\,\mathbf{b}\cdot\nabla\alpha_j = \bar{C}_{ii}^l\left(\bar{f}_{i1}\right) + \bar{S}_{i1}. \tag{8}$$

The driving terms, A_j, are the radial gradients:

$$A_1 = \frac{N_i'}{N_i} + \frac{T_i'}{T_i}, \qquad A_2 = \frac{T_i'}{T_i} \qquad \text{and} \qquad A_3 = \frac{\omega'}{\omega},$$

where a prime on a scalar quantity indicates the derivative with respect to Ψ, and the coefficients, α_j, are given by:

$$\alpha_1 = \frac{m_i}{e}\left[\frac{I v_\parallel}{B} + \omega R^2\right], \qquad \alpha_2 = \left(\frac{H}{T_i} - \frac{5}{2}\right)\alpha_1 \qquad \text{and} \qquad \alpha_3 = \frac{m_i^2\omega}{2eT_i}\left[\left(\frac{I v_\parallel}{B} + \omega R^2\right)^2 + \frac{\mu|\nabla\Psi|^2}{m_i B}\right].$$

B. Radial Transport

Our aim is to determine the effect on radial bulk ion transport of the strong source term. The radial magnetic surface averaged fluxes of particles, energy and toroidal angular momentum carried by the bulk ions are defined in the laboratory frame, respectively, by (see Ref. [7]):

$$\Gamma_i = \left\langle \int d^3v\,\mathbf{v}\cdot\nabla\Psi f_i \right\rangle, \tag{9}$$

$$Q_i = \left\langle \int d^3v \left(\frac{1}{2} m_i v^2 + e\tilde{\Phi}\right)\mathbf{v}\cdot\nabla\Psi f_i \right\rangle \tag{10}$$

and

$$\Pi_i = \left\langle \int d^3v\, m_i R v_\phi \mathbf{v}\cdot\nabla\Psi f_i \right\rangle. \tag{11}$$

We further define the heat flux q, such that $q = Q - \omega\Pi - \frac{5}{2}T\Gamma$. Here $v_\phi = \mathbf{v}\cdot\hat{e}_\phi$ and the angled brackets represent the usual flux surface averaging operation, defined by:

$$\langle A\rangle = \oint A\frac{dl_p}{B_p} / \oint \frac{dl_p}{B_p} \equiv \oint A\frac{dl_p}{B_p}/V', \tag{12}$$

where dl_p is an element of length poloidally around the flux surface.

Remembering that we assume all changes which occur on the fast, t_0, timescale are complete, the procedure followed by Hinton and Wong(7) shows that the leading order contributions to the cross-field bulk ion fluxes are still second order in the parameter δ, as usual. As is pointed out by Hinton and Wong, these higher order particle, energy and toroidal angular momentum fluxes may be more conveniently evaluated using, respectively, the $m_i v_\phi R, m_i v^2 v_\phi R/2$ and $m_i v_\phi^2 R^2/2$ flux surface averaged velocity moments of the Fokker-Planck equation, which give:

$$\frac{\partial}{\partial t_1}\langle m_i n_{i0} u_{0\phi} R\rangle = e\Gamma_{i2} + \left\langle \int d^3v m_i v_\phi R \left(C_{ii}^l\left(f_{i1}\right) + S_{i1}\right)\right\rangle, \tag{13}$$

$$\begin{aligned}\frac{\partial}{\partial t_1}\left\langle \int d^3v \frac{1}{2} m_i v^2 R v_\phi f_{i0}\right\rangle = & - \frac{e}{m_i c}\left(Q_{i2} + \omega\Pi_{i2}\right) \\ & + \left\langle \int d^3v v_\phi R\left(\frac{1}{2}m_i v^2 + e\tilde{\Phi}_0\right)\left(C_{ii}^l\left(f_{i1}\right) + S_{i1}\right)\right\rangle\end{aligned} \tag{14}$$

and

$$\frac{\partial}{\partial t_1}\left\langle \int d^3v \frac{1}{2} m_i v_\phi^2 R^2 f_{i0}\right\rangle = \frac{e}{m_i}\Pi_{i2} + \left\langle \int d^3v \frac{1}{2} m_i v_\phi^2 R^2\left(C_{ii}^l\left(f_{i1}\right) + S_{i1}\right)\right\rangle. \tag{15}$$

The time-derivative terms on the left-hand side of Eqs. (13) - (15) may be evaluated using the zeroth-order Maxwellian, Eq. (3). This is done in the next section. Note that, in the case of a weak source, the time derivatives on the t_1 timescale may be set to zero and $S_{i1} = 0$, so Eqs. (13) - (15) reduce to flux-friction relations, as given in Ref. [7].

III. NEUTRAL BEAM INJECTION AS A SOURCE TERM

Now we will specialise to the case in which the source term in the kinetic equation represents neutral beam injection. We formally take the fast ion density to be sufficiently low, that the leading order effect of the source is first order in the expansion parameter δ. After approximately a slowing down time, the fast ions act as a first order particle source(12). The deflection time is much shorter than the slowing down time, so the particle source will have an isotropic distribution in velocity space. We therefore describe the effect of the fast ions in the bulk ion kinetic equation by a time dependent source term, $S_p(v,t)$, plus the bulk-fast ion collision operator, C_{if}. (Throughout a subscript f will refer to properties of the fast ions.) Thus, $S_{i1} = S_p(v,t) + C_{if}\left(f_{i0}(t), f_{f0}(t)\right)$. Due to the low fast ion density, the two arguments of the collision operator are the lowest order distribution functions of both the bulk ions, f_{i0}, and of the fast ions when expanded in the fast ion gyroradius, f_{f0}, which is also independent of gyrophase. Note that an equivalent sink, $-S_p$, must appear in the fast ion kinetic equation.

With this form of source term, the expressions for the radial fluxes, Eqs. (13) - (15), and the drift kinetic equation (8), are linear in the first order bulk ion distribution function. It is therefore useful to write $\bar{f}_{i1}$ as the sum of a part driven by radial gradients of f_{i0}, which was previously determined by Hinton and Wong in Ref. [7] and which we shall call f^{HW}, and a part, f^b, which describes the additional structure produced by the NBI: $\bar{f}_{i1} = f^{HW} + f^{b}$. Substituting this into the drift kinetic equation (8), gives the drift kinetic equation for f^{HW} obtained in Ref. [7] (Eq. (110)) and also:

$$\frac{\partial f_{i0}}{\partial t_1} + \frac{m_i}{T_i} v_{\|} \mathbf{b} \cdot \frac{\partial \mathbf{u}_0}{\partial t_1} f_{i0} + v_{\|} \mathbf{b} \cdot \nabla f^b = \bar{C}_{ii}^l \left(f^b\right) + \bar{C}_{if} \left(f_{i0}, f_{f0}\right) + S_p. \tag{16}$$

Using Eqs. (2) and (3), we may write the time dependent terms as variations of the bulk ion parameters and thus obtain the following drift kinetic equation for f^b:

$$v_{\|} \mathbf{b} \cdot \nabla \left(f^b\right) + \left[\frac{\partial \ln n_{i0}}{\partial t_1} + \left(\frac{m_i \left(v'\right)^2}{2T_i} - \frac{3}{2}\right) \frac{\partial \ln T_i}{\partial t_1} + \frac{m_i I v_{\|}}{T_i B} \frac{\partial \omega}{\partial t_1}\right] f_{i0} = \bar{C}_{ii}^l \left(f^b\right) + \bar{C}_{if} \left(f_{i0}, f_{f0}\right) + S_p. \tag{17}$$

By using approximate forms for C_{ii}^l and C_{if}, which are presented in the following subsection, we will solve this equation directly for f^b.

The contributions from $\tilde{f}_{i1}$ and f^{HW} to Eqs. (13) - (15) for the leading order radial fluxes, Γ_{i2}, q_{i2} and Π_{i2}, are the usual fluxes which occur when there is no NBI. They are given in Refs. [7] and [13]. The additional contributions to these radial fluxes, which arise in the presence of the NBI are given by the remaining terms in Eqs. (13) - (15). We denote them by a superscript b:

$$\Gamma^b = -\left\langle \int d^3 v \alpha_1 \left[\bar{C}_{ii}^l \left(f^b\right) + \bar{C}_{if} \left(f_{i0}, f_{f0}\right) + S_p\right] \right\rangle + \frac{m_i}{e} \left\langle n_{i0} \omega R^2 \left[\frac{\partial \ln n_{i0}}{\partial t_1} + \frac{\partial \ln \omega}{\partial t_1}\right] \right\rangle, \tag{18}$$

$$\begin{aligned} q^b = & -\left\langle \int d^3 v T_i \alpha_2 \left[\bar{C}_{ii}^l \left(f^b\right) + \bar{C}_{if} \left(f_{i0}, f_{f0}\right) + S_p\right] \right\rangle \\ & + \frac{m_i}{e} \left\langle p_{i0} \omega R^2 \left[\frac{3}{2} \frac{\partial \ln T_i}{\partial t_1} - \left(1 + M_i^2\right) \frac{\partial \ln n_{i0}}{\partial t_1} - M_i^2 \frac{\partial \ln \omega}{\partial t_1}\right] \right\rangle \end{aligned} \tag{19}$$

and

$$\begin{aligned} \Pi^b = & -\left\langle \int d^3 v \frac{T_i}{\omega} \alpha_3 \left[\bar{C}_{ii}^l \left(f^b\right) + \bar{C}_{if} \left(f_{i0}, f_{f0}\right) + S_p\right] \right\rangle \\ & + \frac{m_i}{e} \left\langle p_{i0} R^2 \left[\frac{1}{2} \frac{\partial \ln T_i}{\partial t_1} + \left(1 + M_i^2\right) \frac{\partial \ln n_{i0}}{\partial t_1} + M_i^2 \frac{\partial \ln \omega}{\partial t_1}\right] \right\rangle. \end{aligned} \tag{20}$$

Here we have used Eq. (3) and converted to velocity variables in the rotating frame to evaluate:

$$\int d^3 v \frac{1}{2} m_i v^2 R v_\phi f_{i0} = \left[5 p_{i0} \omega R + m_i n_{i0} \omega^3 R^3\right] \frac{R}{2} \tag{21}$$

and

$$\int d^3 v \frac{1}{2} m_i v_\phi^2 R^2 f_{i0} = \left[p_{i0} + m_i n_{i0} \omega^2 R^2\right] \frac{R^2}{2}. \tag{22}$$

The ion pressure, p_i, is defined as:

$$\frac{3}{2} p_i = \int d^3 v \frac{1}{2} m_i \left(\mathbf{v} - \mathbf{u}_0\right)^2 f_i = \int d^3 v \frac{1}{2} m_i \left(v'\right)^2 f_i \tag{23}$$

and the square of the toroidal bulk ion Mach number is $M_i^2 = m_i \omega^2 R^2 / 2T_i$. The integrals remaining in Eqs. (18) - (20) have been written in terms of velocity variables in the rotating frame, as was done in Ref. [7], so that the form of f^b resulting from Eq. (17) may be used directly. Note that the first terms in the above expressions for the beam driven fluxes, involving the collision operators, are contributions to the neoclassical fluxes, whilst the remaining, time dependent terms, contain contributions to both the neoclassical and classical fluxes, as they include the classical polarisation current, which will be seen in Sec. IV.

Bulk Ion Collision Operator

The drift kinetic equation (17) may be solved for f^b by using explicit forms for the two collision operators, the gyroaveraged, linearised, bulk ion self-collision operator, $\bar{C}^l_{ii}$, and the bulk-fast ion collision operator, $\bar{C}_{if}\left(f_{i0}, f_{f0}\right)$. We use the gyroaveraged, model self-collision operator of Kovrizhnykh[12], for $\bar{C}^l_{ii}$:

$$\bar{C}^l_{ii}\left(f^b\right) = \nu^{ii}_D \mathcal{L}\left(f^b\right) + \frac{m_i}{T_i}\nu^{ii}_D v_\parallel s_\parallel f_{i0}. \tag{24}$$

The first term represents pitch-angle scattering and the second momentum conservation. The deflection frequency is given by $\nu^{ii}_D = \left(3\pi^{1/2}/4\tau_{ii}\right)\left(\phi\left(x\right) - G\left(x\right)\right)/x^3$, where $x = v'/v_{Ti}$, $\phi(x)$ is the error function, $G(x) = (\phi(x) - x\phi'(x))/2x^2$ is the Chandrasekhar function and $\sqrt{2}\tau_{ii} = \tau_i$, with τ_i the ion-ion collision time defined by Braginskii[14]. We require only the gyrophase independent part of the Lorentz operator:

$$2\mathcal{L} = \frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial}{\partial \xi},$$

where the pitch angle cosine is $\xi = v_\parallel/v'$. The constant vector $\mathbf{s}$ is chosen to ensure momentum is conserved in collisions:

$$\int d^3v m_i \mathbf{v}' C^l_{ii}\left(f^b\right) = 0. \tag{25}$$

A useful form for $\bar{C}_{if}\left(f_{i0}, \bar{f}_{f0}\right)$ may be obtained by expanding the bulk-fast ion collision operator, in the small ratio of the typical bulk and fast ion velocities. The details of the derivation, starting from the Landau expression for the collision operator in the laboratory frame, are given in the Appendix. Defining parallel and perpendicular components of the fast ion velocity in the rotating frame, $\mathbf{w}$, which are analogous to those introduced for the bulk ions in Sec. II, and retaining terms up to the order $1/w^2$, the expanded collision operator may be written conveniently in terms of the n-th order Legendre polynomials, P_n, in the cosines of the bulk ion pitch angle variable, $v_\parallel/v'$:

$$\bar{C}_{if}\left(f_{i0}, f_{f0}\right) = \frac{2m_i}{T_i}\left[\left(\frac{2}{3}x^2 - 1\right)P_0 B_0 + \left(\frac{m_i}{m_b} - 2 + \frac{6}{5}x^2\right)xP_1B_1 - \frac{2}{3}x^2P_2B_2 - \frac{6}{5}x^3P_3B_3\right]f_{i0}. \tag{26}$$

The B_n are velocity moments of the fast ion distribution function:

$$B_n = \gamma_{if}\int d^3w f_{f0}\frac{1}{w}\zeta_n P_n\left(\frac{w_\parallel}{w}\right), \tag{27}$$

where $\gamma_{if} = e_i^2 e_f^2 \ln\Lambda/8\pi\epsilon_0^2 m_i^2$ and the factor $\zeta_n = 1$ for n even and $\zeta_n = v_{Ti}/w$ for n odd. We assume that f_{f0} is known, so these moments may be determined. Note that Hirshman and Sigmar[15] point out that the effects of higher order structure in pitch angle of a distribution function may be significant in determining neoclassical radial transport, due to the presence of the parallel convective term in the kinetic equation. In their more approximate treatment, Hinton and Kim[3] only retained the term in the collision operator describing momentum exchange between the fast and bulk ions, that is the P_1 term of Eq. (26) with the factor $\left(\frac{m_i}{m_f} - 2 + \frac{6}{5}x^2\right)$ replaced by $\left(\frac{m_i}{m_f} + 1\right)$. In some respects this is the most important term, particularly at large aspect ratio as we shall see, but the other terms can also contribute significantly to the transport.

IV. CONSERVATION LAWS

In this section, the changes in the parameters of the zeroth order bulk ion Maxwellian distribution function will be related to velocity moments of the source term, C_{if}. This can be done by considering the conservation equations which must hold on the timescales considered. Conservation laws for particles, momentum and energy during the evolution of the distribution function may be derived by considering a further set of three velocity moments, $1, m_i v_\phi R$ and $m_i v^2/2$, of the Fokker-Planck equation for the bulk ions in the laboratory frame[7]:

$$\frac{\partial \langle n_i \rangle}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial \Psi} V' \Gamma_i = \left\langle \int d^3 v S_i \right\rangle, \tag{28}$$

$$\frac{\partial}{\partial t} \langle m_i n_i u_{i\phi} R \rangle + \frac{1}{V'}\frac{\partial}{\partial \Psi} V' \Pi_i = e\Gamma_i + \left\langle \int d^3 v m_i v_\phi R \left(C_i + S_i\right) \right\rangle \tag{29}$$

and

$$\frac{\partial}{\partial t}\left\langle \frac{3}{2}p_i + \frac{1}{2}m_i n_i u_i^2 + n_i e\tilde{\Phi} \right\rangle + \frac{1}{V'}\frac{\partial}{\partial \Psi} V' Q_i = -e\Gamma_i \frac{\partial \Phi}{\partial \Psi} + \left\langle n_i e \frac{\partial \tilde{\Phi}}{\partial t} \right\rangle + \left\langle \int d^3 v \left(\frac{1}{2} m_i v^2 + e\tilde{\Phi}\right)\left(C_i + S_i\right) \right\rangle . \tag{30}$$

We have neglected the effects of an inductive electric field, as it is taken to be at most second order with respect to δ here. Hinton and Wong discussed the conservation laws which hold during evolution on the t_0 and t_1 timescales in Ref. [7]. The method presented remains valid in the presence of a first order source and we outline here the discussion for the t_1 timescale, as differences appear in this order. Note it has been assumed that no rapid changes in the magnetic field are taking place, thus time derivatives on the t_1 timescale are interchangeable with the flux surface averaging operation.

The first order terms in Eq. (28) give the evolution of n_{i0} and simply describe particle conservation on the first order timescale:

$$\frac{\partial}{\partial t_1} \langle n_{i0} \rangle = \left\langle \int d^3 v S_{i1} \right\rangle = \left\langle \int d^3 v S_p \left(v, t\right) \right\rangle . \tag{31}$$

Note that, because pitch-angle scattering of the fast ions is strong, we may assume that the particle source is independent of poloidal angle and no poloidal dependence of n_{i0} is introduced via the source.

The first order terms in Eq. (29) are:

$$\frac{\partial}{\partial t_1} \langle m_i n_{i0} u_{0\phi} R \rangle = e\Gamma_{i2} + \left\langle \int d^3 v m_i v_\phi R \left(C_{ii}^l \left(f_{i1}\right) + S_{i1}\right) \right\rangle . \tag{32}$$

This equation, with suitable replacements of parameters, such as $m_i \to m_f$, also holds for the electrons and fast ions, as no assumptions regarding the distribution function have been made in obtaining it. The first order part, S_{f1}, of the source term in the fast ion kinetic equation, would then appear, which describes the deposition of angular momentum, as well as the loss of thermalised particles to the bulk and fast-bulk ion collisions.

Taking the electron mass to be negligible in comparison to the ion mass, we see from Eq. (32) that $\Gamma_{e2} = 0$ for the electrons. Therefore the second order radial current density, averaged over a flux surface, is:

$$\langle \mathbf{j}_2 \cdot \nabla \Psi \rangle = e\Gamma_{i2} + z_f e \Gamma_{f2}, \tag{33}$$

where z_f is the charge on a fast ion. The flux surface average of the scalar product of $\nabla\Psi$ with the second order terms in the Maxwell equation:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_2 + \frac{1}{c^2}\frac{\partial \mathbf{E_{-1}}}{\partial t_1}, \tag{34}$$

relates the current density to the changing electric field:

$$\langle \mathbf{j}_2 \cdot \nabla\Psi \rangle = -\epsilon_0 \frac{\partial}{\partial t_1} \langle \mathbf{E}_{-1} \cdot \nabla\Psi \rangle , \tag{35}$$

where, from Sec. I:

$$\langle \mathbf{E}_{-1} \cdot \nabla\Psi \rangle = \omega \left\langle |\nabla\Psi|^2 \right\rangle . \tag{36}$$

Thus, summing Eq. (32) for each of the three plasma species, we obtain the conservation equation for angular momentum on the t_1 timescale:

$$\frac{\partial}{\partial t_1}\left(\langle m_i n_{i0} \mathbf{u}_0 \cdot \hat{e}_\phi R \rangle + \langle m_f n_{f0} \mathbf{u}_{f0} \cdot \hat{e}_\phi R \rangle + \omega \left\langle |\nabla\Psi|^2 \right\rangle \right) = \left\langle \int d^3v m_f v_\phi R S_{f1}^D \right\rangle . \tag{37}$$

Here, $\mathbf{u}_{f0}$ is the zeroth order velocity of the fast ions. Conservation of momentum in collisions means that only that part of the source term describing the (first order) deposition of angular momentum by the charged beam particles, denoted by S_{f1}^D, remains. The storage of angular momentum injected into the system is represented by the terms on the left of Eq. (37). As Hinton and Wong point out in Ref. [7], the first two terms represent the total mechanical angular momentum of the bulk and fast ions, whilst the third term represents the angular momentum in the electromagnetic field, each per unit volume and averaged over the volume between two adjacent magnetic surfaces. Unlike Ref. [7], we have a non-zero angular momentum source term and therefore mechanical angular momentum is not conserved on the t_1 timescale.

The physical processes by which angular momentum is transferred through the system have been discussed previously, for example by Hinton and Rosenbluth(8) or Helander, Akers and Eriksson(16). The motion of the fast ions resulting from the injection, due to their drift velocity, represents a radial displacement current, which sets up a time-varying radial electric field. This induces a radial return 'polarization' current of the bulk plasma ions, which consists of both a classical and a much larger, neoclassical contribution. (The neoclassical current accounts for poloidal inertia of the plasma and thus only the classical part is directly related to the toroidal acceleration of the bulk plasma, via the $\mathbf{j} \times \mathbf{B}$ forces experienced by all of the currents.) As noted by Hinton and Rosenbluth [8], in typical tokamak plasmas the effective dielectric constant of the plasma, $\mu_0 n_{i0} m_i c^2 / B_p^2$, is large, which is equivalent to the statement that the Alfvén speed evaluated using the poloidal magnetic field is much slower than the speed of light. Thus, in Eq. (37), the angular momentum stored in the field is seen to be negligible by comparison to the mechanical angular momentum and we may neglect any imbalance between the displacement and polarization currents in leading order:

$$\Gamma_{i2} \approx -z_f \Gamma_{f2}. \tag{38}$$

Finally, note that a steady state solution on the t_1 timescale would have the left-hand side of Eq. (37) equal to zero and thus a sink term would be required to ensure angular momentum is conserved.

Hinton and Kim, in Ref. [3], assume a turbulent sink is present and go on to determine such a steady state solution. The energy conservation equation follows from the first order terms of Eq. (30) in an analogous manner to that shown above, so we omit the details here.

We may now reconstruct the conservation laws, obtained above from fluid equations, by taking moments of the drift kinetic equation (17). This leads to useful forms, which directly relate velocity moments of the source term, C_{if}, to the evolution of the Maxwellian parameters of the zeroth order bulk ion distribution function on the t_1 timescale. Such expressions are required to solve the drift kinetic equation (17) and to evaluate the cross-field fluxes, Eqs. (18) to (20). We first introduce a generalization of the usual flux surface average defined in Eq. (12), which has meaning in both the trapped and passing domains$^{(13)}$:

$$\langle A \rangle = \frac{1}{\sum_\sigma |\sigma|} \sum_\sigma \int_{-\theta_b}^{\theta_b} A \frac{d\theta}{\mathbf{B}\cdot\nabla\theta} / \int_{-\pi}^{\pi} \frac{d\theta}{\mathbf{B}\cdot\nabla\theta}, \tag{39}$$

Here $\sigma = v_\| / |v_\||$ and the sum over σ is taken over the course of one orbit, noting that σ is constant over a passing orbit. The angle θ measures poloidal position on a flux surface, with $\theta = 0$ at the outboard side and $\pm\theta_b$ defines the bounce points of the particle orbit, so is $\pm\pi$ for passing particles.

To consider particle conservation on the first order timescale, we take the zeroth order velocity moment of the drift kinetic equation (17) and average over a flux surface. Noting that, due to the periodicity of a particle distribution function with respect to the poloidal angle:

$$\left\langle \int d^3v v_\| \mathbf{b}\cdot\nabla f^b \right\rangle = \frac{2\pi}{m_i^2} \int dH \int \left\langle \mathbf{B}\cdot\nabla f^b \right\rangle d\mu = 0, \tag{40}$$

where the use of the generalized flux surface average has allowed us to interchange the order of integration over velocity space and the flux surface average. As particles are conserved in collisions, upon remembering that we may interchange first order time derivatives and the flux surface averaging operation, we recover the fluid equation (31).

The variation of T_i on the t_1 timescale is obtained by taking the H/T_i moment of Eq. (17) and flux surface averaging. As H and T_i are flux surface functions, we have:

$$\left\langle \int d^3v \frac{H}{T_i} v_\| \mathbf{b}\cdot\nabla f^b \right\rangle = \int dH \frac{H}{T_i} \int \left\langle \mathbf{B}\cdot\nabla f^b \right\rangle d\mu = 0. \tag{41}$$

Also, as the particle source term is isotropic in velocity space:

$$\left\langle \int d^3v \frac{H}{T_i} \left[\frac{\partial \ln n_{i0}}{\partial t_1} f_{i0} - S_p \right] \right\rangle = \left\langle \frac{3}{2} \frac{\partial \ln n_{i0}}{\partial t_1} - \int d^3v x^2 S_p \right\rangle + \left\langle \frac{\alpha}{T_i} \left(\frac{\partial n_{i0}}{\partial t_1} - \int d^3v S_p \right) \right\rangle, \tag{42}$$

where $\alpha(\theta) = e\tilde{\Phi}_0 - m_i\omega^2 R^2/2$ and we remember from Sec. III that $x = v'/v_{Ti}$. The first term on the right-hand side is zero by Eq. (31), that is particle conservation. The second term is also zero, as the particles described by the source term are thermalized and therefore do not add energy to the bulk plasma. Thus, remembering energy conservation in like species collisions: $\int d^3v v^2 \bar{C}_{ii}^l \left(f^b\right) = 0$, the equation for the evolution of T_i becomes:

$$\frac{3}{2} \langle n_{i0} \rangle \frac{\partial \ln T_i}{\partial t_1} = \left\langle \int d^3v x^2 \bar{C}_{if} \left(f_{i0}, f_{f0}\right) \right\rangle. \tag{43}$$

To determine the variation of the toroidal rotation frequency, we consider the flux surface average of the $m_i \bar{v}_\phi R$ moment of Eq. (17), where $\bar{v}_\phi$ is the gyroaverage of v_ϕ, writing $m_i \bar{v}_\phi R = m_i \left(\frac{I v_\parallel}{B} + \omega R^2 \right)$ in terms of variables in the rotating frame. The terms which remain after symmetry in $v_\parallel$ and the particle and momentum conservation properties of the collision operators are taken into account are:

$$\left\langle \int d^3v \left[m_i \left(\frac{I v_\parallel}{B} + \omega R^2 \right) v_\parallel \mathbf{b} \cdot \nabla \left(f^b \right) + m_i \omega R^2 \left(\frac{\partial \ln n_{i0}}{\partial t_1} f_{i0} - S_p \right) + \frac{m_i^2 I^2}{T_i B^2} v_\parallel^2 \frac{\partial \omega}{\partial t_1} f_{i0} \right] \right\rangle = \left\langle \int d^3v \frac{m_i I}{B} v_\parallel \bar{C}_{if} \left(f_{i0}, f_{f0} \right) \right\rangle. \quad (44)$$

Noting that Hinton and Wong[7] determine the radial component of the bulk ion guiding centre drift velocity in the rotating frame to be $\mathbf{v}_D \cdot \nabla \Psi = \frac{m_i}{e} v_\parallel \mathbf{b} \cdot \nabla \left(\frac{I v_\parallel}{B} + \omega R^2 \right)$, the first term may be re-written, upon integrating by parts, as:

$$\left\langle \int d^3v m_i \left(\frac{I v_\parallel}{B} + \omega R^2 \right) v_\parallel \mathbf{b} \cdot \nabla \left(f^b \right) \right\rangle = -e \left\langle \int d^3v \mathbf{v}_D \cdot \nabla \Psi f^b \right\rangle. \quad (45)$$

By definition, this is the negative of the radial current carried by the bulk ion guiding centres, which we may write as $- \langle \mathbf{j}_{gc} \cdot \nabla \Psi \rangle$.

The polarization current discussed above originates from the second and third terms in Eq. (44). Interchanging the time derivative and the flux surface average, we may write:

$$\left\langle m_i n_{i0} \omega R^2 \frac{\partial \ln n_{i0}}{\partial t_1} \right\rangle + \left\langle m_i n_{i0} \frac{I^2}{B^2} \frac{\partial \omega}{\partial t_1} \right\rangle = \frac{\partial}{\partial t_1} \left\langle m_i n_{i0} \omega R^2 \right\rangle - \left\langle m_i n_{i0} \frac{|\nabla \Psi|^2}{B^2} \frac{\partial \omega}{\partial t_1} \right\rangle. \quad (46)$$

Using Eq. (2), we see that the second term is the leading-order classical polarisation current:

$$\langle \mathbf{j}_p \cdot \nabla \Psi \rangle = \left\langle m_i n_{i0} \frac{1}{B^2} |\nabla \Psi|^2 \frac{\partial \omega}{\partial t_1} \right\rangle = \left\langle m_i n_{i0} \frac{1}{B^2} \frac{\partial \mathbf{E}_{-1}}{\partial t_1} \cdot \nabla \Psi \right\rangle. \quad (47)$$

Thus, the equation for the conservation of bulk ion toroidal angular momentum takes the form:

$$\frac{\partial}{\partial t_1} \left\langle m_i n_{i0} \omega R^2 \right\rangle = \langle (\mathbf{j}_{gc} + \mathbf{j}_p) \cdot \nabla \Psi \rangle + \left\langle \int d^3v m_i \left(\frac{I v_\parallel}{B} \bar{C}_{if} \left(f_{i0}, f_{f0} \right) + \omega R^2 S_p \right) \right\rangle \quad (48)$$

and the correspondence to the fluid conservation equation, (32), may be seen. The physical interpretation of this equation is clear: the angular momentum of the bulk plasma increases due to the $\mathbf{j} \times \mathbf{B}$ torque experienced by the bulk ions and collisional friction with the fast ions, as well as the deposition of angular momentum by the particle source.

Assuming that the dielectric constant of the plasma is large, we may use Eq. (38) to substitute $-z_f \Gamma_{f2}$ for $\Gamma_{i2} = \langle (\mathbf{j}_{gc} + \mathbf{j}_p) \cdot \nabla \Psi \rangle$. Upon expanding the first term of Eq. (48) and remembering particle conservation as given by Eq. (31), the required relation for the change of the toroidal rotation frequency, ω, on the t_1 timescale is obtained:

$$\left\langle n_{i0} R^2 \right\rangle \frac{\partial \omega}{\partial t_1} = -\frac{z_f e}{m_i} \Gamma_{f2} + \left\langle \int d^3v \frac{I}{B} v_\parallel \bar{C}_{if} \left(f_{i0}, \bar{f}_{f0} \right) \right\rangle. \quad (49)$$

Finally in this section, we evaluate the velocity moments of the bulk-fast ion collision operator required in Eqs. (43) and (49), which is done most readily using the form of the bulk-fast ion collision operator given in Eq. (26):

$$\left\langle \int d^3v x^2 \bar{C}_{if} \left(f_{i0}, f_{f0} \right) \right\rangle = 2 \frac{m_i}{T_i} \langle n_{i0} B_0 \rangle \quad (50)$$

and

$$\left\langle \int d^3v \frac{v_\parallel}{B} \bar{C}_{if}\left(f_{i0}, f_{f0}\right) \right\rangle = 2\left(1 + \frac{m_i}{m_f}\right) \left\langle \frac{n_{i0} B_1}{B v_{Ti}} \right\rangle . \tag{51}$$

V. NEOCLASSICAL FLUXES DRIVEN BY NBI

We may now proceed to solve the drift kinetic equation, (17), for f^b and evaluate the additional radial fluxes, Eqs. (18) - (20), arising in the presence of NBI. We consider the case of a low-collisionality, "banana regime", bulk plasma and thus we may perform a subsidary expansion of Eq. (17) in the small ratio of the bulk ion collision frequency to their typical bounce frequency, in the usual manner. We assume that the beam density is sufficiently low that both the source term and the time dependent terms are of the same order as the bulk ion self-collision term, hence the bulk ions establish basic drift orbits. Indicating the order of expansion with a bracketed superscript, the zeroth order equation of the expansion shows that $f^{b(0)}$ is a constant along a field line, thus is a function only of H, μ, σ and Ψ. The first order equation is a constraint which will allow $f^{b(0)}$ to be determined explicitly:

$$v_\parallel \mathbf{b} \cdot \nabla f^{b(1)} + \left[\frac{\partial \ln n_{i0}}{\partial t_1} + \left(x^2 - \frac{3}{2}\right) \frac{\partial \ln T_i}{\partial t_1} + \frac{m_i I v_\parallel}{T_i B} \frac{\partial \omega}{\partial t_1} \right] f_{i0} = \bar{C}^l_{ii}\left(f^{b(0)}\right) + \bar{C}_{if}\left(f_{i0}, f_{f0}\right) + S_p. \tag{52}$$

Equation (52) is solved for $\partial f^{b(0)}/\partial \mu$ in the usual way[12]. In the passing regime we multiply by $B/v_\parallel$ and take the flux surface average. This annihilates the first term and the constraint equation is therefore:

$$\begin{aligned} m_i \frac{\partial}{\partial \mu} \mu \left\langle \nu_D^{ii} v_\parallel \right\rangle \frac{\partial f^{b(0)}}{\partial \mu} + \frac{m_i}{T_i} \left\langle B \nu_D^{ii} w_\parallel f_{i0} \right\rangle &= \left\langle \frac{B}{v_\parallel} \left[\frac{\partial \ln n_{i0}}{\partial t_1} - \frac{S_p}{f_{i0}} + \left(x^2 - \frac{3}{2}\right) \frac{\partial \ln T_i}{\partial t_1} + \frac{m_i I v_\parallel}{T_i B} \frac{\partial \omega}{\partial t_1} \right] f_{i0} \right\rangle \\ &- \left\langle \frac{B}{v_\parallel} \bar{C}_{if}\left(f_{i0}, f_{f0}\right) \right\rangle . \end{aligned} \tag{53}$$

This must be integrated over μ, from $\mu = 0$ to an arbitrary $\mu < \mu_c(H)$, where $\mu_c(H)$ defines the trapped-passing boundary and is a function of the particle energy. The order of the flux surface averaging operation and the integral over μ may be interchanged, as μ is constant on a flux surface. Using the bulk-fast ion collision operator given by Eq. (26), the integrals are conveniently evaluated when expressed in terms of the variable ξ, introduced in Sec. III, noting that:

$$\left.\frac{\partial \xi}{\partial \mu}\right|_H = -\frac{B}{m_i v^2 \xi} \tag{54}$$

and the integration limit $\mu = 0$ becomes $\xi = \sigma$, whilst the upper limit, μ, becomes $\xi(\mu)$, with $\mu = \left(1 - \xi^2\right) m_i v^2 / 2B$.

In the trapped region, we integrate Eq. (52) with respect to time over a closed orbit. Again, the first term is annihilated, $\oint v_\parallel \mathbf{b} \cdot \nabla f^{b(1)} dt = 0$, as are terms which are odd with respect to σ. Thus, using the generalised flux surface average given in Eq. (39), the constraint equation becomes:

$$\begin{aligned} m_i \frac{\partial}{\partial \mu} \mu \left\langle \nu_D^{ii} |v_\parallel| \right\rangle \frac{\partial f^{b(0)}}{\partial \mu} &= \left\langle \frac{B}{|v_\parallel|} \left[\frac{\partial \ln n_{i0}}{\partial t_1} - \frac{S_p}{f_{i0}} + \left(x^2 - \frac{3}{2}\right) \frac{\partial \ln T_i}{\partial t_1} \right.\right. \\ &- \left.\left. \frac{4 m_i}{3 T_i} \left[\left(x^2 - \frac{3}{2}\right) P_0 B_0 - x^2 P_2 B_2 \right] \right] f_{i0} \right\rangle . \end{aligned} \tag{55}$$

We integrate over μ to determine $\partial f^{b(0)}/\partial\mu$, now taking an indefinite integral such that $\mu_c < \mu < \mu_{max}(\theta, H)$. Here $\mu_{max}(\theta, H) = xT/B$ is the maximum value of μ possible at the position θ on the flux surface, for a particle with energy H. Although the limits of the generalized flux surface average depend on θ, we may still interchange the order of the integral over μ and this average, as it is taken over a complete orbit.

Upon matching the two solutions, using the requirement that the flux in velocity space is continuous across the trapped-passing boundary:

$$\sum_\sigma \left.\frac{\partial f^{b(0)}}{\partial\mu}\right|_{\mu_c^-} = \sum_\sigma \left.\frac{\partial f^{b(0)}}{\partial\mu}\right|_{\mu_c^+}, \tag{56}$$

the equation for $\partial f^{b(0)}/\partial\mu$, valid in both the passing and trapped regimes is found to be:

$$\begin{aligned}\frac{\partial f^{b(0)}}{\partial\mu} = & -\frac{\sigma H(\mu_c-\mu)}{\langle\nu_D^{ii}|v_\parallel|\rangle}\frac{1}{T_i}\left\langle\left[\nu_D^{ii}Bs_\parallel - I\frac{\partial\omega}{\partial t_1} + \frac{2B}{v_{Ti}}\left[\left(\frac{m_i}{m_f}-2+\frac{6}{5}x^2\right)B_1 + \frac{3}{10}x^2\left(1-5\xi^2\right)B_3\right]\right]f_{i0}\right\rangle \\ & + \frac{1}{\langle\nu_D^{ii}|v_\parallel|\rangle}\left\langle\left[\frac{\partial\ln n_{i0}}{\partial t_1} - \frac{S_p}{f_{i0}} + \left(x^2-\frac{3}{2}\right)\frac{\partial\ln T_i}{\partial t_1} - \frac{4m_i}{3T_i}\left(x^2-\frac{3}{2}\right)B_0\right]\frac{v}{\mu}\left(1-\frac{|v_\parallel|}{v}\right)f_{i0}\right. \\ & + \left.\frac{2m_i}{3T_i^2}B|v_\parallel|B_2 f_{i0}\right\rangle\end{aligned} \tag{57}$$

where H is the Heaviside step function.

To evaluate the fluxes, we require $s_\parallel$, which may be determined using Eq. (25), the conservation of momentum in bulk ion self-collisions in the rotating frame. To lowest order, the parallel component of Eq. (25), which gives the parallel bulk ion self-friction, $R_{ii\parallel}$, is:

$$R_{ii\parallel} = \int d^3v m_i v_\parallel \nu_D^{ii}\left(-f^{b(0)} + \frac{m_i v_\parallel}{T_i}s_\parallel f_{i0}\right) = 0. \tag{58}$$

Only the part of $f^{b(0)}$ which is odd with respect to σ, which we denote as $f_{odd}^{b(0)}$, contributes to the first term in Eq. (58). Noting that the measure $d^3v = \sum_\sigma\left(2\pi B/m_i^2|v_\parallel|\right)dHd\mu$, integrating by parts with respect to μ gives:

$$\int d^3v m_i v_\parallel \nu_D^{ii} f^{b(0)} = -n_{i0}m_i\eta_1\frac{u(1)}{\tau_{ii}}, \tag{59}$$

where $\eta_1 = \sqrt{2} - \ln\left(1+\sqrt{2}\right)$ and we have defined the functional u such that:

$$\eta_1 u(A) = -\frac{3\pi^{3/2}B}{n_{i0}m_i^2}\int_{\alpha(\theta)}^{\infty} dHA\frac{\phi(x)-G(x)}{x^3}\int_0^{\mu_c(H)}\mu\sigma\frac{\partial f_{odd}^{b(0)}}{\partial\mu}d\mu. \tag{60}$$

Remembering that ν_D^{ii} depends only on the magnitude of v and using Eq. (3) for f_{i0}, the second term in Eq. (58) is seen to be equal to $n_{i0}m_i\eta_1 s_\parallel/\tau_{ii}$. Thus, Eq. (25) indicates that:

$$s_\parallel = u(1). \tag{61}$$

We may now evaluate the additional radial fluxes occuring in the presence of NBI, which are given by Eqs. (18) - (20).

The terms in Eq. (18) containing the bulk ion self-collision operator are zero, due to particle and momentum conservation in like species collisions, so:

$$\Gamma^b = -\frac{m_i}{e}\left\langle\int d^3v\left(\frac{I}{B}v_\parallel\bar{C}_{if}(f_{i0}, f_{f0}) + \omega R^2 S_p\right)\right\rangle + \frac{m_i}{e}\left\langle n_{i0}\omega R^2\left[\frac{\partial\ln n_{i0}}{\partial t_1} + \frac{\partial\ln\omega}{\partial t_1}\right]\right\rangle. \tag{62}$$

Upon substituting for the time dependent terms from Eqs. (31) and (49), we obtain:

$$\Gamma^b = -z_f \Gamma_{f2}, \tag{63}$$

as expected from Eq. (38).

To evaluate the heat flux, given by Eq. (19), we note that by using the functional $u(A)$ defined in Eq. (60), we may write:

$$\int d^3 v x^2 v_\parallel \bar{C}^l_{ii}\left(f^{b(0)}\right) = -\frac{n_{i0}}{\tau_{ii}}\left(\eta_1 u\left(x^2\right) + \frac{1}{\sqrt{2}} u(1)\right). \tag{64}$$

Remembering that energy is conserved in like particle collisions and using the form of the fast-bulk ion collision operator given in Eq. (26), the heat flux is seen to be:

$$\begin{aligned}\frac{q^b}{T_i} = &- \frac{m_i}{e}\left[\frac{2m_i}{T_i}\left\langle n_{i0}\omega R^2 B_0\right\rangle + I\left\langle \frac{n_{i0}}{B}\left[2\left(3+\left(1+\frac{m_i}{m_f}\right)\frac{\alpha}{T_i}\right)\frac{B_1}{v_{Ti}} - \left\{\eta_1 u\left(x^2\right) - \frac{1}{\sqrt{2}}u(1)\right\}\frac{1}{\tau_{ii}}\right]\right\rangle\right] \\ &+ \frac{m_i}{e}\left\langle n_{i0}\omega R^2\left[\frac{3}{2}\frac{\partial \ln T_i}{\partial t_1} - \left(1+M_i^2\right)\frac{\partial \ln n_{i0}}{\partial t_1} - M_i^2\frac{\partial \ln \omega}{\partial t_1} - \frac{1}{n_{i0}}\int d^3 v\left(\frac{\alpha}{T_i}-1\right)S_p\right]\right\rangle. \end{aligned}\tag{65}$$

Finally we determine the angular momentum flux, defined by Eq. (20). Noting that this requires:

$$\int d^3 v v_\parallel^2 \bar{C}^l_{ii}\left(f^b\right) = -\frac{8\pi B}{m_i^2}\int_{\alpha(\theta)}^{\infty} dH \nu_D^{ii}\int_0^{\mu_{max}(H,\theta)} \mu |v_\parallel| \frac{\partial f^{b(0)}_{even}}{\partial \mu} d\mu, \tag{66}$$

and again making use of Eq. (26), the angular momentum flux is seen to be:

$$\begin{aligned}\Pi_2^b = &- \frac{m_i^2}{e}\left[\frac{2}{3}\left\langle n_{i0}R^2 B_0\right\rangle + 2I\omega\left(1+\frac{m_i}{m_f}\right)\left\langle\frac{n_{i0}}{B}R^2\frac{B_1}{v_{Ti}}\right\rangle - \frac{2}{3}\left\langle n_{i0}\frac{I^2 - |\nabla\Psi|^2/2}{B^2}B_2\right\rangle\right] \\ &- \frac{2\pi}{e}\left\langle \frac{\left(2I^2 - |\nabla\Psi|^2\right)}{B}\int_{\alpha(\theta)}^{\infty} dH\nu_D^{ii}\int_0^{\mu_{max}(H,\theta)}\mu|v_\parallel|\frac{\partial f^{b(0)}_{even}}{\partial\mu}d\mu\right\rangle \\ &+ \frac{m_i}{e}\left\langle p_{i0}R^2\left[\frac{1}{2}\frac{\partial \ln T_i}{\partial t_1} + \left(1+M_i^2\right)\frac{\partial \ln n_{i0}}{\partial t_1} + M_i^2\frac{\partial \ln\omega}{\partial t_1}\right.\right. \\ &- \left.\left.\frac{1}{n_{i0}}\int d^3 v\left(\frac{x^2}{3} + M_i^2\right)S_p\right]\right\rangle. \end{aligned}\tag{67}$$

Subsonic Toroidal Rotation

The analysis and results presented in the previous section simplify considerably in the limit of subsonic toroidal rotation of the bulk plasma, that is $M_i^2 \ll 1$, where M_i is the toroidal bulk ion Mach number, defined in Sec. III. With this ordering α is negligible and n_{i0} may be taken to be approximately constant around a flux surface. Such a limit is relevant, as most tokamak discharges with NBI have $M_i^2 \sim 0.1$, although Mach numbers up to 1 have been reported in spherical tokamaks[17]. The velocity, v, will now be an approximate constant of motion, and we may use it to replace the energy variable, H, so that the independent variables may be taken to be $(v, \lambda, \sigma, \psi)$, where $\lambda = v_\perp^2/\left(v^2 B\right)$. In this new set of velocity variables, the functional $u(A)$ is written:

$$\eta_1 u(A) = -\frac{3\pi^{3/2}B}{4n_{i0}} m_i v_{Ti}^6 \int_0^\infty dx A x^2 \left(\Phi(x) - G(x)\right)\int_0^{\lambda_c}\sigma\frac{\partial f^{b(0)}_{odd}}{\partial\mu}\lambda d\lambda. \tag{68}$$

The trapped-passing boundary in velocity space is defined by $\lambda_c = 1/B(\theta=\pi) = 1/B_{max}$ and as the variation of the potential around the flux surface can now be neglected, the lower limit of the energy integral is zero. Note that $u(A)$ is now a flux surface function if A is independent of θ.

The equations (31), (43) and (49) determining the time dependencies of the parameters of f_{i0} thus become:

$$\frac{\partial n_{i0}}{\partial t_1} = \int d^3 v S_p, \tag{69}$$

$$\frac{\partial \ln T_i}{\partial t_1} = \frac{4}{3}\frac{m_i}{T_i}\langle B_0 \rangle \tag{70}$$

and

$$\langle R^2 \rangle \frac{\partial \omega}{\partial t_1} = -\frac{z_f e \Gamma_{f2}}{m_i n_{i0}} + 2I\left(1 + \frac{m_i}{m_f}\right)\left\langle \frac{B_1}{B v_{Ti}} \right\rangle. \tag{71}$$

As ν_D^{ii} and f_{i0} are functions only of v, they may now be taken out from under flux surface averages, so Eq. (57) for $\partial f^{b(0)}/\partial \mu$ simplifies to:

$$\begin{aligned}
\frac{\partial f^{b(0)}}{\partial \mu} &= - \frac{\sigma H(\lambda_c - \lambda)}{\nu_D^{ii} v \langle \sqrt{1-\lambda B} \rangle} \left[\left\langle \nu_D^{ii} B u(1) \right\rangle - I \frac{\partial \omega}{\partial t_1} + 2\left[\left(\frac{m_i}{m_f} - 2 + \frac{6}{5}x^2 \right) \left\langle B \frac{B_1}{v_{Ti}} \right\rangle \right.\right. \\
&+ \left.\left. \frac{3}{10} x^2 \left\langle (5\lambda B - 4) B \frac{B_3}{v_{Ti}} \right\rangle \right]\right] \frac{f_{i0}}{T_i} \\
&+ \frac{2}{3\nu_D^{ii}} \frac{1}{\langle \sqrt{1-\lambda B} \rangle} \left[\frac{4}{v^2 \lambda} \left(x^2 - \frac{3}{2} \right) \left\langle (B_0 - \langle B_0 \rangle) \sqrt{1-\lambda B} \right\rangle + \frac{m_i}{T_i} \left\langle B_2 B \sqrt{1-\lambda B} \right\rangle \right] \frac{f_{i0}}{T_i} \\
&+ \frac{2}{\nu_D^{ii} m_i v^2 \lambda} \left(\frac{1}{n_{i0}} \int S_p d^3 v - \frac{S_p}{n_{i0}} \right) \left(\frac{1}{\langle \sqrt{1-\lambda B} \rangle} - 1 \right) f_{i0},
\end{aligned} \tag{72}$$

where we have substituted the relations for the time dependencies of the Maxwellian parameters given by Eqs. (69) and (70).

We may now explicitly evaluate the integrals involving $\partial f^{b(0)}/\partial \mu$, which appear in Eqs. (65) and (67) for the fluxes of heat and angular momentum. For convenience, we define the fraction of trapped particles, $f_t = 1 - f_c$ (5), where:

$$f_c = \frac{3}{4}\langle B^2 \rangle \int_0^{\lambda_c} \frac{\lambda}{\langle \sqrt{1-\lambda B} \rangle} d\lambda, \tag{73}$$

along with the parameter:

$$\lambda_\star = \frac{15 \langle B^2 \rangle}{16 f_c} \int_0^{\lambda_c} \frac{\lambda^2}{\langle \sqrt{1-\lambda B} \rangle} d\lambda \tag{74}$$

and remember that $(\Phi(x) - G(x))/\nu_D^{ii} = (4/3\sqrt{\pi})\, x^3 \tau_{ii}$. For the heat flux we require:

$$\begin{aligned}
\frac{1}{B\tau_{ii}} \left[\eta_1 u(x^2) - \frac{1}{\sqrt{2}} u(1) \right] &= -\frac{f_c}{\langle B^2 \rangle} \left[\left(\frac{5}{2} - \frac{1}{\sqrt{2}\eta_1} \right) I \frac{\partial \omega}{\partial t_1} - 2\left(\frac{11}{2} - \frac{1}{\sqrt{2}\eta_1} + \frac{m_i}{m_f} \left(\frac{5}{2} - \frac{1}{\sqrt{2}\eta_1} \right) \right) \left\langle B \frac{B_1}{v_{Ti}} \right\rangle \right. \\
&- \left. 6\left(\frac{7}{2} - \frac{1}{\sqrt{2}\eta_1} \right) \left\langle (B\lambda_\star - 1) B \frac{B_3}{v_{Ti}} \right\rangle \right].
\end{aligned} \tag{75}$$

The angular momentum flux requires the following integral, which has been recast into the new set of velocity variables:

$$\frac{m_i^3}{4} \int_0^\infty dv \nu_D^{ii} v^6 \int_0^{B^{-1}(\theta)} \frac{\partial f_{even}^{b(0)}}{\partial \mu} \lambda \sqrt{1-\lambda B} d\lambda = n_{i0} \frac{m_i^2}{2\pi} \int_0^{B^{-1}(\theta)} \langle \beta(\lambda) \rangle \frac{\sqrt{1-\lambda B}}{\langle \sqrt{1-\lambda B} \rangle} d\lambda, \tag{76}$$

where, for convenience, we have defined the function:

$$\beta(\lambda) = \left(B_0 - \langle B_0 \rangle + \frac{5}{4} B_2 B \lambda \right) \sqrt{1-\lambda B} + \frac{\pi v_{Ti}^5}{n_{i0}} \left(1 - \sqrt{1-\lambda B} \right) \int_0^\infty \left(\frac{9}{8} - x^4 \right) S_p dx. \tag{77}$$

It is useful to note that, recalling the generalized flux surface average from Eq. (39), for arbitrary functions $h(\theta)$ and $k(\lambda,\theta)$: $\left\langle h(\theta)\int_0^{B^{-1}(\theta)} k(\lambda,\theta)\,d\lambda\right\rangle = \int_0^{B^{-1}_{\min}}\langle hk\rangle\,d\lambda$. Thus, the contribution to the angular momentum flux may be written as:

$$\begin{aligned}\frac{2\pi}{e}&\left\langle\frac{2I^2-|\nabla\Psi|^2}{B}\frac{m_i^3}{4}\int_0^\infty dv\nu_D^{ii}v^6\int_0^{B^{-1}(\theta)}\frac{\partial f_{even}^{b(0)}}{\partial\mu}\lambda\sqrt{1-\lambda B}d\lambda\right\rangle\\ &=\frac{m_i^2n_{i0}}{e}\int_0^{B^{-1}_{min}}\left\langle\frac{2I^2-|\nabla\Psi|^2}{B}\sqrt{1-\lambda B}\right\rangle\langle\beta\rangle\frac{d\lambda}{\left\langle\sqrt{1-\lambda B}\right\rangle}.\end{aligned}\tag{78}$$

So, applying the assumed ordering to Eqs. (63), (65) and (67), and substituting the time dependencies given by Eqs. (69) - (70), the additional radial fluxes which occur in a plasma with subsonic toroidal rotation, in the presence of neutral beam injection, are:

$$\Gamma^b=\frac{m_in_{i0}}{e}\left[-2I\left(1+\frac{m_i}{m_f}\right)\left\langle\frac{B_1}{Bv_{Ti}}\right\rangle+\left\langle R^2\right\rangle\frac{\partial\omega}{\partial t_1}\right]=-z_f\Gamma_{f2},\tag{79}$$

$$\begin{aligned}\frac{q^b}{T_i}=&-2\frac{m_in_{i0}}{e}\left[\omega\frac{m_i}{T_i}\left\langle\left(R^2-\left\langle R^2\right\rangle\right)B_0\right\rangle+3I\left\langle\frac{B_1}{Bv_{Ti}}\right\rangle\right.\\ &-I\left[\frac{11}{2}-\frac{1}{\sqrt{2}\eta_1}+\frac{m_i}{m_f}\left(\frac{5}{2}-\frac{1}{\sqrt{2}\eta_1}\right)\right]\frac{f_c}{\langle B^2\rangle}\left\langle B\frac{B_1}{v_{Ti}}\right\rangle-3I\left(\frac{7}{2}-\frac{1}{\sqrt{2}\eta_1}\right)\frac{f_c}{\langle B^2\rangle}\left\langle\left(B\lambda_\star-1\right)B\frac{B_3}{v_{Ti}}\right\rangle\\ &\left.+\frac{1}{2}\left[\left\langle M_i^2R^2\right\rangle+I^2\left(\frac{5}{2}-\frac{1}{\sqrt{2}\eta_1}\right)\frac{f_c}{\langle B^2\rangle}\right]\frac{\partial\omega}{\partial t_1}-\frac{\omega}{2n_{i0}}\left\langle M_i^2R^2\right\rangle\int d^3vS_p\right]\end{aligned}\tag{80}$$

and

$$\begin{aligned}\Pi^b=&-\frac{m_i^2n_{i0}}{e}\left[\frac{2}{3}\left\langle\left(R^2-\left\langle R^2\right\rangle\right)B_0\right\rangle+2I\omega\left(1+\frac{m_i}{m_f}\right)\left\langle\frac{R^2B_1}{Bv_{Ti}}\right\rangle-\frac{1}{3}\left\langle\frac{2I^2-|\nabla\Psi|^2}{B^2}B_2\right\rangle\right.\\ &+\int_0^{B^{-1}_{min}}\left\langle\frac{2I^2-|\nabla\Psi|^2}{B}\sqrt{1-\lambda B}\right\rangle\langle\beta\rangle\frac{d\lambda}{\left\langle\sqrt{1-\lambda B}\right\rangle}-\frac{T_i}{m_i}\left\langle M_i^2R^2\right\rangle\frac{\partial\ln\omega}{\partial t_1}\\ &\left.-\frac{T_i}{2m_in_{i0}}\left\langle R^2\right\rangle\int d^3vS_p\right].\end{aligned}\tag{81}$$

The above expressions are valid for arbitrary flux surface geometry. We may now consider the limit addressed by Hinton and Kim, that is large aspect ratio and circular flux surfaces. We take R_0 to be the major radius of the magnetic axis, so $R=R_0(1+\epsilon\cos\theta)$, where the inverse aspect ratio $\epsilon\ll 1$, $B_p/B_\phi\approx\epsilon/q$, where q is the safety factor and $f_t\approx 1.46\sqrt{\epsilon}$. We also define $M_{i0}=M_i(R_0)$. The normalised radial fluxes then take the following forms, where we have assumed $m_i=m_f$ for clarity and retained only the leading order of each driving term:

$$\begin{aligned}\frac{q^b}{T_i}\approx-2\frac{m_in_{i0}}{e}&\left[\frac{4M_{i0}^2}{\omega}\epsilon\left\langle B_0\cos\theta\right\rangle+2M_{i0}^2\frac{R_0}{v_{Ti}}\langle B_1\rangle+2.41\frac{R_0}{v_{Ti}}\sqrt{\epsilon}\langle B_3\rangle\right.\\ &\left.-\left(0.59+\frac{M_{i0}^2}{2}\right)\frac{z_fe\Gamma_{f2}}{m_in_{i0}}+\frac{\omega R_0^2M_{i0}^2}{2n_{i0}}\int d^3vS_p\right]\end{aligned}\tag{82}$$

and

$$\begin{aligned}\frac{\Pi^b}{m_i\omega R_0^2}\approx-\frac{m_in_{i0}}{e}&\left[\frac{8}{3\omega}\epsilon\left\langle B_0\cos\theta\right\rangle+2\frac{R_0}{v_{Ti}}\langle B_1\rangle-\frac{2}{3\omega}\epsilon\left\langle B_2\cos\theta\right\rangle+\frac{1}{2}\frac{z_fe\Gamma_{f2}}{m_in_{i0}}\right.\\ &\left.-\frac{2}{3\omega}\frac{\pi v_{Ti}^5}{n_{i0}}\int_0^\infty\left(x^4+\frac{9}{8}\right)S_pdx\right].\end{aligned}\tag{83}$$

The approximate form of the bulk-fast ion collision operator used by Hinton and Kim in Ref. [3] consisted of only the B_1 term of Eq. (26), neglecting its energy dependent part. We recognise that with such an approximation, only the following beam driven contribution to the heat flux arises:

$$2\frac{m_i n_{i0}}{e} I \left(1 + \frac{m_i}{m_f}\right)\left(\frac{5}{2} - \frac{1}{\sqrt{2}\eta_1}\right)\frac{f_c}{\langle B^2\rangle}\left\langle B\frac{B_1}{v_{Ti}}\right\rangle,$$

which may be seen to be contained within the third term of Eq. (80). This reduces to Hinton and Kim's result in the large aspect ratio limit considered:

$$\frac{q^{b,HK}}{T_i} = 2.34\left(1 + \frac{m_i}{m_f}\right)\frac{m_i n_{i0} R_0}{e v_{Ti}}\langle B_1\rangle . \tag{84}$$

The additional terms involving B_1 present in Eq. (80) arise as a result of retaining the energy dependence of the B_1 term in the bulk-fast ion collision operator.

We see that, when higher order velocity space structure of the fast ion distribution is retained, the B_1 moment is no longer the primary driving force of the heat flux, in the limit we have considered. The drive is predominantly due to the radial current Γ_{f2}. Furthermore, the B_0 moment, which is larger than the B_1 moment by approximately v_{Tf}/v_{Ti} and was neglected by Hinton and Kim, contributes to the heat and angular momentum fluxes. In the limit we have considered, of large aspect ratio and circular flux surfaces, the terms should not be neglected unless $\epsilon \ll v_{Ti}/v_{Tf}$. These results will be discussed further in Sec. VIII.

Finally, we may clarify the restrictions under which **the beam may be considered to be a strong source and its effects be taken to dominate over the usual neoclassical effects, such as the neoclassical viscosity, which do not produce evolution on the t_1 timescale.** The contribution, f^b, to the bulk ion distribution function arising in the presence of NBI was formally taken to be of order δf_{i0}, where $\delta \ll 1$. Considering the term involving B_0 in Eq. (72), we see that $f^b \sim (v_{Ti}/v_{Tf})(n_{f0}/n_{i0}) f_{i0}$, where we have used $B_0 \sim \gamma_{if} n_{f0}/v_{Tf}$ and $\gamma_{if}/\nu_D^{ii} \sim v_{Ti}^3/n_{i0}$. Thus we strictly require that: $\delta \sim n_{f0} v_{Ti}/n_{i0} v_{Tf} \ll 1$. A lower limit may be determined by considering our retention of beam driven effects in Eq. (30), and thus Eq. (70), whilst neglecting the term involving Q_i, which describes the usual neoclassical diffusive evolution of the temperature occurring without NBI. An expression for this Q_i, which we denote as Q_i^{NC}, is given by Eq. (97) of Ref. [11], for large aspect ratio and subsonic toroidal rotation. Thus, the neoclassical contribution scales as:

$$\frac{1}{V'}\frac{\partial}{\partial\Psi}V' Q_i^{NC} \sim \frac{1}{V'}\frac{\partial}{\partial\Psi}V'\left[\epsilon^{1/2}\frac{n_{i0}T_i}{m_i\tau_i}\left(\frac{m_i R_0}{e}\right)^2\frac{\partial T_i}{\partial\Psi}\right] \sim \frac{1}{\sqrt{\epsilon}}\left(\frac{\rho_i^{ban}}{L_T}\right)^2\frac{n_{i0}T_i}{\tau_i}. \tag{85}$$

We have defined the radial temperature gradient scale length, L_T, and noted that $\rho^{ban} \sim q\rho/\sqrt{\epsilon}$ is the typical width of the banana orbit of a particle with gyroradius ρ and $B/B_p \sim q/\epsilon$. Comparing this to the beam driven evolution, that is the term on the right-hand side of Eq. (70) multiplied by $n_{i0}T_i$, and using the approximations introduced above, we find:

$$\frac{n_{f0}}{n_{i0}}\frac{v_{Ti}}{v_{Tf}} \gg \frac{1}{\sqrt{\epsilon}}\left(\frac{\rho_i^{ban}}{L_T}\right)^2, \tag{86}$$

to justify the **assumption that the beam driven evolution dominates over the effect of the usual small neoclassical terms and therefore their neglect in Eq. (30).** We may treat Eq.

(29) similarly, comparing the effects of neoclassical viscosity, given by the term involving Π_i, to the beam driven evolution, given by the terms,which are taken to be comparable, on the right-hand side of Eq. (71). The required expression for Π_i, which we denote as Π_i^{NC}, is given by Eq. (159) of Ref. [7], for large aspect ratio and subsonic toroidal rotation. Thus the viscous contribution scales as:

$$\frac{1}{V'}\frac{\partial}{\partial\Psi}V'\Pi_i^{NC} \sim \frac{1}{V'}\frac{\partial}{\partial\Psi}V'\left[\epsilon^2\frac{n_{i0}T_i}{\tau_i}\left(\frac{m_iR_0^2}{e}\right)^2\frac{\partial\omega}{\partial\Psi}\right] \sim \epsilon\left(\frac{\rho_i^{ban}}{L_\omega}\right)^2\frac{n_{i0}m_i}{\tau_i}\omega R_0^2. \tag{87}$$

where we have defined the radial scale length of toroidal rotation frequency, L_ω. We compare this to the beam driven evolution, which is characterised by the term on the right-hand side of Eq. (71) involving B_1, multiplied by $n_{i0}m_i$. Using the approximations introduced above and noting that $B_1 \sim \gamma_{if}n_{f0}v_{Ti}/v_{Tf}^2$, we find that, to justify the neglect of the neoclassical viscosity terms in comparison to the beam driven contribution:

$$\frac{n_{f0}}{n_{i0}}\left(\frac{v_{Ti}}{v_{Tf}}\right)^2 \gg \epsilon\left(\frac{\rho_i^{ban}}{L_\omega}\right)^2 M_i. \tag{88}$$

If these inequalities are not satisfied, the effects of the beam are diminished and will appear alongside the usual neoclassical effects on plasma evolution.

VI. BULK ION POLOIDAL VELOCITY

Finally, in this section we consider the additional poloidal velocity of the bulk ions induced by neutral beam injection. The contribution of the zeroth order, Maxwellian part of the distribution function, f_{i0}, to the bulk ion poloidal velocity, V_{pol}, is zero, so to lowest order:

$$n_{i0}V_{pol} = \int d^3v\left(\bar{f}_{i1}+\tilde{f}_{i1}\right)\left(v_\parallel\mathbf{b}+\mathbf{v}_\perp\right)\cdot\hat{\theta}, \tag{89}$$

where $\hat{\theta}$ is a unit vector in the poloidal direction and we have used velocity coordinates in the rotating frame. In Sec. III, $\bar{f}_{i1}$ was written as the sum of a part determined by Hinton and Wong[(7)], f^{HW}, which is driven by radial gradients of f_{i0}, and a part which arises in response to the NBI, f^b. Thus, the poloidal velocity may be considered to be the sum:

$$V_{pol} = V_{pol}^{HW} + V_{pol}^{b}, \tag{90}$$

where V_{pol}^{HW} is produced by $\tilde{f}_{i1} + f^{HW}$ and is known. The poloidal rotation driven by the beams is:

$$V_{pol}^b = \frac{1}{n_{i0}}\int d^3v f^{b(0)}v_\parallel\frac{B_p}{B}, \tag{91}$$

which can be evaluated using the results of the previous section. It may then be compared to the poloidal rotation, V_{pol}^{HW}, present without NBI.

Integrating by parts, the poloidal velocity driven by the beam is:

$$n_{i0}V_{pol}^b = \frac{B_p}{B}\int d^3v\sigma|v_\parallel|f^{b(0)} = -\frac{4\pi}{m_i^2}B_p\int_{\alpha(\theta)}^{\infty}dH\int_0^{\mu_c(H)}\sigma\mu\frac{\partial f_{odd}^{b(0)}}{\partial\mu}d\mu. \tag{92}$$

This is valid for arbitrary values of the bulk plasma toroidal rotation velocity. To proceed analytically, we again restrict to the limit of subsonic toroidal rotation, as defined in Sec. V, and substitute the form of $\partial f_{odd}^{b(0)}/\partial\mu$ given in Eq. (72). Defining the velocity space average of a function of velocity: $\{F(v)\} = \frac{8}{3\sqrt{\pi}}\int_0^\infty F(x)\, x^4 e^{-x^2} dx$, we may perform the integral over velocity, noting that $u(1)$ is a flux surface function. The beam driven poloidal velocity of the bulk ions is thus seen to be:

$$\begin{aligned}\frac{V_{pol}^b}{\tau_{ii}B_p} = & \frac{f_c}{\langle B^2\rangle}\left[2\left(\frac{f_c}{\eta_1 f_t}\left(1+\frac{m_i}{m_f}\right)+\frac{1}{\tau_{ii}}\left\{\left(\frac{m_i}{m_f}-2+\frac{6}{5}x^2\right)\frac{1}{\nu_D^{ii}}\right\}\right)\left\langle B\frac{B_1}{v_{Ti}}\right\rangle\right. \\ + & \left. 12\left(\frac{f_c}{2\eta_1 f_t}+\frac{1}{5\tau_{ii}}\left\{\frac{x^2}{\nu_D^{ii}}\right\}\right)\left\langle\left(B\lambda_\star - 1\right)B\frac{B_3}{v_{Ti}}\right\rangle - \left(\frac{f_c}{\eta_1 f_t}+\frac{1}{\tau_{ii}}\left\{\frac{1}{\nu_D^{ii}}\right\}\right)I\frac{\partial\omega}{\partial t_1}\right]. \quad (93)\end{aligned}$$

The velocity space averages may be evaluated numerically: $\tau_{ii}^{-1}\left\{\nu_D^{ii\,-1}\right\} = 4.05$ and $\tau_{ii}^{-1}\left\{x^2\nu_D^{ii\,-1}\right\} = 15.51$.

In the case of large aspect ratio and circular flux surfaces, as was considered by Hinton and Kim and defined in the previous section, this expression reduces to:

$$\frac{V_{pol}^b}{\tau_{ii}B_p} = \frac{1}{R_0 B_A}\left[\left(\frac{1.29}{\sqrt{\epsilon}}+0.30\right)\frac{z_f e\Gamma_{f2}}{m_i n_{i0}} + 12.92\frac{R_0}{v_{Ti}}\langle B_1\rangle - 1.52\frac{R_0}{v_{Ti}}\langle B_3\rangle\right]. \quad (94)$$

We have again taken $m_i = m_f$ for clarity, defined the quantity $B_A^2 = \left\langle B^2\right\rangle$ and retained each driving term to leading order only. We note that with the approximate form of the bulk-fast ion collision operator used by Hinton and Kim, only the following beam driven poloidal velocity arises:

$$2\left(1+\frac{m_i}{m_f}\right)\left[\frac{f_c}{\eta_1 f_t}+4.05\right]\frac{f_c}{\langle B^2\rangle}\left\langle B\frac{B_1}{v_{Ti}}\right\rangle,$$

which may be recognised in the second term of Eq. (93). Again, this reduces to Hinton and Kim's result in the limit of large aspect ratio considered:

$$\frac{V_{pol}^{b,HK}}{\tau_{ii}B_p} = \frac{2.57}{B_A v_{Ti}}\left(1+\frac{m_i}{m_f}\right)\frac{\langle B_1\rangle}{\sqrt{\epsilon}}. \quad (95)$$

We see that, upon allowing for toroidal acceleration of the plasma by the neutral beam, the poloidal velocity is driven primarily by the radial current Γ_{f2}, rather than the B_1 moment as found by Hinton and Kim. This may be understood in terms of the parallel force balance in the accelerating frame and will be discussed further in the next section.

For comparison, we give the contribution to the bulk ion poloidal velocity, V_{pol}^{HW}, which has been evaluated explicitly by Helander and Sigmar in Ref. [12]. It is directly proportional to the bulk ion temperature gradient:

$$V_{pol}^{HW} = B_p\frac{f_c}{\langle B^2\rangle}\frac{IT_i}{e}\frac{\eta_2}{\eta_1}A_2, \quad (96)$$

where $\eta_2 = (5/2)\,\eta_1 - \left(1/\sqrt{2}\right)$. In the case of large aspect ratio, with circular flux surfaces, and assuming that the two contributions to the toroidal acceleration in Eq. (71) are comparable, **we see that the ratio of the beam driven poloidal velocity to this standard neoclassical value is:**

$$\frac{V_{pol}^b}{V_{pol}^{HW}} \sim \left(\frac{n_{f0}v_{Ti}}{n_{i0}v_{Tf}}\right)\left(\frac{v_{Ti}}{v_{Tf}}\right)\frac{L_T}{\rho_f^{ban}}. \quad (97)$$

The beam driven and the usual temperature-gradient driven contributions to the poloidal bulk ion velocity may therefore be comparable in the presence of sufficiently high beam density, lower beam energy and small fast ion banana widths.

VII. DISCUSSION

The presence of fast ions, resulting from neutral beam injection into a tokamak plasma, has been seen previously to drive additional radial transport[(1),(2),(3)]. In this paper we have derived forms for the additional radial bulk ion transport of particles, heat and toroidal angular momentum induced by the NBI, which are valid for arbitrary flux surface geometry and fast ion distribution functions. We have also allowed for stronger toroidal rotation, that is, at a velocity greater than the diamagnetic velocity, and toroidal acceleration, both of which may occur in an NBI heated discharge. However, the analytical forms of the fluxes, given in Eqs. (79) - (81) and discussed below, are restricted to the case of subsonic bulk ion toroidal rotation. In the analysis, we represented the effect of the fast ions by a Fokker-Planck collision operator, expanded in the small ratio of the thermal bulk ion velocity to that of the fast ions up to second order, plus a source term representing fast ions which have slowed down and joined the thermal plasma. Hinton and Kim retained only the part of the expansion describing momentum transfer between the species, neglecting its energy dependence, as was noted in Sec. I, which is proportional to the parallel velocity moment of the fast ion distribution function, f_{f0}, and equivalent to the B_1 moment used here, defined by Eq. (27).

We have found that, for toroidal rotation velocities larger than the diamagnetic velocity, the lowest order velocity moment of the fast ion distribution function, describing the isotropic nature of f_{f0} and defined by B_0 in Eq. (27), contributes to the heat flux, Eq. (80). This moment is larger, by approximately the ratio of the fast ion to bulk ion thermal velocities, than the B_1 moment, and was neglected by Hinton and Kim. In the limit of large aspect ratio, circular flux surfaces, which they considered, we see that this is only justified if $\epsilon \ll v_{Ti}/v_{Tf}$. At tight aspect ratio, the drive from the B_0 term may become more significant. By retaining higher order velocity space structure of the fast ions, associated with the B_1 moment, than Hinton and Kim (see Sec. III), and the effects of toroidal acceleration, we have seen that the net drive from the B_1 moment, in the limit considered, is proportional to M_{i0}^2, which we have taken to be small. The heat flux, in this limit, is therefore driven predominantly by the component of the toroidal acceleration related to the bulk ion return current, or equivalently Γ_{f2}. Higher order pitch-angle structure of f_{f0}, described by the B_3 moment, also contributes to the heat flux. In the case of nearly tangential or perpendicular NBI, $\langle B_3 \rangle$ may be of the order of $\langle B_1 \rangle$, so it should not be neglected as, at large aspect ratio with circular flux surfaces, the two driving terms can be comparable for $\sqrt{\epsilon} \sim M_{i0}^2$. Again, the drive from the B_3 term may become more significant in a spherical tokamak. The overall radial direction of the heat flux will depend on the system considered.

The B_0, and higher order B_2, moments contribute to the flux of toroidal angular momentum and, for roughly parallel or perpendicular NBI, the two moments can be comparable. However, at large aspect ratio, with circular flux surfaces, the drive from the B_1 moment dominates when $\epsilon \ll v_{Ti}/v_{Tf}$. Again, the radial direction of the flux depends on the system and at tighter aspect ratio, all of these drives may be comparable.

Finally, we find that the higher order B_3 moment, as well as the B_1 moment retained by Hinton and Kim, drives the bulk ion poloidal velocity. The contributions have opposite sign and in the cases where the two moments are comparable, the B_1 drive dominates. Hinton and Kim found that the poloidal velocity at large aspect ratio, with circular flux surfaces, was proportional to $\langle B_1 \rangle / \sqrt{\epsilon}$. The origin of this form can be simply understood by considering the parallel force balance which must hold, in the absence of toroidal acceleration, between the parallel friction due to the fast ions, which drives parallel bulk ion flow, and the parallel self-collisional friction with trapped bulk ions, which acts to suppress the parallel flow. However, upon allowing for time dependence introduced by the NBI and therefore toroidal acceleration, we have seen that it is the fast ion radial current, and thus essentially the effect of the bulk ion polarization current, which now provides the dominant drive for the poloidal velocity. This form can again be understood by considering the parallel force balance, but now it must be done in an accelerating reference frame. An apparent force, proportional to the rate of change of the frame velocity given by Eq. (71), is therefore introduced, acting against the parallel flow. The toroidal acceleration is driven partly by friction with the fast ions, described by a weighted average of the B_1 moment. The parallel flow is driven directly by a different weighted average of the B_1 moment (see Eq. (93)), but at large aspect ratio the two coincide to leading order and the effects therefore cancel. Thus, the poloidal velocity is determined, in leading order, by a balance between the apparent force due to toroidal acceleration induced by the fast ion radial current and the parallel self-collisional friction with trapped bulk ions, and is therefore proportional to $\Gamma_{f2}/\sqrt{\epsilon}$.

For the case of large aspect ratio, with circular flux surfaces, we have compared the beam driven poloidal velocity given in Eq. (93) to the neoclassical value given by Eq. (96). They may be comparable when the beam density is large, the beam energy is low and the fast ion banana width is small, so the ratio will depend strongly on the parameters of the particular system.

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APPENDIX

In this Appendix we give details of the expansion of the bulk-fast ion collision operator, starting from the Landau expression for the collision operator in the laboratory frame[12]:

$$C_{if}\left(f_i, f_f\right) = -\gamma_{if}\frac{\partial}{\partial \mathbf{v}} \cdot \int \mathsf{U} \cdot \left[\frac{m_i}{m_f} f_i\left(v\right) \frac{\partial f_f\left(w\right)}{\partial \mathbf{w}} - f_f\left(w\right) \frac{\partial f_i\left(v\right)}{\partial \mathbf{v}}\right] d^3 w. \tag{A1}$$

Here $\gamma_{if} = e_i^2 e_f^2 \ln\Lambda / 8\pi\epsilon_0^2 m_i^2$, $\mathbf{v}$ is the bulk ion velocity, $\mathbf{w}$ is the fast ion velocity, $\mathbf{g} = \mathbf{v} - \mathbf{w}$ is the relative velocity of the ions, with modulus $g = |\mathbf{g}|$ and the dyad $\mathsf{U} = \nabla_g \nabla_g g = g^{-1}\mathsf{I} - g^{-3}\mathbf{gg}$, where I

is the identity tensor.

As $\nabla_g \cdot \mathsf{U} = -2\mathbf{g}g^{-3} = 2\nabla_g g^{-1}$, the first integral in Eq. (A1) becomes:

$$\begin{aligned}\int d^3w \nabla_g \nabla_g g \cdot \nabla_w f_f &= -\int d^3w f_f \nabla_w \cdot (\nabla_g \nabla_g g) \\ &= \int d^3w f_f \nabla_g \cdot (\nabla_g \nabla_g g) \\ &= 2\int d^3w f_f \nabla_g \left(\frac{1}{g}\right),\end{aligned} \tag{A2}$$

where we have used the fact that $f_f \to 0$ as $|w| \to \infty$ to integrate by parts. For most collisions between fast and bulk ions, $|\mathbf{v}| << |\mathbf{w}|$. Upon expanding in this ratio, to leading order Eq. (A2) becomes:

$$\int d^3w \nabla_g \nabla_g g \cdot \nabla_w f_f \approx -2\int d^3w f_f \nabla_w \left(\frac{1}{w}\right) = 2\int d^3w f_f \frac{\mathbf{w}}{(w)^3}. \tag{A3}$$

For the second term of Eq. (A1), we must expand U in the same small ratio:

$$\nabla_g \nabla_g g = \nabla_w \nabla_w w - \mathbf{v} \cdot \nabla_w \nabla_w \nabla_w w + \ldots. \tag{A4}$$

Thus, retaining terms up to the order $1/w^2$:

$$C_{if} \approx \gamma_{if} \nabla_v \cdot \left[\left[\int d^3w f_f \nabla_w \nabla_w w - \mathbf{v} \cdot \int d^3w f_f \nabla_w \nabla_w \nabla_w w\right] \cdot \nabla_v f_i - 2\frac{m_i}{m_f} f_i \int d^3w f_f \frac{\mathbf{w}}{(w)^3}\right]. \tag{A5}$$

We note at this point that the collision operator is Galilean invariant and proceed in the rotating frame. The bulk ion velocity in the rotating frame is $\mathbf{v}' = v_\parallel \mathbf{b} + v_\perp (\hat{e}_1 \cos\zeta + \hat{e}_2 \sin\zeta)$ and we take $\mathbf{w}$ to be the fast ion velocity in the rotating frame from hereon, similar in form to $\mathbf{v}'$, rather than introducing another primed quantity. We require the form of the collision operator acting on the Maxwellian bulk ion distribution function, f_{i0}, given by Eq. (3). In this case, Eq. (A5) may be written as:

$$C_{if}(f_{i0}, f_f) = \gamma_{if} \nabla_v \cdot (\mathbf{C} f_{i0}), \tag{A6}$$

where

$$\mathbf{C} f_{i0} = -\left(\mathbf{A} + \mathsf{D} \cdot \mathbf{v}' - \mathbf{v}' \cdot \hat{\mathsf{T}} \cdot \mathbf{v}'\right) f_{i0} \tag{A7}$$

and we have defined the following quantities, which are independent of v:

$$\mathbf{A} = 2\frac{m_i}{m_f} \int d^3w f_f \frac{\mathbf{w}}{(w)^3},$$

$$\mathsf{D} = \frac{m_i}{T_i} \int d^3w f_f \nabla_w \nabla_w w$$

and the triad

$$\hat{\mathsf{T}} = \frac{m_i}{T_i} \int d^3w f_f \nabla_w \nabla_w \nabla_w w.$$

Upon defining the notation $\mathbf{abc} \therefore \mathbf{def} = (\mathbf{a} \cdot \mathbf{f})(\mathbf{b} \cdot \mathbf{e})(\mathbf{c} \cdot \mathbf{d})$ and noting that the identity tensor has the form $\mathsf{I} = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \mathbf{bb}$, performing the gradient operation gives:

$$\nabla_v \cdot (\mathbf{C} f_{i0}) = \frac{m_i}{T_i} \left(\mathbf{A} \cdot \mathbf{v}' + \mathbf{v}' \cdot \mathsf{D} \cdot \mathbf{v}' - \mathbf{v}'\mathbf{v}'\mathbf{v}' \therefore \hat{\mathsf{T}}\right) f_{i0} - \left(\mathsf{D} : \mathsf{I} - \mathsf{I} : \hat{\mathsf{T}} \cdot \mathbf{v}' - \mathbf{v}' \cdot \hat{\mathsf{T}} : \mathsf{I}\right) f_{i0}. \tag{A8}$$

The gyroaverage over $\mathbf{v}'$ must now be taken and for this we note that:

$$\overline{\mathbf{v}' \cdot \mathsf{D} \cdot \mathbf{v}'} = \left[\frac{1}{2} w_\perp^2 \left(\mathsf{I} - \mathbf{bb}\right) + w_\parallel^2 \mathbf{bb}\right] : \mathsf{D},$$

$$\overline{\mathsf{I} : \hat{\mathsf{T}} \cdot \mathbf{v}' + \mathbf{v}' \cdot \hat{\mathsf{T}} : \mathsf{I}} = 2\overline{\mathbf{v}' \cdot \hat{\mathsf{T}} : \mathsf{I}} = 2 w_\parallel \mathbf{b} \cdot \hat{\mathsf{T}} : \mathsf{I}$$

and

$$\overline{\mathbf{v}'\mathbf{v}'\mathbf{v}'} \therefore \hat{\mathsf{T}} = w_\parallel^3 \mathbf{bbb} + \frac{w_\parallel w_\perp^2}{2} \left[\mathbf{b}\left(\mathsf{I} - \mathbf{bb}\right) + \left(\mathsf{I} - \mathbf{bb}\right)\mathbf{b} + \hat{e}_1 \mathbf{b} \hat{e}_1 + \hat{e}_2 \mathbf{b} \hat{e}_2\right] \therefore \nabla_w \nabla_w \nabla_w w.$$

Also, as $\nabla_w \nabla_w w = w^{-1}\mathsf{I} - w^{-3}\mathbf{ww}$, we see that, with $\hat{e}_3 = \mathbf{b}$:

$$\mathbf{bb} : \nabla_w \nabla_w w = w_\perp^2 / w^3,$$

$$\nabla_w \nabla_w \nabla_w w = \frac{3}{w^5} \mathbf{www} - \frac{1}{w^3} \left[\mathbf{w}\mathsf{I} + \mathsf{I}\mathbf{w} + \sum_{j=1}^{3} \hat{e}_j \mathbf{w} \hat{e}_j\right]$$

and hence

$$\mathbf{bbb} \therefore \nabla_w \nabla_w \nabla_w w = -3 w_\parallel w_\perp^2 / w^5.$$

Thus, upon grouping terms with the same functional dependence on the fast ion velocity, the expanded form of the required collision operator is seen to be:

$$\begin{aligned} \bar{C}_{if}\left(f_{i0}, f_{f0}\right) &= \gamma_{if} \frac{m_i}{T_i} \left[\left(\frac{m_i}{T_i} v_\perp^2 - 2\right) \int d^3 w f_{f0} \frac{1}{w} + 2\left(\frac{m_i}{m_f} - 2 + \frac{3 m_i}{2 T_i} v_\perp^2\right) v_\parallel \int d^3 w f_{f0} \frac{w_\parallel}{w^3} \right. \\ &+ \left. 3 \frac{m_i}{T_i} \left(v_\parallel^2 - \frac{3}{2} v_\perp^2\right) v_\parallel \int d^3 w f_{f0} \frac{w_\parallel^2 w_\perp^2}{w^5} + \frac{m_i}{T_i} \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) \int d^3 w f_{f0} \frac{w_\perp^2}{w^3}\right] f_{i0}, \quad \text{(A9)} \end{aligned}$$

which can also be written in the form shown in Eq. (26) of Sec. III.

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