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Guazzotto, L.,* Freidberg, J.P. and Betti, R.**

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* University of Padua, Italy
** University of Rochester, NY

Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge MA 02139 USA

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A General Formulation of MHD Stability Including Flow and a Resistive Wall

L. Guazzotto, J. P. Freidberg
MIT Plasma Science and Fusion Center

R. Betti
University of Rochester

Abstract

A general formulation is presented for determining the ideal MHD stability of an axisymmetric toroidal magnetic configuration including the effects of an arbitrary equilibrium flow velocity and a resistive wall. The system is inherently non self adjoint with the eigenvalue appearing in both the equations and the boundary conditions. Even so, after substantial analysis we show that the stability problem can be recast in the form of a standard eigenvalue problem: $\omega \mathsf{A} \cdot \mathbf{z} = \mathsf{B} \cdot \mathbf{z}$ which is highly desirable for numerical computation.

I. Introduction

In recent years the study of the linear ideal MHD stability of fusion plasmas has continued to advance with the inclusion of additional physical effects of important practical relevance. Of particular interest are the effects of an equilibrium flow velocity and a resistive wall.

The inclusion of flow and a resistive wall substantially increases the complexity of the formulation with respect to the development of fast, reliable, and accurate numerical solvers. To understand the complexity recall that ideal MHD without flow and a resistive wall reduces to a self adjoint eigenvalue problem for ω^2, a standard numerical problem. The self adjointness guarantees that ω^2 is always purely real. Adding flow by itself leads to a non self adjoint system. The basic eigenvalue becomes ω (not ω^2) and in general ω is complex. However, there is some degree of symmetry in that if ω is an eigenvalue then so is ω^*. A resistive wall by itself also introduces non self adjointness and complex eigenvalues although in this case the symmetry implies that if ω is an eigenvalue, then so is $-\omega^*$. With both flow and a resistive wall the system is non self adjoint and has complex eigenvalues with no general symmetry between the real and imaginary parts.

In the present work a general formulation is derived for determining the ideal MHD stability of an axisymmetric toroidal magnetic configuration including the effects of an arbitrary equilibrium flow velocity and a resistive wall. In spite of the fact that the system is inherently non self adjoint with the eigenvalue appearing in both the differential equation and the resistive wall boundary condition, we show that the stability problem can, by means of a Galerkin procedure, be cast in the form of a standard eigenvalue problem: $\omega \mathsf{A} \cdot \mathbf{z} = \mathsf{B} \cdot \mathbf{z}$. This is a highly desirable form for numerical computation and is the main contribution of the paper.

A key feature of the formulation is the analytic solution of the vacuum/resistive wall region by means of Green's theorem. This allows us to project the outer solutions onto the plasma surface so that only the plasma interior is required for the computational domain. To obtain this analytic solution requires an extensive amount of linear algebra, which, however, is straightforward computationally. It is also necessary to exploit the commonly used "thin wall" approximation for the resistive wall. The result is that for a given geometry the matrices A and B are of comparable size to the corresponding matrices for an ideal plasma without flow and a resistive wall. Also, they have a similar banded structure with most of the elements being zero, which is advantageous from a computer memory point of view since it allows for sparse-matrix memory allocation.

To put our work in perspective we note there are many MHD codes already in existence. The early codes mainly treated plasmas without flow or a resistive wall. Later codes added the effects of flow but not a resistive wall. One notable example is the upgraded version of CASTOR [1] named PHOENIX [2] which includes both toroidal and poloidal flows as well as gravity. Its main applications are to astrophysics. Other advanced MHD codes, aimed primarily at fusion research, include both toroidal flow and a resistive wall. These are the MARS-F [3,4], CASTOR_FLOW [5], and CARMA [6] codes. The MARS-F code has an eigenvalue formulation of the problem but requires the construction of a grid in both the inner and outer vacuum regions. The CASTOR_FLOW code requires multiple iterations to determine stability because of the nonlinear appearance of the eigenvalue. The CARMA code combines the MARS-F code with the CARIDDI [7] wall code, thereby treating the more realistic case of a 3-D wall. Although the effects of flow are potentially available to CARMA because of MARS-F, it is typically run with zero flow for computational efficiency – it is run as an MHD code with a resistive wall but no flow. Based on this summary, the present formulation can thus be viewed as a next step of progress along the path to the more efficient and realistic computation of MHD stability in toroidal geometries: a computationally efficient MHD formulation including both toroidal and poloidal flows as well as a resistive wall.

The paper is organized as follows. To begin, an analysis is presented of a simple model problem that has the same features as the more general 2-D problem of interest. This analysis demonstrates the steps necessary to convert the problem into the desired form without the need for large amounts of mathematics. With this calculation in hand, we next generalize the analysis to a 2-D axisymmetric toroidal system. The generalization is relatively straightforward except for the complicated resistive wall boundary condition. Most of the text is then devoted to recasting the resistive wall boundary condition into a form suitable for use in a standard eigenvalue problem. Finally, the results are combined, leading to the desired formulation.

II. Model problem

In this Section we consider a simple model problem that incorporates the critical features of the more general problem under consideration. The goal is to show how the model differential equation, which qualitatively includes the effects of an equilibrium flow velocity and a resistive wall, can be cast in the form of a standard linear algebra eigenvalue problem. The value of the model problem is that the analysis is greatly simplified leading to a more transparent understanding of the steps to be taken in the general case.

The model differential equation is given by

$$\left(f\xi'\right)' + \left[(\omega-\Omega)^2 + g\right]\xi = 0 \tag{1}$$

where $\Omega = \Omega(x)$ represents the flow velocity, $f = f(x)$, $g = g(x)$ represent the stabilizing and destabilizing MHD drives, and ω is the eigenvalue. The boundary conditions are as follows. At $x = 0$ we assume a regularity condition given by

$$\xi(0) = 0 \tag{2}$$

At $x = a$ we assume a resistive wall boundary condition of the form

$$\left(i\omega K_1 + K_2\right)\xi(a) + \left(i\omega K_3 + K_4\right)\xi'(a) = 0 \tag{3}$$

with the K_j being constants. The essential features of this condition are that (1) it involves a linear homogenous combination of $\xi(a)$ and $\xi'(a)$, (2) the eigenvalue ω appears explicitly and not higher than a linear power, and (3) the K_j are all real. It is shown shortly that the resistive wall boundary condition for the general case is of the identical form.

Note that for a static problem (i.e. $\Omega = 0$) with an ideal conducting wall (i.e. $K_1 = K_3 = 0$) the system is real and self adjoint. The formulation reduces to that of a standard self adjoint eigenvalue problem with ω^2 being the eigenvalue. This is a classic problem in numerical analysis.

If flow is introduced (i.e. $\Omega \neq 0$) the system loses its self adjointness. Even so a simple substitution converts the equations into the form of a standard eigenvalue problem where ω (rather than ω^2) is the eigenvalue. For this case ω is in general complex.

The main difficulty is associated with the resistive wall (i.e. $K_1 \neq 0, K_3 \neq 0$), which also causes the system to lose its self adjointness, resulting in complex eigenvalues. The problem is the appearance of ω in the boundary condition, which complicates the analysis. For an arbitrary ω dependence in the boundary condition there is no obvious way to transform the system into the form of a standard eigenvalue problem. However, the fact that ω appears linearly in the boundary condition for the thin-wall case makes it possible to reduce the problem to standard form.

The points just discussed can be demonstrated explicitly by showing how the model problem can be cast in the form of a standard eigenvalue problem. There are three steps involved. First, convert the second order (in ω) system into two first order systems. Second, solve the equations using, for instance, a Galerkin expansion procedure. Third, apply the boundary conditions at each end point showing how the resistive wall boundary condition can be incorporated into the linear algebraic equations. The end result is an eigenvalue problem in standard form: $\omega \mathsf{A} \cdot \mathbf{z} = \mathsf{B} \cdot \mathbf{z}$.

We start by converting the system into a set of two first order equations in ω by introducing a new dependent variable $u(x) = (\omega - \Omega)\xi(x)$. The model reduces to

$$
\begin{aligned}
\omega\xi &= \Omega\xi + u \\
\omega u &= -[(f\xi')' + g\xi] + \Omega u
\end{aligned}
\qquad (4)
$$

Next, following the Galerkin procedure we form a quadratic integral L by multiplying the first equation by $\hat{u}$, the second by $\hat{\xi}$, adding the equations together and then integrating over the plasma domain $0 \le x \le a$.

$$
L = \int_0^a \left[\omega(\hat{u}\xi + u\hat{\xi}) - f\hat{\xi}'\xi' + g\hat{\xi}\xi - \Omega(\hat{u}\xi + u\hat{\xi}) - \hat{u}u\right]dx + \left[f\hat{\xi}\xi'\right]_a \qquad (5).
$$

The solutions for ξ and u are represented by an expansion in a set of basis functions $\phi_j(x)$ with expansion coefficients ξ_j, u_j:

$$
\begin{aligned}
\xi &= \sum_0^N \xi_j \phi_j(x) \qquad & d\xi/dx &= \sum_0^N \xi_j \phi_j'(x) \\
u &= \sum_0^N u_j \phi_j(x) \qquad & du/dx &= \sum_0^N u_j \phi_j'(x)
\end{aligned}
\qquad (6)
$$

The boundary condition at the origin is satisfied by setting $\xi_0 = 0$ and assuming that $\phi_j(0) = 0$ for all j. The first equation in Eq. (4) also implies that $u_0 = 0$..

At $x = a$ there is a mixed homogeneous boundary condition involving both ξ and ξ'. A convenient way to treat this type of boundary condition is to select a set of compact basis functions (e.g. tent functions, parabolic functions, etc.) with the following properties:

$$\begin{array}{lll} \phi_j(x_j) = 1 & \phi_j(x_{j-1}) = \phi_j(x_{j+1}) = 0 & 1 \le j \le N-1 \\ \phi_j(x_j) = 1 & \phi_j(x_{j-1}) = 0 & j = N \end{array} \tag{7}$$

With this choice of basis functions we see that the values of ξ and u at the node point $x = x_j$ depend only on the coefficients ξ_j and u_j.

Now, in general, we expect from basic mathematics that the value of $\xi'(a)$ should be independent of the value of $\xi(a)$. The relationship between these two quantities is determined by the externally imposed boundary condition. This is a slightly subtle point but one that is crucial to the analysis. As an explicit demonstration, consider the finite element expansion described above. The value of $\xi'(x_j)$ depends on ξ_j but in addition couples to the neighboring node point coefficients $\xi_{j\pm1}$. This is a straightforward evaluation in the interior of the plasma. However, at the boundary $x_N = a$ the value of $\xi'(a)$ depends upon the coefficients ξ_{N-1}, ξ_N and "ξ_{N+1}" where x_{N+1} represents the node point of a hypothetical ghost element just outside the boundary. Note that "ξ_{N+1}" is not one of the actual unknowns in the problem even though it affects the value of $\xi'(a)$. (A similar situation arises with a finite difference formulation)

The implication is that it is incorrect to determine the value of $\xi'(a)$ by simply differentiating the expression for $\xi(x)$ given in Eq. (6) and then evaluating $\xi'(a)$ using only the coefficients ξ_{N-1} and ξ_N: that is, the quantity $\xi'(a) \equiv \xi'_a$ (or equivalently ξ_{N+1}) must be viewed as an additional independent unknown in the problem. This extra degree of freedom is essential in order to be able to apply the boundary condition at $x = a$. A similar discussion applies in principle to $u'(a)$ although in practice there is no issue here since $u'(x)$ appears neither in the integrand nor the boundary condition. To summarize, there are a total of $2N+1$ unknown coefficients in the problem represented by the eigenvector $\mathbf{z} = [\mathbf{u}, \boldsymbol{\xi}, \xi_{\mathbf{a}}']$ where $\mathbf{u} = [u_1, \ldots u_N]$ and $\boldsymbol{\xi} = [\xi_1, \ldots \xi_N]$.

We now derive a set of linear algebraic equations for the unknown coefficients by choosing two sequences of test functions for $\hat{\xi}$ and $\hat{u}$, and then evaluating the corresponding integrals. Following the Galerkin procedure we choose the first set of test functions as follows.

$$\hat{u}_i(x) = \phi_i(x) \qquad \hat{\xi}_i(x) = 0 \qquad 1 \le i \le N \tag{8}$$

Carrying out the integrals leads to a set of N linear algebraic equations

$$\omega \mathsf{D} \cdot \boldsymbol{\xi} = \mathsf{U} \cdot \boldsymbol{\xi} + \mathsf{D} \cdot \mathbf{u} \tag{9}$$

The matrices are square and symmetric, having dimensions $(N \times N)$ with elements

$$\begin{aligned} D_{ij} &= \int_0^a \phi_i \phi_j dx \\ U_{ij} &= \int_0^a \Omega \phi_i \phi_j dx \end{aligned} \tag{10}$$

This procedure is repeated for a second set of test functions defined by

$$\hat{u}_i(x) = 0 \qquad \hat{\xi}_i(x) = \phi_i(x) \qquad 1 \le i \le N \tag{11}$$

In this case we obtain a set of N linear equations, which because of the boundary term, contains $N+1$ unknowns

$$\omega \mathsf{D} \cdot \mathbf{u} = \mathsf{W} \cdot \boldsymbol{\xi} + \mathsf{U} \cdot \mathbf{u} + f_a \xi_a' \delta_{i-N} \delta_{j-N-1} \tag{12}$$

Here, W is an $(N \times N)$ symmetric square matrix with elements

$$W_{ij} = \int_0^a \left(f\phi_i'\phi_j' - g\phi_i\phi_j \right) dx \tag{13}$$

One more equation is needed to close the system. This equation corresponds to the boundary condition which can be written in terms of the finite element expansion as

$$\left(i\omega K_1 + K_2 \right) \xi_N + \left(i\omega K_3 + K_4 \right) \xi'(a) = 0 \tag{14}$$

The final step in the procedure is to collect all the algebraic equations. The result is an eigenvalue problem in standard form

$$\omega \mathsf{A} \cdot \mathbf{z} = \mathsf{B} \cdot \mathbf{z} \tag{15}$$

where A and B are given by

$$\mathsf{A} = \left| \begin{array}{c:c:c} \mathbf{0} & \mathsf{D} & \\ \hdashline \mathsf{D} & \mathbf{0} & 0 \\ \hdashline & iK_1 & iK_3 \end{array} \right| \qquad \mathsf{B} = \left| \begin{array}{c:c:c} \mathsf{D} & \mathsf{U} & \\ \hdashline \mathsf{U} & \mathsf{W} & f_a \\ \hdashline & K_2 & K_4 \end{array} \right| \tag{16}$$

Equation (15) is the desired result. Observe that the crucial point that allows the transformation of the original problem into a standard linear algebra problem is the fact that the model resistive wall boundary condition contains only constant and linear terms in ω. A major part of the analysis that follows involves showing that the actual resistive wall boundary condition for a 2-D axisymmetric torus is of the same form as that of the model problem.

III. General axisymmetric toroidal formulation

The analysis just described is generalized in this section to the problem of ideal MHD stability in an arbitrary 2-D axisymmetric toroidal geometry including the effects of flow and a resistive wall. The end goal is to show that the 2-D problem can be transformed into a standard linear algebra problem similar in form to Eq. (15). To achieve this goal requires the following steps: (1) describe the 2-D equilibria of interest, (2) derive the general stability equations, (3) transform the stability equations into a linear algebra problem with a temporarily unspecified boundary condition at the plasma-vacuum interface, (4) derive an explicit form for the resistive wall boundary condition, and (5) combine the results to obtain the final standard eigenvalue problem. We note that the analysis has also been extended to arbitrary 3-D configurations. The plasma analysis remains essentially the same. The main difference is a slightly more complicated form of the resistive wall analysis because of the 3-D wall geometry. However, the structure of the resistive wall boundary condition remains unchanged.

A. Equilibrium

The analysis presented here relies heavily on the early work of Frieman and Rotenberg [8] and at this point is valid for an arbitrary 3-D geometry. The starting point is the assumption that an ideal MHD equilibrium has been calculated, either analytically, or more probably numerically, that satisfies the following equations,

$$\begin{aligned} &\nabla\cdot(\rho\mathbf{V})=0 \\ &\nabla\times(\mathbf{V}\times\mathbf{B})=0 \\ &\nabla\cdot(p\mathbf{V}/\rho^{\gamma-1})=0 \\ &\rho\mathbf{V}\cdot\nabla\mathbf{V}=\mathbf{J}\times\mathbf{B}-\nabla p \end{aligned} \tag{17}$$

Here, ρ, p, $\mathbf{B}$, $\mathbf{J}$ and γ have their usual definitions (from static ideal MHD) and $\mathbf{V}$ is the equilibrium flow velocity which has both toroidal and poloidal components.

The goal now is to formulate the stability problem by linearizing about this equilibrium.

B. Stability

For the linear stability analysis we expand all dependent variables as follows: $Q(\mathbf{r},t) = Q_0(\mathbf{r}) + \tilde{Q}(\mathbf{r})\exp(-i\omega t)$ with $\tilde{Q} \ll Q_0$. Here Q_0 is the equilibrium solution and $\tilde{Q}$ is a small perturbation. The $\exp(-i\omega t)$ dependence implies that we are carrying out a normal mode stability analysis. For simplicity, the zero subscript is hereafter suppressed on all equilibrium quantities.

The stability problem is formulated in terms of the perturbed displacement vector $\boldsymbol{\xi}$. For a system with an equilibrium flow velocity the relationship between $\boldsymbol{\xi}$ and the perturbed velocity $\tilde{\mathbf{v}}$ is defined as

$$\tilde{\mathbf{v}} \equiv -i\omega\boldsymbol{\xi} + \mathbf{V}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{V} \tag{18}$$

With this definition it can be shown that the perturbed density, pressure, magnetic field, and current density can be written as

$$\begin{aligned}
&\tilde{\rho} = -\nabla\cdot(\rho\boldsymbol{\xi}) && \text{Mass conservation}\\
&\tilde{p} = -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi} && \text{Adiabatic equation of state}\\
&\tilde{\mathbf{B}} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) && \text{Faraday plus Ohm's laws}\\
&\tilde{\mathbf{J}} = \nabla\times\nabla\times(\boldsymbol{\xi}\times\mathbf{B}) && \text{Ampere's law}
\end{aligned} \tag{19}$$

Note that these are the identical relations as if there were no flow. Also, here and below we set $\mu_0 = 1$ in Ampere’s law to simplify the notation.

The last remaining equation corresponds to the linearized conservation of momentum and is given by

$$\left(\rho \frac{d\mathbf{v}}{dt}\right)_1 = \tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \tilde{\mathbf{B}} - \nabla \tilde{p} \tag{20}$$

After a short calculation the inertial term can be linearized leading to the following perturbed momentum equation

$$\begin{aligned} -\omega^2 \rho \boldsymbol{\xi} - 2i\omega\rho(\mathbf{V} \cdot \nabla \boldsymbol{\xi}) + \rho (\mathbf{V} \cdot \nabla)^2 \boldsymbol{\xi} - \nabla \cdot \left[\boldsymbol{\xi} (\rho \mathbf{V} \cdot \nabla \mathbf{V})\right] \\ = \tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \tilde{\mathbf{B}} - \nabla \tilde{p} \end{aligned} \tag{21}$$

This is the basic equation describing the linear stability of the plasma.

Observe that the stability equation is second order in ω. It can be easily rewritten as two first order equations by introducing a new dependent variable $\mathbf{u} = \omega \boldsymbol{\xi} + i \mathbf{V} \cdot \nabla \boldsymbol{\xi}$. The system of equations can then be rewritten as

$$\begin{aligned} \omega \rho \boldsymbol{\xi} &= -i\rho \mathbf{V} \cdot \nabla \boldsymbol{\xi} + \rho \mathbf{u} \\ \omega \rho \mathbf{u} &= -\mathbf{F}(\boldsymbol{\xi}) - i\rho \mathbf{V} \cdot \nabla \mathbf{u} \end{aligned} \tag{22}$$

where

$$\mathbf{F}(\boldsymbol{\xi}) = \tilde{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \tilde{\mathbf{B}} - \nabla \tilde{p} + \nabla \cdot \left[\boldsymbol{\xi} (\rho \mathbf{V} \cdot \nabla \mathbf{V})\right] \tag{23}$$

The equations now have the same form as that of the test problem. See Eq. (4). Note that this form is similar to, but not identical to, the one given by Frieman and Rotenberg. The difference is that the term $i\rho \mathbf{V} \cdot \nabla \mathbf{u}$ is not incorporated in our definition of $\mathbf{F}(\boldsymbol{\xi})$ while it is in Frieman and Rotenberg. Our definition is shown to be useful theoretically in maximizing the symmetry that arises in the matrix formulation of the problem.

C. Intermediate linear algebra equations

The next step is to transform Eq. (22) into an intermediate weak-form linear algebra problem using the Galerkin procedure. Here, "intermediate" refers to the fact that the resistive wall boundary condition has not as yet been specified.

A Galerkin integral L is formed by multiplying the first equation by $\hat{\mathbf{u}}$, the second equation by $\hat{\boldsymbol{\xi}}$, adding them together, and integrating over the plasma volume. Here, $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\xi}}$ serve as the Galerkin test functions. A short calculation yields

$$\begin{aligned} L = \int \Big\{ & \omega\rho\left(\hat{\mathbf{u}}\cdot\boldsymbol{\xi} + \hat{\boldsymbol{\xi}}\cdot\mathbf{u}\right) - \rho\hat{\mathbf{u}}\cdot\mathbf{u} \\ & + i\rho\left[\hat{\boldsymbol{\xi}}\cdot(\mathbf{V}\cdot\nabla)\mathbf{u} + \hat{\mathbf{u}}\cdot(\mathbf{V}\cdot\nabla\boldsymbol{\xi})\right] + \hat{\boldsymbol{\xi}}\cdot\mathbf{F}(\boldsymbol{\xi})\Big\} d\mathbf{r} \end{aligned} \tag{24}$$

Here, $\mathbf{F}(\boldsymbol{\xi})$ is the MHD force operator including the effects of flow:

$$\begin{aligned} \mathbf{F}(\boldsymbol{\xi}) &= \tilde{\mathbf{J}}\times\mathbf{B} + \mathbf{J}\times\tilde{\mathbf{B}} - \nabla\tilde{p} + \nabla\cdot(\boldsymbol{\xi}\,\mathbf{K}) \\ \mathbf{K} &= \rho\mathbf{V}\cdot\nabla\mathbf{V} \end{aligned} \tag{25}$$

The MHD force term can be rewritten in several alternate forms that are inherently symmetric by construction, with the exception of a boundary term arising from several integrations by parts. As usual, obtaining these forms requires a large amount of seemingly mindless algebra. In any event two such forms are given below. Each can be written as

$$\int \hat{\boldsymbol{\xi}}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r} = -\int W_F\,d\mathbf{r} - \int S_B\,dS \tag{26}$$

The first form is a modification of the so called "intuitive form" of the MHD potential energy [9] which includes the contributions due to flow. It is given by

$$
\begin{aligned}
W_F = & \hat{\mathbf{Q}}_\perp \cdot \mathbf{Q}_\perp && \text{line bending} \\
& + B^2(\nabla\cdot\hat{\boldsymbol{\xi}}_\perp + 2\hat{\boldsymbol{\xi}}_\perp\cdot\boldsymbol{\kappa})(\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) && \text{magnetic compression} \\
& + \gamma p(\nabla\cdot\hat{\boldsymbol{\xi}})(\nabla\cdot\boldsymbol{\xi}) && \text{plasma compression} \\
& + \frac{1}{2}\mathbf{J}_\parallel\cdot\left[\hat{\boldsymbol{\xi}}_\perp\times\mathbf{Q}_\perp + \boldsymbol{\xi}_\perp\times\hat{\mathbf{Q}}_\perp\right] && \text{current driven modes} \\
& - \mathbf{J}_\perp\times\mathbf{B}\cdot\left[(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp)\hat{\boldsymbol{\xi}}_\perp + (\boldsymbol{\kappa}\cdot\hat{\boldsymbol{\xi}}_\perp)\boldsymbol{\xi}_\perp\right] && \text{pressure driven modes} \\
& + \frac{1}{2}\mathbf{K}\cdot\left[\nabla\cdot\left(\hat{\boldsymbol{\xi}}_\parallel\boldsymbol{\xi} + \boldsymbol{\xi}_\parallel\hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\xi}}\boldsymbol{\xi}_\perp + \boldsymbol{\xi}\hat{\boldsymbol{\xi}}_\perp\right)\right] && \text{flow driven modes} \\
S_B = & (\mathbf{n}\cdot\hat{\boldsymbol{\xi}})(\tilde{p} + \mathbf{B}\cdot\mathbf{Q}) && \text{boundary term}
\end{aligned}
\tag{27}
$$

Here, $\mathbf{Q} \equiv \tilde{\mathbf{B}}$, $\mathbf{b} = \mathbf{B}/B$ and we have assumed for simplicity that $p = \nabla p = \mathbf{J} = \mathbf{K} = 0$ on the plasma surface.

The second form is considerably more compact. It can be written as

$$
\begin{aligned}
W_F = & \gamma p(\nabla\cdot\hat{\boldsymbol{\xi}})(\nabla\cdot\boldsymbol{\xi}) \\
& + (\mathbf{B}\cdot\nabla\hat{\boldsymbol{\xi}} - \mathbf{B}\nabla\cdot\hat{\boldsymbol{\xi}})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}\nabla\cdot\boldsymbol{\xi}) \\
& + (\nabla p^*)\cdot(\hat{\boldsymbol{\xi}}\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\nabla\cdot\hat{\boldsymbol{\xi}}) + \frac{1}{2}(\hat{\boldsymbol{\xi}}\boldsymbol{\xi} + \boldsymbol{\xi}\hat{\boldsymbol{\xi}}) : \nabla\nabla p^* \\
S_B = & (\mathbf{n}\cdot\hat{\boldsymbol{\xi}})(\tilde{p} + \mathbf{B}\cdot\mathbf{Q})
\end{aligned}
\tag{28}
$$

where $p^* = p + B^2/2$. Interestingly, the flow does not explicitly appear in this form. It only appears implicitly through the equilibrium quantities. Note that for incompressible MHD (i.e. $\nabla\cdot\boldsymbol{\xi} = 0$) the form simplifies even further as follows

$$
W_F = (\mathbf{B}\cdot\nabla\hat{\boldsymbol{\xi}})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\xi}) + \frac{1}{2}(\hat{\boldsymbol{\xi}}\boldsymbol{\xi} + \boldsymbol{\xi}\hat{\boldsymbol{\xi}}) : \nabla\nabla p^* \tag{29}
$$

This surprisingly compact form is valid for a general 3-D geometry including arbitrary toroidal and poloidal flow although one must make sure that the incompressibility constraint $\nabla\cdot\boldsymbol{\xi} = 0$ is satisfied when substituting trial functions for $\boldsymbol{\xi}$.

While Eqs. (28) and (29) are elegant theoretically, they are not very efficient computationally since third order “radial derivatives” on the equilibrium flux function are required to numerically evaluate $\nabla\nabla p^*$. In practice it makes computational sense to substitute the equilibrium relation $\nabla p^* = \mathbf{B}\cdot\nabla\mathbf{B} - \mathbf{K}$ to eliminate one “radial derivative”.

The next step is to introduce expansions for $\boldsymbol{\xi}$ and $\mathbf{u}$. To do this it is necessary to choose (1) a coordinate system, (2) a set of unit projection vectors, (3) normalizations for the components of $\boldsymbol{\xi}$ and $\mathbf{u}$, (4) a convenient form of the force integral W_F, and (5) an appropriate set of basis functions. Clearly there is a great deal of freedom in these choices. However, once the choices have been made a large part of the problem becomes standard, although requiring extensive algebra. The one non-standard feature is the resistive wall boundary condition.

For present purposes it suffices to outline the standard part of the analysis for one choice of expansion options as described in (1) –(4) above and to then focus on the resistive wall boundary condition. To demonstrate the procedure we begin by introducing a system of flux coordinates ψ,χ,ϕ where $\psi = \psi(R,Z)$ is the equilibrium flux function satisfying $\mathbf{B}\cdot\nabla\psi = 0$, ϕ is the usual toroidal angle, and $\chi = \chi(R,Z)$ is an arbitrarily defined poloidal angle. Also, following Goedbloed [10], we introduce a set of unit orthogonal projection vectors $\mathbf{n} = \nabla\psi/|\nabla\psi|$, $\mathbf{b} = \mathbf{B}/B$ and $\boldsymbol{\tau} = \mathbf{b}\times\mathbf{n}$. The unknowns $\boldsymbol{\xi}$ and $\mathbf{u}$ can thus be written as

$$\begin{aligned} \boldsymbol{\xi} &= \xi_\psi\,\mathbf{n} + i\xi_\tau\,\boldsymbol{\tau} + i\xi_\parallel\,\mathbf{b} \\ \mathbf{u} &= u_\psi\,\mathbf{n} + iu_\tau\,\boldsymbol{\tau} + iu_\parallel\,\mathbf{b} \end{aligned} \tag{30}$$

One common normalization [10] for the components of $\boldsymbol{\xi}$ and $\mathbf{u}$ is given by

$$
\begin{aligned}
X &= RB_p\xi_\psi \qquad Y = \frac{B\xi_\tau}{RB_p} \qquad Z = \frac{\xi_\parallel}{B} \\
\overline{X} &= RB_p u_\psi \qquad \overline{Y} = \frac{Bu_\tau}{RB_p} \qquad \overline{Z} = \frac{u_\parallel}{B}
\end{aligned}
\tag{31}
$$

Expansions for the normalized variables are introduced by Fourier analyzing in χ and ϕ, and assuming a set of appropriate finite element basis functions in ψ. Specifically, we assume that

$$
\begin{aligned}
\overline{Z} &= e^{in\phi}\sum_{l=1}^{L}\overline{Z}_l\, h_j(\psi)\, e^{im\chi} & \overline{Z}' &= e^{in\phi}\sum_{l=1}^{L}\overline{Z}_l\, h_j'(\psi)\, e^{im\chi} \\
\overline{Y} &= e^{in\phi}\sum_{l=1}^{L}\overline{Y}_l\, g_j(\psi)\, e^{im\chi} & \overline{Y}' &= e^{in\phi}\sum_{l=1}^{L}\overline{Y}_l\, g_j'(\psi)\, e^{im\chi} \\
\overline{X} &= e^{in\phi}\sum_{l=1}^{L}\overline{X}_l\, f_j(\psi)\, e^{im\chi} & \overline{X}' &= e^{in\phi}\sum_{l=1}^{L}\overline{X}_l\, f_j'(\psi)\, e^{im\chi} \\
\\
Z &= e^{in\phi}\sum_{l=1}^{L}Z_l\, h_j(\psi)\, e^{im\chi} & Z' &= e^{in\phi}\sum_{l=1}^{L}Z_l\, h_j'(\psi)\, e^{im\chi} \\
Y &= e^{in\phi}\sum_{l=1}^{L}Y_l\, g_j(\psi)\, e^{im\chi} & Y' &= e^{in\phi}\sum_{l=1}^{L}Y_l\, g_j'(\psi)\, e^{im\chi} \\
X &= e^{in\phi}\sum_{l=1}^{L}X_l\, f_j(\psi)\, e^{im\chi} & X' &= e^{in\phi}\sum_{l=1}^{L}X_l\, f_j'(\psi)\, e^{im\chi}
\end{aligned}
\tag{32}
$$

Here, prime denotes $\partial/\partial\psi$ and

$$
\sum_{l=1}^{L} \equiv \sum_{j=1}^{J}\sum_{m=-M}^{M} \tag{33}
$$

with $l=(2M+1)j-M+m$ a unique identifier for each combination m,j. The range of l is $1\le l\le L=(2M+1)J$. Also, the order of the summation corresponds to fixing a value for j and then summing over all m before proceeding to the next j. The magnetic axis corresponds to the index $j=0$. Consequently the first amplitude coefficients (e.g. X_0) and all basis functions are assumed to satisfy the regularity condition on axis and are therefore known. They are suppressed from the summation. The first unknown coefficients (e.g. X_1) correspond to the second radial point.

In analogy with the test problem we select a set of compact basis functions with the following properties:

$$\begin{aligned}
&f_j(\psi_j) = g_j(\psi_j) = h_j(\psi_j) = 1 && 1 \le j \le J \\
&f_j(\psi_{j\pm1}) = g_j(\psi_{j\pm1}) = h_j(\psi_{j\pm1}) = 0 && 1 \le j \le J-1 \\
&f_j(\psi_{j-1}) = g_j(\psi_{j-1}) = h_j(\psi_{j-1}) = 0 && j = J
\end{aligned} \tag{34}$$

The number of unknowns defined by the expansion equations is equal to $6L$. Again in analogy with the test problem we point out that the derivative of each harmonic on the boundary $\psi = \psi_B$ is an independent free constant (because the amplitude of each harmonic ghost element is not included in the summation). It is shown shortly that the only radial derivative appearing in the boundary term is proportional to $\partial X / \partial \psi$. Therefore, there are an additional $2M+1$ unknowns denoted by $X'_m(\psi_B) \equiv X'_m$. The conclusion is that all told there are $6L+2M+1$ unknown coefficients to be determined.

In accordance with the Galerkin procedure we derive a set of linear equations for the unknown coefficients by substituting a sequence of test functions into the quadratic integral given by Eq. (24). Normalized test functions $\hat{X}, \hat{Y}, \hat{Z}$ and $\hat{\hat{X}}, \hat{\hat{Y}}, \hat{\hat{Z}}$ are defined by

$$\begin{aligned}
\hat{\boldsymbol{\xi}} &= \frac{\hat{X}}{RB_p}\mathbf{n} - i\frac{RB_p\hat{Y}}{B}\boldsymbol{\tau} - iB\hat{Z}\,\mathbf{b} \\
\hat{\mathbf{u}} &= \frac{\hat{\hat{X}}}{RB_p}\mathbf{n} - i\frac{RB_p\hat{\hat{Y}}}{B}\boldsymbol{\tau} - iB\hat{\hat{Z}}\,\mathbf{b}
\end{aligned} \tag{35}$$

Again, in analogy with the test problem, we choose six sequences as follows.

$$
\begin{array}{lll}
\text{Sequence 1} & \hat{\bar{Z}} = e^{-in\phi - im'\chi} h_{j'}(\psi) & 1 \le l' \le L \\
\text{Sequence 2} & \hat{\bar{Y}} = e^{-in\phi - im'\chi} g_{j'}(\psi) & 1 \le l' \le L \\
\text{Sequence 3} & \hat{\bar{X}} = e^{-in\phi - im'\chi} f_{j'}(\psi) & 1 \le l' \le L \\
 & & \\
\text{Sequence 4} & \hat{Z} = e^{-in\phi - im'\chi} h_{j'}(\psi) & 1 \le l' \le L \\
\text{Sequence 5} & \hat{Y} = e^{-in\phi - im'\chi} g_{j'}(\psi) & 1 \le l' \le L \\
\text{Sequence 6} & \hat{X} = e^{-in\phi - im'\chi} f_{j'}(\psi) & 1 \le l' \le L
\end{array}
\tag{36}
$$

In each sequence only the component of test function listed is non-zero. The resulting integrals give rise to $6L$ linear equations. The remaining $2M+1$ equations are determined from the resistive wall boundary condition.

In its present form the linear algebra problem can be written as

$$
\begin{aligned}
L_1 &= \int \hat{\mathbf{u}} \cdot \left[\omega\rho\boldsymbol{\xi} - \rho\mathbf{u} + i\rho(\mathbf{V}\cdot\nabla\boldsymbol{\xi})\right] d\mathbf{r} = 0 \\
&= \omega \mathsf{D}\cdot\mathbf{x} - \mathsf{D}\cdot\mathbf{y} - \mathsf{U}\cdot\mathbf{x} = 0 \\
\\
L_2 &= \int \hat{\boldsymbol{\xi}} \cdot \left[\omega\rho\mathbf{u} + i\rho(\mathbf{V}\cdot\nabla\mathbf{u}) + \mathbf{F}(\boldsymbol{\xi})\right] d\mathbf{r} = 0 \\
&= \omega \mathsf{D}\cdot\mathbf{y} - \mathsf{U}\cdot\mathbf{y} - \mathsf{W}\cdot\mathbf{x} - \mathsf{S}_Y\cdot\mathbf{Y} - \mathsf{S}_X\cdot\mathbf{X} - \mathsf{S}'\cdot\mathbf{X}'_B = 0
\end{aligned}
\tag{37}
$$

Here, $\mathbf{y} = [\bar{\mathbf{Z}}, \bar{\mathbf{Y}}, \bar{\mathbf{X}}]$ and $\mathbf{x} = [\mathbf{Z}, \mathbf{Y}, \mathbf{X}]$ with $\bar{\mathbf{Z}} = [\bar{Z}_1, \ldots \bar{Z}_L]$, $\bar{\mathbf{Y}} = [\bar{Y}_1, \ldots \bar{Y}_L]$, $\bar{\mathbf{X}} = [\bar{X}_1, \ldots \bar{X}_L]$, $\mathbf{Z} = [Z_1, \ldots Z_L]$, $\mathbf{Y} = [Y_1, \ldots Y_L]$, and $\mathbf{X} = [X_1, \ldots X_L]$. Also, $\mathbf{X}'_B = [X'_{-M}, \ldots X'_M]$. Note that $\mathbf{x}, \mathbf{y}$ each has a length $3L$ while $\mathbf{X}'_B$ has a length $2M+1$. The combination of these three vectors represent the total number of the unknown amplitudes in the problem equal to $6L + 2M + 1$.

The matrices D, U, and W have dimensions $(3L)\times(3L)$ and can be calculated in a straightforward but tedious manner by evaluating the integrals using the appropriate test functions. The smaller matrices S_Y, S_X, and S' arise from the surface boundary terms and (are shown to) have dimensions $(2M+1)\times(2M+1)$.

Also, the matrices S_Y and S_X act only on the boundary elements of $\mathbf{X}$ and $\mathbf{Y}$, corresponding to $l=(2M+1)J-M+m$ where $-M\le m\le M$. For purposes of illustration we rewrite all the matrices appearing in Eq. (37) in terms of flux coordinates for the case of purely toroidal flow in Appendix A. Once these coordinates are introduced it is then straightforward to explicitly determine the matrix elements. Hereafter, we assume that the matrix elements $\mathsf{D}, \mathsf{U}, \mathsf{W}, \mathsf{S}_Y, \mathsf{S}_X,$ and S' defined above are known.

To close the system, we require $2M+1$ additional equations which arise from the resistive wall boundary condition. This is the next task.

D. The resistive wall boundary condition

Consider now the resistive wall boundary condition. The geometry of interest is illustrated in Fig. 1 and four steps are required to accomplish our goal: (1) solve for the fields in the vacuum regions using Green's theorem [11,12] , (2) solve for the fields within the resistive wall using the thin wall approximation, ultimately converting this solution into a set of jump conditions to connect the vacuum fields across the wall, (3) combine these results and project the solutions back onto the plasma surface, and (4) express the projected fields in terms of the plasma displacement $\boldsymbol{\xi}$ using the plasma-vacuum jump conditions on the plasma surface. The end result is an expression for the resistive wall boundary condition in terms of $\boldsymbol{\xi}$, which is shown to be linear in ω.

Calculation of the boundary condition is the critical component of the analysis and requires a lengthy calculation. For the sake of continuity, we simply state the result here and give the details in Appendix B. Specifically, in Appendix B it is shown that the resistive wall boundary condition can be written as

$$\left[i\omega\mathsf{K}_1-\mathsf{K}_2\right]\cdot\mathbf{X}+\left[i\omega\mathsf{K}_3-\mathsf{K}_4\right]\cdot\mathbf{Y}+\left[i\omega\mathsf{K}_5-\mathsf{K}_6\right]\cdot\mathbf{X}'_B=0 \qquad (38)$$

where the K_j are known $(2M+1)\times(2M+1)$ matrices depending only on the geometry of the plasma surface and the resistive wall. The matrices $\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3, \mathsf{K}_4$ act only on the boundary elements as described above.

With the calculation of the K_j, all of the individual contributions to the analysis have been evaluated. They must now be collected and assembled into the final formulation. This is the task of the final Section.

IV. Summary of the formulation

Following the procedure outlined in the model problem we can cast the 2-D axisymmetric toroidal problem into the form of a standard eigenvalue problem. By adding the resistive wall boundary condition to the equations describing the plasma interior [i.e. Eq. (37)] we arrive at a standard linear algebra problem for the eigenvalue ω. The formulation can be written as

$$\omega\,\mathsf{A}\cdot\mathbf{z} = \mathsf{B}\cdot\mathbf{z} \tag{39}$$

Here, the eigenvector $\mathbf{z} = [\mathbf{y}\,,\mathbf{x}\,,\mathbf{X}_B'] = [\bar{\bar{\mathbf{Z}}}, \bar{\bar{\mathbf{Y}}}, \bar{\bar{\mathbf{X}}}\,, \mathbf{Z}, \mathbf{Y}, \mathbf{X}\,, \mathbf{X}_B']$. The matrices are separated into to two contributions, one arising from the plasma interior and the other from the boundary terms: $\mathsf{A} = \mathsf{A}_P + i\mathsf{A}_B$, $\mathsf{B} = \mathsf{B}_P + \mathsf{B}_B$. These are given as follows.

$$
\mathsf{A}_P = \left|\begin{array}{c|c|c} \mathbf{0} & \mathsf{D} & \mathbf{0} \\ \hline \mathsf{D} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right| \qquad \mathsf{A}_B = \left|\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathsf{K}_3 \quad \mathsf{K}_1 & \mathsf{K}_5 \end{array}\right|
$$

$$
\mathsf{B}_P = \left|\begin{array}{c|c|c} \mathsf{D} & \mathsf{U} & \mathbf{0} \\ \hline \mathsf{U} & \mathsf{W} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right| \qquad \mathsf{B}_B = \left|\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathsf{S}_Y \quad \mathsf{S}_X & \mathsf{S}' \\ \hline \mathbf{0} & \mathsf{K}_4 \quad \mathsf{K}_2 & \mathsf{K}_6 \end{array}\right| \tag{40}
$$

This is the desired formulation of the problem which should be highly convenient for numerical computation.

We conclude by emphasizing that the critical feature that has allowed us to achieve our goal is the fact that ω appears linearly in the resistive wall boundary condition in the thin wall approximation.

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Appendix A

The desired relations for L_1 and L_2 are obtained by using the projection vectors and eigenfunction normalizations suggested by Goedbloed [10]. We begin with the density normalization matrix D which appears in several terms. The matrix elements can be determined from any of these terms, for instance

$$\omega \int \hat{\mathbf{u}} \cdot \boldsymbol{\xi} \, d\mathbf{r} = \omega \, \mathsf{D} \cdot \mathbf{x} \tag{A1}$$

From the definitions of $\hat{\mathbf{u}}$ and $\boldsymbol{\xi}$ it immediately follows that

$$\begin{aligned} &\int \rho \hat{\mathbf{u}} \cdot \boldsymbol{\xi} \, d\mathbf{r} = 2\pi \int J d\psi d\chi \left(\frac{\rho}{R^2 B_p^2} \hat{\bar{X}} X + \frac{\rho R^2 B_p^2}{B^2} \hat{\bar{Y}} Y + \rho B^2 \hat{\bar{Z}} Z \right) \\ &J = \frac{1}{\mathbf{B}_p \cdot \nabla \chi} \end{aligned} \tag{A2}$$

Consider next the flow matrix U which can be evaluated from the definition

$$i \int \rho \, \hat{\mathbf{u}} \cdot (\mathbf{V} \cdot \nabla \boldsymbol{\xi}) \, d\mathbf{r} = -\mathsf{U} \cdot \mathbf{x} \tag{A3}$$

For the case of purely toroidal flow, $\mathbf{V} = \Omega(\psi) R \, \mathbf{e}_\phi$. A short calculation yields

$$\begin{aligned} &i \int \rho \, \hat{\mathbf{u}} \cdot (\mathbf{V} \cdot \nabla \boldsymbol{\xi}) \, d\mathbf{r} = 2\pi \int J d\psi d\chi \, \rho \, \Omega \, (U_1 + U_2 + U_3) \\ &U_1 = \hat{\bar{X}} \left(\frac{n}{R^2 B_p^2} X - \frac{\mathrm{n}_R \tau_\phi}{B} Y - \frac{B \, \mathrm{n}_R \mathrm{b}_\phi}{R B_p} Z \right) \\ &U_2 = \hat{\bar{Y}} \left(-\frac{\mathrm{n}_R \tau_\phi}{B} X + \frac{n R^2 B_p^2}{B^2} Y + i R B_p \, \mathrm{n}_Z Z \right) \\ &U_3 = \hat{\bar{Z}} \left(-\frac{B \, \mathrm{n}_R \mathrm{b}_\phi}{R B_p} X - i R B_p \, \mathrm{n}_Z Y + n B^2 Z \right) \end{aligned} \tag{A4}$$

Here,

$$\begin{aligned} \mathrm{n}_R &= \mathbf{n}\cdot\mathbf{e}_R \\ \mathrm{n}_Z &= \mathbf{n}\cdot\mathbf{e}_Z \\ \mathrm{b}_\phi &= \mathbf{b}\cdot\mathbf{e}_\phi \\ \tau_\phi &= \boldsymbol{\tau}\cdot\mathbf{e}_\phi \end{aligned} \tag{A5}$$

The next relation of interest involves the W matrix, defined by

$$\int W_F \, d\mathbf{r} = \mathsf{W}\cdot\mathbf{x} \tag{A6}$$

The desired relation can be written as

$$\int W_F \, d\mathbf{r} = 2\pi\int J d\psi d\chi\left(W_1 + W_2 + W_3 + W_4 + W_5 + W_6\right) \tag{A7}$$

After substantial algebra the W_j can be evaluated. In the same sequence as in Eq. (27), the first five W_j are given by

$$\begin{aligned} W_1 &= \frac{1}{R^2B_p^2}(F*\hat{X})(FX) + \left[\frac{B\alpha_\tau}{RB_p}\hat{X} + \frac{RB_p}{B}F*\hat{Y}\right]\left[\frac{B\alpha_\tau}{RB_p}X + \frac{RB_p}{B}FY\right] \\ W_2 &= B^2\left[\left(H + \frac{2\kappa_n B_p}{R}\right)\frac{\hat{X}}{B_p^2} - \frac{1}{B^2}G*\hat{Y}\right]\left[\left(H + \frac{2\kappa_n B_p}{R}\right)\frac{X}{B_p^2} - \frac{1}{B^2}GY\right] \\ W_3 &= \gamma p\left[H\frac{\hat{X}}{B_p^2} - G*\frac{\hat{Y}}{B^2} - F*\hat{Z}\right]\left[H\frac{X}{B_p^2} - G\frac{Y}{B^2} - FZ\right] \\ W_4 &= -\frac{J_\parallel}{2B}\left[\hat{X}\left(\frac{B^2\alpha_\tau}{R^2B_p^2}X + FY\right) - \hat{Y}FX\right]\left[X\left(\frac{B^2\alpha_\tau}{R^2B_p^2}\hat{X} + F*\hat{Y}\right) - YF*\hat{X}\right] \\ W_5 &= -\frac{J_\perp B}{R^2B_p^2}\left[\hat{X}\left(\kappa_n X + i\frac{R^2B_p^2\kappa_\tau}{B}Y\right)\right]\left[X\left(\kappa_n\hat{X} - i\frac{R^2B_p^2\kappa_\tau}{B}\hat{Y}\right)\right] \end{aligned} \tag{A8}$$

where

$$
\begin{aligned}
F &= -i\,\mathbf{B}\cdot\nabla = -i\left(\frac{1}{J}\frac{\partial}{\partial\chi} + \frac{B_\phi}{R}\frac{\partial}{\partial\phi}\right) \\
G &= -i\,\mathbf{B}\times\nabla\psi\cdot\nabla = -i\left(B_p^2\frac{\partial}{\partial\phi} - \frac{RB_\phi}{J}\frac{\partial}{\partial\chi}\right) \\
H &= \nabla\cdot\left(\frac{\nabla\psi}{R^2}...\right) = B_p^2\frac{\partial}{\partial\psi} + \frac{\nabla\psi\cdot\nabla\chi}{R^2}\frac{\partial}{\partial\chi} - \frac{J_\phi}{R} \\
\alpha_\tau &= \boldsymbol{\tau}\cdot\nabla\times\boldsymbol{\tau} \\
\kappa_n &= \mathbf{n}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \\
\kappa_\tau &= \boldsymbol{\tau}\cdot(\mathbf{b}\cdot\nabla\mathbf{b})
\end{aligned}
\tag{A9}
$$

The flow term is more complicated and can be written in the following form $W_6 = w_1 + w_1^\dagger + w_2 + w_2^\dagger$

$$
\begin{aligned}
w_1 &= \hat{\xi}_\parallel\left(-i\xi_\psi\beta_{31} + \xi_\tau\beta_{32} + \xi_\parallel\beta_{33}\right) - i\mathrm{n}_R\mathbf{b}\cdot\nabla\left(\hat{\xi}_\parallel\xi_\psi\right) + \tau_R\mathbf{b}\cdot\nabla\left(\hat{\xi}_\parallel\xi_\tau\right) + \mathrm{b}_R\mathbf{b}\cdot\nabla\left(\hat{\xi}_\parallel\xi_\parallel\right) \\
w_2 &= \hat{\xi}_\psi\left(\xi_\psi\beta_{11} + i\xi_\tau\beta_{12}\right) - i\hat{\xi}_\tau\left(\xi_\psi\beta_{21} + i\xi_\tau\beta_{22}\right) - i\hat{\xi}_\psi\left(\xi_\psi\beta_{31} + i\xi_\tau\beta_{32}\right) \\
&\quad + \mathrm{n}_R\left[\mathbf{n}\cdot\nabla\left(\hat{\xi}_\psi\xi_\psi\right) - i\boldsymbol{\tau}\cdot\nabla\left(\hat{\xi}_\tau\xi_\psi\right) - i\mathbf{b}\cdot\nabla\left(\hat{\xi}_\parallel\xi_\parallel\right)\right] \\
&\quad + i\tau_R\left[\mathbf{n}\cdot\nabla\left(\hat{\xi}_\psi\xi_\tau\right) - i\boldsymbol{\tau}\cdot\nabla\left(\hat{\xi}_\tau\xi_\tau\right) - i\mathbf{b}\cdot\nabla\left(\hat{\xi}_\parallel\xi_\tau\right)\right]
\end{aligned}
\tag{A10}
$$

The geometric factors appearing are given by

$$
\begin{aligned}
\tau_R &= \boldsymbol{\tau}\cdot\mathbf{e}_R \\
\mathrm{b}_R &= \mathbf{b}\cdot\mathbf{e}_R \\
\beta_{ij} &= \mathbf{e}_R\cdot\nabla\cdot\left(\mathbf{e}_i\mathbf{e}_j\right)
\end{aligned}
\tag{A11}
$$

with $\mathbf{e}_1 = \mathbf{n}, \mathbf{e}_2 = \boldsymbol{\tau},$ and $\mathbf{e}_3 = \mathbf{b}$. Also $w_1^\dagger\left(\hat{Z}, X\right) = w_1 * \left(Z, \hat{X}\right)$ and so on.

The last relation of interest involves the boundary term defined by

$$
\int S_B dS = \mathsf{S}_Y\cdot\mathbf{Y} + \mathsf{S}_X\cdot\mathbf{X} + \mathsf{S}'\cdot\mathbf{X}_B' \tag{A12}
$$

This expression can be written as

$$\int S_B dS = 2\pi \int J d\chi\, S \tag{A13}$$

where

$$\begin{aligned} S &= \left[\hat{X}(\mathbf{B}\cdot\mathbf{Q})\right]\Big|_{\psi_B} \\ &= -\hat{X}\left[\left(B_p^2 \frac{\partial}{\partial\psi} + \frac{\nabla\psi\cdot\nabla\chi}{R^2}\frac{\partial}{\partial\chi}\right)\left(\frac{B^2}{B_p^2}X\right) + \left(i\frac{B_\phi^2}{q}\frac{\partial}{\partial\chi} + nB_p^2\right)(Y)\right]_{\psi_B} \end{aligned} \tag{A14}$$

and for purposes of demonstration we have assumed straight field line coordinates: $(\mathbf{B}\cdot\nabla\phi)/(\mathbf{B}\cdot\nabla\chi) = q(\psi)$.

From these relations it is straightforward to directly read off the matrix elements for $\mathsf{D},\mathsf{U},\mathsf{W},\mathsf{S}_Y,\mathsf{S}_X,$ and S' .

Appendix B

The calculation of the resistive wall boundary condition proceeds as follows.

1. The vacuum fields

The perturbed magnetic field in each vacuum region satisfies $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. Thus for each region we can write $\mathbf{B} = \nabla V$ with V satisfying $\nabla^2 V = 0$. The potential V is actually only needed on the plasma surface and the wall surfaces. It is conveniently determined by means of Green's theorem which can be written as

$$\alpha V = \sum_S \int \left(V' \frac{\partial G}{\partial \rho'} - G \frac{\partial V'}{\partial \rho'} \right) J' dv' \, d\phi' \tag{B1}$$

Here, G is the 3-D free space Green's function

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{1}{\left| \mathbf{r} - \mathbf{r}' \right|} \tag{B2}$$

Each surface has been parameterized in terms of an arbitrary poloidal angle v: $R = R(v), Z = Z(v)$. In terms of these coordinates

$$J' \equiv R' Q' = R' \left(R_{v'}^2 + Z_{v'}^2 \right)^{1/2} \tag{B3}$$

where as usual primed and unprimed quantities represent the integration and observation coordinates respectively and the subscript v' denotes $\partial / \partial v'$. Also we introduce ρ', the distance in the outward (away from the plasma) normal direction off any surface by the definition $\partial / \partial \rho' \equiv \mathbf{n}' \cdot \nabla'$. The value of the parameter α depends on the location of the observation point and is defined as follows

$$\alpha = \begin{cases} 1 & \text{observation point within the volume} \\ 1/2 & \text{observation point on the surface} \\ 0 & \text{observation point outside the volume} \end{cases} \qquad \text{(B4)}$$

Lastly, note that the summation includes both the plasma and inner wall surfaces for the inner vacuum solution but only the outer wall surface for the outer vacuum solution since this solution vanishes at infinity.

The required solutions are obtained by applying Green's theorem three times, once for the observation point on the plasma surface, once on the inner wall surface, and once on the outer wall surface. This leads to the following three equations

$$\begin{aligned} V_1 &= -2\int_{S_1}\left(V_1'\frac{\partial H_{11}}{\partial \rho_1'} - H_{11}\frac{\partial V_1'}{\partial \rho_1'}\right)J'dv' + 2\int_{S_2}\left(V_2'\frac{\partial H_{12}}{\partial \rho_2'} - H_{12}\frac{\partial V_2'}{\partial \rho_2'}\right)J'dv' \\ V_2 &= -2\int_{S_1}\left(V_1'\frac{\partial H_{21}}{\partial \rho_1'} - H_{21}\frac{\partial V_1'}{\partial \rho_1'}\right)J'dv' + 2\int_{S_2}\left(V_2'\frac{\partial H_{22}}{\partial \rho_2'} - H_{22}\frac{\partial V_2'}{\partial \rho_2'}\right)J'dv' \qquad \text{(B5)} \\ V_3 &= -2\int_{S_3}\left(V_3'\frac{\partial H_{33}}{\partial \rho_3'} - H_{33}\frac{\partial V_3'}{\partial \rho_3'}\right)J'dv' \end{aligned}$$

where for an axisymmetric system the ϕ' integral involves only the Green's function which can be carried out analytically [13], yielding

$$\begin{aligned} H\left(v,v'\right) &= \int_0^{2\pi} G e^{in(\phi'-\phi)} d\phi' \\ &= -\frac{(2n-1)!!}{2^n n!}\frac{z^{n+1/2}}{\left(4RR'\right)^{1/2}} F\left(1/2, n+1/2; n+1; z^2\right) \end{aligned} \qquad \text{(B6)}$$

with

$$z = \frac{k^2}{2 - k^2 + 2\left(1 - k^2\right)^{1/2}}$$
$$k^2 = \frac{4RR'}{\left(R + R'\right)^2 + \left(Z - Z'\right)^2} \tag{B7}$$

Here, F is the hypergeometric function which can be evaluated using any standard mathematical package (e.g. Mathematica). For low n the integral can be simply expressed in terms of a few elliptic integrals. Note also that in Eq. (B5) all normal vectors point radially outward.

This set of equations should be viewed as three coupled integral equations. We assume that $\partial V_1' / \partial \rho_1'$ is a known quantity determined from the plasma-vacuum jump conditions. The specific relation is derived shortly. Thus, there are five unknowns in the problem: V_1', V_2', V_3', $\partial V_2' / \partial \rho_2'$, and $\partial V_3' / \partial \rho_3'$. Two more relations are required to close the system and these are determined by matching to the solutions within the wall. Ultimately our goal is to eliminate unknowns leading to a single relation between V_1' and $\partial V_1' / \partial \rho_1'$.

2. The fields within the wall

The fields within the wall can be found using standard techniques which exploit the "thin" wall approximation. Here, we assume the wall is axisymmetric with a thickness d and a characteristic minor radius b. The thin wall approximation assumes that $\delta \equiv d / b \ll 1$. For resistive wall modes the appropriate ordering for the various quantities is as follows

$$\begin{aligned}
&\frac{1}{b}\frac{\partial}{\partial \chi} \sim \frac{1}{R}\frac{\partial}{\partial \phi} \sim \delta \frac{\partial}{\partial \rho} \\
&\omega \sim 1 / \mu_0 \sigma b d \\
&\mathbf{n} \cdot \tilde{\mathbf{B}} \equiv \tilde{B}_\rho(\rho, \chi, \phi) \approx \overline{B}(\chi, \phi) + \tilde{B}(\rho, \chi, \phi) \\
&\tilde{B} / \overline{B} \sim \delta
\end{aligned} \tag{B8}$$

where the range of ρ, the normal distance measured from the inside of the wall, is $0 \le \rho \le d$.

In the thin wall analysis we only need the equation for the normal component of perturbed magnetic field within the resistive wall. Under the assumption of small δ this equation reduces to

$$\frac{\partial^2 \tilde{B}}{\partial \rho^2} \approx -i\omega\mu_0\sigma\overline{B} \tag{B9}$$

The solution, based on the usual "constant ψ" analysis, is given by

$$\tilde{B}_\rho(\rho,\chi,\phi) \approx \overline{B}(\chi,\phi)\left(1 - i\frac{\omega\mu_0\sigma}{2}\rho^2\right) \tag{B10}$$

From this solution it follows that the jumps in $\tilde{B}_\rho$ and the normal derivative of $\tilde{B}_\rho$ across the resistive wall are given by

$$\begin{aligned} \left[\!\left[\tilde{B}_\rho\right]\!\right]_0^d &\approx 0 \\ \left[\!\left[\partial\tilde{B}_\rho / \partial\rho\right]\!\right]_0^d &\approx -i\omega\mu_0\sigma d\overline{B} \end{aligned} \tag{B11}$$

Next, since no ideal surface currents flow on the inner and outer surfaces of the wall (i.e. at $\rho = 0$ and $\rho = d$ respectively) the jump conditions given by Eq. (B11) translate into an equivalent set of jump conditions on the vacuum fields. Specifically, on each wall surface we have

$$\begin{aligned} \left[\!\left[\tilde{B}_\rho\right]\!\right]_{0_-}^{0_+} &= 0 \qquad & \left[\!\left[\partial\tilde{B}_\rho / \partial\rho\right]\!\right]_{0_-}^{0_+} &= 0 \\ \left[\!\left[\tilde{B}_\rho\right]\!\right]_{d_-}^{d_+} &= 0 \qquad & \left[\!\left[\partial\tilde{B}_\rho / \partial\rho\right]\!\right]_{d_-}^{d_+} &= 0 \end{aligned} \tag{B12}$$

Combining Eqs. (B11) and (B12) enables us to write the $\left[\!\left[\tilde{B}_\rho\right]\!\right]$ conditions as

$$\frac{\partial V_3'}{\partial \rho_3'} = \frac{\partial V_2'}{\partial \rho_2'} \tag{B13}$$

This relation is in the desired form since $\partial V_3' / \partial \rho_3'$ and $\partial V_2' / \partial \rho_2'$ are two of the basic unknowns appearing in the coupled integral equations.

The second relation is more complicated.

$$\frac{\partial^2 V_3'}{\partial \rho_3'^2} = \frac{\partial^2 V_2'}{\partial \rho_2'^2} - i\omega\mu_0\sigma d \frac{\partial V_2'}{\partial \rho_2'} \tag{B14}$$

The difficulty is that the condition is expressed in terms of $\partial^2 V' / \partial \rho'^2$ whereas the basic unknowns in the problem involve V'. A simple relation can be obtained between $\partial^2 V / \partial \rho^2$ and V by making use of the fact that $\nabla^2 V = 0$ and defining an angle $\chi = \chi(v)$ to correspond to a normalized arc length-like variable. Specifically we define

$$\begin{aligned} d\chi &= \frac{R_0}{R}\frac{Q}{b} dv \\ b &= \frac{1}{2\pi} \oint \frac{R_0}{R} Q dv \end{aligned} \tag{B15}$$

After a slightly lengthy calculation, Laplace's equation can be evaluated on both wall surfaces in terms of the arc length variable leading to the following jump condition relation

$$\frac{R^2}{R_0^2} \left[\!\left[\frac{\partial^2 V}{\partial \rho^2} \right]\!\right] = \left[\!\left[\frac{n^2}{R_0^2} V - \frac{1}{b^2} \frac{\partial^2 V}{\partial \chi^2} \right]\!\right] \tag{B16}$$

This equation can be easily solved by means of Fourier analysis. We expand

$$V'_j = \sum_{m'} V_{m'}^{(j)} e^{im'\chi'}$$

$$\frac{R'^2}{R_0^2}\frac{\partial^2 V'_j}{\partial \rho_j'^2} = \frac{1}{b^2}\sum_{m'} \ddot{V}_{m'}^{(j)} e^{im'\chi'} \tag{B17}$$

where $j = 2,3$. The Fourier coefficients $V_{m'}^{(j)}, \ddot{V}_{m'}^{(j)}$ are related to each other through Eq. (B16).

$$\ddot{V}_{m'}^{(3)} - \ddot{V}_{m'}^{(2)} = k_{m'}^2 b^2 \left[V_{m'}^{(3)} - V_{m'}^{(2)}\right] \qquad k_{m'}^2 = \frac{m'^2}{b^2} + \frac{n^2}{R_0^2} \tag{B18}$$

The final desired relations are obtained by Fourier expanding the normal derivative

$$\frac{R'^2}{R_0^2}\frac{\partial V'_j}{\partial \rho'_j} = \frac{1}{b}\sum_{m'} \dot{V}_{m'}^{(j)} e^{im'\chi'} \tag{B19}$$

and substituting into the jump conditions given by Eq. (B13) and (B14). This yields

$$\dot{V}_{m'}^{(3)} = \dot{V}_{m'}^{(2)}$$

$$V_{m'}^{(3)} = V_{m'}^{(2)} - i\frac{\omega \tau_w}{k_{m'}^2 b^2}\dot{V}_{m'}^{(2)} \tag{B20}$$

where $\tau_w = \mu_0 \sigma b d$ is the characteristic resistive wall diffusion time.

These two relations, combined with the three integral relations given by Eq. (B5) represent a closed set of five equations that ultimately enables us to express V'_1 in terms of $\partial V'_1 / \partial \rho'_1$. This is the next step in the procedure.

3. Solving the set of linear equations

The relation between V_1' and $\partial V_1'/\partial\rho_1'$ can be found by a relatively straightforward but somewhat lengthy linear algebra analysis. The first step is to substitute Eq. (B20) into the third integral relation given in Eq. (B5). This leads to an explicit relationship between the Fourier amplitudes $V_m^{(2)}$ and $\dot{V}_m^{(2)}$ which can be written as

$$\left[\mathsf{I}+\dot{\mathsf{H}}^{(22)}\right]\cdot\mathbf{V}^{(2)}=\left[\mathsf{H}^{(22)}+i\omega\tau_w\dot{\mathsf{M}}^{(22)}\right]\cdot\dot{\mathbf{V}}^{(2)} \qquad \text{(B21)}$$

where the matrix elements are given by

$$\begin{aligned}\dot{H}_{mm'}^{(22)}&=\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}e^{-im\chi+im'\chi'}\,\frac{R'^2}{R_0^2}\left(bR_0\,\frac{\partial H_{22}}{\partial\rho_2'}\right)d\chi\,d\chi'\\ H_{mm'}^{(22)}&=\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}e^{-im\chi+im'\chi'}\left(R_0H_{22}\right)d\chi\,d\chi'\\ \dot{M}_{mm'}^{(22)}&=\frac{1}{k_{m'}^2b^2}\left[\delta_{mm'}+\dot{H}_{mm'}^{(22)}\right]\end{aligned} \qquad \text{(B22)}$$

Note also that on the wall surface we no longer have to distinguish between ρ_3' and ρ_2' or H_{33} and H_{22} because of the thin wall approximation.

The second step focuses on the plasma surface. We choose the angle v on S_1 to coincide with the poloidal angle χ defined within the plasma; that is, on S_1 we set $v=\chi$. Next, the unknowns on S_1 are Fourier expanded as follows

$$\begin{aligned}V_1'&=\sum_{m'}V_{m'}^{(1)}e^{im'\chi'}\\ J'\frac{\partial V_1'}{\partial\rho'}&=R'\left(R_{\chi'}'^2+Z_{\chi'}'^2\right)^{1/2}\frac{\partial V_1'}{\partial\rho'}=R_0\sum_{m'}\dot{V}_{m'}^{(1)}e^{im'\chi'}\end{aligned} \qquad \text{(B23)}$$

and then substituted into the first two integral relations given in Eq. (B5). This leads to a set of two matrix equations coupling the unknown Fourier coefficients $V_{m'}^{(1)},\dot{V}_{m'}^{(1)},V_{m'}^{(2)},\dot{V}_{m'}^{(2)}$

$$
\begin{aligned}
\left[\mathsf{I}+\dot{\mathsf{H}}^{(11)}\right]\cdot \mathbf{V}^{(1)} - \mathsf{H}^{(11)}\cdot \dot{\mathbf{V}}^{(1)} &= \dot{\mathsf{H}}^{(12)}\cdot \mathbf{V}^{(2)} - \mathsf{H}^{(12)}\cdot \dot{\mathbf{V}}^{(2)} \\
\left[\mathsf{I}-\dot{\mathsf{H}}^{(22)}\right]\cdot \mathbf{V}^{(2)} + \mathsf{H}^{(22)}\cdot \dot{\mathbf{V}}^{(2)} &= -\dot{\mathsf{H}}^{(21)}\cdot \mathbf{V}^{(1)} + \mathsf{H}^{(21)}\cdot \dot{\mathbf{V}}^{(1)}
\end{aligned}
\tag{B24}
$$

Here, the undefined matrix elements are given by

$$
\begin{aligned}
H^{(11)}_{mm'} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-im\chi+im'\chi'}\left(R_0 H_{11}\right) d\chi\, d\chi' \\
\dot{H}^{(11)}_{mm'} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-im\chi+im'\chi'}\left(J'\,\frac{\partial H_{11}}{\partial \rho_1'}\right) d\chi\, d\chi' \\
H^{(21)}_{mm'} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-im\chi+im'\chi'}\left(R_0 H_{21}\right) d\chi\, d\chi' \\
\dot{H}^{(21)}_{mm'} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-im\chi+im'\chi'}\left(J'\,\frac{\partial H_{21}}{\partial \rho_1'}\right) d\chi\, d\chi' \\
H^{(12)}_{mm'} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-im\chi+im'\chi'}\left(R_0 H_{12}\right) d\chi\, d\chi' \\
\dot{H}^{(12)}_{mm'} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-im\chi+im'\chi'}\,\frac{R'^2}{R_0^2}\left(bR_0\,\frac{\partial H_{12}}{\partial \rho_2'}\right) d\chi\, d\chi'
\end{aligned}
\tag{B25}
$$

The last step in the procedure is to eliminate the matrix elements $V^{(2)}_{m'}$ and $\dot{V}^{(2)}_{m'}$ from Eq. (B24) by making use of Eq. (B21). The key point to keep in mind is that this elimination must be accomplished without ever having to calculate a matrix inverse that includes the eigenvalue ω. This is the crucial step in the analysis. Several sub-steps are required. We begin by eliminating $\mathbf{V}^{(2)}$ in terms of $\dot{\mathbf{V}}^{(2)}$ in Eq. (B24) by means of Eq. (B21).

$$
\begin{aligned}
\left[\mathsf{I}+\dot{\mathsf{H}}^{(11)}\right]\cdot \mathbf{V}^{(1)} - \mathsf{H}^{(11)}\cdot \dot{\mathbf{V}}^{(1)} &= \mathsf{S}^{(1)}\cdot \dot{\mathbf{V}}^{(2)} + i\omega\tau_w \mathsf{S}^{(2)}\cdot \dot{\mathbf{V}}^{(2)} \\
-\dot{\mathsf{H}}^{(21)}\cdot \mathbf{V}^{(1)} + \mathsf{H}^{(21)}\cdot \dot{\mathbf{V}}^{(1)} &= \dot{\mathsf{S}}^{(1)}\cdot \dot{\mathbf{V}}^{(2)} + i\omega\tau_w \dot{\mathsf{S}}^{(2)}\cdot \dot{\mathbf{V}}^{(2)}
\end{aligned}
\tag{B26}
$$

where

$$
\begin{aligned}
\mathsf{S}^{(1)} &= \dot{\mathsf{H}}^{(12)} \cdot \left[\mathsf{I} + \dot{\mathsf{H}}^{(22)}\right]^{-1} \cdot \mathsf{H}^{(22)} - \mathsf{H}^{(12)} \\
\mathsf{S}^{(2)} &= \dot{\mathsf{H}}^{(12)} \cdot \left[\mathsf{I} + \dot{\mathsf{H}}^{(22)}\right]^{-1} \cdot \dot{\mathsf{M}}^{(22)} \\
\dot{\mathsf{S}}^{(1)} &= \left[\mathsf{I} - \dot{\mathsf{H}}^{(22)}\right] \cdot \left[\mathsf{I} + \dot{\mathsf{H}}^{(22)}\right]^{-1} \cdot \mathsf{H}^{(22)} + \mathsf{H}^{(22)} \\
\dot{\mathsf{S}}^{(2)} &= \left[\mathsf{I} - \dot{\mathsf{H}}^{(22)}\right] \cdot \left[\mathsf{I} + \dot{\mathsf{H}}^{(22)}\right]^{-1} \cdot \dot{\mathsf{M}}^{(22)}
\end{aligned}
\tag{B27}
$$

The next sub-step is to eliminate the $i\omega\tau_w$ terms in Eq. (B26) by left multiplying the first equation by $\left[\mathsf{S}^{(2)}\right]^{-1}$, the second equation by $-\left[\dot{\mathsf{S}}^{(2)}\right]^{-1}$, and adding. The resulting equation can then be explicitly solved for $\dot{\mathbf{V}}^{(2)}$ by left multiplying by $\mathsf{U} = \left[\left(\mathsf{S}^{(2)}\right)^{-1} \cdot \left(\mathsf{S}^{(1)}\right) - \left(\dot{\mathsf{S}}^{(2)}\right)^{-1} \cdot \left(\dot{\mathsf{S}}^{(1)}\right)\right]^{-1}$. We obtain

$$
\begin{aligned}
\dot{\mathbf{V}}^{(2)} &= \dot{\mathsf{T}} \cdot \mathbf{V}^{(1)} - \mathsf{T} \cdot \dot{\mathbf{V}}^{(1)} \\
\dot{\mathsf{T}} &= \mathsf{U} \cdot \left[\left(\mathsf{S}^{(2)}\right)^{-1} \cdot \left(\mathsf{I} + \dot{\mathsf{H}}^{(11)}\right) + \left(\dot{\mathsf{S}}^{(2)}\right)^{-1} \cdot \left(\dot{\mathsf{H}}^{(21)}\right)\right] \\
\mathsf{T} &= \mathsf{U} \cdot \left[\left(\mathsf{S}^{(2)}\right)^{-1} \cdot \left(\mathsf{H}^{(11)}\right) + \left(\dot{\mathsf{S}}^{(2)}\right)^{-1} \cdot \left(\mathsf{H}^{(21)}\right)\right]
\end{aligned}
\tag{B28}
$$

In the final sub-step we left multiply the first equation in Eq. (B26) by $\left[\mathsf{S}^{(1)}\right]^{-1}$, the second equation by $-\left[\dot{\mathsf{S}}^{(1)}\right]^{-1}$, and add. Substituting for $\dot{\mathbf{V}}^{(2)}$ then yields the desired boundary relation for the vacuum fields on the plasma surface

$$
\left[\dot{\mathsf{C}}^{(1)} - i\omega\tau_w \dot{\mathsf{C}}^{(2)}\right] \cdot \mathbf{V}^{(1)} - \left[\mathsf{C}^{(1)} - i\omega\tau_w \mathsf{C}^{(2)}\right] \cdot \dot{\mathbf{V}}^{(1)} = 0
\tag{B29}
$$

Here,

$$
\begin{aligned}
\dot{\mathsf{C}}^{(1)} &= \left(\mathsf{S}^{(1)}\right)^{-1} \cdot \left(\mathsf{I} + \dot{\mathsf{H}}^{(11)}\right) + \left(\dot{\mathsf{S}}^{(1)}\right)^{-1} \cdot \left(\dot{\mathsf{H}}^{(21)}\right) \\
\mathsf{C}^{(1)} &= \left(\mathsf{S}^{(1)}\right)^{-1} \cdot \left(\mathsf{H}^{(11)}\right) + \left(\dot{\mathsf{S}}^{(1)}\right)^{-1} \cdot \left(\mathsf{H}^{(21)}\right) \\
\dot{\mathsf{C}}^{(2)} &= \left[\left(\mathsf{S}^{(1)}\right)^{-1} \cdot \left(\mathsf{S}^{(2)}\right) - \left(\dot{\mathsf{S}}^{(1)}\right)^{-1} \cdot \left(\dot{\mathsf{S}}^{(2)}\right)\right] \cdot \dot{\mathsf{T}} \\
\mathsf{C}^{(2)} &= \left[\left(\mathsf{S}^{(1)}\right)^{-1} \cdot \left(\mathsf{S}^{(2)}\right) - \left(\dot{\mathsf{S}}^{(1)}\right)^{-1} \cdot \left(\dot{\mathsf{S}}^{(2)}\right)\right] \cdot \mathsf{T}
\end{aligned}
\tag{B30}
$$

Admittedly a large amount of linear algebra has been required including the evaluation of a number of matrix inverses. However, the linear algebra is straightforward computationally and the matrices are not very large since we are only evaluating elements on a surface rather than in a volume. Furthermore, each matrix depends only on the geometry of the wall and the plasma surface. Thus, once the geometry is set there is no need to re-compute the matrices as the plasma parameters change.

The analysis requires one more step for completion, relating the vacuum fields on the plasma surface $\mathbf{V}^{(1)}$ and $\dot{\mathbf{V}}^{(1)}$ to the plasma displacement $\boldsymbol{\xi}$.

4. Relating the vacuum fields to the plasma displacement

Recall that our ultimate goal is to derive an expression for the resistive wall boundary condition in terms of $\boldsymbol{\xi}$ and its normal derivative. This final relation should have only constant and linear terms in ω.

The derivation of the desired relation begins by noting that the normal and tangential components of magnetic field on the vacuum side of the plasma-vacuum interface can be conveniently related to the following two plasma quantities:

$$\begin{aligned} &\mathbf{n}\cdot\boldsymbol{\xi} = \xi_\psi = X / RB_p \\ &\mathbf{B}\cdot\mathbf{Q} = -\nabla\cdot\left(B^2\boldsymbol{\xi}_\perp\right) = -\nabla\cdot\left[B^2\left(\xi_\psi\,\mathbf{n} + i\xi_\tau\boldsymbol{\tau}\right)\right] \end{aligned} \qquad \text{(B31)}$$

As in Appendix A the expression for the perturbed parallel magnetic field can be rewritten in terms of flux coordinates as follows

$$\mathbf{B}\cdot\mathbf{Q} = -\left(B_p^2\frac{\partial}{\partial\psi} + \frac{\nabla\psi\cdot\nabla\chi}{R^2}\frac{\partial}{\partial\chi}\right)\left(\frac{B^2}{B_p^2}X\right) + \left(i\frac{B_\phi^2}{q}\frac{\partial}{\partial\chi} + nB_p^2\right)(Y) \qquad \text{(B32)}$$

where, as before, we have assumed straight field line coordinates: $(\mathbf{B}\cdot\nabla\phi)/(\mathbf{B}\cdot\nabla\chi) = q(\psi)$.

To determine the desired relations we use the following jump conditions across the plasma-vacuum interface.

$$\begin{aligned} [\![\mathbf{n}\cdot\mathbf{B}_1]\!] = 0 \quad &\rightarrow \quad \nabla\psi\cdot\nabla V_1 = \mathbf{B}\cdot\nabla(RB_p\mathbf{n}\cdot\boldsymbol{\xi}) \\ [\![\mathbf{B}\cdot\mathbf{B}_1]\!] = 0 \quad &\rightarrow \quad \mathbf{B}\cdot\nabla V_1 = \mathbf{B}\cdot\mathbf{Q} \end{aligned} \tag{B33}$$

The next step is to carry out Fourier analysis. A straightforward calculation shows that the first jump condition can be written as

$$\begin{aligned} \dot{\mathbf{V}}^{(1)} &= i\,\mathsf{k}\cdot\mathbf{X} \\ k_{l'l} &= \frac{nq+m}{R_0}\,\delta_{m'-m} \end{aligned} \tag{B34}$$

Here, k is a diagonal matrix with dimensions $(2M+1)\times(2M+1)$ that operates only on the surface elements of $\mathbf{X}$ corresponding to $l=(2M+1)J-M+m$ with $-M\le m\le M$. The second jump condition is slightly more complicated and is given by

$$\mathbf{V}^{(1)} = i\,\mathsf{L}_X\cdot\mathbf{X} + i\,\mathsf{L}_B\cdot\mathbf{X}_B' + i\,\mathsf{L}_Y\cdot\mathbf{Y} \tag{B35}$$

with

$$\begin{aligned} L_{l'l}^{(X)} &= \frac{1}{2\pi}\frac{1}{m+nq}\int_0^{2\pi}\frac{qR}{B_\phi}\left[B_p^2\frac{\partial}{\partial\psi}\frac{B_\phi^2}{B_p^2} + \frac{\nabla\psi\cdot\nabla\chi}{R^2}\left(im\frac{B^2}{B_p^2} + \frac{\partial}{\partial\chi}\frac{B_\phi^2}{B_p^2}\right)\right]e^{i(m-m')\chi}d\chi \\ L_{l'l}^{(X_B')} &= \frac{1}{2\pi}\frac{1}{m+nq}\int_0^{2\pi}\frac{qRB^2}{B_\phi}e^{i(m-m')\chi}d\chi \\ L_{l'l}^{(Y)} &= \frac{1}{2\pi}\frac{1}{m+nq}\int_0^{2\pi}\frac{R}{B_\phi}\left(mB_\phi^2 - nqB_p^2\right)e^{i(m-m')\chi}d\chi \end{aligned} \tag{B36}$$

As above, each of these matrices has dimensions $(2M+1)\times(2M+1)$.

In the last step we use Eqs. (B34) and (B35) to form the combination of quantities given in Eq. (B29). This yields

$$\left[i\omega \mathsf{K}_1 - \mathsf{K}_2\right]\cdot \mathbf{X} + \left[i\omega \mathsf{K}_3 - \mathsf{K}_4\right]\cdot \mathbf{Y} + \left[i\omega \mathsf{K}_5 - \mathsf{K}_6\right]\cdot \mathbf{X}_B' = 0 \qquad \text{(B37)}$$

where

$$\begin{aligned}
\mathsf{K}_1 &= \tau_w[\dot{\mathsf{C}}^{(2)}\cdot \mathsf{L}_X - \mathsf{C}^{(2)}\cdot \mathsf{k}] \\
\mathsf{K}_2 &= \dot{\mathsf{C}}^{(1)}\cdot \mathsf{L}_X - \mathsf{C}^{(1)}\cdot \mathsf{k} \\
\mathsf{K}_3 &= \tau_w \dot{\mathsf{C}}^{(2)}\cdot \mathsf{L}_Y \\
\mathsf{K}_4 &= \dot{\mathsf{C}}^{(1)}\cdot \mathsf{L}_Y \\
\mathsf{K}_5 &= \tau_w \dot{\mathsf{C}}^{(2)}\cdot \mathsf{L}_B \\
\mathsf{K}_6 &= \dot{\mathsf{C}}^{(1)}\cdot \mathsf{L}_B
\end{aligned} \qquad \text{(B38)}$$

Equation (B37) is the desired expression for the resistive wall boundary condition. It consists of the $2M+1$ coupled linear algebraic equations required for closure of the overall MHD stability analysis and is the analog of Eq. (3) for the model problem. Note the linear dependence on ω.

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Figure Captions

Fig. 1 Resistive wall geometry