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Transport of momentum in full f gyrokinetics

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Full f gyrokinetic formulations employ two gyrokinetic equations, one for ions and the other for electrons, and quasineutrality to obtain the ion and electron distribution functions and the electrostatic potential. This article shows with several examples that the long wavelength radial electric field obtained with full f approaches is extremely sensitive to errors in the ion and electron density since small deviations in density give rise to large, non-physical deviation in the conservation of toroidal angular momentum. For typical tokamak values, a relative error of 10^{-7} in the ion or electron densities is enough to obtain the incorrect toroidal rotation. Based on the insights gained with the examples in this article, three simple tests to check transport of toroidal angular momentum in full f simulations are proposed.

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I. INTRODUCTION

Gyrokinetic δf simulations [1–4] have proven very useful in the study of micro-turbulence. In these simulations, the lowest order distribution function is a slowly evolving Maxwellian, and only the fluctuating piece, small in a gyroradius over scale length expansion, is solved for. This separation is advantageous because the time derivatives of the two pieces of the distribution function are of very different order. The fluctuating piece, f_i^{tb}, is of order $\delta_i f_{Mi}$, where f_{Mi} is the zeroth order Maxwellian, and $\delta_i = \rho_i/a \ll 1$ is the ratio of the ion gyroradius ρ_i over the tokamak minor radius a. Its time derivative is of the order of the drift wave frequency that for wavelengths comparable to the ion gyroradius is $\omega^* \sim v_i/a$, with $v_i = \sqrt{2T_i/M}$ the ion thermal speed, giving

$$\frac{\partial f_i^{\mathrm{tb}}}{\partial t} \sim \omega^* f_i^{\mathrm{tb}} \sim \delta_i f_{Mi} \frac{v_i}{a}. \qquad (1)$$

On the other hand, the time derivative of the slowly evolving Maxwellian is related to the transport time scale. Assuming that the transport is of gyroBohm order, $\partial/\partial t \sim D_{gB}/a^2 \sim \delta_i^2 v_i/a$, where $D_{gB} \sim \delta_i \rho_i v_i$ is the gyroBohm diffusion coefficient. Then,

$$\frac{\partial f_{Mi}}{\partial t} \sim \delta_i^2 f_{Mi} \frac{v_i}{a} \ll \frac{\partial f_i^{\mathrm{tb}}}{\partial t} \sim \delta_i f_{Mi} \frac{v_i}{a}, \qquad (2)$$

where we have used the estimate in Eq. (1). Any equation that includes both the slow Maxwellian f_{Mi} and the fast fluctuating piece f_i^{tb} at the same time must treat terms that differ by $\delta_i = 0.5 \times 10^{-2}$ in most relevant tokamaks. Then, we need at least 3 digits of accuracy to solve a full f equation that contains both pieces of the distribution function. This problem is avoided in δf simulations by not evolving f_{Mi}. The density and temperature of f_{Mi} can be calculated using conservation equations [5].

The problem of full f simulations becomes even more acute with the transport of toroidal angular momentum.

It was realized 40 years ago [6, 7] that neoclassical transport in axisymmetric devices is intrinsically ambipolar, i.e., quasineutrality is independent of the long wavelength radial electric field to a very high order. We recently proved that intrinsic ambipolarity also applies to turbulent tokamaks [8]. Due to the weak dependence on the radial electric field, any small error in the calculation of the charge density leads to large, non-physical deviations in the radial electric field. These deviations lead to a toroidal rotation profile that does not satisfy the correct conservation equation for the toroidal angular momentum. In Ref. 8 we showed that the radial current density must be good to order $\delta_i^3(V_i/v_i)en_e v_i$ to obtain the correct transport of momentum by employing the quasineutrality equation. The required accuracy depends on the size of the ion mean velocity V_i that in tokamaks ranges from being of the order of the ion thermal speed v_i with neutral beam injection, to the low flow ordering $V_i \sim \delta_i v_i$ without external momentum input. In terms of accuracy in the gyrokinetic equation, we need to keep ion flows $n_i \mathbf{V}_i$ to order $\delta_i^3(V_i/v_i)n_i v_i$ that correspond to ion density time derivatives of order $\partial n_i/\partial t = -\nabla \cdot (n_i \mathbf{V}_i) \sim \delta_i^3(V_i/v_i)n_i(v_i/a)$. Thus, in the gyrokinetic Fokker-Planck equation it is necessary to keep terms of order

$$\frac{f_{Mi}}{n_i}\frac{\partial n_i}{\partial t} \sim \delta_i^3 \frac{V_i}{v_i} f_{Mi} \frac{v_i}{a} \ll \frac{\partial f_i^{\mathrm{tb}}}{\partial t} \sim \delta_i f_{Mi} \frac{v_i}{a}, \qquad (3)$$

where we have employed the estimate in Eq. (1). Notice that in a full f simulation with $\delta_i = 0.5 \times 10^{-2}$, we need 5 digits of accuracy to obtain the correct transport of toroidal angular momentum if $V_i \sim v_i$, and 7 digits if $V_i \sim \delta_i v_i$.

The requirements on the gyrokinetic Fokker-Planck equation are much reduced if instead of using gyrokinetic quasineutrality to obtain the long wavelength radial electric field, we employ the conservation equation for toroidal angular momentum. Under these circumstances,

the gyrokinetic Fokker-Planck equation need only be correct to order $\delta_i(V_i/v_i)f_{Mi}v_i/a$. This result is well known in drift kinetic theory [9], where it is used for both high flow, $V_i \sim v_i$ [10–13], and low flow, $V_i \sim \delta_i v_i$ [14–20].

In this article, we propose a series of tests to validate transport of momentum in full f simulations. Unless these tests are passed satisfactorily, the rotation profiles obtained in full f simulations will remain unreliable.

The remainder of this article is organized as follows. First, in section II we revisit the derivation in Ref. 8 where we showed that, in order to recover the transport of toroidal angular momentum from quasineutrality, it is necessary to obtain the gyrokinetic equation to order $\delta_i^3(V_i/v_i)f_{Mi}v_i/a$. We rederive the same result using a more general procedure to help the reader understand the next sections. Then, in sections III and IV we work out several examples that we consider relevant cases in simulations. In section III we show that even in the simple drift kinetic ordering, problems arise with momentum transport. It is possible to prove that employing the most common drift kinetic equation for ions and electrons in combination with quasineutrality is equivalent to including a non-physical momentum source in the plasma. In section IV we study the gyrokinetic Fokker-Planck and quasineutrality equations in a slab. We have already shown in Ref. 21 that the classic formulation by Dubin *et al* [22] is equivalent to introducing a non-physical source of momentum. We now explore a different case where the non-physical momentum source is eliminated, but the problem is now that the momentum transport is incorrect. Finally, based on these examples, we propose three checks for full f simulations in section V.

II. TRANSPORT OF TOROIDAL ANGULAR MOMENTUM

In this section, we derive an equation for the flux surface averaged charge density employing the full Fokker-Planck equation. This equation will explicitly show that imposing quasineutrality leads to a momentum conservation equation. It will also give the order to which the time derivative of the charge density must be obtained so that

transport of toroidal angular momentum is recovered.

We assume that the magnetic field is axisymmetric, i.e.,

$$\mathbf{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi. \tag{4}$$

We use magnetic coordinates, with ψ the poloidal flux function, ζ the toroidal angle and θ the poloidal angle. The gradient $\nabla\zeta = \hat{\zeta}/R$, where $\hat{\zeta}$ is the unit vector in the toroidal direction and R is the distance to the axis of symmetry of the tokamak.

We prove now that there is a relation between the quasineutrality equation and the transport of toroidal angular momentum. First, we write the relation of the flux surface averaged charge density with the radial current. Then, we use the total momentum conservation to obtain the radial current as a function of the transport of toroidal angular momentum. Importantly, we will not use the quasineutrality equation in the derivation because we want to show the relation between charge density evolution and toroidal angular momentum conservation. This relation, crucial to obtain the correct radial electric field, will only be satisfied if the Fokker-Planck equation is solved to high enough order.

Integrating the ion and electron full Fokker-Planck equations over velocity space, subtracting, and flux surface averaging gives

$$\frac{\partial}{\partial t}\langle Zen_i - en_e\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial\psi}V'\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi, \tag{5}$$

where $\mathbf{J}$ is the current density, $\langle\ldots\rangle_\psi = (V')^{-1}\int d\theta d\zeta(\ldots)/(\mathbf{B}\cdot\nabla\theta)$ is the flux surface average, and $V' \equiv dV/d\psi = \int d\theta d\zeta(\mathbf{B}\cdot\nabla\theta)^{-1}$ is the volume of the flux surface. Here Ze and e are the ion and electron charge magnitude.

To obtain the radial current $\langle\mathbf{J}\cdot\nabla\psi\rangle_\psi$, we employ the total momentum conservation equation. Taking the $M\mathbf{v}$ and $m\mathbf{v}$ moments of the ion and electron Fokker-Planck equations, with M and m the ion and electron masses, adding, dotting by $R\hat{\zeta}$ and flux surface averaging gives total conservation of toroidal angular momentum to be

$$\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi = c\left\langle (Zen_i - en_e)\frac{\partial\phi}{\partial\zeta}\right\rangle_\psi + c\left(\frac{\partial}{\partial t}\langle Rn_i M\mathbf{V}_i\cdot\hat{\zeta}\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle R\hat{\zeta}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\right\rangle_\psi\right), \tag{6}$$

where $n_i\mathbf{V}_i = \int d^3v\, \mathbf{v}f_i$ is the ion flow, and $\overleftrightarrow{\boldsymbol{\pi}}_i = M\int d^3v\, f_i[\mathbf{v}\mathbf{v} - (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - v_\parallel^2\hat{\mathbf{b}}\hat{\mathbf{b}}]$ is the ion viscosity that includes Reynolds stresses, gyroviscosity and perpendicular viscosity. Notice that we have neglected the electron viscosity and the time derivative of the elec-

tron momentum because they are smaller than the other terms by a mass ratio m/M. We also employ that $\nabla(R\hat{\zeta}) = \nabla R\hat{\zeta} - \hat{\zeta}\nabla R$ is antisymmetric and we use Eq. (4) to write $R(\mathbf{J}\times\mathbf{B})\cdot\hat{\zeta} = \mathbf{J}\cdot\nabla\psi$. Substituting Eq. (6) into Eq. (5) finally gives

$$\frac{\partial}{\partial t}\langle Zen_i - en_e\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial \psi}V'\left\langle c\frac{\partial \phi}{\partial \zeta}(Zen_i - en_e)\right\rangle_\psi = -\frac{c}{V'}\frac{\partial}{\partial \psi}V'\left(\frac{\partial}{\partial t}\langle Rn_iM\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial \psi}V'\left\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\right\rangle_\psi\right).$$

$$(7)$$

In this equation, the second term in the left side corresponds to the $\mathbf{E}\times\mathbf{B}$ transport of charge.

Equation (7) shows that imposing quasineutrality $Zn_i = n_e$ in the ion and electron full Fokker-Planck equations leads to conservation of toroidal angular momentum

$$\frac{\partial}{\partial t}\langle Rn_iM\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi}V'\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\rangle_\psi. \quad (8)$$

Also, from Eq. (7) we can estimate the accuracy needed to recover the correct momentum transport (8). Assuming that the transport of toroidal angular momentum is gyroBohm, $\partial\langle Rn_iM\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi/\partial t \sim (D_{gB}/a^2)Rn_iMV_i$, giving

$$\frac{\partial}{\partial t}\langle Rn_iM\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi \sim \delta_i^2 Rn_iMV_i\frac{v_i}{a}, \quad (9)$$

where we have used the gyroBohm diffusion coefficient $D_{gB} = \delta_i\rho_iv_i$. Using this estimate in Eq. (7), we find that $\partial\langle Zen_i - en_e\rangle_\psi/\partial t$ must be obtained to order $\delta_i^3(V_i/v_i)en_ev_i/a$. This justifies the estimate in Eq. (3) of section I.

In general, the ion and electron full Fokker-Planck equations are asymptotically expanded in the small parameter $\delta_i = \rho_i/a \ll 1$ to obtain the gyrokinetic equations for ions and electrons. This expansion usually neglects terms of order $\delta_i^2 f_{Mi}v_i/a$, although terms of this order can be retained in simplified geometries. Thus, most gyrokinetic simulations will not be high enough order to recover the correct transport of toroidal angular momentum.

The failure to reproduce the correct toroidal rotation manifests itself in the long wavelength radial electric field. There is a strong relation between the toroidal rotation and the long wavelength radial electric field found by using the radial and parallel momentum balance [8, 23] to write

$$n_i\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}} = -cn_iR\left(\frac{\partial \phi}{\partial \psi} + \frac{1}{Zen_i}\frac{\partial p_i}{\partial \psi}\right) + IU(\psi). \quad (10)$$

In up-down symmetric tokamaks, the flux function $U(\psi)$ is given by neoclassical theory [24, 25] if the ion-ion collision frequency ν_{ii} satisfies $qR\nu_{ii}/v_i \gg \delta_i^2$ [8, 23], valid in most tokamaks. Relation (10) will be satisfied in gyrokinetic simulations at long wavelengths if the simulations are run for times longer than the collisional time ν_{ii}^{-1}. However, the long wavelength radial electric field, calculated using the quasineutrality equation, will be incorrect, resulting in a toroidal velocity that does not satisfy the momentum conservation equation.

In the next two sections, we work out several examples that are illustrative of the problems of quasineutrality.

In Ref. 21, we employed the classic gyrokinetic model for a slab by Dubin *et al* [22] to show that gyrokinetic equations may introduce non-physical sources of momentum. In section III, we use the most common drift kinetic equations to show that a similar problem arises in tokamaks due to the magnetic geometry. After that, in section IV, we analyze again gyrokinetics in a slab with a different model derived from a variational method [26, 27]. We will show that this approach has its own problems.

III. DRIFT KINETIC EXAMPLE

In this section, we examine a system composed of two drift kinetic equations, one for ions and the other for electrons, and quasineutrality. This is not a complete model of the drift wave turbulence in tokamaks, but it is a good example in which we can demonstrate the shortcomings of quasineutrality. First, in subsection III A, we obtain the correct transport of toroidal angular momentum in the low flow ordering, $V_i \sim \delta_iv_i$, in the limit $B_p/B \ll 1$, where $B_p = |\nabla\psi|/R$ is the poloidal component of the magnetic field [23]. Then, in subsection III B, we show that using quasineutrality along with the most common drift kinetic equation is equivalent to introducing a non-physical momentum source. At the end of subsection III B we will discuss a drift kinetic equation that does not have this problem.

A. Toroidal angular momentum transport in drift kinetics

In this subsection, we calculate the off-diagonal component $\langle R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\boldsymbol{\pi}}_i\cdot\nabla\psi\rangle_\psi$ of the stress tensor to the necessary order. We make two crucial assumptions so that the drift kinetic model remains valid. We obviously need to assume that the turbulence has characteristic wavelengths much longer than the ion gyroradius. Additionally, we work in a magnetic field with $B_p/B \ll 1$. This assumption is needed for several reasons, but the main one is that in this limit the toroidal rotation is determined exclusively by the parallel mean velocity. It is then possible to neglect the gyrophase dependent piece of the distribution functions. Requiring wavelengths to be much longer than the ion gyroradius is not physically relevant, but this example is only meant to illustrate the problems with quasineutrality.

We follow the orderings in Ref. 23, where the low flow, $B_p/B \ll 1$ limit was carefully studied. We assume that the ion and electron lowest order distribution functions,

f_{Mi} and f_{Me}, are Maxwellians that spatially depend only on ψ. We allow turbulent fluctuations in the potential such that the perpendicular gradient is $\nabla_\perp \phi \sim T_e/ea$, whereas the parallel gradient is smaller, $\hat{\mathbf{b}} \cdot \nabla \phi \sim \delta_i T_e/ea$. Then, $\partial\phi/\partial\zeta = R\hat{\boldsymbol{\zeta}} \cdot \nabla\phi \sim (R/a)(B_p/B)T_e/e$.

In Ref. 23, we argued that to first order in δ_i there are two pieces of the ion distribution function that scale differently with B_p/B, namely the neoclassical long wavelength piece, $f_{i1}^{\mathrm{nc}} \sim (B/B_p)\delta_i f_{Mi}$, and the turbulent short wavelength components, $f_{i1}^{\mathrm{tb}} \sim \delta_i f_{Mi}$. Based on this distinction, it is possible to show that the lowest order gyrokinetic equation (and hence the lowest order drift kinetic equation) is accurate enough to find the next order corrections in δ_i, including the neoclassical piece $f_{i2}^{\mathrm{nc}} \sim (B/B_p)^2\delta_i^2 f_{Mi}$, and the turbulent contribu-

tion $f_{i2}^{\mathrm{tb}} \sim (B/B_p)\delta_i^2 f_{Mi}$. For $B_p/B \sim 1$, a higher order gyrokinetic or drift kinetic equation would be needed to obtain the higher order contributions $f_{i2} \sim \delta_i^2 f_{Mi}$.

The higher order contributions f_{i2}^{nc} and f_{i2}^{tb} are needed to obtain the correct momentum transport in the low flow ordering. In Refs. 8 and 23, we argued that the fast time averaged, long wavelength piece of $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi\rangle_\psi$ must be of order $R|\nabla\psi|\delta_i^3 p_i$ in the low flow ordering [this result can also be obtained from Eq. (9)]. To obtain $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi\rangle_\psi$, we use a moment approach developed in drift kinetics [28]. The final result is presented in Eq. (37) of Ref. 23. Using $B_p/B \ll 1$ to write $R\mathbf{v} \cdot \hat{\boldsymbol{\zeta}} \simeq (I/B)v_{||}$, it simplifies to

$$\left\langle \left\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla\psi\right\rangle_\psi\right\rangle_{\mathrm{T}} = Mc\left\langle\left\langle\frac{I}{B}\frac{\partial\phi}{\partial\zeta}\int d^3v\,\overline{f}_i v_{||}\right\rangle_\psi\right\rangle_{\mathrm{T}} + \frac{M^2c^2}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\left\langle\frac{I^2}{B^2}\frac{\partial\phi}{\partial\zeta}\int d^3v\,\overline{f}_i v_{||}^2\right\rangle_\psi\right\rangle_{\mathrm{T}}$$
$$+\frac{Mc}{2Ze}\left\langle\frac{I^2}{B^2}\right\rangle_\psi\frac{\partial p_i}{\partial t} - \frac{M^2c}{2Ze}\left\langle\frac{I^2}{B^2}\int d^3v\,\langle C_{ii}\{\overline{f}_i\}\rangle_{\mathrm{T}} v_{||}^2\right\rangle_\psi - \frac{M^3c^2}{6Z^2e^2}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\frac{I^3}{B^3}\int d^3v\,\langle C_{ii}\{\overline{f}_i\}\rangle_{\mathrm{T}} v_{||}^3\right\rangle_\psi. \quad (11)$$

Here $\overline{f}_i$ is the gyrophase independent piece of the distribution function, and $\langle\ldots\rangle_{\mathrm{T}} = (\Delta t\,\Delta\psi)^{-1}\int_{\Delta t} dt\int_{\Delta\psi} d\psi(\ldots)$ is the "transport" or coarse grain average that averages over a radial distance longer than the radial correlation length of the turbulence and over times longer than the turbulence correlation time. Since we are using the drift kinetic equation, we have assumed that $k_\perp\rho_i$ is small enough that the contribution of the gyrophase dependent piece of the ion distribution function to Eq. (11) can be neglected. Notice that in Eq. (11), the first and fourth term are formally of lower order in δ_i [23], thus requiring the second order pieces f_{i2}^{tb} and f_{i2}^{nc}.

Equation (11) is the correct expression for the transport of toroidal angular momentum in the drift kinetic limit. Any result obtained from the quasineutrality equation must be compared to this result.

B. Toroidal angular momentum transport from drift kinetic quasineutrality

In this subsection, we calculate the ion and electron gyrophase independent pieces of the distribution function, $\overline{f}_i(\mathbf{r},\varepsilon_i,\mu_0,t)$ and $\overline{f}_e(\mathbf{r},\varepsilon_e,\mu_0,t)$, using two drift kinetic equations. Here $\varepsilon_i = v^2/2 + Ze\phi/M$ and $\varepsilon_e = v^2/2 - e\phi/m$ are the ion and electron total energies, and $\mu_0 = v_\perp^2/2B$ is the lowest order magnetic moment. The drift kinetic equations that we will use are accurate enough to obtain the second order pieces of the ion distribution function in the limit $B_p/B \ll 1$ and consequently

they are good enough to evaluate Eq. (11). Nevertheless, we will show that for the most common version of the drift kinetic equation the long wavelength electric field obtained by imposing $Z\int d^3v\,\overline{f}_i = e\int d^3v\,\overline{f}_e$ is incorrect. In this case, employing quasineutrality is equivalent to introducing a non-physical source of momentum and a modification to the transport of toroidal angular momentum. At the end of this section, we will briefly discuss a slightly modified version of the drift kinetic equation that gives the correct radial electric field in the limit $B_p/B \ll 1$ by employing the quasineutrality equation.

The most common ion and electron drift kinetic equations are

$$\frac{\partial}{\partial t}(J_j\overline{f}_j) + \overline{\nabla}_j \cdot \left(J_j\overline{f}_j\dot{\mathbf{r}}_j\right) + \frac{\partial}{\partial\varepsilon_j}\left(J_j\overline{f}_j\dot{\varepsilon}_j\right) = J_j C_j\{\overline{f}_j\},$$
(12)

with $j = i,e$. Here $\overline{\nabla}_j$ is the gradient holding ε_j, μ_0 and t fixed, and $J_j = B/|v_{||}|$ is the lowest order velocity space Jacobian, giving $d^3v = 2\pi J_j d\varepsilon_j d\mu_0$. The lowest order drifts are

$$\dot{\mathbf{r}}_j = v_{||}\hat{\mathbf{b}} + \mathbf{v}_{dj},$$
(13)

where the parallel velocity is obtained by using $v_{||} = \sqrt{2(\epsilon_i - \mu_0 B - Ze\phi/M)} = \sqrt{2(\epsilon_e - \mu_0 B + e\phi/m)}$, and the perpendicular drifts are usually written as

$$\mathbf{v}_{di} = \frac{\mu_0}{\Omega_i}\hat{\mathbf{b}}\times\nabla B + \frac{v_{||}^2}{\Omega_i}\hat{\mathbf{b}}\times\boldsymbol{\kappa} - \frac{c}{B}\nabla\phi\times\hat{\mathbf{b}}$$
$$= \frac{v_{||}}{\Omega_i}\overline{\nabla}_i\times(v_{||}\hat{\mathbf{b}}) - \frac{v_{||}^2}{\Omega_i}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} \quad (14)$$

and

$$\mathbf{v}_{de} = -\frac{\mu_0}{\Omega_e}\hat{\mathbf{b}} \times \nabla B - \frac{v_{||}^2}{\Omega_e}\hat{\mathbf{b}} \times \boldsymbol{\kappa} - \frac{c}{B}\nabla\phi \times \hat{\mathbf{b}}$$

$$= -\frac{v_{||}}{\Omega_e}\overline{\nabla}_e \times (v_{||}\hat{\mathbf{b}}) + \frac{v_{||}^2}{\Omega_e}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}. \quad (15)$$

Here, $\Omega_i = ZeB/Mc$ and $\Omega_e = eB/mc$ are the ion and electron gyrofrequencies, $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}$ is the curvature of the magnetic field line, and in the second equalities, the curl $\overline{\nabla}_j \times (\dots)$ is taken holding ε_j, μ_0 and t fixed. Finally, the time derivatives of the total energies are

$$\dot{\varepsilon}_i = \frac{Ze}{M}\frac{\partial\phi}{\partial t} \quad (16)$$

and

$$\dot{\varepsilon}_e = -\frac{e}{m}\frac{\partial\phi}{\partial t}. \quad (17)$$

The quasineutrality condition in drift kinetics is

$$Zn_i \equiv Z\int d^3v\,\overline{f}_i = \int d^3v\,\overline{f}_e \equiv n_e. \quad (18)$$

Notice that the polarization density is not included because this is a drift kinetic limit. It can be neglected because we are assuming that $B_p/B \ll 1$ and the effect of the polarization density on the transport of toroidal angular momentum will be small by that factor.

We will study the time derivative of Eq. (18) up to order $\delta_i^4 en_e v_i/a$, as required by a low flow ordering. The time evolution of the flux surface average ion density is obtained from Eq. (12), giving

$$\frac{\partial}{\partial t}\langle n_i\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\,\overline{f}_i\mathbf{v}_{di} \cdot \nabla\psi\right\rangle_\psi = 0. \quad (19)$$

The equation for n_e is similar. Subtracting one from the other, imposing quasineutrality $Zn_i = n_e$ and integrating once in ψ, we find

$$0 = M\left\langle\int d^3v\,\overline{f}_i v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}_i\left(\frac{Iv_{||}}{B}\right)\right\rangle_\psi$$

$$+m\left\langle\int d^3v\,\overline{f}_e v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}_e\left(\frac{Iv_{||}}{B}\right)\right\rangle_\psi. \quad (20)$$

where we have used Eqs. (14) and (15) to write $\mathbf{v}_{di} \cdot \nabla\psi = v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}_i(Iv_{||}/\Omega_i) + c(\partial\phi/\partial\zeta)$ and $\mathbf{v}_{de} \cdot \nabla\psi = -v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla}_e(Iv_{||}/\Omega_e) + c(\partial\phi/\partial\zeta)$. Equation (20) must be found to order $\delta_i^3 Rp_i/a$ to obtain the relevant physics, i.e., the gyroBohm transport of momentum. The details of the derivation are in Appendix A. The final result is

$$\frac{\partial}{\partial t}\left\langle Rn_i M\mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}}\right\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial\psi}(V'\Pi) = F_\zeta, \quad (21)$$

where we have "transport" or coarse-grain averaged, $\langle\dots\rangle_T = (\Delta t\,\Delta\psi)^{-1}\int_{\Delta t}dt\int_{\Delta\psi}d\psi(\dots)$. The transport of momentum is given by

$$\Pi = \left\langle\left\langle\frac{MI}{B}\int d^3v\,\overline{f}_i v_{||}\mathbf{v}_{di} \cdot \nabla\psi\right\rangle_\psi\right\rangle_T$$

$$= \left\langle\langle R\hat{\boldsymbol{\zeta}}\cdot\overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi\right\rangle_T + \tilde{\Pi}, \quad (22)$$

and there is a non-physical torque

$$F_\zeta = M\left\langle\left\langle\int d^3v\,\overline{f}_i\mathbf{v}_{di} \cdot \overline{\nabla}_i\left(\frac{Iv_{||}}{B}\right)\right\rangle_\psi\right\rangle_T. \quad (23)$$

Notice that the transport of momentum Π differs from the real transport of momentum $\langle\langle R\hat{\boldsymbol{\zeta}}\cdot\overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi\rangle_T$ given in Eq. (11) by the quantity

$$\tilde{\Pi} = -\frac{M^2c}{2Ze}\left\langle\left\langle\int d^3v\,\overline{f}_i\mathbf{v}_{di} \cdot \overline{\nabla}_i\left(\frac{I^2v_{||}^2}{B^2}\right)\right\rangle_\psi\right\rangle_T. \quad (24)$$

The size of both the non-physical torque F_ζ and the non-physical transport of momentum $\tilde{\Pi}$ can be estimated to be $F_\zeta \sim \delta_i^2 Rp_i/a$ and $\tilde{\Pi} \sim \delta_i^2 p_i R|\nabla\psi|$. To obtain these estimates it is enough to realize that $\mathbf{v}_{di} \cdot \overline{\nabla}_i(Iv_{||}/B) \sim \delta_i v_i^2(R/a)$. Then, $F_\zeta \sim \delta_i^2 Rp_i/a$ because the lowest order Maxwellian contribution vanishes (the integrand is odd in $v_{||}$). In $\tilde{\Pi}$, the lowest order Maxwellian piece contributes giving $\tilde{\Pi} \sim \delta_i^2 p_i R|\nabla\psi|$. According to their order of magnitude estimate, F_ζ and $\tilde{\Pi}$ will not only contribute to the transport of momentum equation (21) but they will be the dominant terms, larger by a factor δ_i^{-1}, determining an incorrect radial electric field and hence the incorrect toroidal rotation. We must point out that these terms will tend to be smaller than their formal order of magnitude because they are small in β, the ratio between the plasma pressure and the magnetic field pressure. Indeed, employing Eq. (14) and using the expression (see Appendix B)

$$\frac{v_{||}}{\Omega_i}\overline{\nabla}_i \times (v_{||}\hat{\mathbf{b}}) \cdot \overline{\nabla}_i\left(\frac{Iv_{||}}{B}\right) = -\frac{cv_{||}}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}\frac{\partial\phi}{\partial\zeta}, \quad (25)$$

we find that $\mathbf{v}_{di} \propto \hat{\mathbf{b}}\cdot\nabla \times \hat{\mathbf{b}} \propto \beta$. However, even with the extra factor of β these non-physical contributions will be at least comparable to the rest of the terms in Eq. (21).

Interestingly, it is possible to rewrite the drift kinetic equation in a form that prevents this problem. Instead of using the Jacobian J_i and the drift motion $\dot{\mathbf{r}}_i$, we can employ $J_i^* = B_i^*/|v_{||}|$ and

$$\dot{\mathbf{r}}_i = \frac{v_{||}}{B_i^*}\mathbf{B} + \mathbf{v}_{di}^*, \quad (26)$$

with

$$\mathbf{v}_{di}^* = \frac{Mcv_{||}}{ZeB_i^*}\overline{\nabla}_i \times (v_{||}\hat{\mathbf{b}}) \quad (27)$$

and

$$B_i^* = B + \frac{Mcv_{||}}{Ze}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}. \qquad (28)$$

The modified expressions for the electrons are obvious. Using these new drift kinetic equations, and following the same procedure as in Appendix A, we find that the transport of momentum derived from quasineutrality is now given by the correct equation (8), with $\langle\langle R\hat{\boldsymbol{\zeta}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\nabla\psi\rangle_\psi\rangle_\mathrm{T}$ as in Eq. (11). To obtain this result, use $\mathbf{v}_{di}^* \cdot \nabla\psi = (v_{||}/B_{||}^*)\mathbf{B} \cdot \overline{\nabla}_i(Iv_{||}/\Omega_i) + (cB/B_{||}^*)(\partial\phi/\partial\zeta)$, and $\mathbf{v}_{di}^* \cdot \overline{\nabla}_i(Iv_{||}/B) = (Ze/MB_{||}^*)(B - B_{||}^*)(\partial\phi/\partial\zeta)$ [this last expression is easily derived from Eq. (25)].

Finally, even though we were able to find a drift kinetic formulation where quasineutrality would reproduce the correct behavior of the radial electric field at long wavelengths when $B_p/B \ll 1$, we still see two problems with this procedure. First, we are not aware of any gyrokinetic formulation that has been proven satisfactory even in the simpler $B_p/B \ll 1$ limit. Including finite gyroradius effects complicates the calculations enormously. Second, even if such a formulation is found, there is still the numerical problem. For such a formulation to work, it would be necessary to achieve 5 digit accuracy for the high flow regime, and 7 digit accuracy for the low flow ordering with $\delta_i = 0.5 \times 10^{-2}$. ITER would be even more challenging. On the other hand, the evaluation of transport of toroidal angular momentum via Eq. (11) is much less sensitive to numerical noise and resolution.

IV. GYROKINETIC EXAMPLE

In slab gyrokinetics, the magnetic field $\mathbf{B}$ is constant. We assume that the lowest order ion and electron distribution functions, f_{is} and f_{es}, and the lowest order potential, ϕ_s, only vary spatially in x (one of the directions perpendicular to $\mathbf{B}$) with a characteristic length of variation L much larger than the ion gyroradius. The lowest order distribution functions f_{is} and f_{es} need not be Maxwellians for very low collisionality. The turbulent fluctuations that can vary in the direction y, perpendicular to both x and $\mathbf{B}$, are ordered as small in $\delta_i = \rho_i/L$.

The interesting feature of the slab is that it is symmetric along y in an average sense (the turbulence obviously breaks the strict symmetry). Then, assuming that there is no large scale current $\langle\mathbf{J} \cdot \hat{\mathbf{x}}\rangle_x$ across the slab, y-momentum is conserved for reasons similar to those employed to prove conservation of toroidal angular momentum in a tokamak. Here $\langle\ldots\rangle_x = (\Delta y\,\Delta z)^{-1} \int_{\Delta y} dy \int_{\Delta z} dz(\ldots)$ is the flux surface average in a slab, $\hat{\mathbf{x}}$ is the unit vector along x, and z is the coordinate along the magnetic field. The conservation of y-momentum is

$$\frac{\partial}{\partial t}\langle n_i M\mathbf{V}_i \cdot \hat{\mathbf{y}}\rangle_x = -\frac{\partial}{\partial x}\langle\hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x, \qquad (29)$$

with $\hat{\mathbf{y}} = \hat{\mathbf{b}} \times \hat{\mathbf{x}}$. As in tokamaks, it is possible to obtain the long wavelength contribution to $\langle\hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x$ to very high order without requiring a higher order ion distribution function via moments of the full Fokker-Planck equation [28]. This was done for the low flow ordering without collisions in Ref. 21. The transport of momentum $\langle\hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x$ must be found to order $\delta_i^3 p_i$ in the low flow ordering, and it is given by

$$\langle\hat{\mathbf{x}}\cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot\hat{\mathbf{y}}\rangle_x = -\left\langle \frac{c}{B}\frac{\partial\phi}{\partial y}\int d^3v\, f_i M\mathbf{v} \cdot \hat{\mathbf{y}}\right\rangle_x - \frac{1}{2\Omega_i}\frac{\partial p_{i\perp}}{\partial t}$$
$$+ \frac{1}{2\Omega_i}\frac{\partial}{\partial x}\left\langle\frac{c}{B}\frac{\partial\phi}{\partial y}\int d^3v\, f_i M(\mathbf{v}\cdot\hat{\mathbf{y}})^2\right\rangle_x. \qquad (30)$$

The first term in this expression is formally of lower order than the rest. Consequently the distribution function and the potential are needed to order $\delta_i^2 f_{is}$ and $\delta_i^2 T_e/e$.

Since the long wavelength electric field is related to the y velocity through the $\mathbf{E} \times \mathbf{B}$ flow, it is necessary to correctly solve for the transport of y-momentum to self-consistently calculate the long wavelength electric field. As in tokamaks, the relation between quasineutrality and conservation of momentum is automatic if the exact Fokker-Planck equation is used, but it will not be satisfied if the Fokker-Planck equation is not solved up to order $\delta_i^3(V_i/v_i)f_{is}v_i/L$. Usual gyrokinetic formulations in slab geometry assume low flows, $V_i \sim \delta_i v_i$, needing the Fokker-Planck equation up to order $\delta_i^4 f_{is}v_i/L$. Even though slab gyrokinetics can be obtained to higher order in δ_i than in more complex magnetic geometries, the slab gyrokinetic equation is only good to order $\delta_i^2 f_{is}v_i/L$; not enough to recover the correct transport of toroidal angular momentum. In Ref. 21 we proved that using the classic formulation by Dubin *et al* [22] was equivalent to introducing a non-physical momentum source in the y direction, invalidating then any long wavelength velocity profiles obtained using this method. However, the formulation in Ref. 22 is not the only one for a slab. In this section, we will study the variational formulations [26, 27]. These formulations are able to conserve y-momentum in the slab, but the transport of momentum they predict is incorrect. To see this, we will first review some of the results in Ref. 21 that will help us explain the problems that arise in the variational formulation. At the end of the section we will prove that even though the variational formulation conserves y-momentum, it transports momentum from one flux surface to the next incorrectly.

The gyrokinetic variables are defined up to second order in δ_i. As an example of variables, we use the gyrocenter position $\mathbf{R} = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}} + \mathbf{R}_2$, the gyrokinetic parallel velocity $u = v_{||} + u_2$ and the magnetic moment $\mu = \mu_0 + \mu_1 + \mu_2$ (the exact definitions of $\mathbf{R}_2$, u_2, μ_1 and μ_2 can be found in Ref. 21). In the absence of collisions, the distribution function written in these variables will be independent of gyrophase. Once the distribution function is found, it must be integrated over velocities to obtain densities, temperatures or the transport of momentum in Eq. (30), but this integral must be done holding

the real position of the particle $\mathbf{r} = \mathbf{R} - \mathbf{R}_1 - \mathbf{R}_2$ fixed. It is then customary to Taylor expand around the variables $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, $v_{||}$ and μ_0 to obtain, for the ions, $f_i(\mathbf{R}, u, \mu, t) \simeq f_{ig} + f_{ip}$, where $f_{ig} \equiv f_i(\mathbf{R}_g, v_{||}, \mu_0, t)$ is obtained by replacing $\mathbf{R}$, u and μ in $f_i(\mathbf{R}, u, \mu, t)$ by $\mathbf{R}_g$, $v_{||}$ and μ_0, and

$$f_{ip} = \mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig} + u_2 \frac{\partial f_{ig}}{\partial v_{||}} + (\mu_1 + \mu_2)\frac{\partial f_{ig}}{\partial \mu_0} + \frac{\mu_1^2}{2}\frac{\partial^2 f_{ig}}{\partial \mu_0^2}.$$

(31)

The subindex p indicates that the integral over velocity of f_{ip} is the polarization density. The expression $f_i = f_{ig} + f_{ip}$ is good to order $\delta_i^2 f_{is}$ if f_{ig} is found to that order, but it is not valid to higher order because we have not obtained the next order corrections to the gyrokinetic variables. Similarly, if we were to keep only the first order correction $f_{ip} \simeq \mu_1(\partial f_{ig}/\partial \mu_0)$, we would be missing the rest of the terms $\mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_{ig} + \ldots \sim \delta_i^2 f_{is}$ even for f_{ig} good to order $\delta_i^2 f_{is}$.

Once f_{ip} is known as a function of f_{ig}, it is necessary to obtain f_{ig} to the correct order. The gyrokinetic Fokker-Planck equation that determines $f_i(\mathbf{R}, u, \mu, t)$ can be used by simply replacing $\mathbf{R}$, u and μ by $\mathbf{R}_g$, $v_{||}$ and μ_0 to obtain

$$\frac{\partial f_{ig}}{\partial t} + \dot{\mathbf{R}}_g \cdot \nabla_{\mathbf{R}_g} f_{ig} + \dot{v}_{||}\frac{\partial f_{ig}}{\partial v_{||}} = 0.$$

(32)

Full f gyrokinetics obtains the electrostatic potential from quasineutrality. To simplify the problem we can assume that the electrons are determined by a drift kinetic equation

$$\frac{\partial \overline{f}_e}{\partial t} + \left(v_{||}\hat{\mathbf{b}} - \frac{c}{B}\nabla\phi \times \hat{\mathbf{b}}\right)\cdot\nabla\overline{f}_e + \frac{e}{m}\hat{\mathbf{b}}\cdot\nabla\phi\frac{\partial \overline{f}_e}{\partial v_{||}} = 0,$$

(33)

leaving then the simple quasineutrality equation

$$Ze\int d^3v\, f_{ig} + Ze\int d^3v\, f_{ip} = e\int d^3v\, \overline{f}_e.$$

(34)

Here it is where different gyrokinetic formalisms differ. Usually, the quasineutrality equation is written as

$$Ze\int d^3v\, f_{ig} + Zen_{ip} = e\int d^3v\, \overline{f}_e,$$

(35)

where the polarization density $n_{ip} = \int d^3v\, f_{ip}^*$ is different from the integral $\int d^3v\, f_{ip}$. Notice that employing the incorrect polarization density $n_{ip} = \int d^3v\, f_{ip}^*$ leads to errors in the fluctuating potential, errors that may become important in Eq. (30) where the potential is needed to order $\delta_i^2 T_e/e$.

In Ref. 21, the quasineutrality equation (35) was studied for the formulation of Dubin *et al*, characterized by a particular gyrokinetic equation (32) and a particular choice of $n_{ip} = \int d^3v\, f_{ip}^*$. The procedure developed in Ref. 21 is easy to extend to a general gyrokinetic equation (32) and a general f_{ip}^*. Taking the time derivative of Eq. (35) gives

$$\frac{\partial}{\partial t}\left\langle Ze\int d^3v\, f_{ip}\right\rangle_x - \frac{\partial}{\partial x}\left\langle Ze\int d^3v\, f_{ig}\dot{\mathbf{R}}_g \cdot \hat{\mathbf{x}}\right\rangle_x$$
$$= \frac{\partial}{\partial x}\left\langle \frac{Zec}{B}\frac{\partial \phi}{\partial y}\int d^3v\,(f_{ig} + f_{ip}^*)\right\rangle_x, \quad (36)$$

where we have used Eqs. (32) and (33) to rewrite the time derivatives $\partial f_{ig}/\partial t$ and $\partial \overline{f}_e/\partial t$, and in the right side of the equation we have employed the quasineutrality equation (35) to write $\int d^3v\, \overline{f}_e = \int d^3v\,(f_{ig} + f_{ip}^*)$.

In Ref. 21, we found that, at long wavelengths,

$$\frac{\partial}{\partial t}\left\langle Ze\int d^3v\, f_{ip}\right\rangle_x - \frac{\partial}{\partial x}\left\langle Ze\int d^3v\, f_{ig}\dot{\mathbf{R}}_g \cdot \hat{\mathbf{x}}\right\rangle_x = -\frac{\partial\Gamma}{\partial x} + \frac{c}{B}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\langle n_i M\mathbf{V}_i \cdot \hat{\mathbf{y}}\rangle_x - \frac{1}{2\Omega_i}\frac{\partial^2 p_{i\perp}}{\partial x\partial t}\right], \quad (37)$$

$$\left\langle \frac{Zec}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_{ig}\right\rangle_x = \Gamma_g + \frac{c}{B}\frac{\partial}{\partial x}\left[\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_{ig}M\mathbf{v} \cdot \hat{\mathbf{y}}\right\rangle_x - \frac{1}{2\Omega_i}\frac{\partial}{\partial x}\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_{ig}M(\mathbf{v} \cdot \hat{\mathbf{y}})^2\right\rangle_x\right] \quad (38)$$

and

$$\left\langle \frac{Zec}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_{ip}^*\right\rangle_x = \Gamma_p + \frac{c}{B}\frac{\partial}{\partial x}\left[\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_{ip}^*M\mathbf{v} \cdot \hat{\mathbf{y}}\right\rangle_x - \frac{1}{2\Omega_i}\frac{\partial}{\partial x}\left\langle \frac{c}{B}\frac{\partial \phi}{\partial y}\int d^3v\, f_{ip}^*M(\mathbf{v} \cdot \hat{\mathbf{y}})^2\right\rangle_x\right]. \quad (39)$$

Here Γ, Γ_g and Γ_p are quantities with the dimensions of current whose exact expressions are not relevant; Γ

depends on the exact form of $\dot{\mathbf{R}}_g$ in Eq. (32) and Γ_p depends on the particular choice of f_{ip}^*. Substituting the long wavelength results (37), (38) and (39) into Eq. (36) and integrating once in x gives

$$\frac{\partial}{\partial t}\langle n_i M \mathbf{V}_i \cdot \hat{\mathbf{y}}\rangle_x = -\frac{\partial}{\partial x}\Pi\{f_{ig} + f_{ip}^*, \phi\} + F_y, \quad (40)$$

where there is a non-physical force

$$F_y = \frac{B}{c}(\Gamma + \Gamma_g + \Gamma_p) \quad (41)$$

and the transport of toroidal angular momentum $\Pi\{f_i, \phi\}$ is the functional of f_i and ϕ defined by Eq. (30). If f_i and ϕ are the correct distribution function and potential up to order $\delta_i^2 f_{is}$ and $\delta_i^2 T_e/e$, $\Pi\{f_i, \phi\}$ will be the physical transport of momentum $\langle \hat{\mathbf{x}}\cdot \overset{\leftrightarrow}{\boldsymbol{\pi}}_i \cdot \hat{\mathbf{y}}\rangle_x$.

In the work of Dubin $et\ al$ [22], the polarization density $n_{ip}^{\text{Dubin}} = \int d^3v\, f_{ip}^{\text{Dubin}} \equiv \int d^3v\, f_{ip}$ is obtained as in Eq. (31), i.e., by Taylor expansion around the lowest order gyrokinetic variables. The resulting $n_i = \int d^3v\, f_{ig} + \int d^3v\, f_{ip}^{\text{Dubin}}$ and ϕ are correct up to order $\delta_i^2 n_e$ and $\delta_i^2 T_e/e$, good enough to calculate the transport of y-momentum using Eq. (30). Unfortunately, problems arise when using the quasineutrality equation (34). In Ref. 21, we proved that the current densities Γ, Γ_g and

Γ_p in Eqs. (37), (38) and (39) do not cancel, thereby giving a non-physical force F_y in Eq. (40).

In Ref. 21, we suggested that in order to make the non-physical force F_y vanish, it would be necessary to add a higher order correction to the ion gyrokinetic drifts. There is another procedure that also cancels the non-physical force, namely variational gyrokinetics [26, 27]. In the variational methodology, the polarization density $n_{ip}^{\text{var}} = \int d^3v\, f_{ip}^{\text{var}}$ is not defined by Taylor expanding around the lowest order gyrokinetic variables, but it is obtained by extremizing an action with respect to the potential $\phi(\mathbf{r}, t)$. The same action is extremized with respect to the gyrocenter trajectories to obtain the gyrocenter motion. It is then possible to use Noether's theorem to prove that certain quantities are conserved; among them, y-momentum. Comparing this method with the study in Ref. 21, it becomes obvious that the variational approach provides a different polarization density $n_{ip}^{\text{var}} = \int d^3v\, f_{ip}^{\text{var}}$ than Dubin $et\ al$ [22]. Using the same notation as in Ref. 21, the variational approach gives

$$f_{ip}^{\text{var}} = \frac{Ze\tilde{\phi}_g}{MB}\frac{\partial f_{ig}}{\partial \mu_0} - \frac{c}{B\Omega_i}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_g \times \hat{\mathbf{b}})\cdot \nabla_{\mathbf{R}_g} f_{ig}, \quad (42)$$

while the approach by Dubin $et\ al$ has additional second order pieces,

$$f_{ip}^{\text{Dubin}} = \frac{Ze\tilde{\phi}_g}{MB}\frac{\partial f_{ig}}{\partial \mu_0} - \frac{c}{B\Omega_i}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_g \times \hat{\mathbf{b}})\cdot \nabla_{\mathbf{R}_g} f_{ig} + \frac{Ze}{MB}\left[-\frac{Ze\tilde{\phi}_g}{MB}\frac{\partial \langle\phi\rangle_g}{\partial \mu_0} + \frac{c}{B\Omega_i}(\nabla_{\mathbf{R}_g}\tilde{\Phi}_g \times \hat{\mathbf{b}})\cdot \nabla_{\mathbf{R}_g}\langle\phi\rangle_g \right.$$
$$\left. + \frac{Ze}{2MB}\frac{\partial}{\partial\mu_0}\langle\tilde{\phi}_g^2\rangle + \frac{c}{2B\Omega_i}\langle(\nabla_{\mathbf{R}_g}\tilde{\phi}_g \times \hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}_g}\tilde{\Phi}_g\rangle \right]\frac{\partial f_{ig}}{\partial\mu_0} + \frac{Z^2 e^2 \tilde{\phi}_g^2}{2M^2 B^2}\frac{\partial^2 f_{ig}}{\partial\mu_0^2}. \quad (43)$$

Following the same reasoning as in Ref. 21 with f_{ip}^{var} instead of f_{ip}^{Dubin}, it is easy to prove that F_y in Eq. (40) vanishes, leading to conservation of y-momentum. Unfortunately, the transport of momentum $\Pi\{f_{ig} + f_{ip}^{\text{var}}, \phi^{\text{var}}\}$ is now missing pieces of order $\delta_i^3 p_i$ due to the differences between f_{ip}^{var} and the physical definition of f_{ip} in Eq. (31), and the differences between the fluctuating potential calculated using the variational method ϕ^{var}, and the physical fluctuating potential ϕ, found to $O(\delta_i^2 T_e/e)$ using the correct polarization density $\int d^3v\, f_{ip}$. Thus, even though having some form of conservation of momentum may be arguably better than not having any, as in the formulation of Dubin $et\ al$ [22], the variational approach has not solved the problem of the long wavelengths electric field. It is necessary to obtain the gyrokinetic equation to painfully high orders when using quasineutrality, and even variational approaches cannot escape that fact. Indeed, using a variational formulation may only hide the problem in an incorrect transport of momentum. On the

other hand, variational formulations may be a possible brute force approach to find the long wavelength electric field using quasineutrality in slab geometry.

Finally, we emphasize that any improvements to gyrokinetics obtained by working only in slab geometry are not easy to translate to more complicated geometries. We believe that the solution to this problem does not lie in gyrokinetic formulations that split magnetic geometry and fluctuations (for which they use the slab results), calculate the gyrokinetic Lagrangian for each independently, and add them as if the problem were linear. To obtain the correct radial electric field from quasineutrality, the gyrokinetic equation has to be derived to an order as high as $\delta_i^3 (V_i/v_i) f_{Mi} v_i/a$. Cross terms that involve both magnetic geometry and fluctuations will most likely matter.

V. TOROIDAL ANGULAR MOMENTUM TESTS

In Ref. 21 and in sections III and IV we have demonstrated with several examples that employing the quasineutrality equation at long wavelengths leads to problems in the conservation of momentum. These calculations are not meant to cover all possible cases exhaustively, but to show the extreme sensitivity of the toroidal rotation to small errors in the gyrokinetic formulation.

It is possible to distinguish several cases:

1. There are gyrokinetic formulations that do not conserve momentum, like the formulation by Dubin *et al* [22], studied in Ref. 21, or the drift kinetic approach studied in section III. These approaches introduce a non-physical source of momentum that will spin the plasma to an incorrect toroidal rotation profile.

2. Some gyrokinetic approaches conserve momentum, that is, the toroidal velocity satisfies an equation of the form $\partial \langle Rn_i M \mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi / \partial t = -(V')^{-1} \partial (V'\Pi) / \partial \psi$, but the transport of momentum Π will not be the physical quantity $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla \psi \rangle_\psi$. Here, there are two possibilities, namely

   (a) the momentum transport Π is clearly different from $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla \psi \rangle_\psi$, as in Eq. (22); or

   (b) the transport of momentum Π is seemingly the same as $\langle R\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\boldsymbol{\pi}}_i \cdot \nabla \psi \rangle_\psi$, i.e., Π is given by the integrals in Eq. (37) of Ref. 23 [or its simplified version in Eq. (11)] of some approximate f_i and ϕ; however, the transport of momentum Π is non-physical because the approximate distribution function and potential are not calculated to the required order. A good example of this last case are the variational approaches studied in section IV.

With these cases in mind, we propose three tests for full f simulations that will help determine if they achieve the correct conservation of momentum. Passing these tests is a necessary, but not sufficient, condition to obtain the correct transport of momentum.

The three tests follow the evolution of toroidal angular momentum in the volume contained between two flux surfaces ψ_1 and ψ_2 separated by more than an ion gyroradius, $(\psi_2 - \psi_1)/aRB_p \gg \delta_i$. The volume should not include regions of the simulation where extraneous effects are used to damp numerical oscillations. According to Eq. (8), the time derivative of the total toroidal angular momentum in the volume must be equal to the toroidal angular momentum incoming through the boundaries. If the difference between these two terms grows secularly in time, either there is a non-physical source of momentum or the transport of momentum through the boundaries is incorrect. The three tests differ on how to determine the transport of momentum through the boundaries.

The first test is the easiest. It is more convenient for particle codes in which the trajectories of particles are known. Then, it is possible to count the particles that cross the boundaries and the momentum they carry. The momentum balance must be performed taking into account the finite gyroradius size, in particular the variation of f_i along the gyromotion. Indeed, the ion gyrokinetic distribution function is $f_i = f_{ig} + f_{ip}$, where f_{ip}, defined in Eq. (31), is the piece of the distribution function that contains the effect of the potential variations on gyroradius scales. The correction f_{ip} is important because upon integration over velocity space it gives the polarization density and the $\mathbf{E} \times \mathbf{B}$ flow.

The second test is more involved, but works for both particle and continuum codes. It will also provide more useful information. Instead of obtaining the transport of toroidal angular momentum from the motion of the particles through the boundaries ψ_1 and ψ_2, we propose to use the expression in Eq. (37) of Ref. 23 [or its simplified version in Eq. (11)]. The integrals are to be taken using the numerical solutions for f_i and ϕ.

Importantly, some variational formulations might pass these first two tests and yet predict the wrong transport of toroidal angular momentum because they do not obtain f_i and ϕ to high enough order (see the end of section IV). In these cases, another possible test (although probably unfeasible) is to extend the formulation to the next order in δ_i. The new, seemingly negligible contributions would have an $O(1)$ effect on the long wavelength rotation profile. Since going to higher order is rarely possible, for these cases we propose comparing with accurate δf simulations that employ Eq. (37) of Ref. 23 [or its simplified version in Eq. (11)] to obtain the transport of toroidal angular momentum; δf simulations are now available in the high flow ordering [30], and are underway for low flows [23]. In the third test, the time derivative of the total angular momentum in the volume between ψ_1 and ψ_2 is compared to the difference between the transport of momentum evaluated by δf simulations at each of the two boundaries. This is probably the most robust test. The comparison must be obviously performed in regions of the plasma where the δf approach is reliable, i.e., outside of transport barriers or pedestals.

To summarize, the accuracy required to recover the correct transport of toroidal angular momentum from quasineutrality is daunting. The gyrokinetic equation must be obtained to order $\delta_i^3 (V_i/v_i) f_{Mi} v_i/a$, requiring up to 5 digit accuracy in the high flow ordering, and 7 digits for low flow. We have shown several cases in which the analytical model is not accurate enough to determine the correct rotation profile, demonstrating that even in the simple cases of drift kinetics and slab geometry, the conservation of momentum is very sensitive to errors. The errors may manifest themselves as non-physical sources of momentum, or incorrect transport of momentum from one flux surface to the next. Accordingly, we have proposed three tests that will help decide if full f simulations are giving the correct toroidal rotation. Notice that these

tests are not only benchmarking the gyrokinetic analytical model, but they also check if the stringent numerical resolution requirements are satisfied.

Acknowledgments

APPENDIX A: DETAILS OF THE CALCULATION OF TRANSPORT OF TOROIDAL ANGULAR MOMENTUM IN DRIFT KINETICS

In this Appendix, we calculate the terms $M\langle \int d^3v\, \overline{f}_i v_{||} \hat{\mathbf{b}}\cdot\overline{\nabla}_i(Iv_{||}/B)\rangle_\psi$ and $m\langle \int d^3v\, \overline{f}_e v_{||} \hat{\mathbf{b}}\cdot\overline{\nabla}_e(Iv_{||}/B)\rangle_\psi$ in Eq. (20) to order $\delta_i^3 Rp_i/a$. To do so, we will integrate by parts three times, and then use Eq. (12) to write $\overline{\nabla}_i\cdot(J_i\overline{f}_i v_{||}\hat{\mathbf{b}})$ and $\overline{\nabla}_e\cdot(J_e\overline{f}_e v_{||}\hat{\mathbf{b}})$ as a function of the other terms in the equation.

We will do the calculation for the ions only. The electrons are equivalent. For $M\langle \int d^3v\, \overline{f}_i v_{||} \hat{\mathbf{b}}\cdot\overline{\nabla}_i(Iv_{||}/B)\rangle_\psi$, the first integration by parts is done employing $d^3v = 2\pi J_i d\varepsilon_i d\mu_0$,

$$
M\left\langle \int d^3v\, \overline{f}_i v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}_i\left(\frac{Iv_{||}}{B}\right)\right\rangle_\psi = -\left\langle \frac{MI}{B}\int 2\pi d\varepsilon_i d\mu_0\, v_{||}\overline{\nabla}_i\cdot\left(J_i\overline{f}_i v_{||}\hat{\mathbf{b}}\right)\right\rangle_\psi
$$
$$
= \left\langle \frac{MI}{B}\int 2\pi d\varepsilon_i d\mu_0\, v_{||}\left[\frac{\partial}{\partial t}(J_i\overline{f}_i)+\overline{\nabla}_i\cdot\left(J_i\overline{f}_i\mathbf{v}_{di}\right)+\frac{\partial}{\partial\varepsilon_i}(J_i\overline{f}_i\dot{\varepsilon}_i)-J_i C_i\{\overline{f}_i\}\right]\right\rangle_\psi, \tag{A1}
$$

where we have used Eq. (12) to obtain the second equality. From this form, it is straightforward to obtain

$$
M\left\langle \int d^3v\, \overline{f}_i v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}_i\left(\frac{Iv_{||}}{B}\right)\right\rangle_\psi = \frac{\partial}{\partial t}\left\langle \frac{I}{B}n_i M\mathbf{V}_i\cdot\hat{\mathbf{b}}\right\rangle_\psi + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \frac{MI}{B}\int d^3v\, \overline{f}_i v_{||}\mathbf{v}_{di}\cdot\nabla\psi\right\rangle_\psi
$$
$$
- M\left\langle \int d^3v\, \overline{f}_i \mathbf{v}_{di}\cdot\overline{\nabla}_i\left(\frac{Iv_{||}}{B}\right)\right\rangle_\psi - \left\langle \frac{MI}{B}\int d^3v\, v_{||}C_i\{\overline{f}_i\}\right\rangle_\psi. \tag{A2}
$$

Realizing that electrons have a very similar form, and using that $\int d^3v\, C_i\{\overline{f}_i\}Mv_{||}+\int d^3v\, C_e\{\overline{f}_e\}mv_{||}=0$, we find Eq. (21). To obtain the final expression in Eq. (21), we have neglected terms small by the mass ratio m/M, and we have used the lowest order result $R\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\simeq(I/B)\mathbf{V}_i\cdot\hat{\mathbf{b}}$.

Next, again by integration by parts and using Eq. (12), the term $\Pi=\langle(MI/B)\int d^3v\, \overline{f}_i v_{||}\mathbf{v}_{di}\cdot\nabla\psi\rangle_\psi$ is obtained to order $\delta_i^3 p_i R|\nabla\psi|$. Using that $\mathbf{v}_{di}\cdot\nabla\psi=v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}_i(Iv_{||}/\Omega_i)+c(\partial\phi/\partial\zeta)$, we find

$$
\Pi = Mc\left\langle \frac{I}{B}\frac{\partial\phi}{\partial\zeta}\int d^3v\, \overline{f}_i v_{||}\right\rangle_\psi + \frac{M^2 c}{2Ze}\left\langle \int d^3v\, \overline{f}_i v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}_i\left(\frac{I^2 v_{||}^2}{B^2}\right)\right\rangle_\psi. \tag{A3}
$$

Integrating $\langle \int d^3v\, \overline{f}_i v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}_i(I^2 v_{||}^2/B^2)\rangle_\psi$ by parts and using Eq. (12), we find

$$
\Pi = Mc\left\langle \frac{I}{B}\frac{\partial\phi}{\partial\zeta}\int d^3v\, \overline{f}_i v_{||}\right\rangle_\psi + \frac{M^2 c}{2Ze}\frac{\partial}{\partial t}\left\langle \frac{I^2}{B^2}\int d^3v\, \overline{f}_i v_{||}^2\right\rangle_\psi + \frac{M^2 c}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \frac{I^2}{B^2}\int d^3v\, \overline{f}_i v_{||}^2\mathbf{v}_{di}\cdot\nabla\psi\right\rangle_\psi
$$
$$
- \frac{M^2 c}{2Ze}\left\langle \int d^3v\, \overline{f}_i\mathbf{v}_{di}\cdot\overline{\nabla}_i\left(\frac{I^2 v_{||}^2}{B^2}\right)\right\rangle_\psi - \frac{M^2 c}{2Ze}\left\langle \frac{I^2}{B^2}\int d^3v\, v_{||}^2 C_i\{\overline{f}_i\}\right\rangle_\psi. \tag{A4}
$$

"Transport" or coarse-grain averaging this expression gives

$$
\Pi = Mc\left\langle \left\langle \frac{I}{B}\frac{\partial\phi}{\partial\zeta}\int d^3v\, \overline{f}_i v_{||}\right\rangle_\psi\right\rangle_{\mathrm{T}} + \frac{Mc}{2Ze}\left\langle \frac{I^2}{B^2}\right\rangle_\psi\frac{\partial p_i}{\partial t} + \frac{M^2 c}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\left\langle \frac{I^2}{B^2}\int d^3v\, \overline{f}_i v_{||}^2\mathbf{v}_{di}\cdot\nabla\psi\right\rangle_\psi\right\rangle_{\mathrm{T}}
$$

$$-\frac{M^2 c}{2Ze}\left\langle\left\langle\int d^3v\,\overline{f}_i\mathbf{v}_{di}\cdot\overline{\nabla}_i\left(\frac{I^2 v_{||}^2}{B^2}\right)\right\rangle_\psi\right\rangle_T - \frac{M^2 c}{2Ze}\left\langle\frac{I^2}{B^2}\int d^3v\,v_{||}^2\langle C_i\{\overline{f}_i\}\rangle_T\right\rangle_\psi. \quad (A5)$$

Here we have used that only the time derivative of the total ion pressure is large enough at long time scales that cannot be neglected. The fast time derivatives due to turbulence average to zero under the "transport" average.

To finish the calculation, the term $\langle\langle(I^2/B^2)\int d^3v\,\overline{f}_i v_{||}^2\mathbf{v}_{di}\cdot\nabla\psi\rangle_\psi\rangle_T$ must be found to order $\delta_i^2 n_i v_i^3 R^2|\nabla\psi|$. Employing $\mathbf{v}_{di}\cdot\nabla\psi = v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}_i(I v_{||}/\Omega_i) + c(\partial\phi/\partial\zeta)$ gives

$$\left\langle\left\langle\frac{I^2}{B^2}\int d^3v\,\overline{f}_i v_{||}^2\mathbf{v}_{di}\cdot\nabla\psi\right\rangle_\psi\right\rangle_T = c\left\langle\left\langle\frac{I^2}{B^2}\frac{\partial\phi}{\partial\zeta}\int d^3v\,\overline{f}_i v_{||}^2\right\rangle_\psi\right\rangle_T + \frac{Mc}{3Ze}\left\langle\left\langle\int d^3v\,\overline{f}_i v_{||}\hat{\mathbf{b}}\cdot\nabla\left(\frac{I^3 v_{||}^3}{B^3}\right)\right\rangle_\psi\right\rangle_T. \quad (A6)$$

Integrating by parts and using Eq. (12), this result reduces to

$$\left\langle\left\langle\frac{I^2}{B^2}\int d^3v\,\overline{f}_i v_{||}^2\mathbf{v}_{di}\cdot\nabla\psi\right\rangle_\psi\right\rangle_T = c\left\langle\left\langle\frac{I^2}{B^2}\frac{\partial\phi}{\partial\zeta}\int d^3v\,\overline{f}_i v_{||}^2\right\rangle_\psi\right\rangle_T - \frac{Mc}{3Ze}\left\langle\frac{I^3}{B^3}\int d^3v\,v_{||}^3\langle C_i\{\overline{f}_i\}\rangle_T\right\rangle_\psi$$
$$+\frac{Mc}{3Ze}\left\langle\frac{\partial}{\partial t}\left\langle\frac{I^3}{B^3}\int d^3v\,\overline{f}_i v_{||}^3\right\rangle_\psi\right\rangle_T + \frac{Mc}{3Ze}\left\langle\left\langle\frac{I^3}{B^3}\int 2\pi d\varepsilon_i d\mu_0\,v_{||}^3\overline{\nabla}_i\cdot(J_i\overline{f}_i\mathbf{v}_{di})\right\rangle_\psi\right\rangle_T. \quad (A7)$$

The last two terms in this equation vanish due to the "transport" average. In the time derivative, only the secular terms with $\partial/\partial t \sim D_{gB}/a^2 \sim \delta_i^2 v_i/a$ contribute. In addition, the $v_{||}^3$ moment of the lowest order Maxwellian vanishes, giving $(Mc/3Ze)\langle\partial\langle(I^3/B^3)\int d^3v\,\overline{f}_i v_{||}^3\rangle_\psi/\partial t\rangle_T \sim \delta_i^4 n_i v_i^3 R^2|\nabla\psi|$, negligible compared to the terms of order $\delta_i^2 n_i v_i^3 R^2|\nabla\psi|$. The integral $(Mc/3Ze)\langle\langle(I^3/B^3)\int 2\pi d\varepsilon_i d\mu_0\,v_{||}^3\overline{\nabla}_i\cdot(J_i\overline{f}_i\mathbf{v}_{di})\rangle_\psi\rangle_T$ vanishes as well. Here only the term $\overline{\nabla}\cdot(J_i f_{Mi}\mathbf{v}_{di}) \sim \delta_i f_{Mi}v_i/a$ could give a relevant contribution to order $\delta_i^2 n_i v_i^3 R^2|\nabla\psi|$. The long wavelength contributions of rest of the terms, including the turbulent term $-\overline{\nabla}_i\cdot[J_i(\overline{f}_i - f_{Mi})(c/B)(\nabla\phi\times\hat{\mathbf{b}})] \sim \delta_i^2 J_i f_{Mi}v_i/a$ [8, 23], will be negligible, thus vanishing under the "transport" average. Since the contribution of the Maxwellian to $(Mc/3Ze)\langle\langle(I^3/B^3)\int 2\pi d\varepsilon_i d\mu_0\,v_{||}^3\overline{\nabla}_i\cdot(J_i\overline{f}_i\mathbf{v}_{di})\rangle_\psi\rangle_T$ is zero because the Maxwellian is even in $v_{||}$, the last term of Eq. (A7) can also be neglected.

Taking into account the cancellations of the last two terms in Eq. (A7) and substituting Eq. (A7) into Eq. (A5) gives Eq. (21).

APPENDIX B: CALCULATION OF $(v_{||}/\Omega_i)\overline{\nabla}_i\times(v_{||}\hat{\mathbf{b}})\cdot\overline{\nabla}_i(I v_{||}/B)$

The expression $(v_{||}/\Omega_i)\overline{\nabla}_i\times(v_{||}\hat{\mathbf{b}})\cdot\overline{\nabla}_i(I v_{||}/B)$ can be written as $(v_{||}/\Omega_i)\overline{\nabla}_i\cdot[v_{||}\hat{\mathbf{b}}\times\overline{\nabla}_i(I v_{||}/B)]$. Using Eq. (4), this expression becomes

$$\frac{v_{||}}{\Omega_i}\overline{\nabla}_i\times(v_{||}\hat{\mathbf{b}})\cdot\overline{\nabla}_i\left(\frac{I v_{||}}{B}\right) = \frac{v_{||}}{\Omega_i}\overline{\nabla}_i\cdot\left[\frac{I v_{||}}{B}\nabla\zeta\times\overline{\nabla}_i\left(\frac{I v_{||}}{B}\right)\right] + \frac{v_{||}}{\Omega_i}\overline{\nabla}_i\cdot\left[\frac{v_{||}}{B}(\nabla\zeta\times\nabla\psi)\times\overline{\nabla}_i\left(\frac{I v_{||}}{B}\right)\right]. \quad (B1)$$

Employing $\overline{\nabla}_i\cdot[(I v_{||}/B)\nabla\zeta\times\overline{\nabla}_i(I v_{||}/B)] = \overline{\nabla}_i\cdot[\nabla\zeta\times\overline{\nabla}_i(I^2 v_{||}^2/2B^2)] = 0$, we rewrite this expression as

$$\frac{v_{||}}{\Omega_i}\overline{\nabla}_i\times(v_{||}\hat{\mathbf{b}})\cdot\overline{\nabla}_i\left(\frac{I v_{||}}{B}\right) = \frac{v_{||}}{\Omega_i}\overline{\nabla}_i\cdot\left[\frac{v_{||}}{B}\nabla\psi\nabla\zeta\cdot\overline{\nabla}_i\left(\frac{I v_{||}}{B}\right)\right] - \frac{v_{||}}{\Omega_i}\overline{\nabla}_i\cdot\left[\frac{v_{||}}{B}\nabla\zeta\nabla\psi\cdot\overline{\nabla}_i\left(\frac{I v_{||}}{B}\right)\right]$$
$$= \frac{R^2 I v_{||}^2}{B^2\Omega_i}\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)\nabla\zeta\cdot\overline{\nabla}_i v_{||} - \frac{v_{||}^2}{B^2\Omega_i}(\nabla\psi\cdot\nabla I)\nabla\zeta\cdot\overline{\nabla}_i v_{||}. \quad (B2)$$

Finally, employing that the magnetic field in Eq. (4) satisfies $\mathbf{B}\cdot\nabla\times\mathbf{B} = I\nabla\cdot(\nabla\psi/R^2) - \nabla I\cdot\nabla\psi/R^2$, we find Eq. (25), where we have used that $R^2\nabla\zeta\cdot\overline{\nabla}_i v_{||} = -(Ze/Mv_{||})(\partial\phi/\partial\zeta)$.

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