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Fluid and Drift-Kinetic Description of a Magnetized Plasma with Low Collisionality and Slow Dynamics Orderings. II: Ion Theory

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Abstract

The ion side of a closed, fluid and drift-kinetic theoretical model to describe slow and macroscopic plasma processes in a fusion-relevant, low collisionality regime is presented. It follows the ordering assumptions and the methodology adopted in the companion electron theory[1]. To reach the frequency scale where collisions begin to play a role, the drift-kinetic equation for the ion distribution function perturbation away from a Maxwellian must be accurate to the second order in the Larmor radius. The macroscopic density, flow velocity and temperature are accounted for in the Maxwellian, and are evolved by a fluid system which includes consistently the gyroviscous part of the stress tensor and second-order contributions to the collisionless perpendicular heat flux involving non-Maxwellian fluid moments. The precise compatibility among these coupled high-order fluid and drift-kinetic equations is made manifest by showing that the evolution of the non-Maxwellian part of the distribution function is such that its first three velocity moments remain equal to zero.

I. Introduction.

A closed fluid and drift-kinetic description of magnetic confinement plasmas in a low collisionality regime, applicable to slow dynamics with length scales larger than the ion Larmor radius, was introduced in Ref.1 where the electron side of the system was analyzed. The present second part of the series completes the theory by developing its ion side. This theory relies on low collisionality and small mass ratio orderings whereby the ratios between the ion collision and cyclotron frequencies and between the electron and ion masses are second-order, compared to the first-order ratio between the ion Larmor radius and the macroscopic lenght scales; it also assumes macroscopic flows of the order of the diamagnetic drifts. Accordingly, asymptotic expansions are systematically carried out based on uniform powers of a single small parameter:

$$\delta \sim \rho_\iota / L \sim \nu_\iota L / v_{th\iota} \sim (m_e/m_\iota)^{1/2} \sim u_\iota / v_{th\iota} \ll 1 \ , \tag{1}$$

where ρ_ι, ν_ι, $v_{th\iota}$ and u_ι stand for the ion Larmor radius, collision frequency, thermal velocity and macroscopic flow velocity, respectively, and L represents any macroscopic length or mode wavelength without additional geometrical assumptions. These orderings (which for the ions imply a collisionality lower than in the conventional neoclassical banana regime) represent a best attempt towards a realistic simulation of core plasmas in fusion-relevant tokamak experiments, as argued in Refs.1-2. Similar orderings were proposed and argued for earlier by H. Weitzner in the context of axisymmetric equilibrium and transport theory[3,4]. The present theory is fully dynamical, 3-dimensional and electromagnetic but, as intended to be applied to slowly evolving excursions from a well confined equilibrium such as the "neoclassical tearing" modes[5,6], it assumes near-Maxwellian distribution functions. For the ions, the non-Maxwellian perturbation is first-order and the distribution function is represented as

$$f_\iota = f_{M\iota} + f_{NM\iota} = \frac{n}{(2\pi)^{3/2}\ v_{th\iota}^3} \exp\left(- \frac{|\mathbf{v} - \mathbf{u}_\iota|^2}{2\ v_{th\iota}^2} \right) + f_{NM\iota} \ , \tag{2}$$

with $f_{NM\iota} \sim \delta f_{M\iota}$. The Maxwellian part, $f_{M\iota}$, is referred to the moving frame of the ion macroscopic flow $\mathbf{u}_\iota$, and the thermal velocity is defined as $v_{th\iota}^2 \equiv T_\iota / m_\iota \equiv p_\iota/(m_\iota n)$, where T_ι and p_ι are the mean ion temperature and pressure; n is the particle number density which, for the assumed quasineutral plasma with a single ion species of unit charge, is the same as that of the electrons. Thus, in

Chapman-Enskog-like fashion[7–9], the density, flow velocity and temperature are carried entirely by the Maxwellian and will be determined by the fluid equations. The 1, $\mathbf{v} - \mathbf{u}_\iota$ and $|\mathbf{v} - \mathbf{u}_\iota|^2$ velocity moments of $f_{NM\iota}$ (which will provide the higher-rank fluid moment closures) are then required to vanish. In addition, consistency of this near-Maxwellian form with an asymptotic solution of the ion kinetic equation under the assumed low collisionality ordering, requires the small ion parallel temperature gradient $\mathbf{B} \cdot \nabla T_\iota \sim \delta \; BT_\iota / L$ [there is an inconsequential error in Eq.(5) of Ref.1 which implies that the parallel temperature gradients are second-order in δ for both species: they are second-order for the electrons but first-order for the ions, same as the respective orderings of the non-Maxwellian perturbations relative to the Maxwellians]. On the other hand, the parallel density gradient remains arbitrary.

With the adopted low colisionality and close to Maxwellian orderings, collisions begin to influence the dynamics at the frequency scale of order

$$\delta \nu_\iota \sim \delta^2 \nu_e \sim \delta \omega_D \sim \delta^2 v_{th\iota}/L \sim \delta^3 v_{the}/L \sim \delta^3 \Omega_{c\iota} \; , \tag{3}$$

where ω_D is the diamagnetic drift frequency. As with the electrons, this will be the smallest frequency scale the analysis will be carried to. In the case of the ions, this means that the non-Maxwellian part of the distribution function has to be accounted for to the accuracy of $f_{NM\iota} = O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota})$ and that the drift-kinetic equation which determines its gyrophase average, $\bar{f}_{NM\iota}$, has to be accurate to the second order in the ion Larmor radius. Proper second-order drift-kinetic equations have only been obtained recently[10,11] and such derivations pose a significant analytical challenge. The derivation of Ref.10 was carried out in the laboratory reference frame, whereas Ref.11 used the frame of the $\mathbf{E} \times \mathbf{B}$ drift. Yet another independent method will be used here, based on the reference frame of the complete macroscopic flow velocity, $\mathbf{u}_\iota$, and first devised to obtain a first-order drift-kinetic equation with sonic flows and far-from-Maxwellian distribution functions[12]. The ensuing second-order ion drift-kinetic equation is the main new result in this paper. It bears little resemblance to either of the expressions given in Refs.10-11, but its form is more compact and explicit, involving only conventional fluid and magnetic geometry variables and cylindrical velocity space coordinates with a simple Jacobian, which should facilitate its coupling to a fluid simulation code. Besides, the fact that its solution for $\bar{f}_{NM\iota}$ is obtained in the reference frame of the macroscopic flow, allows a direct evaluation of the gyrotropic closure variables (i.e. the moments of $\bar{f}_{NM\iota}$ such as the pressure anisotropy and the parallel heat flux) avoiding cumbersome and error prone substractions of mean flows. Most important, the present form of the second-order drift-kinetic equation makes possible the explicit proof given in Sec.V that the evolution of $\bar{f}_{NM\iota}$ preserves automatically the condition that its 1, $v_\| - u_{\iota\|}$ and $|\mathbf{v} - \mathbf{u}_\iota|^2$ moments remain always equal to zero.

The finite-Larmor-radius (FLR) ion fluid equations are given in Sec.II. There, because of the low collisionality ordering, the non-gyrotropic closure variables (i.e. the moments of the gyrophase-dependent part of the distribution function) are needed only in their collisionless limit. Hence, explicit expressions for these are available from earlier FLR collisionless fluid theory[13,14]. The only closures that require a kinetic evaluation are the gyrotropic moments of $\bar{f}_{NM\iota}$, which obeys the drift-kinetic equation derived in Sec.III. The collision operators are discussed in Sec.IV. Again, because of the low collisionality and close to Maxwellian orderings, they are needed only in their linearized version. Otherwise, complete Fokker-Planck-Landau[15] forms are used as was done in the case of the electrons[1].

II. Ion fluid equations.

In addition to the quasineutral Maxwell and continuity equations, the fluid part of the ion description includes the momentum conservation and temperature moment equations. Expanding with the presently adopted orderings (1) the $m_\iota \mathbf{v}$ moment of the ion Vlasov-Boltzmann equation (see. e.g. Ref.2) and retaining terms to $O(\delta^2 n m_\iota v_{th\iota}^2/L)$ while neglecting $O(\delta^3 n m_\iota v_{th\iota}^2/L)$, one gets

$$m_\iota n \left[\frac{\partial \mathbf{u}_\iota}{\partial t} + (\mathbf{u}_\iota \cdot \nabla)\mathbf{u}_\iota\right] - en(\mathbf{E} + \mathbf{u}_\iota \times \mathbf{B}) + \nabla(nT_\iota) + \nabla \cdot \left[(p_{\iota\|} - p_{\iota\perp})(\mathbf{bb} - \mathsf{I}/3) + \mathsf{P}_\iota^{GV}\right] = 0 \ , \quad (4)$$

where $p_{\iota\|}$ and $p_{\iota\perp}$ are the parallel and perpendicular pressures, $nT_\iota \equiv p_\iota \equiv (p_{\iota\|} + 2p_{\iota\perp})/3$ is the mean pressure, $\mathbf{b} \equiv \mathbf{B}/B$ is the magnetic unit vector, I is the identity tensor and P_ι^{GV} is the collision-independent gyroviscous stress tensor. The collisional friction force between ions and electrons and the non-gyrotropic collisional ion viscosity are of the order of $\delta^3 n m_\iota v_{th\iota}^2/L$, therefore neglected. After a similar expansion of the temperature moment equation, retaining terms to $O(\delta^2 n m_\iota v_{th\iota}^3/L)$ while neglecting $O(\delta^3 n m_\iota v_{th\iota}^3/L)$, one gets

$$\frac{3n}{2}\left(\frac{\partial T_\iota}{\partial t}+\mathbf{u}_\iota\cdot\nabla T_\iota\right)+nT_\iota\nabla\cdot\mathbf{u}_\iota+(p_{\iota\|}-p_{\iota\perp})(\mathbf{bb}-\mathsf{I}/3):(\nabla\mathbf{u}_\iota)+\nabla\cdot(q_{\iota\|}\mathbf{b}+\mathbf{q}_{\iota\perp})-G_\iota^{coll}=0\ , \quad (5)$$

where $q_{\iota\|}\mathbf{b}$ and $\mathbf{q}_{\iota\perp}$ are the parallel and perpendicular components of the heat flux and G_ι^{coll} is the collisional heat source that will later be shown to account for the temperature equilibration between ions and electrons, $G_\iota^{coll}=(2/\pi)^{1/2}\nu_e n(m_e/m_\iota)(T_e-T_\iota)=O(\delta^2 n m_\iota v_{th\iota}^3/L)$. The term $\mathsf{P}_\iota^{GV}:(\nabla\mathbf{u}_\iota)$ has been neglected because it is of the order of $\delta^3 n m_\iota v_{th\iota}^3/L$, but it could be reinstated if one wants to ensure an exact energy conservation law. The collisional part of the perpendicular heat flux is also negligible within the retained accuracy.

The non-gyrotropic closure terms in Eqs.(4-5), derivable from FLR fluid theory, are the collision-independent gyroviscosity and perpendicular heat flux. The appropriate form of the gyroviscous stress tensor is obtained by expanding the general result of Ref.14 for the present orderings (1), to the accuracy of $O(\delta^2 n m_\iota v_{th\iota}^2)$, which yields

$$\mathsf{P}_{\iota,jk}^{GV}\ =\ \frac{1}{4}\,\epsilon_{jlm}b_l\ \mathsf{K}_{\iota,mn}\left(\delta_{nk}+3b_nb_k\right)\ +\ (j\leftrightarrow k) \quad (6)$$

with

$$\mathsf{K}_{\iota,jk}\ =\ \frac{m_\iota}{eB}\left[nT_\iota\,\frac{\partial u_{\iota,k}}{\partial x_j}+\frac{\partial(q_{\iota T\|}b_k)}{\partial x_j}+(2q_{\iota B\|}-3q_{\iota T\|})\,b_j\kappa_k+\frac{\partial}{\partial x_j}\left(\frac{nT_\iota}{eB}\,\epsilon_{klm}\,b_l\,\frac{\partial T_\iota}{\partial x_m}\right)\right]\ +\ (j\leftrightarrow k). \quad (7)$$

Here, $q_{\iota B\|}$ and $q_{\iota T\|}$ are, respectively, the parallel fluxes of parallel and perpendicular heat (such that $q_{\iota\|}=q_{\iota B\|}+q_{\iota T\|}$) and $\boldsymbol{\kappa}$ is the magnetic curvature. This form of the gyroviscous stress extends the Braginskii form[16] for high collisionality and sonic flows, which corresponds to the first term in (7). It also extends the Mikhailowskii-Tsypin form[17,18] for high collisionality and diamagnetic flows, which corresponds to the limit $q_{\iota B\|}=3q_{\iota\|}/5,\ q_{\iota T\|}=2q_{\iota\|}/5$, such that the third term in (7) vanishes and the second and fourth terms combine into $(2/5)\partial q_{\iota,k}/\partial x_j$, where $q_{\iota,k}$ are the components of the first-order total ion heat flux. The appropriate form of the ion perpendicular heat flux to be used in (5) is similarly obtained by expanding the result of Ref.13 for the present orderings, keeping the accuracy of $O(\delta n m_\iota v_{th\iota}^3)+O(\delta^2 n m_\iota v_{th\iota}^3)$, which yields

$$\mathbf{q}_{\iota\perp}=\frac{\mathbf{b}}{eB}\times\left\{\frac{5}{2}nT_\iota\nabla T_\iota+\frac{5}{6}T_\iota\nabla(p_{\iota\|}-p_{\iota\perp})+T_\iota(p_{\iota\|}-p_{\iota\perp})\left[\frac{1}{3}\nabla\ln(nT_\iota)-\frac{5}{2}\boldsymbol{\kappa}\right]+\nabla\hat{r}_{\iota\perp}+(\hat{r}_{\iota\|}-\hat{r}_{\iota\perp})\boldsymbol{\kappa}\right\}. \quad (8)$$

Besides the familiar first term[16], this expression has the additional terms of the order of $\delta^2 n m_\iota v_{th\iota}^3$ which involve the non-Maxwellian even moments $(p_{\iota\|} - p_{\iota\perp})$, $\hat{r}_{\iota\|}$ and $\hat{r}_{\iota\perp}$, the last two being the fourth-rank gyrotropic moments whose definition is given below.

The task of closing the ion fluid system has now been left to the specification of a set of gyrotropic moments. These, along with their required accuracies, are:

$$(p_{\iota\|} - p_{\iota\perp}) \; = \; \frac{m_\iota}{2} \int d^3\mathbf{v} \; \left\{ 3[\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_\iota)]^2 - |\mathbf{v}-\mathbf{u}_\iota|^2 \right\} \; \bar{f}_{NM\iota} \; = \; O(\delta n m_\iota v_{th\iota}^2) \; + \; O(\delta^2 n m_\iota v_{th\iota}^2) \; , \quad (9)$$

$$q_{\iota\|} = \; \frac{m_\iota}{2} \int d^3\mathbf{v} \; [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_\iota)] \; |\mathbf{v}-\mathbf{u}_\iota|^2 \; \bar{f}_{NM\iota} \; = \; O(\delta n m_\iota v_{th\iota}^3) \; + \; O(\delta^2 n m_\iota v_{th\iota}^3) \; , \qquad (10)$$

$$q_{\iota B\|} = \; \frac{m_\iota}{2} \int d^3\mathbf{v} \; [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_\iota)]^3 \; \bar{f}_{NM\iota} \; = \; O(\delta n m_\iota v_{th\iota}^3) \; , \qquad (11)$$

$$q_{\iota T\|} \; = \; q_{\iota\|} \; - \; q_{\iota B\|} \; = \; O(\delta n m_\iota v_{th\iota}^3) \; , \qquad (12)$$

$$\hat{r}_{\iota\|} = \; \frac{m_\iota^2}{2} \int d^3\mathbf{v} \; [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_\iota)]^2 \; |\mathbf{v}-\mathbf{u}_\iota|^2 \; \bar{f}_{NM\iota} \; = \; O(\delta n m_\iota^2 v_{th\iota}^4) \qquad (13)$$

and

$$\hat{r}_{\iota\perp} = \; \frac{m_\iota^2}{4} \int d^3\mathbf{v} \; \left\{ |\mathbf{v}-\mathbf{u}_\iota|^2 - [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_\iota)]^2 \right\} \; |\mathbf{v}-\mathbf{u}_\iota|^2 \; \bar{f}_{NM\iota} \; = \; O(\delta n m_\iota^2 v_{th\iota}^4) \; . \qquad (14)$$

These can be extracted from a solution for the gyrophase average of the non-Maxwellian part of the ion distribution function in the reference frame of its macroscopic flow, correct to the accuracy of $\bar{f}_{NM\iota} \; = \; O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota})$. The drift-kinetic equation to provide such a solution will be derived in the next Section. Notice that $\hat{r}_{\iota\|}$ and $\hat{r}_{\iota\perp}$ are defined here as fourth-rank moments of the difference between the actual distribution function and the isotropic Maxwellian, whereas the variables $\tilde{r}_{\iota\|}$ and $\tilde{r}_{\iota\perp}$ of Ref.13 were defined as the corresponding moments of the difference between the actual distribution function and a two-temperature bi-Maxwellian. Accordingly, the relationships $\tilde{r}_{\iota\|} = \hat{r}_{\iota\|} - 7T_\iota(p_{\iota\|} - p_{\iota\perp})/3$ and $\tilde{r}_{\iota\perp} = \hat{r}_{\iota\perp} + 7T_\iota(p_{\iota\|} - p_{\iota\perp})/6$ must be used when deriving Eq.(8) from the results of Ref.13.

III. Second-order ion drift-kinetic equation.

The derivation of the second-order ion drift-kinetic equation in the reference frame of its mean flow will follow the recursive operator method introduced in Ref.12, adapted to the present low collisionality, slow dynamics and close to Maxwellian orderings. In the macroscopic flow reference frame, defined by the space-time dependent Galilean transformation from the laboratory frame

$$t = t \ , \qquad \mathbf{x} = \mathbf{x} \ , \qquad \mathbf{v} = \mathbf{v}' + \mathbf{u}_\iota(\mathbf{x}, t) \ , \tag{15}$$

the ion kinetic equation is[19,20]

$$\frac{\partial f_\iota(\mathbf{v}', \mathbf{x}, t)}{\partial t} + (\mathbf{v}' + \mathbf{u}_\iota) \cdot \frac{\partial f_\iota(\mathbf{v}', \mathbf{x}, t)}{\partial \mathbf{x}} + \left[\Omega_{c\iota} \mathbf{v}' \times \mathbf{b} + \frac{\mathbf{F}_\iota}{m_\iota n} - (\mathbf{v}' \cdot \nabla) \mathbf{u}_\iota \right] \cdot \frac{\partial f_\iota(\mathbf{v}', \mathbf{x}, t)}{\partial \mathbf{v}'} = \sum_{s=\iota,e} C_{\iota s}[f_\iota, f_s]. \tag{16}$$

Here, $C_{\iota s}[f_\iota, f_s]$ are the collision operators which will be discussed in the next Section, $\Omega_{c\iota} = eB/m_\iota$ is the ion cyclotron frequency and $\mathbf{F}_\iota$ is the force density

$$\mathbf{F}_\iota(\mathbf{x}, t) \ = \ en(\mathbf{E} + \mathbf{u}_\iota \times \mathbf{B}) \ - \ m_\iota n \left[\frac{\partial \mathbf{u}_\iota}{\partial t} + (\mathbf{u}_\iota \cdot \nabla) \mathbf{u}_\iota \right] \tag{17}$$

which combines the electric field force in the moving frame with an inertial force that arises from the transformation to such accelerating frame. Using the momentum conservation equation (4), it becomes simply

$$\mathbf{F}_\iota(\mathbf{x}, t) \ = \ \nabla \cdot \mathsf{P}_\iota \ = \ \nabla \cdot \left(\mathsf{P}_\iota^{CGL} + \mathsf{P}_\iota^{GV} \right) , \tag{18}$$

where P_ι is the full stress tensor made of the gyrotropic (Chew-Goldberger-Low) part

$$\mathsf{P}_\iota^{CGL} \ = \ (nT_\iota)\mathsf{I} + (p_{\iota\parallel} - p_{\iota\perp})(\mathbf{bb} - \mathsf{I}/3) \tag{19}$$

and the non-gyrotropic (gyroviscous) part P_ι^{GV} (6-7). So, in the favored reference frame of the ion macroscopic flow, an exact algebraic elimination of the electric field with the momentum equation takes place, after which only the divergence of the stress tensor remains. Then, carrying out the change of variables to cylindrical coordinate systems in velocity space locally aligned with the magnetic field in which $\mathbf{b}(\mathbf{x}, t)$, $\mathbf{e}_1(\mathbf{x}, t)$ and $\mathbf{e}_2(\mathbf{x}, t)$ form right-handed sets of mutually orthogonal unit vectors,

$$t = t \ , \qquad \mathbf{x} = \mathbf{x} \ , \qquad \mathbf{v}' = v'_\parallel \ \mathbf{b}(\mathbf{x}, t) \ + \ v'_\perp \ [\cos\alpha \ \mathbf{e}_1(\mathbf{x}, t) + \sin\alpha \ \mathbf{e}_2(\mathbf{x}, t)] \ , \tag{20}$$

the ion kinetic equation becomes of the form

$$\Omega_{c\iota} \frac{\partial f_\iota(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t)}{\partial \alpha} = \sum_{l=-2}^{2} e^{il\alpha} \left[\Lambda_l f_\iota + \lambda_l \frac{\partial f_\iota}{\partial \alpha} \right] - \sum_{s=\iota,e} C_{\iota s}[f_\iota, f_s] , \tag{21}$$

where $\Lambda_l(\partial/\partial v'_\parallel, \partial/\partial v'_\perp, \partial/\partial \mathbf{x}, \partial/\partial t, v'_\parallel, v'_\perp, \mathbf{x}, t) = \Lambda^*_{-l}$ are gyrophase-independent operators and $\lambda_l(v'_\parallel, v'_\perp, \mathbf{x}, t) = \lambda^*_{-l}$ are gyrophase-independent functions, whose complete expressions are given in Ref.12. Here they will be expanded according to the orderings (1) followed in this work as $\Lambda_l = \sum_j \Lambda_l^{(j)}$ and $\lambda_l = \sum_j \lambda_l^{(j)}$, with $\Lambda_l^{(j)} \sim \lambda_l^{(j)} \sim \delta^j v_{th\iota}/L$. The required terms are listed in Appendix A. Those expressions apply to general 3-dimensional magnetic geometry and, in them, Faraday's law has been substituted for the time derivative of the magnetic field, with the electric field eliminated algebraically with the momentum equation (4), so they also apply to fully electromagnetic dynamics.

Introducing the Fourier series representation in harmonics of the gyrophase,

$$f_\iota(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t) = \sum_{l=-\infty}^{\infty} e^{il\alpha} f_{\iota,l}(v'_\parallel, v'_\perp, \mathbf{x}, t) , \tag{22}$$

equation (21) yields

$$il\Omega_{c\iota} f_{\iota,l} = \sum_{l'=-2}^{2} \left[\Lambda_{l'} f_{\iota,l-l'} + i(l-l')\lambda_{l'} f_{\iota,l-l'} \right] - \sum_{s=\iota,e} \langle e^{-il\alpha} C_{\iota s}[f_\iota, f_s] \rangle_\alpha \tag{23}$$

where the shorthand notation for the gyrophase average, $\langle ... \rangle_\alpha \equiv (2\pi)^{-1} \oint d\alpha (...)$, has been used. In the adopted asymptotic ordering scheme this system admits a recursive solution which, with the desired second-order accuracy, has the form

$$f_{\iota,0} = f_{M\iota} + \bar{f}_{NM\iota} = f_{M\iota} + O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota}) + ... ,$$

$$f_{\iota,\pm 1} = O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota}) + ... , \qquad f_{\iota,\pm 2} = O(\delta^2 f_{M\iota}) + ... , \qquad ... , \tag{24}$$

the ellipses indicating terms that need not be retained. Neglecting such unnecessary higher-order terms, the $l = 0, 1, 2$ components of the system (23) yield

$$2\Re \left\{ \left(\Lambda_{-2}^{(0)} + 2i\lambda_{-2}^{(0)} \right) f_{\iota,2} + \left[\Lambda_{-1}^{(0)} + \Lambda_{-1}^{(1)} + i \left(\lambda_{-1}^{(0)} + \lambda_{-1}^{(1)} \right) \right] f_{\iota,1} \right\} +$$

$$+ \left(\Lambda_0^{(0)} + \Lambda_0^{(1)} \right) (f_{M\iota} + \bar{f}_{NM\iota}) + \Lambda_0^{(2)} f_{M\iota} = \sum_{s=\iota,e} \langle C_{\iota s}^{(2)}[f_\iota, f_s] \rangle_\alpha \tag{25}$$

where $\Re$ indicates the real part,

$$f_{\iota,1} = \frac{1}{i\Omega_{c\iota}}\left[\left(\Lambda_0^{(0)} + i\lambda_0^{(0)}\right)\left(\frac{1}{i\Omega_{c\iota}}\Lambda_1^{(0)} f_{M\iota}\right) + \right.$$

$$\left. +\Lambda_1^{(0)}(f_{M\iota} + \bar{f}_{NM\iota}) + \Lambda_1^{(1)} f_{M\iota} + \left(\Lambda_2^{(0)} - i\lambda_2^{(0)}\right)\left(\frac{i}{\Omega_{c\iota}}\Lambda_{-1}^{(0)} f_{M\iota}\right)\right] \tag{26}$$

and

$$f_{\iota,2} = \frac{1}{2i\Omega_{c\iota}}\left[\left(\Lambda_1^{(0)} + i\lambda_1^{(0)}\right)\left(\frac{1}{i\Omega_{c\iota}}\Lambda_1^{(0)} f_{M\iota}\right) + \Lambda_2^{(0)}\bar{f}_{NM\iota} + \Lambda_2^{(1)} f_{M\iota}\right] , \tag{27}$$

the latter reflecting the property that $\Lambda_2^{(0)} f_{M\iota} = 0$. The collision operators are needed only in their lowest non-vanishing order, $C_{\iota s}^{(2)}[f_\iota, f_s] \sim \delta^2(v_{th\iota}/L) f_{M\iota}$ and they matter only in Eq.(25). In that equation, $\Lambda_0^{(0)} f_{M\iota} = (v'_\parallel/2)(v'^2/v_{th\iota}^2 - 3)(\mathbf{b}\cdot\nabla \ln T_\iota) f_{M\iota}$ and the small parallel temperature gradient ordering $\mathbf{b}\cdot\nabla \ln T_\iota \sim \delta/L$ guarantees that there are no unbalanced terms of the order of $(v_{th\iota}/L) f_{M\iota}$.

The time derivative of the Maxwellian appears in the $\Lambda_0^{(1)} f_{M\iota}$ term of (25) which, after differentiating $f_{M\iota}$, becomes

$$\Lambda_0^{(1)} f_{M\iota} \;=\; \left\{\left(\frac{\partial}{\partial t} + \mathbf{u}_\iota\cdot\nabla\right)\ln n \;+\; \frac{1}{2}\left(\frac{v'^2}{v_{th\iota}^2} - 3\right)\left(\frac{\partial}{\partial t} + \mathbf{u}_\iota\cdot\nabla\right)\ln T_\iota \;-\; v'_\parallel\,\mathbf{b}\cdot\ln T_\iota \;- \right.$$

$$\left. -\;\frac{v'_\parallel}{nT_\iota}\,\mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel} - p_{\iota\perp}) - \left(p_{\iota\parallel} - p_{\iota\perp}\right)\nabla\ln B\right] \;+\; \frac{v'^2_\perp}{2v_{th\iota}^2}\,\nabla\cdot\mathbf{u}_\iota \;+\; \frac{2v'^2_\parallel - v'^2_\perp}{2v_{th\iota}^2}\,(\mathbf{bb}):(\nabla\mathbf{u}_\iota)\right\} f_{M\iota}\,. \tag{28}$$

Substituting the continuity equation $\partial n/\partial t + \nabla\cdot(n\mathbf{u}_\iota) = 0$ and the temperature evolution Eq.(5) for the time derivatives of the density and the temperature, one gets the expression

$$\Lambda_0^{(1)} f_{M\iota} \;=\; \hat{\Lambda}_0^{(1)} f_{M\iota} \;+\; \frac{1}{3nT_\iota}\left(\frac{v'^2}{v_{th\iota}^2} - 3\right) G_\iota^{coll} f_{M\iota} \tag{29}$$

where the collisional term has been singled out and $\hat{\Lambda}_0^{(1)} f_{M\iota}$ contains the terms that do not depend explicitly on the collisions:

$$\hat{\Lambda}_0^{(1)} f_{M\iota} \;=\; \left\{-\frac{1}{3nT_\iota}\left(\frac{v'^2}{v_{th\iota}^2} - 3\right)\left[(p_{\iota\parallel} - p_{\iota\perp})(\mathbf{bb} - \mathbf{I}/3):(\nabla\mathbf{u}_\iota) + \nabla\cdot(q_{\iota\parallel}\mathbf{b} + \mathbf{q}_{\iota\perp})\right] -\; v'_\parallel\,\mathbf{b}\cdot\ln T_\iota \;- \right.$$

$$\left. -\;\frac{v'_\parallel}{nT_\iota}\,\mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel} - p_{\iota\perp}) - \left(p_{\iota\parallel} - p_{\iota\perp}\right)\nabla\ln B\right] \;+\; \frac{2v'^2_\parallel - v'^2_\perp}{2v_{th\iota}^2}\,(\mathbf{bb} - \mathbf{I}/3):(\nabla\mathbf{u}_\iota)\right\} f_{M\iota}\,. \tag{30}$$

Now, defining

$$\mathcal{Q}_\iota^{coll} \;\equiv\; \sum_{s=\iota,e} \langle C^{(2)}_{\iota s}[f_\iota, f_s]\rangle_\alpha \;-\; \frac{1}{3nT_\iota}\left(\frac{v'^2}{v^2_{th\iota}} - 3\right) G_\iota^{coll} f_{M\iota}\ , \tag{31}$$

equation (25) can be rewritten as

$$2\Re\left\{\left(\Lambda^{(0)}_{-2} + 2i\lambda^{(0)}_{-2}\right) f_{\iota,2} + \left[\Lambda^{(0)}_{-1} + \Lambda^{(1)}_{-1} + i\left(\lambda^{(0)}_{-1} + \lambda^{(1)}_{-1}\right)\right] f_{\iota,1}\right\} +$$

$$+\left(\Lambda^{(0)}_0 + \Lambda^{(1)}_0\right) \bar{f}_{NM\iota} + \left(\Lambda^{(0)}_0 + \hat{\Lambda}^{(1)}_0 + \Lambda^{(2)}_0\right) f_{M\iota} \;=\; \mathcal{Q}_\iota^{coll}\ . \tag{32}$$

The second-order drift-kinetic equation for $\bar{f}_{NM\iota}$ is obtained by substituting in (32) the solutions (26-27) for $f_{\iota,1}$ and $f_{\iota,2}$. It can be expressed in the form

$$\frac{d_\iota \bar{f}_{NM\iota}}{dt} \;=\; D_\iota f_{M\iota} \;+\; \mathcal{Q}_\iota^{coll} \tag{33}$$

where the collision-independent streaming operator acting on $\bar{f}_{NM\iota}$ is

$$\frac{d_\iota}{dt} \;=\; 2\Re\left[\left(\Lambda^{(0)}_{-2} + 2i\lambda^{(0)}_{-2}\right)\frac{1}{2i\Omega_{c\iota}}\Lambda^{(0)}_2 + \left(\Lambda^{(0)}_{-1} + i\lambda^{(0)}_{-1}\right)\frac{1}{i\Omega_{c\iota}}\Lambda^{(0)}_1\right] + \Lambda^{(0)}_0 + \Lambda^{(1)}_0 \tag{34}$$

and the action of the collision-independent streaming on the Maxwellian has been moved to the right-hand-side as the driving term

$$D_\iota f_{M\iota} \;=\; -2\Re\Bigg\{\left(\Lambda^{(0)}_{-2} + 2i\lambda^{(0)}_{-2}\right)\frac{1}{2i\Omega_{c\iota}}\left[\left(\Lambda^{(0)}_1 + i\lambda^{(0)}_1\right)\frac{1}{i\Omega_{c\iota}}\Lambda^{(0)}_1 + \Lambda^{(1)}_2\right] +$$

$$+\left[\Lambda^{(0)}_{-1} + \Lambda^{(1)}_{-1} + i\left(\lambda^{(0)}_{-1} + \lambda^{(1)}_{-1}\right)\right]\frac{1}{i\Omega_{c\iota}}\Bigg[\left(\Lambda^{(0)}_0 + i\lambda^{(0)}_0\right)\frac{1}{i\Omega_{c\iota}}\Lambda^{(0)}_1 + \Lambda^{(0)}_1 + \Lambda^{(1)}_1 +$$

$$+\left(\Lambda^{(0)}_2 - i\lambda^{(0)}_2\right)\frac{i}{\Omega_{c\iota}}\Lambda^{(0)}_{-1}\Bigg]\Bigg\} f_{M\iota} - \left(\Lambda^{(0)}_0 + \hat{\Lambda}^{(1)}_0 + \Lambda^{(2)}_0\right) f_{M\iota}\ . \tag{35}$$

It is now a matter of straightforward if somewhat lengthy algebra to work out the explicit forms of the operator d_ι/dt (34) and the function D_ι (35), using Eq.(30) and the expressions for $\Lambda^{(j)}_l$ and $\lambda^{(j)}_l$ given in Appendix A. These depend on the auxiliary unit vectors $\mathbf{e}_1(\mathbf{x}, t)$ and $\mathbf{e}_2(\mathbf{x}, t)$ that establish the origin of the gyrophase (20), but the final result is independent of them and involves only the intrinsic geometry of the magnetic field. Since $\bar{f}_{NM\iota} = O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota})$, the operator acting on it (34) retains only the first-order accuracy, $d_\iota/dt = O(v_{th\iota}/L) + O(\delta v_{th\iota}/L)$. So, Eq.(34) is just a special case of the general first-order result of Ref.12, namely its slow flow and close to Maxwellian limit:

$$\frac{d_\iota}{dt} \;=\; \frac{\partial}{\partial t} \;+\; \dot{\mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{x}} \;+\; \dot{v}'_\parallel\,\frac{\partial}{\partial v'_\parallel} \;+\; \dot{v}'_\perp\,\frac{\partial}{\partial v'_\perp} \tag{36}$$

where the coefficient functions are

$$\dot{\mathbf{x}} \;=\; v'_\parallel \mathbf{b} \;+\; \mathbf{u}_\iota - \mathbf{u}_{D\iota} \;+\; \frac{v'^2_\perp}{2}\nabla\times\left(\frac{\mathbf{b}}{\Omega_{c\iota}}\right) \;+\; \left(v'^2_\parallel - \frac{v'^2_\perp}{2}\right)\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_{c\iota}} \tag{37}$$

$$\dot{v}'_\parallel \;=\; \frac{\mathbf{b}\cdot(\nabla\cdot\mathsf{P}^{CGL}_\iota)}{m_\iota n} \;-\; \frac{v'^2_\perp}{2}\mathbf{b}\cdot\nabla\ln B \;-\; v'_\parallel(\mathbf{bb}):[\nabla(\mathbf{u}_\iota - \mathbf{u}_{D\iota})] \;+\; \frac{v'_\parallel v'^2_\perp}{2}\nabla\cdot\left(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_{c\iota}}\right) \tag{38}$$

and

$$\dot{v}'_\perp = \frac{v'_\perp}{2}\;\left\{v'_\parallel\mathbf{b}\cdot\nabla\ln B \;+\; (\mathbf{bb}-\mathsf{I}):[\nabla(\mathbf{u}_\iota - \mathbf{u}_{D\iota})] \;-\; v'^2_\parallel\nabla\cdot\left(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_{c\iota}}\right)\right\} \tag{39}$$

with $\mathbf{u}_{D\iota} = \mathbf{b}\times\nabla(nT_\iota)/(m_\iota n\Omega_{c\iota})$, the lowest-order diamagnetic drift velocity. It is immediately verified that (37-39) fulfill the phase-space volume conservation condition

$$\frac{\partial}{\partial \mathbf{x}}\cdot\dot{\mathbf{x}} \;+\; \frac{\partial \dot{v}'_\parallel}{\partial v'_\parallel} \;+\; \frac{1}{v'_\perp}\frac{\partial(v'_\perp\dot{v}'_\perp)}{\partial v'_\perp} \;=\; 0\;, \tag{40}$$

so the phase-space advection of $\bar{f}_{NM\iota}$ can be expressed in Liouville theorem form:

$$\frac{d_\iota\bar{f}_{NM\iota}}{dt} \;=\; \frac{\partial\bar{f}_{NM\iota}}{\partial t} \;+\; \frac{\partial}{\partial\mathbf{x}}\cdot(\bar{f}_{NM\iota}\dot{\mathbf{x}}) \;+\; \frac{\partial}{\partial v'_\parallel}\left(\bar{f}_{NM\iota}\dot{v}'_\parallel\right) \;+\; \frac{1}{v'_\perp}\,\frac{\partial}{\partial v'_\perp}\,(v'_\perp\bar{f}_{NM\iota}\dot{v}'_\perp)\;. \tag{41}$$

Turning now to the collision-independent driving term, the result from (35) has the desired accuracy of $D_\iota = O(\delta v_{th\iota}/L) + O(\delta^2 v_{th\iota}/L)$. It is convenient to write $D_\iota = D^{even}_\iota + D^{odd}_\iota$, splitting it into its even and odd parts with respect to $v'_\parallel$. The even part is

$$D^{even}_\iota \;=\; \frac{(2v'^2_\parallel - v'^2_\perp)}{2v^2_{th\iota}}(\mathsf{I}/3 - \mathbf{bb}):(\nabla\mathbf{u}_\iota) \;+\; \frac{1}{3nT_\iota}\left(\frac{v'^2}{v^2_{th\iota}} - 3\right)\nabla\cdot(q_{\iota\parallel}\mathbf{b}) \;-$$

$$
-\frac{(2v_\parallel'^2-v_\perp'^2)}{6\Omega_{c\iota}T_\iota}(\mathbf{b}\times\nabla\ln n)\cdot\nabla T_\iota \;-\; \frac{1}{2m_\iota\Omega_{c\iota}}\left[\frac{(v_\parallel'^4+v_\parallel'^2v_\perp'^2)}{v_{th\iota}^4} \;-\; \frac{5(4v_\parallel'^2+v_\perp'^2)}{3v_{th\iota}^2} \;+\; 5\right](\mathbf{b}\times\boldsymbol{\kappa})\cdot\nabla T_\iota \;-
$$

$$
-\;\frac{1}{2m_\iota\Omega_{c\iota}}\left[\frac{(v_\parallel'^2v_\perp'^2+v_\perp'^4)}{2v_{th\iota}^4} \;-\; \frac{5(2v_\parallel'^2+5v_\perp'^2)}{6v_{th\iota}^2} \;+\; 5\right](\mathbf{b}\times\nabla\ln B)\cdot\nabla T_\iota \;+
$$

$$
+\;\frac{1}{6n}\left[\frac{(5v_\parallel'^2+2v_\perp'^2)}{v_{th\iota}^2}-15\right]\;\nabla\cdot\left\{\frac{\mathbf{b}}{m_\iota\Omega_{c\iota}}\times\left[\frac{1}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})-(p_{\iota\parallel}-p_{\iota\perp})\boldsymbol{\kappa}\right]\right\}\;+
$$

$$
+\frac{1}{3nT_\iota}\left(\frac{v'^2}{v_{th\iota}^2}-3\right)\nabla\cdot\left\{\frac{\mathbf{b}}{m_\iota\Omega_{c\iota}}\times\left[\nabla\hat{r}_{\iota\perp}+(\hat{r}_{\iota\parallel}-\hat{r}_{\iota\perp})\boldsymbol{\kappa}\right]\right\}+\frac{(2v_\parallel'^2-v_\perp'^2)}{6\Omega_{c\iota}T_\iota}\left[\mathbf{b}\times\left(\frac{1}{3}\nabla\ln n-\boldsymbol{\kappa}\right)\right]\cdot\nabla\left(\frac{p_{\iota\parallel}-p_{\iota\perp}}{n}\right)+
$$

$$
+\;\frac{1}{3nT_\iota}\left(\frac{v'^2}{v_{th\iota}^2}-3\right)(p_{\iota\parallel}-p_{\iota\perp})(\mathbf{bb}-\mathbf{I}/3):\left[\nabla(\mathbf{u}_\iota-\mathbf{u}_{D\iota})\right] \tag{42}
$$

and the odd part is

$$
D_\iota^{odd} \;=\; \frac{v_\parallel'}{2}\left(5-\frac{v'^2}{v_{th\iota}^2}\right)\mathbf{b}\cdot\nabla\ln T_\iota \;+\; \frac{v_\parallel'}{nT_\iota}\;\mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})-(p_{\iota\parallel}-p_{\iota\perp})\nabla\ln B\right]\;+
$$

$$
+\;\frac{v_\parallel'}{nT_\iota}\;\mathbf{b}\cdot\left(\nabla\cdot\mathbf{P}_\iota^{GV}\right)\;-\;\frac{v_\parallel'v_\perp'^2}{2nT_\iota v_{th\iota}^2}\;\nabla\cdot\left\{\frac{nT_\iota}{\Omega_{c\iota}}\;\mathbf{b}\times\left[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+\mathbf{b}\times(\nabla\times\mathbf{u}_\iota)\right]\right\}\;-
$$

$$
-\;\frac{v_\parallel'}{\Omega_{c\iota}v_{th\iota}^2}\left\{(v_\parallel'^2-v_\perp'^2)\;(\mathbf{b}\times\boldsymbol{\kappa})\cdot\left[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+\mathbf{b}\times(\nabla\times\mathbf{u}_\iota)\right]\;+\;\frac{v_\perp'^2}{4}\;\mathbf{M}_\times:(\nabla\mathbf{u}_\iota)\right\}\;+
$$

$$
+\frac{v_\parallel'}{\Omega_{c\iota}}\left\{\left(1-\frac{v_\perp'^2}{2v_{th\iota}^2}\right)\left[\mathbf{b}\times\nabla\ln(nT_\iota)\right]\cdot\left[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+\mathbf{b}\times(\nabla\times\mathbf{u}_\iota)\right]+\left(\frac{v'^2}{v_{th\iota}^2}-5\right)(\mathbf{b}\times\nabla\ln T_\iota)\cdot\left[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\right]\right\}-
$$

$$
-\;\frac{v_\parallel'v_\perp'^2}{2nT_\iota^2}\;\nabla\cdot\left\{\frac{nT_\iota}{\Omega_{c\iota}^2}\;\mathbf{b}\times\left[(\mathbf{b}\cdot\nabla\ln n)\mathbf{b}\times\nabla T_\iota\right]\right\}\;+\;\frac{v_\parallel'v_\perp'^2}{4nT_\iota^2}\left(\frac{v'^2}{v_{th\iota}^2}-5\right)\nabla\cdot\left(\frac{nT_\iota\tau}{\Omega_{c\iota}^2}\;\mathbf{b}\times\nabla T_\iota\right)\;+
$$

$$
+\frac{v_\parallel'}{\Omega_{c\iota}^2}\left\{(\mathbf{b}\cdot\nabla\ln n)\left[(v_\perp'^2-v_\parallel'^2)\mathbf{b}\times\boldsymbol{\kappa}+\left(v_{th\iota}^2-\frac{v_\perp'^2}{2}\right)\mathbf{b}\times\nabla\ln n+\left(\frac{v_\perp'^2v'^2}{4v_{th\iota}^2}+\frac{v_\parallel'^2}{2}-\frac{7v_\perp'^2}{4}-\frac{3v_{th\iota}^2}{2}\right)\mathbf{b}\times\nabla\ln T_\iota\right]-\right.
$$

$$
\left.-\;\frac{\tau}{2}\left[\left(\frac{v'^2}{v_{th\iota}^2}-5\right)(v_\parallel'^2-v_\perp'^2)\boldsymbol{\kappa}+\left(\frac{v_\perp'^2v'^2}{2v_{th\iota}^2}-v_\parallel'^2-\frac{9v_\perp'^2}{2}+5v_{th\iota}^2\right)\nabla\ln n\right]\right\}\cdot(\mathbf{b}\times\nabla\ln T_\iota)\;-
$$

$$
-\frac{v_\parallel'v_\perp'^2}{8\Omega_{c\iota}^2}\mathbf{M}:\left\{\left(\frac{v'^2}{v_{th\iota}^2}-5\right)\frac{\Omega_{c\iota}}{nT_\iota^2}\nabla\left(\frac{nT_\iota}{\Omega_{c\iota}}\nabla T_\iota\right)+\left[\left(\frac{v'^4}{2v_{th\iota}^4}-\frac{8v'^2}{v_{th\iota}^2}+\frac{49}{2}\right)\nabla\ln T_\iota-\left(\frac{v'^2}{v_{th\iota}^2}-7\right)\nabla\ln n\right]\nabla\ln T_\iota\right\}. \tag{43}
$$

These explicit forms of the drift-kinetic coefficient functions (37-39, 42-43) involve only the cylindrical random velocity coordinates $v'_\parallel$ and $v'_\perp$ (with $v'^2 = v'^2_\parallel + v'^2_\perp$), the conventional fluid variables of Sec.II and the magnetic field geometry. So, they are well suited to be coupled to the fluid part of the system in a simulation code. The variables that characterize the intrinsic magnetic geometry are the two scalars $\nabla \cdot \mathbf{b} = -\mathbf{b} \cdot \nabla \ln B$ and $\tau \equiv \mathbf{b} \cdot (\nabla \times \mathbf{b})$, the curvature vector $\boldsymbol{\kappa} \equiv (\mathbf{b} \cdot \nabla)\mathbf{b}$ and the second-rank symmetric tensor M which is the traceless and perpendicular (in the sense that $\mathbf{b} \cdot \mathsf{M} = 0$) projection of the symmetrized $\nabla \mathbf{b}$ tensor. Its Cartesian component representation is given in Appendix B, along with that of the associated ("crossed with $\mathbf{b}$") tensor $\mathsf{M}_\times$. As the consequence of including the contribution of the gyroviscosity to the parallel electric field, Eq.(43) contains the term $\mathbf{b} \cdot (\nabla \cdot \mathsf{P}_\iota^{GV})$ whose explicit form based on the tensors (6-7) is also given in Appendix B. The final form of Eq.(42) includes the contribution of the second-order perpendicular heat flux (8), that was substituted in $\hat{\Lambda}_0^{(1)}$ (30). All the second-order terms in D_ι^{even} (42) and a good part of those in D_ι^{odd} (43) are so because of the slow flow and near-Maxwellian orderings, but they are only zeroth-order or first-order in the Larmor radius as evidenced by their inverse powers of $\Omega_{c\iota}$. Hence, they are derivable as a special limit of the result of Ref.12 that applies to fast flows, far-from-Maxwellian distribution functions and first-order in the Larmor radius. The only terms that require the more difficult, truly second-order analysis in the Larmor radius are those, in D_ι^{odd}, inversely proportional to $\Omega_{c\iota}^2$ and their final expression appears remarkably compact and transparent here.

IV. Collisional terms.

To complete the theory, one needs to evaluate the collisional source $\mathcal{Q}_\iota^{coll}$ (31) in the drift-kinetic Eq.(33), including the moment G_ι^{coll} that also enters as a source in the fluid temperature Eq.(5). Since the collision operators are needed only in their lowest non-vanishing order $C_{\iota s}^{(2)}[f_\iota, f_s] \sim \delta^2 (v_{th\iota}/L) f_{M\iota}$ for the present low collisionality, small mass ratio and close to Maxwellian asymptotics, the ion-ion operator can be linearized and the ion-electron operator needs to keep only the contribution of the lowest-order Maxwellians, with comparable but distinct temperatures:

$$\sum_{s=\iota,e} \langle C_{\iota s}^{(2)}[f_\iota, f_s] \rangle_\alpha \;=\; \langle C_{\iota\iota}[f_{M\iota}, f_{NM\iota}] + C_{\iota\iota}[f_{NM\iota}, f_{M\iota}] \rangle_\alpha \;+\; \langle C_{\iota e}^{(2)}[f_{M\iota}, f_{Me}] \rangle_\alpha \; . \tag{44}$$

Also, the mean flow difference between species $|\mathbf{u}_\iota - \mathbf{u}_e| = O(\delta v_{th\iota})$ can be neglected in the required lowest-order form of the ion-electron operator. Then, by virtue of their Galilean invariance, the laboratory frame expressions of the Fokker-Planck-Landau collision operators can be trivially translated to the moving reference frame this work uses. Finally, the small electron to ion mass ratio allows a Taylor expansion of the electron Maxwellian for $v'/v_{the} \sim v_{th\iota}/v_{the} \ll 1$. Taking all this into account, the following lowest-order form of the ion-electron collision operator is obtained:

$$C^{(2)}_{\iota e}[f_{M\iota}, f_{Me}] \;=\; \frac{2\nu_e m_e}{3(2\pi)^{1/2} m_\iota}\left(\frac{T_e}{T_\iota} - 1\right)\left(\frac{v'^2}{v^2_{th\iota}} - 3\right) f_{M\iota} \;=\; O\left(\delta^2 \frac{v_{th\iota}}{L} f_{M\iota}\right) \; . \tag{45}$$

By definition, the collisional heat source in the fluid temperature equation is

$$G^{coll}_\iota \;\equiv\; \frac{m_\iota}{2}\int d^3\mathbf{v}'\; v'^2\; C_{\iota e}[f_\iota, f_e] \tag{46}$$

and, evaluating it with the lowest-order expression (45), one gets

$$G^{coll}_\iota \;=\; \frac{2\nu_e n m_e}{(2\pi)^{1/2} m_\iota}(T_e - T_\iota) \tag{47}$$

so that

$$C^{(2)}_{\iota e}[f_{M\iota}, f_{Me}] \;=\; \frac{G^{coll}_\iota}{3nT_\iota}\left(\frac{v'^2}{v^2_{th\iota}} - 3\right) f_{M\iota} \; . \tag{48}$$

Therefore, in the definition of $\mathcal{Q}^{coll}_\iota$ (31), the terms $\langle C^{(2)}_{\iota e}[f_\iota, f_e]\rangle_\alpha$ and $(G^{coll}_\iota/3nT_\iota)(v'^2/v^2_{th\iota} - 3)f_{M\iota}$, which should in principle cancel only in their 1 and v'^2 moments, cancel completely within the presently required accuracy and so $\mathcal{Q}^{coll}_\iota$ reduces to

$$\mathcal{Q}^{coll}_\iota \;=\; \langle C_{\iota\iota}[f_{M\iota}, f_{NM\iota}] + C_{\iota\iota}[f_{NM\iota}, f_{M\iota}]\rangle_\alpha \; . \tag{49}$$

Here, the standard linearized Fokker-Planck-Landau collision operator for like particles[15,21,22] (also reviewed in detail in Ref.1 for the electrons) is to be used. From its particle, momentum and energy conservation properties, it follows that

$$\int d^3\mathbf{v}'\; (1, v'_\parallel, v'^2)\; \mathcal{Q}^{coll}_\iota \;=\; 0 \; . \tag{50}$$

V. Moments of the ion drift-kinetic equation.

From the definition of $\mathbf{v}'$ as relative to the ion mean flow and the adopted Chapman-Enskog-like representation whereby the macroscopic density, flow velocity and temperature are carried entirely by the Maxwellian, the gyrophase average of the non-Maxwellian distribution function perturbation must satisfy the conditions

$$\int d^3\mathbf{v}' \ (1, v'_\parallel, v'^2) \ \bar{f}_{NM\iota} \ = \ 0 \ . \tag{51}$$

It is important to prove explicitly that the dynamical evolution of $\bar{f}_{NM\iota}$ preserves these conditions, so that the consistency of the hybrid fluid and drift-kinetic system is guaranteed.

Assuming (51) to hold, bringing the expressions (37-39) for $\dot{\mathbf{x}}$, $\dot{v}'_\parallel$, $\dot{v}'_\perp$ to the Liouville theorem form (41), integrating by parts and using the definitions (9-14) of the gyrotropic moments, one obtains

$$\int d^3\mathbf{v}' \ \frac{d\bar{f}_{NM\iota}}{dt} \ = \ \nabla\cdot\left\{\frac{\mathbf{b}}{m_\iota\Omega_{c\iota}}\times\left[-\frac{1}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})+(p_{\iota\parallel}-p_{\iota\perp})\boldsymbol{\kappa}\right]\right\} \ , \tag{52}$$

$$m_\iota\int d^3\mathbf{v}' \ v'_\parallel \ \frac{d\bar{f}_{NM\iota}}{dt} \ = \ \mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})-(p_{\iota\parallel}-p_{\iota\perp})\nabla\ln B\right] \ +$$
$$+ \ \nabla\cdot\left\{\frac{\mathbf{b}}{\Omega_{c\iota}}\times\left[\nabla q_{\iota T\parallel}+2(q_{\iota B\parallel}-q_{\iota T\parallel})\boldsymbol{\kappa}\right]\right\} \ + \ \left(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_{c\iota}}\right)\cdot\nabla q_{\iota T\parallel} \tag{53}$$

and

$$\frac{m_\iota}{2}\int d^3\mathbf{v}' \ v'^2 \ \frac{d\bar{f}_{NM\iota}}{dt} \ = \ \nabla\cdot(q_{\iota\parallel}\mathbf{b}) \ + \ \nabla\cdot\left\{\frac{\mathbf{b}}{m_\iota\Omega_{c\iota}}\times\left[\nabla\hat{r}_{\iota\perp}+(\hat{r}_{\iota\parallel}-\hat{r}_{\iota\perp})\boldsymbol{\kappa}\right]\right\} \ +$$
$$+ \ (p_{\iota\parallel}-p_{\iota\perp})(\mathbf{bb}-\mathbf{I}/3):[\nabla(\mathbf{u}_\iota-\mathbf{u}_{D\iota})] \ . \tag{54}$$

As shown in the previous Section, the 1, $v'_\parallel$ and v'^2 moments of $\mathcal{Q}_\iota^{coll}$ vanish. The corresponding moments of $D_\iota f_{M\iota}$, evaluated after Eqs.(42-43), are

$$\int d^3\mathbf{v}' \ D_\iota^{even} \ f_{M\iota} \ = \ \nabla\cdot\left\{\frac{\mathbf{b}}{m_\iota\Omega_{c\iota}}\times\left[-\frac{1}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})+(p_{\iota\parallel}-p_{\iota\perp})\boldsymbol{\kappa}\right]\right\} \ , \tag{55}$$

$$m_\iota \int d^3\mathbf{v}'\ v'_\parallel\ D_\iota^{odd}\ f_{M\iota}\ =\ \mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})-(p_{\iota\parallel}-p_{\iota\perp})\nabla\ln B\right]\ +$$

$$+\ \mathbf{b}\cdot\left(\nabla\cdot\mathsf{P}_\iota^{GV}\right)\ -\ \nabla\cdot\left\{\frac{nT_\iota}{\Omega_{c\iota}}\ \mathbf{b}\times[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+\mathbf{b}\times(\nabla\times\mathbf{u}_\iota)]\right\}\ -$$

$$-\ \frac{nT_\iota}{\Omega_{c\iota}}\left\{(\mathbf{b}\times\boldsymbol{\kappa})\cdot[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+\mathbf{b}\times(\nabla\times\mathbf{u}_\iota)]+\ \frac{1}{2}\ \mathsf{M}_\times:(\nabla\mathbf{u}_\iota)\right\}\ -\ \frac{1}{2m_\iota\Omega_{c\iota}}\mathsf{M}:\left[\nabla\left(\frac{nT_\iota}{\Omega_{c\iota}}\nabla T_\iota\right)\right]\ -$$

$$-\ \nabla\cdot\left\{\frac{nT_\iota}{\Omega_{c\iota}^2}\ \mathbf{b}\times[(\mathbf{b}\cdot\nabla\ln n)\mathbf{b}\times\nabla T_\iota-\tau\nabla T_\iota]\right\}\ -\ \frac{nT_\iota}{\Omega_{c\iota}^2}\ (\mathbf{b}\times\boldsymbol{\kappa})\cdot[(\mathbf{b}\cdot\nabla\ln n)\mathbf{b}\times\nabla T_\iota-\tau\nabla T_\iota] \quad (56)$$

and

$$\frac{m_\iota}{2}\int d^3\mathbf{v}'\ v'^2\ D_\iota^{even}\ f_{M\iota}\ =\ \nabla\cdot(q_{\iota\parallel}\mathbf{b})\ +\ \nabla\cdot\left\{\frac{\mathbf{b}}{m_\iota\Omega_{c\iota}}\times\left[\nabla\hat{r}_{\iota\perp}+(\hat{r}_{\iota\parallel}-\hat{r}_{\iota\perp})\boldsymbol{\kappa}\right]\right\}\ +$$

$$+\ (p_{\iota\parallel}-p_{\iota\perp})(\mathbf{bb}-\mathsf{I}/3):[\nabla(\mathbf{u}_\iota-\mathbf{u}_{D\iota})]\ . \quad (57)$$

One can see that the right hand sides of Eqs.(55) and (57) are respectively identical to those of Eqs.(52) and (54). Equation (56) has the parallel gyroviscous force term $\mathbf{b}\cdot(\nabla\cdot\mathsf{P}_\iota^{GV})$. After substituting the explicit result given in Eq.(B.17), it becomes

$$m_\iota \int d^3\mathbf{v}'\ v'_\parallel\ D_\iota^{odd}\ f_{M\iota}\ =\ \mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})-(p_{\iota\parallel}-p_{\iota\perp})\nabla\ln B\right]\ +$$

$$+\ \nabla\cdot\left\{\frac{\mathbf{b}}{\Omega_{c\iota}}\times\left[\nabla q_{\iota T\parallel}+2(q_{\iota B\parallel}-q_{\iota T\parallel})\boldsymbol{\kappa}\right]\right\}\ +\ \left(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_{c\iota}}\right)\cdot\nabla q_{\iota T\parallel}\ , \quad (58)$$

that is, also identical to Eq.(53).

In conclusion, assuming (51) to hold, the 1, $v'_\parallel$ and v'^2 moments of the second-order ion drift-kinetic equation are satisfied identically:

$$\int d^3\mathbf{v}'\ (1,v'_\parallel,v'^2)\ \frac{d_\iota\bar{f}_{NM\iota}}{dt}\ \equiv\ \int d^3\mathbf{v}'\ (1,v'_\parallel,v'^2)\ (D_\iota f_{M\iota}+\mathcal{Q}_\iota^{coll})\ . \quad (59)$$

This means that, if the initial value for $\bar{f}_{NM\iota}$ satisfies the conditions (51), its drift-kinetic evolution equation ensures automatically that those conditions remain satisfied at all times.

VI. Summary.

This article completes the theoretical model initiated in Ref.1 to describe slow macroscopic processes in low collisionality, magnetic confinement plasmas. The hybrid fluid and drift-kinetic formulation followed a systematic asymptotic expansion, based on small but finite Larmor radii, low collisionality, small electron to ion mass ratio, diamagnetic scale flows and close to Maxwellian distribution functions. The analysis reached to the frequency scale where collisions begin to influence the dynamics, which turns out to be one order smaller than the diamagnetic drift frequency scale. This requires high-order FLR fluid and drift-kinetic equations and the chosen approach emphasizes the precise consistency among them. The fluid part of the system evolves the macroscopic density, flow velocities and temperatures, which are carried entirely by the Maxwellian part of the distribution functions in Chapman-Enskog-like fashion. The moments of the non-Maxwellian parts yield only the higher-rank fluid closures. Of these, the non-gyrotropic moments of the gyrophase-dependent distribution function terms, namely the perpendicular heat fluxes and the ion gyroviscosity, are deduced from FLR fluid results without the recourse to kinetic theory. Only the gyrophase averages of the non-Maxwellians, which provide the remaining gyrotropic closures, require a kinetic solution and novel forms of drift-kinetic equations for them have been derived. Key to this approach is the use of the reference frames of the mean macroscopic flows, which facilitates naturally the rigorous treatment of the electric field and the evaluation of the fluid closure moments. One of the main payoffs is the explicit proof that the drift-kinetic equations preserve the required conditions that the first three velocity moments of the non-Maxwellian parts of the gyro-averaged distribution functions remain equal to zero through their dynamical evolution. In the case of the ions, such proof is given to the second order in the Larmor radius and hinges on the use of the appropriate second-order forms of the gyroviscosity (6-7) and the perpendicular heat flux (8) in the coupled fluid part of the system.

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Appendix A: Coefficients in the random velocity cylindrical coordinate representation of the ion kinetic equation.

The expression (21) of the ion kinetic equation in the random velocity cylindrical coordinates $(v'_\parallel, v'_\perp, \alpha)$ is written in terms of the set of gyrophase-independent operators Λ_l and gyrophase-independent functions λ_l that were introduced in Ref.12. For the purposes of the present work, they are expanded according to the orderings (1) as $\Lambda_l = \sum_j \Lambda_l^{(j)}$ and $\lambda_l = \sum_j \lambda_l^{(j)}$, with $\Lambda_l^{(j)} \sim \lambda_l^{(j)} \sim \delta^j v_{th\iota}/L$. The needed objects are:

$$\Lambda_0^{(0)} = v'_\parallel \mathbf{b}\cdot\frac{\partial}{\partial\mathbf{x}} - \frac{v'_\perp v'_\parallel}{2}\nabla\cdot\mathbf{b}\,\frac{\partial}{\partial v'_\perp} + \left(\frac{T_\iota}{m_\iota}\mathbf{b}\cdot\nabla\ln n + \frac{v'^2_\perp}{2}\nabla\cdot\mathbf{b}\right)\frac{\partial}{\partial v'_\parallel}\ , \tag{A.1}$$

$$\begin{aligned}\Lambda_0^{(1)} &= \frac{\partial}{\partial t} + \mathbf{u}_\iota\cdot\frac{\partial}{\partial\mathbf{x}} + \frac{v'_\perp}{2}(\mathbf{bb}-\mathsf{I}):(\nabla\mathbf{u}_\iota)\frac{\partial}{\partial v'_\perp} + \\ &+\left\{\frac{\mathbf{b}\cdot\nabla T_\iota}{m_\iota} + \frac{1}{m_\iota n}\mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp}) - (p_{\iota\parallel}-p_{\iota\perp})\nabla\ln B\right] - v'_\parallel(\mathbf{bb}):(\nabla\mathbf{u}_\iota)\right\}\frac{\partial}{\partial v'_\parallel}\ ,\end{aligned} \tag{A.2}$$

$$\Lambda_0^{(2)} = \frac{1}{m_\iota n}\mathbf{b}\cdot\left(\nabla\cdot\mathsf{P}_\iota^{GV}\right)\frac{\partial}{\partial v'_\parallel}\ , \tag{A.3}$$

$$\Lambda_1^{(0)} = \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\frac{\partial}{\partial\mathbf{x}} + \frac{v'_\perp v'_\parallel}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\boldsymbol{\kappa}\,\frac{\partial}{\partial v'_\parallel} + \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\left[\frac{\nabla(nT_\iota)}{m_\iota n} - v'^2_\parallel\boldsymbol{\kappa}\right]\frac{\partial}{\partial v'_\perp}\ , \tag{A.4}$$

$$\begin{aligned}\Lambda_1^{(1)} &= \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\left[\frac{\nabla\ln n\times\nabla T_\iota}{m_\iota\Omega_{c\iota}} - \mathbf{b}\times(\nabla\times\mathbf{u}_\iota)\right]\frac{\partial}{\partial v'_\parallel} + \\ &+\frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\left\{\frac{1}{m_\iota n}\left[-\frac{1}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp}) + (p_{\iota\parallel}-p_{\iota\perp})\boldsymbol{\kappa}\right] - v'_\parallel\left[\frac{\nabla\ln n\times\nabla T_\iota}{m_\iota\Omega_{c\iota}} + 2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\right]\right\}\frac{\partial}{\partial v'_\perp},\end{aligned} \tag{A.5}$$

$$\Lambda_2^{(0)} = \frac{iv'_\perp}{4}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot[\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)]\left(v'_\perp\frac{\partial}{\partial v'_\parallel} - v'_\parallel\frac{\partial}{\partial v'_\perp}\right)\ , \tag{A.6}$$

$$\Lambda_2^{(1)} = -\frac{v'_\perp}{4}[(\mathbf{e}_1 - i\mathbf{e}_2)(\mathbf{e}_1 - i\mathbf{e}_2)]:(\nabla\mathbf{u}_\iota)\frac{\partial}{\partial v'_\perp}\ , \tag{A.7}$$

$$\lambda_0^{(0)} \;=\; \frac{v'_\parallel}{2}\left\{\mathbf{e}_1\cdot[(\mathbf{b}\cdot\nabla)\mathbf{e}_2] \;-\; \mathbf{e}_2\cdot[(\mathbf{b}\cdot\nabla)\mathbf{e}_1] \;-\; \mathbf{b}\cdot(\nabla\times\mathbf{b})\right\} \;, \tag{A.8}$$

$$\lambda_1^{(0)} \;=\; \frac{i}{2v'_\perp}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\left[\frac{\nabla(nT_\iota)}{m_\iota n} - v_\parallel'^2\boldsymbol{\kappa}\right] \;-\; \frac{v'_\perp}{2}\mathbf{b}\cdot[\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)] \;, \tag{A.9}$$

$$\lambda_1^{(1)} = \frac{i}{2v'_\perp}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\left\{\frac{1}{m_\iota n}\left[-\frac{1}{3}\nabla(p_{\iota\parallel} - p_{\iota\perp}) + (p_{\iota\parallel} - p_{\iota\perp})\boldsymbol{\kappa}\right] - v'_\parallel\left[\frac{\nabla\ln n\times\nabla T_\iota}{m_\iota\Omega_{c\iota}} + 2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\right]\right\}, \tag{A.10}$$

$$\lambda_2^{(0)} \;=\; \frac{v'_\parallel}{4}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot[\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)] \;. \tag{A.11}$$

In these expressions, Faraday's law has been substituted for the time derivative of the magnetic field, with the electric field eliminated algebraically with the momentum conservation equation. They apply to general space and time variations of the magnetic field.

Appendix B: Differential magnetic geometry and the parallel gyroviscous force.

For general 3-dimensional magnetic line configurations (consistent with $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{B} = \mathbf{j}$), their differential geometry is characterized by a set of intrinsic variables which comprises the divergence of the unit vector ($\mathbf{b}\equiv\mathbf{B}/B$)

$$\nabla\cdot\mathbf{b} \;=\; -\mathbf{b}\cdot\nabla\ln B \;, \tag{B.1}$$

the curvature vector

$$\boldsymbol{\kappa} \;\equiv\; (\mathbf{b}\cdot\nabla)\mathbf{b} \;=\; -\mathbf{b}\times(\nabla\times\mathbf{b}) \;, \tag{B.2}$$

the "twist" scalar related to the parallel current

$$\tau \;\equiv\; \mathbf{b}\cdot(\nabla\times\mathbf{b}) \;=\; j_\parallel/B \;, \tag{B.3}$$

and the second-rank symmetric tensor M, which is the traceless and perpendicular (in the sense that $\mathbf{b}\cdot\mathsf{M} = 0$) projection of the symmetrized $\nabla\mathbf{b}$, defined in Cartesian component representation by

$$\mathsf{M}_{jk} \;\equiv\; \frac{1}{2}\Big[(\delta_{jl} - b_j b_l)(\delta_{km} - b_k b_m) - \epsilon_{jnl}b_n\epsilon_{kpm}b_p\Big]\left(\frac{\partial b_m}{\partial x_l} + \frac{\partial b_l}{\partial x_m}\right) \;. \tag{B.4}$$

In terms of these, the symmetric part of the magnetic gradient tensor is

$$\frac{\partial b_k}{\partial x_j} + \frac{\partial b_j}{\partial x_k} \;=\; (\nabla \cdot \mathbf{b})\;(\delta_{jk} - b_j b_k) \;+\; (b_j \kappa_k + \kappa_j b_k) \;+\; \mathsf{M}_{jk} \tag{B.5}$$

and its antisymmetric part is

$$\frac{\partial b_k}{\partial x_j} - \frac{\partial b_j}{\partial x_k} \;=\; (b_j \kappa_k - \kappa_j b_k) \;+\; \tau\;\epsilon_{jkl}\;b_l\;. \tag{B.6}$$

The space of second-rank, symmetric, traceless and perpendicular tensors is bidimensional and another independent tensor in that space is $\mathsf{M}_\times$, obtained by taking a "cross product" of M with $\mathbf{b}$

$$\mathsf{M}_{\times jk} \;\equiv\; \mathsf{M}_{jl}\;\epsilon_{lkm}\;b_m \tag{B.7}$$

and whose Cartesian components are

$$\mathsf{M}_{\times jk} \;=\; \frac{1}{2}\Big[(\delta_{jl} - b_j b_l)\epsilon_{knm} b_n + (\delta_{kl} - b_k b_l)\epsilon_{jnm} b_n\Big]\left(\frac{\partial b_m}{\partial x_l} + \frac{\partial b_l}{\partial x_m}\right)\;. \tag{B.8}$$

Besides $\mathsf{M}_{jj} = \mathsf{M}_{\times jj} = b_j \mathsf{M}_{jk} = b_j \mathsf{M}_{\times jk} = 0$, these tensors have the properties

$$\mathsf{M}_{jk} \;=\; -\;\mathsf{M}_{\times jl}\;\epsilon_{lkm}\;b_m \tag{B.9}$$

and

$$\mathsf{M}_{\times jk}\;\left(\frac{\partial b_k}{\partial x_j} + \frac{\partial b_j}{\partial x_k}\right) \;=\; \mathsf{M}_{\times jk}\;\mathsf{M}_{jk} \;=\; 0\;. \tag{B.10}$$

The ion drift-kinetic equation involves the parallel component of the gyroviscous force, $\mathbf{b}\cdot(\nabla\cdot\mathsf{P}_\iota^{GV})$, due to its contribution to the parallel electric field in the momentum conservation equation. After partial integration, it is

$$\mathbf{b}\cdot\left(\nabla\cdot\mathsf{P}_\iota^{GV}\right) \;=\; \nabla\cdot\left(\mathbf{b}\cdot\mathsf{P}_\iota^{GV}\right) \;-\; (\nabla\mathbf{b}) : \mathsf{P}_\iota^{GV}\;. \tag{B.11}$$

Moreover, since P_ι^{GV} is symmetric and traceless and satisfies $(\mathbf{bb}) : \mathsf{P}_\iota^{GV} = 0$, recalling (B.5) one can write

$$\mathbf{b}\cdot\left(\nabla\cdot\mathsf{P}_\iota^{GV}\right) \;=\; \nabla\cdot\left(\mathbf{b}\cdot\mathsf{P}_\iota^{GV}\right) \;-\; \boldsymbol{\kappa}\cdot\left(\mathbf{b}\cdot\mathsf{P}_\iota^{GV}\right) \;-\; \frac{1}{2}\;\mathsf{M} : \mathsf{P}_\iota^{GV}\;. \tag{B.12}$$

From Eq.(6), it follows that

$$\mathbf{b} \cdot \mathsf{P}_\iota^{GV} \;=\; \mathbf{b} \times (\mathbf{b} \cdot \mathsf{K}_\iota) \tag{B.13}$$

and, after substituting Eq.(7) for K_ι and using some vector identities,

$$\begin{aligned} \mathbf{b} \cdot \mathsf{P}_\iota^{GV} \;=\; & \frac{\mathbf{b}}{\Omega_{c\iota}} \times \Big(nT_\iota \;\, [(2(\mathbf{b} \cdot \nabla)\mathbf{u}_\iota + \mathbf{b} \times (\nabla \times \mathbf{u}_\iota)] \;+\; \nabla q_{\iota T\|} \;+\; 2(q_{\iota B\|} - q_{\iota T\|})\boldsymbol{\kappa} \Big) \;+ \\ & + \frac{nT_\iota}{m_\iota \Omega_{c\iota}^2} \; \mathbf{b} \times [(\mathbf{b} \cdot \nabla \ln n)\mathbf{b} \times \nabla T_\iota - \tau \nabla T_\iota] \;\; . \end{aligned} \tag{B.14}$$

Similarly, from Eq.(6) and the properties of the tensors M and $\mathsf{M}_\times$, it follows that

$$\mathsf{M} : \mathsf{P}_\iota^{GV} \;=\; -\, \frac{1}{2} \, \mathsf{M}_\times : \mathsf{K}_\iota \tag{B.15}$$

and, after substituting Eq.(7) for K_ι and using again the properties of M and $\mathsf{M}_\times$,

$$\mathsf{M} : \mathsf{P}_\iota^{GV} \;=\; -\, \frac{nT_\iota}{\Omega_{c\iota}} \, \mathsf{M}_\times : (\nabla \mathbf{u}_\iota) \;-\; \frac{1}{m_\iota \Omega_{c\iota}} \, \mathsf{M} : \left[\nabla \left(\frac{nT_\iota}{\Omega_{c\iota}} \; \nabla T_\iota \right) \right] \;\; . \tag{B.16}$$

Finally, collecting all the terms, one gets the expression of the second-order parallel gyroviscous force:

$$\begin{aligned} \mathbf{b} \cdot (\nabla \cdot \mathsf{P}_\iota^{GV}) \;=\; & \nabla \cdot \left\{ \frac{\mathbf{b}}{\Omega_{c\iota}} \times \Big(nT_\iota \;\, [2(\mathbf{b} \cdot \nabla)\mathbf{u}_\iota + \mathbf{b} \times (\nabla \times \mathbf{u}_\iota)] \;+\; \nabla q_{\iota T\|} \;+\; 2(q_{\iota B\|} - q_{\iota T\|})\boldsymbol{\kappa} \Big) \right\} \;+ \\ & + \frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_{c\iota}} \cdot \Big(nT_\iota \;\, [2(\mathbf{b} \cdot \nabla)\mathbf{u}_\iota + \mathbf{b} \times (\nabla \times \mathbf{u}_\iota)] \;+\; \nabla q_{\iota T\|} \Big) \;+\; \frac{nT_\iota}{2\Omega_{c\iota}} \, \mathsf{M}_\times : (\nabla \mathbf{u}_\iota) \;+ \\ & + \nabla \cdot \left\{ \frac{nT_\iota}{m_\iota \Omega_{c\iota}^2} \; \mathbf{b} \times [(\mathbf{b} \cdot \nabla \ln n)\mathbf{b} \times \nabla T_\iota - \tau \nabla T_\iota] \right\} \;+ \\ & + \frac{nT_\iota}{m_\iota \Omega_{c\iota}^2} \; (\mathbf{b} \times \boldsymbol{\kappa}) \cdot [(\mathbf{b} \cdot \nabla \ln n)\mathbf{b} \times \nabla T_\iota - \tau \nabla T_\iota] \;+\; \frac{1}{2 m_\iota \Omega_{c\iota}} \, \mathsf{M} : \left[\nabla \left(\frac{nT_\iota}{\Omega_{c\iota}} \; \nabla T_\iota \right) \right] \;\; . \end{aligned} \tag{B.17}$$

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