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AND THE RELATIVISTIC COLLISION OPERATOR**

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ABSTRACT

A retarded time superposition principle is formulated and proved for two-particle correlation function in a multispecies relativistic, and fully electromagnetic plasma. This principle is used to obtain the relativistic collision operator. Starting from the relativistic Klimontovich equation, the statistical (Liouville) average of the Klimontovich equation yields an expression for the collision operator in terms of the two-time two-point correlation function for two particles, $G_{12}(1, t_1; 2, t_2)$. It is proved that $G_{12}(1, t_1; 2, t_2)$ can be written in a retarded time superposition form in terms of the self correlation $W_{11}(1, t_1; 2, t_2)$ and the discreteness response function $P(1, t_1; 2, t_2)$. The equation for the pair correlation function $G_{12}(1, t_1; 2, t_2)$ excluding triplet or higher order correlations, is thus replaced by the simpler equation for $P(1, t_1; 2, t_2)$. This is the test particle problem, which relates $P(1, t_1; 2, t_2)$ to the discreteness source term $W_{11}(1, t_1; 2, t_2)$. The equations for $P(1, t_1; 2, t_2)$ and $W_{11}(1, t_1; 2, t_2)$ are solved for stationary, homogeneous plasmas without external fields. With these solutions, the collision operator is expressed in terms of the relativistic dielectric properties of the plasma.

I. Introduction

In this paper, the shielded electromagnetic and relativistic collision operator is derived for a multi-species plasma. The unshielded operator for electrostatic interactions was first derived by Landau¹; Balescu² and Lenard³ independently found the shielded operator. In 1961, Silin⁴ derived the collision operator for electromagnetic interactions. This derivation was based upon taking the nonquantum limit of the weak interaction approximation for *one time* pair correlation determined by a quantum mechanical equation previously obtained by Klimontovich and Temko.⁵ In taking the nonquantum limit, Silin also uses a result obtained by Moller.⁶ Silin characterizes his derivation inconsistently as relativistic. The one time pair correction that he uses is written for non-relativistic particles (for which $p = m\mathbf{v}$). It is clear that a consistent relativistic-electromagnetic generalization of the pair correlation should incorporate a self-consistent treatment of the electromagnetic field coupled with the relativistic particle dynamics. This leads to a *retarded time* reformulation of the whole problem of calculating the pair correlations which is inherently different from that considered by Klimontovich and Temko. The collision integral derived by Silin is formally identical to that obtained in this paper, the only difference being in the way the momenta are related to the velocities involved in the final formula. This agreement is only superficial, because Silin's approach is originated from a nonrelativistic treatment of particle dynamics. It should be mentioned that there is much literature⁷ on the solution of the equation for the *one time* pair correlation.

In a recent paper by Bezzerides and Dubois⁸ a very elaborate approach to the relativistic plasmas was developed. The authors combine the Feynman-Schwinger electrodynamics with the Green's function theory of nonequilibrium quantum statistical mechanics for treating ultrarelativistic plasmas. It is clear that quantum effects become very important for such plasmas and therefore that a quantum mechanical treatment should be used. In this paper, the classical limit of the collision term reduces to that given by Silin in addition to non-adiabatic correction term, which becomes important in the case of unstable plasmas modes or for very small plasma emission and absorption rates *in the collisionless limit*. The Bezzerides and Dubois treatment of the non-adiabatic correction is, however, based on an ansatz. This ansatz is used in deriving the equation for the contributions to the degree of excitation of the fields in excess of the adiabatic limit. The model is linear in the sense that it ignores possible collisional modifications of the non-adiabatic contributions. The authors clearly state that an accurate evaluation of the validity of the adiabatic limit requires a collisional treatment of

the damping of the non-adiabatic fluctuations. The decoupling between collisional and non-adiabatic effects, which is introduced by this ansatz, leads to a collision integral separated from the non-adiabatic contribution, in the source term of the covariant kinetic equation. This is in favor of the idea that a quantum electrodynamical treatment of the ultrarelativistic electrons, *excluding* the non-adiabatic contribution, could be simpler, if necessary, leading to the result that Silin heuristically derived. Whether or not a relativistic analysis or, to what extent, a linear treatment of the non-adiabatic contributions should be appropriate is still, to the extent of our knowledge, an open question.

Beliaev and Budker,⁹ and most recently Bernstein,¹⁰ derived the collision integral for relativistic and electromagnetic plasmas in cases where screening effects could be ignored.

In the present paper, we clarify some of these issues using a classical approach. The main point is the use of the "superposition form" for the pair correlation function. This idea was introduced by Rostoker.¹¹ In this paper, Rostoker's result is generalized to include retardation effects and then the validity of the generalization is proved for a multi-species plasma. The proof does not assume the exclusion of external fields, inhomogeneities, and non-stationary behavior, as long as this inclusion conforms with the truncation of the hierarchy of the kinetic equations, i.e., the exclusion of triplet higher order correlations. It is not, however, the purpose of the present paper to pursue this question of consistency; it is rather to demonstrate how the collision integral can be derived on these generalized, but still *classical*, grounds for the case of an infinitely extended homogeneous, external field free and stationary plasma.

In Section II, starting from the microscopic Maxwell-Klimontovich equations, the electromagnetic interaction is formulated in terms of the retarded Green's functions $G(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$. The six dimensional pair (\mathbf{r}, \mathbf{p}) corresponds to a point of the relativistic phase space and \mathbf{p} is related to the velocity \mathbf{v} and the rest mass in the usual manner $\mathbf{p} = m\gamma\mathbf{v}$, $\gamma \equiv (1 - \beta^2)^{-1/2}$, and $\beta^2 \equiv \frac{\mathbf{v}\cdot\mathbf{v}}{c^2}$. The retarded formulation expresses the fact that the particles correlate with each other through retarded electromagnetic interactions.

To find the collision integral two point correlations must be computed. In general, such correlations are linked by an infinite hierarchy of equations to three and more point correlations. In the quiescent plasma theory, this hierarchy is truncated by an expansion in the discreteness parameter and triplet or higher order correlation are ignored. This procedure is followed here so that a closed hierarchy of equations is obtained for the one particle distribution functions, the two point correlation functions, and the two point self correlation

functions. The latter additional function is a consequence of the retarded time formulation and expresses the probability of finding the *same* particle at two different positions in phase space and time. For well-defined particle orbits, the self correlation functions is an irreducible physical quantity. This irreducibility property is the result of the mutual dependence of the relativistic particle dynamics and the space and time evolution of its own electromagnetic field. The two point correlation function, on the other hand, correlates two different particles at two different positions in phase space and time. The two time character of this function is a result of the finite speed of propagation of the interactions.

In Section III, the statistical apparatus needed to compute the observable quantities is developed. These quantities have to be ensemble averaged over all possible microscopic particle orbits. To formalize this average, Liouville functions are used. The one time Liouville functions for one or more species are defined. The two points and two time kinetic functions are also required because of the retarded character of the interaction. Therefore, the two time Liouville functions for one or more species are defined and related to the one time Liouville functions in the equal time limit.

The Liouville functions are used to carry out the ensemble average, denoted by brackets ($\langle \rangle$). The averaged one point distribution for species α is then denoted by $f_\alpha(\mathbf{r}, \mathbf{p}, t)$, and is defined as $\langle F_{N_\alpha}(\mathbf{r}, \mathbf{p}, t) \rangle$, the ensemble average of $F_{N_\alpha}(\mathbf{r}, \mathbf{p}, t)$, the Klimontovich function for N_α particles of species α . The two point functions come from $\langle F_{N_\alpha}(\mathbf{r}, \mathbf{p}, t) F_{N_\beta}(\mathbf{r}', \mathbf{p}', t') \rangle$, the ensemble average of the product of two Klimontovich functions for species α and β ; here $(\mathbf{r}, \mathbf{p}, t)$ and $(\mathbf{r}', \mathbf{p}', t')$, are, in general, two different phase space and time points. This last averaged product expressed in terms of the two point-two time correlations $G_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ and the two point-two time self correlation $W_\alpha(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$. The function $G_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ refers to two different particles of species α and β , (α could be the same as β) at phase space-time points $(\mathbf{r}, \mathbf{p}, t)$ and $(\mathbf{r}', \mathbf{p}', t')$.

The function $W_\alpha(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ accounts for the possibility of looking at the same particle orbiting between two phase space-time points and interacting with its own electromagnetic field. The collision integral is expressed in terms of those correlations. This resulting expression is correct to first order in the discreteness parameter.

Finally, in Section III, the closed system of kinetic equations is obtained. It contains the equations for one particle distribution for $f_\alpha(\mathbf{r}, \mathbf{p}, t)$ where the collision integral enters, the two point correlation $G_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$,

the equal time two point correlation $g_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t)$, and for the self correlation $W_{\alpha}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t)$. The equal time conditions are included as boundary conditions.

The superposition form for $G_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ and correspondingly for $G_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ is introduced in Section IV, and is

$$\begin{aligned}
G_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t') &= \frac{1}{V} \int d^3\mathbf{r}'' d^3\mathbf{p}'' \int_{-\infty}^{t'} dt'' W_{\alpha}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}'', \mathbf{p}'', t'') P_{\beta\alpha}(\mathbf{r}', \mathbf{p}', t'; \mathbf{r}'', \mathbf{p}'', t'') \\
&+ \frac{1}{V} \int d^3\mathbf{r}'' d^3\mathbf{p}'' \int_{-\infty}^t dt'' W_{\beta}(\mathbf{r}', \mathbf{p}', t'; \mathbf{r}'', \mathbf{p}'', t'') P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}'', \mathbf{p}'', t'') \\
&+ \frac{1}{V} \sum_{\gamma} n_{\gamma} \int d^3\mathbf{r}'' d^3\mathbf{p}'' \int d^3\mathbf{r}''' d^3\mathbf{p}''' \int_{-\infty}^t dt''' \int_{-\infty}^{t'} dt'''' \\
&W_{\gamma}(\mathbf{r}'', \mathbf{p}'', t''; \mathbf{r}''', \mathbf{p}''', t''') P_{\alpha\gamma}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}'', \mathbf{p}'', t'') P_{\beta\gamma}(\mathbf{r}', \mathbf{p}', t'; \mathbf{r}''', \mathbf{p}''', t''') \quad (1)
\end{aligned}$$

where V is the volume of the system and n_{γ} the average density for the species γ . This form decouples the kinetic equations. The new function $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$, the discreteness response function, describes the shielding of a test particle of species α at the point $(\mathbf{r}, \mathbf{p}, t)$ by the response of a field particle of species β at the point $(\mathbf{r}', \mathbf{p}', t')$.

The physical meaning of the $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ function and the derivation of the equation it satisfies are considered in Section V. In this section, the plasma is taken to be homogeneous and stationary; the external fields are also excluded. Under these conditions the equation satisfied by the discreteness response function $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ becomes very simple in form. It is then proved that the distribution function perturbation of a Vlasov plasma made of species α induced by a discrete test particle of species β can be expressed as the time history integral of the function $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$.

In Section VI, the collision integral is formulated in terms of the discreteness response function $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ and the self correlation function $W_{\alpha}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$. This reduces the problem to a much simpler one where only the equations for $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ and $W_{\alpha}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ must be solved.

In the solution for $P_{\alpha\beta}(\mathbf{r}, \mathbf{p}, t; \mathbf{r}', \mathbf{p}', t')$ two additional physical quantities are introduced. These are $\hat{n}_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p})$ and $\hat{J}_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p})$, the density and current perturbations, respectively, induced by discreteness. These quantities are related by

$$\hat{n}_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p}) \equiv n_\alpha \int d^3 p' P_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p}', \mathbf{p}) \quad (2a)$$

$$\hat{J}_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p}) \equiv n_\alpha \int d^3 p' P_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p}', \mathbf{p}) \frac{\mathbf{p}'}{m_\alpha \gamma_\alpha} \quad (2b)$$

where the subscripts \mathbf{k}, ω are the Fourier and Laplace transform components in \mathbf{r} and t . These quantities obey a continuity equation which greatly simplified the form of the solution for $P_{\alpha\beta, \mathbf{k}, \omega}(\mathbf{p}', \mathbf{p})$.

In Section VII, the current perturbation $\hat{J}_{\alpha\beta}$ is related to the dispersive properties of the medium characterized by the tensor $Z_{\mathbf{k}, \omega}$. This tensor is related to the dielectric tensor $\epsilon_{\mathbf{k}, \omega}$ by¹¹

$$Z_{\mathbf{k}, \omega} \cdot \Delta_{\mathbf{k}, \omega} = I \quad (3a)$$

and

$$\Delta_{\mathbf{k}, \omega} \equiv \epsilon_{\mathbf{k}, \omega} - \frac{k^2 c^2}{\omega^2} \left(I - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) \quad (3b)$$

where I is the unit tensor. The dielectric tensor $\epsilon_{\mathbf{k}, \omega}$ is the relativistic electromagnetic one. Finally, the collision integral $C_\alpha(f_\alpha(\mathbf{p}_\alpha))$ (which determines the rate of change of the distribution function of species α) is expressed in terms of the tensor Z and the distribution functions of all the species. It is given in Balescu-Lenard form as

$$C_\alpha(f_\alpha(\mathbf{p}_\alpha)) = \sum_\beta 2q_\alpha^2 q_\beta^2 n_\beta \mathcal{Q}_\alpha \cdot \int d^3 p_\beta \int d^3 k \frac{\delta(\mathbf{k} \cdot \mathbf{v}_\alpha - \mathbf{k} \cdot \mathbf{v}_\beta)}{(\mathbf{k} \cdot \mathbf{v}_\alpha)^4} \quad (4)$$

$$|\mathbf{v}_\beta \cdot Z_{\mathbf{k}, \mathbf{k}} \mathbf{v}_\alpha|^2 \mathbf{k}\mathbf{k} \cdot (\partial_{\mathbf{p}_\alpha} - \partial_{\mathbf{p}_\beta}) f_\beta(\mathbf{p}_\beta) f_\alpha(\mathbf{p}_\alpha)$$

where q_α, q_β are the charges for species α and β . Once this collision integral is in generalized Balescu-Lenard form, its conservation properties can be easily demonstrated. This form of the collision integral is also manipulated into another form and it is also shown how it reduces to the form derived by Belyaev-Budker and Bernstein in the absence of shielding.

Finally, in Section VIII, the results obtained here are summarized and the significance and application of these results are discussed.

II. The Microscopic Formulation of the Electromagnetic Interaction Operator

The microscopic dynamics of the system is developed, on the basis of the single species Klimontovich function. The internal electromagnetic forces are expressed in terms of the potentials which are, self-consistently and causally, related back to the particle dynamics they determine. This is done using the retarded Green's function which solves the inhomogeneous wave equations for the potentials. The self-consistent force term in the Klimontovich equation is finally expressed as an integral over all species and all particle orbits in phase space and time of an operator acting on the Klimontovich function. This electromagnetic interaction operator is related to the Green's function.

The Klimontovich function $F_{N\alpha}(\mathbf{r}, \mathbf{p}, t)$ completely specifies the microscopic state of a system of particles of species α

$$F_{N\alpha}(\mathbf{r}, \mathbf{p}, t) \equiv F_{N\alpha}(\mathbf{r}, \mathbf{p}, \{\mathfrak{G}_\alpha(t)\}, t) = \frac{1}{n_\alpha} \sum_{i=1}^{N_\alpha} \delta(\mathbf{r} - \mathbf{r}_{\alpha i}(t)) \delta(\mathbf{p} - \mathbf{p}_{\alpha i}(t)) \quad (5)$$

where n_α is the average density at species α and N_α is the number of particles of species α . Here, $\{\mathfrak{G}_\alpha(t)\}$ denotes the phase space orbits of all particles of species α and $\mathbf{r}_{\alpha i}(t)$ and $\mathbf{p}_{\alpha i}(t)$ are the position and momentum orbits of the i -th particle of species α .

The Klimontovich function obeys the equation,

$$\left[\partial_{t_1} + L_\alpha(\mathbf{l}, t_1) + q_\alpha \left(\mathbf{e} + \frac{\mathbf{p}_1 \times \mathbf{b}}{m_\alpha \gamma_{\alpha 1} c} \right) \cdot \partial_{\mathbf{p}_1} \right] F_{N\alpha}(\mathbf{l}, t_1) = 0 \quad (6)$$

where the notation is that a number by itself denotes the corresponding phase space point

$$\mathbf{l} \equiv (\mathbf{r}_1, \mathbf{p}_1) \quad (7)$$

The operator $L_\alpha(\mathbf{l}, t_1)$ includes part of the unperturbed orbit operator and also accounts for the accelerated motion due to the external fields

$$L_\alpha(\mathbf{l}, t_1) \equiv \frac{\mathbf{p}_1}{m_\alpha \gamma_{\alpha 1}} \cdot \partial_{\mathbf{r}_1} + q_\alpha \left[E_{\text{ext}}(\mathbf{r}_1, t_1) + \frac{\mathbf{p}_1 \times B_{\text{ext}}(\mathbf{r}_1, t_1)}{m_\alpha \gamma_{\alpha 1} c} \right] \cdot \partial_{\mathbf{p}_1} \quad (8)$$

Note that ∂_t is the partial derivative $\frac{\partial}{\partial t}$ and ∂_r is the gradient operator ∇_1 , e and b are the electric and magnetic microfields, respectively, and the prime in Eq. (6) denotes the exclusion of the self-force. The microfields obey Maxwell's equations

$$\partial_r \cdot b = 0 \quad (9a)$$

$$\partial_r \cdot e = 4\pi \sum_a q_a n_a \int F_{Na} d^3p \quad (9b)$$

$$\partial_r \times e = -\frac{1}{c} \partial_t b \quad (9c)$$

$$\partial_r \times b = \frac{1}{c} \partial_t e + \frac{4\pi}{c} \sum_a \frac{q_a n_a}{m_a} \int \frac{p}{\gamma_a} F_{Na} d^3p \quad (9d)$$

Using the Lorentz gauge, the microfields can be expressed in terms of the scalar potential $\phi(r, t)$ and the vector potential $a(r, t)$

$$e = -\partial_r \phi - \frac{1}{c} \partial_t a \quad (10a)$$

$$b = \partial_r \times a \quad (10b)$$

By virtue of Eqs. (9) and (10) ϕ and a satisfy the following equation

$$\left(\partial_t^2 - \frac{1}{c^2} \partial_r^2\right) \phi(r, t) = \sum_a n_a q_a \left[\int f_{Na}(p, r, t) d^3p - 1 \right] \quad (11)$$

$$\left(\partial_r^2 - \frac{1}{c^2} \partial_t^2\right) a(r, t) = \sum_a \frac{n_a q_a}{c} \int \frac{p}{m_a \gamma_a} F_{Na}(p, r, t) d^3p \quad (12)$$

These inhomogeneous wave equations are solved using the Green's function,

$$G^+(r, t; r', t') = \frac{\delta(t' - t + \frac{|r-r'|}{c})}{|r-r'|} \quad (13)$$

where the retarded solution is adopted to enforce causality. The microfields, their gradients, and their time derivatives are assumed to be zero on the boundaries of the system and in the infinite past. The microfields are then expressed in terms of G^+ so that the Klimontovich equation becomes

$$\left[\partial_1 + L_\alpha(1, t_1) - \sum_\beta \int d2 \int_{-\infty}^{t_2} dt_2 F_{N\beta}(2, t_2) V_{\alpha\beta}(1, t_1; 2, t_2) \right] F_{N\alpha}(1, t_1) = 0 \quad (14)$$

where

$$\begin{aligned} V_{\alpha\beta}(1, t_1; 2, t_2) \equiv & \left[n_\beta q_\alpha q_\beta \partial_{r_1} G^+(r_1, t_1; r_2, t_2) \right. \\ & + \frac{n_\beta q_\alpha q_\beta}{m_\beta c^2} \partial_{t_1} G^+(r_1, t_1; r_2, t_2) \frac{p_2}{\gamma_{\beta 2}} \\ & \left. + \frac{n_\beta q_\alpha q_\beta}{m_\alpha m_\beta c^2} \partial_{r_1} G^+(r_1, t_1; r_2, t_2) \times \frac{p_2}{\gamma_{\beta 2}} \times \frac{p_1}{\gamma_{\alpha 1}} \right] \cdot \partial_{p_1} \end{aligned} \quad (15)$$

This operator $V_{\alpha\beta}$ describes the electromagnetic effect of a particle at $(2, t_2)$ on a particle at $(1, t_1)$. In the limit $c \rightarrow \infty$ only the first term on the right side of Eq. (15) contributes, and accounts then for the electrostatic interaction.

III. The Kinetic Equations

The statistical apparatus, which provides the link between the microscopic description of the previous section and the macroscopic description is developed here. The local observable macroscopic quantities in phase space and time are obtained as ensemble averages of microscopic dynamical quantities over the whole N -dimensional space where a one time Liouville function is postulated to exist. The non-local macroscopic observables involving two phase space points are also expressed as ensemble averages of products of microscopic dynamical quantities via the same one time Liouville function. This function provides performing the appropriate integrations, the most restricted Liouville functions for one or more species. The retarded character of the interactions creates non-local correlations in phase space and time which are expressed as ensemble averages of products of microscopic dynamical quantities via a *two-time, two point Liouville function*.

This statistical apparatus is now used to develop a hierarchy of equations for the one species distribution functions and the two-time correlations of every order. The hierarchy is truncated to form, along with the equal time conditions, a closed system of equations. This is done by dropping triplet and higher order correlations. This truncation is based on the smallness of the plasma discreteness parameter and it is appropriate for studying quiescent plasmas, although the external fields, inhomogeneities and non-stationary behavior are being kept throughout, for the sake of generality.

The Klimontovich function is broken up into average and fluctuating pieces. Thus

$$F_{N\alpha}(\mathbf{r}, \mathbf{p}, t) = f_{\alpha}(\mathbf{r}, \mathbf{p}, t) + \delta f_{N\alpha}(\mathbf{r}, \mathbf{p}, t) \quad (16a)$$

where (as discussed previously) $f_{\alpha}(\mathbf{r}, \mathbf{p}, t)$ is

$$f_{\alpha}(\mathbf{r}, \mathbf{p}, t) \equiv \langle F_{N\alpha}(\mathbf{r}, \mathbf{p}, t) \rangle \quad (16b)$$

To formalize the average, the one-time Liouville function $D_1(\{\mathfrak{K}\}, t)$ is introduced. Here $\{\mathfrak{K}\}$ is

$$\{\mathfrak{K}\} \equiv \{1_{\alpha}, 2_{\alpha}, \dots, N_{\alpha}, 1_{\beta}, 2_{\beta}, \dots, N_{\beta}, 1_{\gamma}, 2_{\gamma}, \dots, N_{\gamma}, \dots\} \quad (17)$$

the set of all the phase space points of all the particles of all species α, β, γ , etc. Then, $D_1(\{\mathfrak{K}\}, t)d\{\mathfrak{K}\}$ is the

fraction of systems of all the particles in the statistical ensemble which are in the phase space volume element $d\{\mathfrak{E}\}$

$$d\{\mathfrak{E}\} \equiv d1_\alpha d2_\alpha \dots dN_\alpha d1_\beta d2_\beta \dots dN_\beta d1_\gamma d2_\gamma \dots dN_\gamma \dots \quad (18)$$

about the phase space point $\{\mathfrak{E}\}$ at time t .

The one species, one-time Liouville function for species α is defined as

$$D_{1\alpha}(\{\mathfrak{E}_\alpha\}, t) \equiv \int_{(\beta, \gamma, \dots \neq \alpha)} d\{\mathfrak{E}_\beta\} \int d\{\mathfrak{E}_\gamma\} \int \dots D_1(\{\mathfrak{E}\}, t) \quad (19)$$

Similarly, the two species one-time Liouville function is obtained by integrating the Liouville function overall species expected,

$$D_{1\alpha\beta}(\{\mathfrak{E}_\alpha\}, \{\mathfrak{E}_\beta\}, t) \equiv \int_{(\gamma, \delta, \dots \neq \alpha, \beta)} d\{\mathfrak{E}_\gamma\} \int d\{\mathfrak{E}_\delta\} \int \dots D_1(\{\mathfrak{E}\}, t) \quad (20)$$

The Liouville functions are used to effect the ensemble average of microscopic dynamical quantities. In general, a microscopic dynamical quantity A is of the form $A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\})$, so that the value of A measured depends on the phase-space-time point of observation $(\mathbf{r}, \mathbf{p}, t)$, and on the phase space positions of all the particles in the system, $\{\mathfrak{E}(t)\}$. The ensemble average of $A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\})$ i.e. the corresponding observable field in physical space-time, is then defined as

$$\langle A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\}) \rangle \equiv \int d\{\mathfrak{E}\} D_1(\{\mathfrak{E}\}, t) A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\}) \quad (21)$$

Applying Eq. (21) to the Klimontovich function, Eq. (5) yields

$$\begin{aligned} \langle F_{N\alpha}(\mathbf{l}, t_1) \rangle &\equiv \langle F_{N\alpha}(\mathbf{r}_1, \mathbf{p}_1, t; \{\mathfrak{E}(t_1)\}) \rangle \\ &= V \int d2_\alpha d3_\alpha \dots dN_\alpha D_{1\alpha}(\{1, 2_\alpha, 3_\alpha, \dots, N_\alpha\}, t_1) \\ &\equiv f_\alpha(\mathbf{l}, t_1) \end{aligned} \quad (22)$$

where V is the volume of the system.

To evaluate the ensemble average of $\delta F_{N\alpha}(1, t_1)\delta F_{N\alpha}(2, t_2)$ a two-time, two-point Liouville function $D_2(\{\mathfrak{E}\}, t; \{\mathfrak{E}'\}, t')$ is needed. $D_2(\{\mathfrak{E}\}, t; \{\mathfrak{E}'\}, t')$ is defined as the fraction of all particle systems in the ensemble which are in the phase space volume elements $d\{\mathfrak{E}\}$ about the phase space point $\{\mathfrak{E}\}$ at time t , and in the volume element $d\{\mathfrak{E}'\}$ about the phase space point $\{\mathfrak{E}'\}$ at time t' . In the equal time limit ($t = t'$)

$$D_2(\{\mathfrak{E}\}, t; \{\mathfrak{E}'\}, t) = D_1(\{\mathfrak{E}\}, t)\delta(\{\mathfrak{E}\} - \{\mathfrak{E}'\}) \quad (23)$$

where

$$\delta(\{\mathfrak{E}\} - \{\mathfrak{E}'\}) = \prod_{i=\alpha, \beta, \gamma, \dots} \prod_{j=1}^{N_i} \delta(\mathbf{r}_{ij} - \mathbf{r}'_{ij})\delta(\mathbf{p}_{ij} - \mathbf{p}'_{ij}) \quad (24)$$

Eq. (23) just expresses the fact that a system cannot be at two different places in the generalized $6N$ dimensional ($N \equiv \sum_{\alpha} N_{\alpha}$) phase space at the same time. The two-time, two-species Liouville function is defined as

$$D_{2\alpha\beta}(\{\mathfrak{E}_{\alpha}\}, t; \{\mathfrak{E}'_{\beta}\}, t') \equiv \int_{(\gamma, \delta, \dots \neq \alpha, \beta)} d\{\mathfrak{E}_{\beta}\}d\{\mathfrak{E}_{\gamma}\} \dots \int d\{\mathfrak{E}'_{\alpha}\}d\{\mathfrak{E}'_{\delta}\} \dots D_2(\{\mathfrak{E}\}, t; \{\mathfrak{E}'\}, t') \quad (25)$$

Note that α could be the same as β , ($\alpha = \beta$) in this definition (in that case, Eq. (25) defines the two-time, one-species Liouville function).

In the equal time limit ($t = t'$), Eqs. (25), (20), and (23) yield for unlike species ($\alpha \neq \beta$)

$$D_{2\alpha\beta}(\{\mathfrak{E}_{\alpha}\}, t; \{\mathfrak{E}'_{\beta}\}, t) = D_{1\alpha\beta}(\{\mathfrak{E}_{\alpha}\}, \{\mathfrak{E}'_{\beta}\}, t) \quad (26)$$

and for the same species ($\alpha = \beta$)

$$D_{2\alpha\alpha}(\{\mathfrak{E}_{\alpha}\}, t; \{\mathfrak{E}'_{\alpha}\}, t) = D_{1\alpha}(\{\mathfrak{E}_{\alpha}\}, t)\delta(\{\mathfrak{E}_{\alpha}\} - \{\mathfrak{E}'_{\alpha}\}) \quad (27)$$

The ensemble average of the product of two observable quantities $A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\})$ and $B(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}'(t')\})$ can then be defined as

$$\langle A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\})B(\mathbf{r}', \mathbf{p}', t'; \{\mathfrak{E}'(t')\}) \rangle \equiv \frac{\int d\{\mathfrak{E}\}d\{\mathfrak{E}'\}D_2(\{\mathfrak{E}\}, t; \{\mathfrak{E}'\}, t')}{A(\mathbf{r}, \mathbf{p}, t; \{\mathfrak{E}(t)\})B(\mathbf{r}', \mathbf{p}', t'; \{\mathfrak{E}'(t')\})} \quad (28)$$

This definition must now be applied to the products $F_{N\alpha}(1, t_1)F_{N\beta}(2, t_2)$ and $\delta F_{N\alpha}(1, t_1)\delta F_{N\beta}(2, t_2)$ to compute the collision integral.

A. Unlike Species, $\alpha \neq \beta$

Using Eqs. (28), (25), and (5),

$$\begin{aligned} \langle F_{N\alpha}(X, t)F_{N\beta}(X', t') \rangle &\equiv V^2 \int d2_\alpha d3_\alpha \dots dN_\alpha \int d2'_\beta d3'_\beta \dots dN'_\beta \\ &\quad D_{2\alpha\beta}(\{X, 2_\alpha, 3_\alpha, \dots, N_\alpha\}, t; \{X', 2'_\beta, 3'_\beta, \dots, N'_\beta\}, t') \\ &\equiv f_{\alpha\beta}(X, t; X', t') \end{aligned} \quad (29)$$

where, $f_{\alpha\beta}(X, t; X', t')dXdX'$ is the joint probability of finding a particle of species α in dX around X at t and a particle of species β in dX' around X' at t' . This function can be expanded as the sum of the product of one point functions plus an irreducible part which is the two point correlation

$$f_{\alpha\beta}(X, t; X', t') = f_\alpha(X, t)f_\beta(X', t') + G_{\alpha\beta}(X, t; X', t') \quad (30)$$

Eqs. (16), (29), and (30) imply that

$$G_{\alpha\beta}(X, t; X', t') = \langle \delta F_{N\alpha}(X, t)\delta F_{N\beta}(X', t') \rangle \quad (31)$$

In the equal time limit ($t = t'$), by virtue of Eq. (26)

$$\begin{aligned} \langle F_{N\alpha}(X, t)F_{N\beta}(X', t) \rangle &= V^2 \int d2_\alpha d3_\alpha \dots dN_\alpha \int d2'_\beta d3'_\beta \dots dN'_\beta \\ &\quad D_{1\alpha\beta}(\{X, 2_\alpha, 3_\alpha, \dots, N_\alpha\}, \{X', 2'_\beta, 3'_\beta, \dots, N'_\beta\}; t) \\ &\equiv f_{\alpha\beta}(X, X'; t) \end{aligned} \quad (29')$$

where $f_{\alpha\beta}(X, X'; t)$ is the one-time, two particle distribution which can be expanded as

$$f_{\alpha\beta}(X, X'; t) = f_{\alpha}(X, t)f_{\beta}(X', t) + g_{\alpha\beta}(X, X'; t) \quad (30')$$

so that $g_{\alpha\beta}(X, X'; t)$ is the one-time correlation, which is the equal time limit of $G_{\alpha\beta}(X, t; X', t')$

$$\lim_{t' \rightarrow t} G_{\alpha\beta}(X, t; X', t') = g_{\alpha\beta}(X, X'; t) \quad (31')$$

B. Like Species, $\alpha = \beta$

Using Eqs. (28), (25), (16), and (1) yields

$$\langle \delta F_{N_{\alpha}}(X, t) \delta F_{N_{\beta}}(X', t') \rangle = \frac{1}{N_{\alpha}} W_{\alpha}(X, t; X', t') + G_{\alpha\alpha}(X, t; X', t') \quad (32)$$

Here the function $W_{\alpha}(X, t; X', t')$ is defined by

$$\begin{aligned} W_{\alpha}(X, t; X', t') \equiv & V^2 \int d1_{\alpha} \dots d(i-1)_{\alpha} d(i+1)_{\alpha} \dots dN_{\alpha} \\ & \int d1'_{\alpha} \dots d(j-1)'_{\alpha} d(j+1)'_{\alpha} \dots dN'_{\alpha} \\ & D_{2\alpha\alpha}(\{1_{\alpha} \dots (i-1)_{\alpha}, X, (i+1)_{\alpha} \dots, N_{\alpha}\}, t; \\ & \{1'_{\alpha} \dots (i-1)'_{\alpha}, X', (i+1)'_{\alpha}, X', (j+1)'_{\alpha}, \dots, N'_{\alpha}\}, t') \end{aligned} \quad (33)$$

Physically, $W_{\alpha}(X, t; X', t')$ is the self-correlation which gives the probability that the same particle is at the point X at time t and at point X' at t' . The second term in Eq. (32) accounts for the correlation between two different particles of the same species.

In the equal time limit the Eq. (27) is used instead of Eq. (26). Using Eq. (27) in Eq. (33) then yields

$$W_{\alpha}(X, t; X', t) = V f_{\alpha}(X, t) \delta(X - X') \quad (34)$$

Eq. (32) becomes

$$\langle \delta F_{N\alpha}(X, t) \delta F_{N\alpha}(X', t) \rangle = \frac{1}{n_\alpha} f_\alpha(X, t) \delta(X - X') + g_{\alpha\alpha}(X, X'; t) \quad (35)$$

This completes the formal evaluation of Liouville averaged products of fluctuations. The results may be summarized as follows

$$\langle \delta F_{N\alpha}(X, t) \delta F_{N\beta}(X', t') \rangle = G_{\alpha\beta}(X, t; X', t') + \frac{\delta_{\alpha\beta}}{N_\alpha} W_\alpha(X, t; X', t') \quad (36a)$$

$$\lim_{t \rightarrow t' \rightarrow 0} G_{\alpha\beta}(X, t; X', t') \equiv g_{\alpha\beta}(X, X'; t) \quad (36b)$$

$$\lim_{t \rightarrow t' \rightarrow 0} W_\alpha(X, t; X', t') \equiv V f_\alpha(X, t) \delta(X - X') \quad (36c)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

The Liouville average of the Klimontovich equation can thus be evaluated, using Eqs. (36a)–(36b), and is

$$\begin{aligned} & \left[\partial_{t_1} + L_\alpha(1, t_1) - \sum_\beta \int d^2 \int_{-\infty}^{t_1} dt_2 V_{\alpha\beta}(1, t_1; 2, t_2) f_\beta(2, t_2) \right] f_\alpha(1, t_1) \\ &= \sum_\beta \int d^2 \int_{-\infty}^{t_1} dt_2 V_{\alpha\beta}(1, t_1; 2, t_2) \left[G_{\alpha\beta}(1, t_1; 2, t_2) + \frac{\delta_{\alpha\beta}}{N_\alpha} W_\alpha(1, t_1; 2, t_2) \right] \\ &\equiv C_\alpha(1, t_1) \end{aligned} \quad (37)$$

This equation is correct to all orders in the plasma discreteness parameter. The quantity $C_\alpha(1, t_1)$ is the collision integral for species α . The functions $f_\alpha(1, t_1)$ and $f_\beta(2, t_2)$ in Eq. (37) can be replaced by $F_{N\alpha}(1, t_1) - \delta F_{N\alpha}(1, t_1)$ and $F_{N\alpha}(2, t_2) - \delta F_{N\alpha}(2, t_2)$, respectively. Multiplying both terms of this equation by $\delta F_{N\gamma}(3, t_3)$ Liouville averaging and dropping the triplet correlation, yields the equation for the evolution of $G_{\alpha\beta}(1, t_1; 2, t_2)$

$$\begin{aligned} [\partial_{t_1} + M_\alpha(1, t_1)] G_{\alpha\beta}(1, t_1; 2, t_2) &= \sum_\gamma \int d^3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\gamma}(1, t_1; 3, t_3) \\ & \left[G_{\beta\gamma}(2, t_2; 3, t_3) + \frac{\delta_{\beta\gamma}}{N_\beta} W_\beta(2, t_2; 3, t_3) \right] f_\alpha(1, t_1) \end{aligned} \quad (38)$$

This equation is correct to first order in the plasma discreteness parameter. The operator $M_\alpha(1, t_1)$ is defined as

$$M_\alpha(1, t_1) \equiv L_\alpha(1, t_1) - \sum_\beta \int d2 \int_{-\infty}^{t_1} dt_2 V_{\alpha\beta}(1, t_1; 2, t_2) f_\beta(2, t_2) \quad (39)$$

The operator $M_\alpha(1, t_1)$ contains the effects of inhomogeneity and non-stationary behavior, which are not included in $L_\alpha(1, t_1)$.

The same technique is also applied in obtaining the equation for the evolution of $g_{\alpha\beta}(1, 2; t_1)$. Replacing $f_\alpha(1, t_1)$ and $f_\beta(2, t_2)$ in Eq. (37) as before by $F_{N\alpha}(1, t_1) - \delta F_{N\alpha}(1, t_1)$ and $F_{N\beta}(2, t_2) - \delta F_{N\beta}(2, t_2)$, multiplying by $\delta F_{N\gamma}(3, t_1)$ and continuing as before, taking into account Eq. (31c), yields the equation for $g_{\alpha\beta}(1, 3; t_1)$

$$\begin{aligned} [\partial_{t_1} + M_\alpha(1, t_1) + M_\beta(2, t_1)] g_{\alpha\beta}(1, 2; t_1) &\equiv \sum_\gamma \int d3 \int_{-\infty}^{t_1} dt_3 \{ \\ &\left[G_{\beta\gamma}(2, t_1; 3, t_3) + \frac{\delta_{\beta\gamma}}{N_\beta} W_\beta(2, t_1; 3, t_3) \right] V_{\alpha\gamma}(1, t_1; 3, t_3) f_\alpha(1, t_1) \\ &+ \left[G_{\alpha\gamma}(1, t_1; 3, t_3) + \frac{\delta_{\alpha\gamma}}{N_\alpha} W_\alpha(1, t_1; 3, t_3) \right] V_{\beta\gamma}(2, t_1; 3, t_3) f_\beta(2, t_1) \\ &- \delta(1-2) \frac{\delta_{\alpha\beta}}{n_\alpha} [\partial_{t_1} + M_\alpha(1, t_1)] f_\alpha(1, t_1) \} \end{aligned} \quad (40)$$

which is also correct to first-order in the discreteness parameter.

To complete the system of kinetic equations, the equation of evolution of the self-correlation is needed. The collision integral is here calculated to first order in the plasma discreteness parameter and therefore both $G_{\alpha\beta}(1, t_1; 2, t_2)$ and $\frac{W_\alpha(1, t_1; 2, t_2)}{N_\alpha}$ are required to first order, so that, $W_\alpha(1, t_1; 2, t_2)$ itself need only be correct to zero-th order.

The equation which $W_\alpha(1, t_1; 2, t_2)$ satisfies can be obtained most easily by introducing one-particle Klimontovich function for species α

$$F_{1\alpha}(X, t) \equiv \delta(X - X_{1\alpha}(t)) \quad (41)$$

where $X_{1\alpha}(t)$ is the particle orbit in phase space.

This microdensity clearly satisfies

$$\left[\partial_t + L_\alpha(X, t) - \sum_\beta \int_{-\infty}^t dX'' dt'' F_{N\beta}(X'' t'') V_{\alpha\beta}(X, t; X'', t'') \right] F_{1\alpha}(X, t) = 0 \quad (42)$$

Summing Eq. (42) over all N_α particles of species α yields the N_α particle Eq. (5). Multiplying Eq. (42) by $F_{1\alpha}(X', t')$, Liouville averaging and dropping terms of order $\delta F_{N\alpha}$ yields

$$[\partial_t + M_\alpha(X, t)] \langle F_{1\alpha}(X, t) F_{1\alpha}(X', t') \rangle = 0 \quad (43)$$

Noting that

$$\langle F_{1\alpha}(X, t) F_{1\alpha}(X', t') \rangle = \frac{1}{V^2} W_\alpha(X, t; X', t') \quad (44)$$

then implies that

$$[\partial_t + M_\alpha(X, t)] W_\alpha(X, t; X', t') = 0 \quad (45a)$$

and, in a similar manner

$$[\partial_{t'} + M_\alpha(X', t')] W_\alpha(X, t; X', t') = 0 \quad (45b)$$

Equations (37), (38), (40), (45), and the equal time conditions (36b) and (36c) form a closed system of kinetic equations whose solution is extremely complicated because of the coupling of $G_{\alpha\beta}(1, t_1; 2, t_2)$ to $g_{\alpha\beta}(1, t_1; 2, t_1)$ and $W_\alpha(1, t_1; 2, t_2)$.

IV. Superposition Principle

In this section the superposition form for the pair correlation functions $G_{\alpha\beta}(1, t_1; 2, t_2)$ and $g_{\alpha\beta}(1, 2; t_1)$ is formulated and proved for a multispecies plasma. The use of this form decouples the equations derived in the previous section and reduces the problem to solving two relatively simple and uncoupled equations.

The two-time pair correlation $G_{\alpha\beta}(1, t_1; 2, t_2)$ can be written according to the superposition form as follows

$$\begin{aligned}
 G_{\alpha\beta}(1, t_1; 2, t_2) = & \frac{1}{V} \int d^3 \int_{-\infty}^{t_2} dt_3 W_{\alpha}(1, t_1; 3, t_3) P_{\alpha\beta}(2, t_2; 3, t_3) \\
 & + \frac{1}{V} \int d^3 \int_{-\infty}^{t_1} dt_3 W_{\beta}(2, t_2; 3, t_3) P_{\alpha\beta}(1, t_1; 3, t_3) \\
 & + \frac{1}{V} \sum_{\gamma} n_{\gamma} \int d^3 \int d^4 \int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_2} dt_4 \\
 & W_{\gamma}(3, t_3; 4, t_4) P_{\alpha\gamma}(1, t_1; 3, t_3) P_{\beta\gamma}(2, t_2; 4, t_4)
 \end{aligned} \tag{1'}$$

where $P_{\alpha\beta}(1, t_1; 2, t_2)$ is a discreteness response function. (The equal time limit of Eq. (1') gives the corresponding superposition principle form for $g_{\alpha\beta}(1, 2; t)$.) The properties of $P_{\alpha\beta}(1, t_1; 2, t_2)$ will be closely examined in the next section. This superposition form satisfies the equations of evolution for $G_{\alpha\beta}(1, t_1; 2, t_2)$ and $g_{\alpha\beta}(1, 2; t)$ if $P_{\alpha\beta}(1, t_1; 2, t_2)$ obeys the following equations

$$\{\partial_{t_1} + \partial_{t_2} + M_{\alpha}(1, t_1) + M_{\beta}(2, t_2)\} P_{\alpha\beta}(1, t_1; 2, t_2) \tag{46a}$$

$$= \frac{1}{n_{\beta}} \hat{V}_{\alpha\beta}(1, t_1; 2, t_2) f_{\alpha}(1, t_1).$$

$$P_{\alpha\beta}(1, t_1; 2, t_2) = 0 \text{ for } t_1 < t_2 \tag{46b}$$

where

$$\begin{aligned}
 \hat{V}_{\alpha\beta}(1, t_1; 2, t_2) \equiv & V_{\alpha\beta}(1, t_1; 2, t_2) \\
 & + n_{\beta} \sum_{\gamma} \int d^3 \int_{t_2}^{t_1} dt_3 P_{\gamma\beta}(3, t_3; 2, t_2) V_{\alpha\gamma}(1, t_1; 3, t_3)
 \end{aligned} \tag{46c}$$

The first term on the right-hand side of Eq. (46c) is the "bare" electromagnetic interaction operator defined in Section I. In $P_{\alpha\beta}(1, t_1; 2, t_2)$ the labels $(1, t_1)$ and α correspond to a particle of species α at the phase space point 1 at time t_1 interacting with a particle of species β at the phase space point 2 at the time t_2 which corresponds to the labels $(2, t_2)$ and β . Particles of every species shield each of these particles, affecting the interaction between them. The labels $(3, t_3)$ and γ in Eq. (46c) correspond to such shielding particles of species γ . Summing over all species and integrating over their phase space position and over the time interval t_2 to t_1 thus provides this additional shielding effect which modifies the bare electromagnetic interaction operator $V_{\alpha\beta}(1, t_1; 2, t_2)$ yielding the shielded operator $\hat{V}_{\alpha\beta}(1, t_1; 2, t_2)$.

To prove the superposition form for $G_{\alpha\beta}(1, t_1, 2, t_2)$, Eq. (1') is substituted in Eq. (38), which yields the following eight terms

$$A_1 \equiv \frac{1}{V} \int d^3W_{\beta}(2, t_2; 3, t_1) P_{\alpha\beta}(1, t_1; 3, t_1) \quad (47a)$$

$$A_2 \equiv \frac{1}{V} \sum_{n_{\gamma}} n_{\gamma} \int d^3 \int d^4 \int_{-\infty}^{t_2} dt_4 W_{\gamma}(3, t_1; 4, t_4) P_{\alpha\gamma}(1, t_1; 3, t_1) P_{\beta\gamma}(2, t_2; 4, t_4) \quad (47b)$$

$$A_3 \equiv \frac{1}{V} \int d^3 \int_{-\infty}^{t_1} dt_3 \{ \partial_1 + M_{\alpha}(1, t_1) \} P_{\alpha\beta}(1, t_1; 3, t_3) W_{\beta}(2, t_2; 3, t_3) \quad (47c)$$

$$A_4 \equiv \frac{1}{V} \sum_{\gamma} n_{\gamma} \int d^3 \int d^4 \int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_2} dt_4 W_{\gamma}(3, t_3; 4, t_4) P_{\alpha\beta}(2, t_2; 4, t_4) \{ \partial_1 + M_{\alpha}(1, t_1) \} P_{\alpha\gamma}(1, t_1; 3, t_3) \quad (47d)$$

$$A_5 \equiv - \int d^3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\beta}(1, t_1; 3, t_3) \frac{1}{N_{\beta}} W_{\beta}(2, t_2; 3, t_3) f_{\alpha}(1, t_1) \quad (47e)$$

$$A_6 \equiv - \sum_{\delta} \int d^4 \int_{-\infty}^{t_3} dt_4 \int d^3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\delta}(1, t_1; 3, t_3) f_{\alpha}(1, t_1) \frac{1}{V} W_{\delta}(3, t_3; 4, t_4) P_{\beta\delta}(2, t_2; 4, t_4) \quad (47f)$$

$$A_7 \equiv - \sum_{\delta} \int d^4 \int_{-\infty}^{t_3} dt_4 \int d^3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\delta}(1, t_1; 3, t_3) f_{\alpha}(1, t_1) \frac{1}{V} W_{\beta}(2, t_2; 4, t_4) P_{\delta\beta}(3, t_3; 4, t_4) \quad (47g)$$

$$A_8 \equiv - \sum_{\delta} \sum_{\gamma} \int d^3 \int_{-\infty}^{t_1} dt_3 \int d^4 \int_{-\infty}^{t_2} dt_4 \int d^5 \int_{-\infty}^{t_3} dt_5 V_{\alpha\delta}(1, t_1; 3, t_3) f_{\alpha}(1, t_1) \frac{n_{\gamma}}{V} W_{\gamma}(4, t_4; 5, t_5) P_{\beta\gamma}(2, t_2; 4, t_4) P_{\delta\gamma}(3, t_3; 5, t_5) \quad (47h)$$

where the sum of them, $A_1 + A_2 + \dots + A_8$ must be proved to be zero. Noting that the electromagnetic interaction operator operates, through its first label, on the momentum coordinate of its operand [see Eq. (15)] and

that the M operator operates, also through its first label, on the space coordinate as well as on the momentum coordinate of its operand (see Eqs. (8) and (38)), and using Eq. (46), then yields

$$A_3 = -\frac{1}{V} \int d^3 W_\beta(2, t_2; 3, t_1) P_{\alpha\beta}(1, t_1; 3, t_1) \\ + \frac{1}{V} \int d^3 \int_{-\infty}^{t_1} dt_3 W_\beta(2, t_2; 3, t_3) \frac{\hat{V}_{\alpha\beta}(1, t_1; 3, t_3)}{n_\beta} f_\alpha(1, t_1) \quad (47i)$$

Adding Eq. (47a) to A_1 , A_5 , and A_7 , gives

$$A_1 + A_3 + A_5 + A_7 = \frac{1}{V} \int d^3 \int dt_3 \frac{1}{n_\beta} W_\beta(2, t_2; 3, t_3) \hat{V}_{\alpha\beta}(1, t_1; 3, t_3) f_\alpha(1, t_1) \\ - \frac{1}{V} \int d^3 \int_{-\infty}^{t_1} dt_3 \frac{1}{n_\beta} W_\beta(2, t_2; 3, t_3) \{V_{\alpha\beta}(1, t_1; 3, t_3) + n_\beta \sum_\delta \\ \int d^4 \int_{t_3}^{t_1} dt_4 V_{\alpha\delta}(1, t_1; 4, t_4) P_{\delta\beta}(4, t_4; 3, t_3)\} f_\alpha(1, t_1) \quad (47j)$$

which is identically zero by virtue of the definition of the shielded electromagnetic interaction operator (46b).

Therefore

$$A_1 + A_3 + A_5 + A_7 = 0 \quad (47k)$$

Applying the same facts to the sum $A_2 + A_4 + A_6 + A_8$ yields

$$A_2 + A_4 + A_6 + A_8 = \sum_\gamma n_\gamma \int d^4 \int_{-\infty}^{t_2} dt_4 \\ P_{\beta\gamma}(2, t_2; 4, t_4) \{A_1 + A_3 + A_5 + A_7\} \quad (47l)$$

which is zero by virtue of Eq. (47k). Therefore

$$A_1 + A_2 + \dots + A_7 + A_8 = 0 \quad (47m)$$

which proves the superposition form for $G_{\alpha\beta}$.

The superposition form for $g_{\alpha\beta}$ must now be proved. Substituting Eq. (1') in Eq. (40) yields the following terms

$$a_1 \equiv \frac{1}{V} \int d^3 W_\alpha(1, t_1; 3, t_1) P_{\alpha\beta}(2, t_1; 3, t_1) \quad (48a)$$

$$a_2 \equiv \frac{1}{V} \int d^3 W_\beta(3, t_1; 2, t_1) P_{\alpha\beta}(1, t_1; 3, t_1) \quad (48b)$$

$$a_3 \equiv \sum_\gamma \frac{n_\gamma}{V} \int d^3 \int d^4 \int_{-\infty}^{t_1} dt_3 W_\gamma(3, t_3; 4, t_1) P_{\alpha\gamma}(1, t_1; 3, t_3) P_{\beta\gamma}(2, t_1; 4, t_1) \quad (48c)$$

$$a_4 \equiv \sum_\gamma \frac{n_\gamma}{V} \int d^3 \int d^4 \int_{-\infty}^{t_1} dt_4 W_\gamma(3, t_1; 4, t_4) P_{\alpha\gamma}(1, t_1; 3, t_1) P_{\beta\gamma}(2, t_1; 4, t_4) \quad (48d)$$

$$a_5 \equiv \frac{1}{V} \int d^3 \int_{-\infty}^{t_1} dt_3 W_\alpha(1, t_1; 3, t_3) \{\partial_{t_1} + M_\beta(2, t_1)\} P_{\beta\alpha}(2, t_1; 3, t_3) \quad (48e)$$

$$a_6 \equiv \frac{1}{V} \int d^3 \int_{-\infty}^{t_1} dt_3 W_\beta(3, t_3; 2, t_1) \{\partial_{t_1} + M_\alpha(1, t_1)\} P_{\alpha\beta}(1, t_1; 3, t_3) \quad (48f)$$

$$a_{71} \equiv \sum_\gamma \frac{n_\gamma}{V} \int d^3 \int d^4 \int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_1} dt_4 W_\gamma(3, t_3; 4, t_4) P_{\beta\gamma}(2, t_1; 4, t_4) \{\partial_{t_1} + M_\alpha(1, t_1)\} P_{\alpha\gamma}(1, t_1; 3, t_3) \quad (48g)$$

$$a_{72} \equiv \sum_\gamma \frac{n_\gamma}{V} \int d^3 \int d^4 \int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_1} dt_4 W_\gamma(3, t_3; 4, t_4) P_{\alpha\gamma}(1, t_1; 3, t_3) \{\partial_{t_1} + M_\beta(2, t_1)\} P_{\beta\gamma}(2, t_1; 4, t_4) \quad (48h)$$

$$a_8 \equiv - \sum_\gamma \int d^3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\gamma}(1, t_1, 3, t_3) f_\alpha(1, t_1) \delta_{\beta\gamma} \frac{W_\beta(2, t_1; 3, t_3)}{N_\beta} \quad (48i)$$

$$a_9 \equiv - \sum_\gamma \int d^3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\gamma}(1, t_1; 3, t_3) f_\alpha(1, t_1) \frac{1}{V} \int d^4 \int_{-\infty}^{t_3} dt_4 W_\beta(2, t_1; 4, t_4) P_{\beta\gamma}(3, t_3; 4, t_4) \quad (48j)$$

$$a_{10} \equiv - \sum_{\gamma} \int d3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\gamma}(1, t_1; 3, t_3) f_{\alpha}(1, t_1) \frac{1}{V} \int d4 \int_{-\infty}^{t_1} dt_4 W_{\gamma}(4, t_4; 3, t_3) P_{\beta\gamma}(2, t_1; 4, t_4) \quad (48k)$$

$$a_{11} \equiv - \sum_{\gamma} \int d3 \int_{-\infty}^{t_1} dt_3 V_{\alpha\gamma}(1, t_1; 3, t_3) f_{\alpha}(1, t_1) \sum_{\delta} \frac{n_{\delta}}{V} \int d4 \int d5 \int_{-\infty}^{t_1} dt_4 \int_{-\infty}^{t_3} dt_5 W_{\delta}(4, t_4; 5, t_5) P_{\beta\delta}(2, t_1; 4, t_4) P_{\gamma\delta}(3, t_3; 5, t_5) \quad (48l)$$

$$a_{12} \equiv - \sum_{\gamma} \int d3 \int_{-\infty}^{t_1} dt_3 V_{\beta\gamma}(2, t_1; 3, t_3) f_{\beta}(2, t_1) \frac{\delta_{\alpha\gamma}}{N_{\alpha}} W_{\alpha}(1, t_1; 3, t_3) \quad (48m)$$

$$a_{13} \equiv - \sum_{\gamma} \int d3 \int_{-\infty}^{t_1} dt_3 V_{\beta\gamma}(2, t_1; 3, t_3) f_{\beta}(2, t_1) \frac{1}{V} \int d4 \int_{-\infty}^{t_3} dt_4 W_{\alpha}(1, t_1, 4, t_4) P_{\gamma\alpha}(3, t_3; 4, t_4) \quad (48n)$$

$$a_{14} \equiv - \sum_{\gamma} \int d3 \int_{-\infty}^{t_1} dt_3 V_{\beta\gamma}(2, t_1; 3, t_3) f_{\beta}(2, t_1) \frac{1}{V} \int d4 \int_{-\infty}^{t_1} dt_4 W_{\gamma}(4, t_4; 3, t_3) P_{\alpha\gamma}(1, t_1; 4, t_4) \quad (48o)$$

$$a_{15} \equiv - \sum_{\gamma} \int d3 \int_{-\infty}^{t_1} dt_3 V_{\beta\gamma}(2, t_1; 3, t_3) f_{\beta}(2, t_1) \sum_{\delta} \frac{n_{\delta}}{V} \int d4 \int d5 \int_{-\infty}^{t_1} dt_4 \int_{-\infty}^{t_3} dt_5 W_{\delta}(4, t_4; 5, t_5) P_{\alpha\delta}(1, t_1; 4, t_4) P_{\gamma\delta}(3, t_3; 5, t_5) \quad (48p)$$

where the sum of the a_i 's must be show to be zero.

By taking the equal time limit of Eq. (47m), it is immediately clear that

$$a_2 + a_4 + a_6 + a_{71} + a_8 + a_9 + a_{10} + a_{11} = \{A_1 + A_2 + \dots + A_7 + A_8\}_{t_2=t_1} = 0 \quad (48q)$$

and also that

$$a_1 + a_3 + a_5 + a_{72} + a_{12} + a_{13} + a_{14} + a_{15} = \{A_1 + A_2 + \dots + A_7 + A_8\}_{1 \leftrightarrow 2} = 0 \quad (48r)$$

where the notation $1 \leftrightarrow 2$ means that the indices 1, 2 have been interchanged. Therefore

$$a_1 + a_2 + \dots + a_{14} + a_{15} = 0 \quad (48s)$$

which proves the superposition form for $g_{\alpha,3}(1, 2; t)$.

V. The Discreteness Response Function

The physical interpretation of the discreteness response function $P_{\alpha\beta}(\mathbf{1}, t_1; \mathbf{2}, t_2)$, will be examined in this section. As well as in the preceding sections, homogeneity and stationarity are assumed for the system and external fields are excluded. Under these assumptions, $L_\alpha(\mathbf{1}, t_1)$ and $M_\alpha(\mathbf{1}, t_1)$ simplify to

$$M_\alpha(\mathbf{1}, t_1) = L_\alpha(\mathbf{1}, t_1) = \frac{p_1}{m_\alpha \gamma_{\alpha_1}} \cdot \partial_{\mathbf{r}} \quad (49)$$

Consider first the Vlasov limit of a multi-species plasma, so that Eq. (37) reduces to the Vlasov equation

$$\left\{ \partial_{t_1} + \frac{p_1}{m_\alpha \gamma_{\alpha_1}} \cdot \partial_{\mathbf{r}_1} - \sum_{\beta} \int d\mathbf{2} \int_{-\infty}^{t_1} dt_2 V_{\alpha\beta}(\mathbf{1}, t_1; \mathbf{2}, t_2) \right\} f_\beta(\mathbf{1}, t_1) = 0 \quad (50)$$

The distribution function response for the species α (or β) induced by a test particle of species s , is now derived. Expanding the distribution function around its equilibrium form (the medium is homogeneous and stationary) yields

$$f_\alpha(\mathbf{1}, t_1) = \bar{f}_\alpha(p_1) + \delta f_{\alpha s}(\mathbf{1}, t_1; \mathbf{0}, t_0) + \frac{1}{n_\alpha} \delta(\mathbf{1} - \tau) \delta_{s\alpha} \quad (51)$$

The first term accounts for the equilibrium distribution function. The second one refers to the distribution function perturbation due to a test particle of species s whose initial position and momentum at time t_0 denoted by $\mathbf{0}$, and whose current position in phase space is denoted by τ . The third term accounts for the contribution to the distribution if the test particle happens to be of the species α and at the phase space position $\mathbf{1}$ at the time t_1 . Note that these last two terms are first order in the discreteness parameter. Substituting Eq. (51) in Eq. (50) and dropping terms of second order in the discreteness parameter then yields

$$\begin{aligned} \left\{ \partial_{t_1} + \frac{p_1}{m_\alpha \gamma_{\alpha_1}} \cdot \partial_{\mathbf{r}_1} \right\} \delta f_{\alpha s}(\mathbf{1}, t_1; \mathbf{0}, t_0) &= \sum_{\beta} \int d\mathbf{2} \int_{-\infty}^{t_1} dt_2 \delta f_{\beta s}(\mathbf{2}, t_2; \mathbf{0}, t_0) V_{\alpha\beta}(\mathbf{1}, t_1; \mathbf{2}, t_2) \bar{f}_\alpha(p_1) \\ &+ \sum_{\beta} \frac{1}{n_\beta} \int d\mathbf{2} \int dt_2 \delta(\mathbf{2} - \tau) \delta_{s\beta} V_{\alpha\beta}(\mathbf{1}, t_1; \mathbf{2}, t_2) \bar{f}_\alpha(p_1) \end{aligned} \quad (52)$$

The perturbation induced in the distribution function by the test particle can also be formally expressed as

$$\delta f_{\alpha s}(1, t_1; 0, t_0) = \int_{-\infty}^{t_1} dt_T \prod_{\alpha T} (1, t_1; \tau, t_T) \quad (53)$$

where

$$\prod_{\alpha T} (1, t_1; \tau, t_T) \equiv \frac{d}{dt_T} \delta f_{\alpha s}(1, t_1; \tau, t_T) \quad (54)$$

Physically, this expresses the fact that the test particle interacts over its whole orbit history with the species α , inducing a rate of response in f_α given by the quantity $\prod_{\alpha T}(1, t_1; \tau, t_T)$. Substituting Eq. (53) in Eq. (52) for $\delta f_{\alpha s}$ and $\delta f_{\beta s}$ then gives

$$\begin{aligned} \left(\partial_{t_1} + \frac{p_1}{m_\alpha \gamma_{\alpha 1}} \cdot \partial_{r_1} \right) \int_{-\infty}^{t_1} dt_T \prod_{\alpha s} (1, t_1; \tau, t_T) &= \sum_{\beta} \int d^2 \int_{-\infty}^{t_1} dt_T \int_{t_T}^{t_1} dt_2 \prod_{\beta s} (2, t_2; \tau, t_T) \\ &V_{\alpha\beta}(1, t_1; 2, t_2) \bar{f}_\alpha(p_1) + \frac{1}{n_s} \int_{-\infty}^{t_1} dt_T V_{\alpha s}(1, t_1; \tau t_T) \bar{f}_\alpha(p_1) \end{aligned} \quad (55)$$

Taking into account Eq. (46c), Eq. (55) can be rewritten as

$$\begin{aligned} \int_{-\infty}^{t_1} dt_T \left[\left(\partial_{t_1} + \partial_{t_T} + \frac{p_1}{m_\alpha \gamma_{\alpha 1}} \cdot \partial_{r_1} + \frac{p_T}{m_s \gamma_{s T}} \cdot \partial_{r_T} \right) \prod_{\alpha s} (1, t_1; \tau, t_T) \right. \\ \left. - \frac{\hat{V}_{\alpha s}(1, t_1; \tau, t_T)}{n_s} \bar{f}_\alpha(p_1) \right] = 0 \end{aligned} \quad (56)$$

which is true for arbitrary value of t_1 and so implies that

$$\begin{aligned} \left\{ \partial_{t_1} + \partial_{t_T} + \frac{p_1}{m_\alpha \gamma_{\alpha 1}} \cdot \partial_{r_1} + \frac{p_T}{m_s \gamma_{s T}} \cdot \partial_{r_T} \right\} \prod_{\alpha s} (1, t_1; \tau t_T) \\ = \frac{\hat{V}_{\alpha s}(1, t_1; \tau, t_T)}{n_s} \bar{f}_\alpha(p_1) \end{aligned} \quad (57)$$

Comparing Eq. (57) to Eq. (46) then yields, by virtue of Eq. (54)

$$P_{\alpha\beta}(1, t_1; 2, t_2) = \frac{d}{dt_2} \delta f_{\alpha\beta}(1, t_1; 2, (t_2)t_2) \quad (58)$$

which says that the discreteness response function $P_{\alpha\beta}(1, t_1; 2, t_2)$ can be identified as the total rate of change of the perturbation of the equilibrium distribution function of species α at 1 at time t_1 , induced by a test particle of species β whose orbit is $2(t_2)$. The total time derivative d/dt is equivalent to the time derivative in the test particle rest frame.

VI. Solution of the Kinetic Equations

The collision integral is expressed in terms of the discreteness response and self correlation functions. These functions satisfy relatively simple and most important, uncoupled equations of evolution in the absence of external fields, inhomogeneities and non-stationary behavior. The time evolution of the self correlation function involves causal and acausal propagators. The solution of the equation for the discreteness response function is facilitated by utilizing a continuity relationship between two of its moments, i.e., the density and current response functions induced by discreteness. The discreteness response function is, however finally expressed in terms of the induced current response function but this is precisely the form which will be used in the next section for the calculation of the generalized collision integral.

Substituting Eq. (1') in Eq. (37) yields for the collision integral

$$C_\alpha(1, t_1) = \sum_\beta \frac{1}{N_\beta} \int d2 \int_{-\infty}^{t_1} dt_2 \hat{V}_{\alpha\beta}(1, t_1; 2, t_2) \left[\delta_{\alpha\beta} W_\beta(1, t_1; 2, t_2) \right. \\ \left. + n_\beta \int d3 \int_{-\infty}^{t_1} dt_3 P_{\alpha\beta}(1, t_1; 3, t_3) W_\beta(3, t_3; 2, t_2) \right] \quad (59)$$

The first term accounts for the contributions to the collision integral due to the retarded interaction of a particle of species α at 1 at time t_1 with its own field; this interaction is shielded due to the presence of all the other particles. The second term is the contribution due to the shielded retarded interaction of a particle of species α at 1 at time t_1 with the fields of the rest of the particles. The equation for $P_{\alpha\beta}(1, t_1; 2, t_2)$, under the assumptions of Section V, becomes

$$\left[\partial_{t_1} + \partial_{t_2} + \frac{p_1}{m_\alpha \gamma_{\alpha 1}} \cdot \partial_{r_1} + \frac{p_2}{m_\beta \gamma_{\beta 2}} \cdot \partial_{r_2} \right] P_{\alpha\beta}(1, t_1; 2, t_2)$$

$$= \frac{1}{n_\beta} \hat{V}_{\alpha\beta}(1, t_1; 2, t_2) f_\alpha(p_1) \quad (60)$$

Eqs. (45a) and (45b) for the self correlation function $W_\alpha(1, t_1; 2, t_2)$, and its equal time limit Eq. (36c), also simplify to

$$\left[\partial_{t_1} + \frac{p_1}{m_\alpha \gamma_{\alpha 1}} \cdot \partial_{r_1} \right] W_\alpha(1, t_1; 2, t_2) = 0 \quad (61a)$$

$$\left[\partial_{t_2} + \frac{p_2}{m_\alpha \gamma_{\alpha 2}} \cdot \partial_{r_2} \right] W_\alpha(1, t_1; 2, t_2) = 0 \quad (61b)$$

$$W_\alpha(1, t_1; 2, t_1) = V f_\alpha(p_1) \delta(1 - 2) \quad (61c)$$

Eqs. (60) and (61) with Eq. (59), as the new form for the collision integral, form the new system of kinetic equations to be solved.

The advantage of this system over the previous one is apparent in the simplicity of the equation for the discreteness response function. There are only two equations to be solved and, most importantly, they are not coupled. Equations (61a) and (61) consist of an initial and a final value problem. Equations (61a) and (61b) are true for times $t_1 > t_2$ and $t_2 > t_1$, respectively. Therefore Eq. (61a) can be solved as an initial value problem and Eq. (61b) as a final value problem.

Introducing the one-sided functions $W_\alpha^\pm(1, t_1; 2, t_2)$

$$W_\alpha^\pm(1, t_1; 2, t_2) = W_\alpha(1, t_1; 2, t_2) H(\pm(t_1 - t_2)) \quad (62a)$$

where

$$H(t) \equiv \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad (62b)$$

It is evident that the functions W_{α}^{+} and W_{α}^{-} satisfy (61a) and (61b), respectively, and that both satisfy the equal time condition (61c).

The Fourier transform of the self correlation function W_{α} in the difference variables in space and time (which is appropriate since the system is homogeneous and stationary) satisfies

$$W_{\alpha,k,\omega}(p_1, p_2) = W_{\alpha,k,\omega}^{+}(p_1, p_2) + W_{\alpha,k,\omega}^{-}(p_1, p_2) \quad (63)$$

Here the vector k labels the Fourier transform in $r_1 - r_2$, and the scalar ω the Laplace transform in $t_1 - t_2$. Fourier transforming Eqs. (54a) - (54b) and taking into account Eq. (62) gives

$$W_{\alpha,k,\omega}^{+}(p_1, p_2) = \frac{Vf(p_1)\delta(p_1 - p_2)}{-i\omega + ik \cdot v_{1\alpha}} \quad (64a)$$

$$W_{\alpha,k,\omega}^{-}(p_1, p_2) = \frac{Vf(p_1)\delta(p_1 - p_2)}{-i\omega + ik \cdot v_{1\alpha}} \quad (64b)$$

Taking into account the causal (anticausal) properties of the function $W_{\alpha,k,\omega}^{+}(W_{\alpha,k,\omega}^{-})$, Eqs. (63) and (64) finally yield

$$W_{\alpha,k,\omega}(p_1, p_2) = 2\pi Vf(p_1)\delta(p_1 - p_2)\delta(\omega - k \cdot v_{1\alpha}) \quad (65)$$

which gives the solution for the self correlation function $W_{\alpha}(1, t_1; 2, t_2)$ in Fourier space.

The next step is to solve Eq. (60) for the discreteness response function $P_{\alpha\beta}(1, t_1; 2, t_2)$. Homogeneity implies that the function $P_{\alpha\beta}(1, t_1; 2, t_2)$ is only a function of $r_1 - r_2$. In general, $P_{\alpha\beta}(1, t_1; 2, t_2)$ can be expressed as a function of $t_1 - t_2$ and $t_1 + t_2$. Furthermore, $f_{\alpha}(1, t_1)$ and $f_{\beta}(2, t_2)$ evolve on a time scale large compared to the time scale on which the discreteness response function $P_{\alpha\beta}(1, t_1; 2, t_2)$ evolves (this is the usual Bogoliubov ansatz). Fourier transforming in space ($r_1 - r_2 \rightarrow k$) and Laplace transforming in time ($t_1 + t_2 \rightarrow \omega_s; t_1 - t_2 \rightarrow \omega_d$) then yields

$$\begin{aligned}
& \left[-i\omega_s + i\mathbf{k} \cdot (\mathbf{v}_{1\alpha} - \mathbf{v}_{2\beta}) \right] \cdot \left[-i\omega_s P_{\alpha\beta} k_{\omega_s, \omega_d}(\mathbf{p}_1, \mathbf{p}_2) \right] - \left[-i\omega_s P_{\alpha\beta} k_{\omega_s, \omega_d}(\mathbf{p}_1, \mathbf{p}_2; t_1 + t_2 = 0) \right] \\
& = V_{\alpha\beta} k_{\omega_s, \omega_d}(\mathbf{p}_1, \mathbf{p}_2) \frac{f_\alpha(\mathbf{p}_1)}{n_\beta} + \sum_\gamma \int d^3 p_3 \int_{-\infty+ic}^{-\omega+ic} \frac{d\omega'_s}{2\pi} \left[-i\omega_s V_{\alpha\gamma} k_{\omega_d, \omega'_s}(\mathbf{p}_1, \mathbf{p}_3) \right. \\
& \quad \left. P_{\gamma\beta} k_{\omega_s - \omega'_s}(\mathbf{p}_3, \mathbf{p}_2) \right] f_\alpha(\mathbf{p}_1) \tag{66}
\end{aligned}$$

Therefore, by employing the final value theorem for the Laplace transform of the function $P_{\alpha\beta}(1, t_1; 2, t_2)$, the time asymptotic, stationary function to which the function $P_{\alpha\beta}(1, t_1; 2, t_2)$ relaxes is obtained. Applying the final value theorem to Eq. (66) then gives

$$\begin{aligned}
& i\mathbf{k} \cdot (\mathbf{v}_{1\alpha} - \mathbf{v}_{2\beta}) P_{\alpha\beta} k_{\omega}(\mathbf{p}_1, \mathbf{p}_2) \\
& + V_{\alpha\beta} k_{\omega}(\mathbf{p}_1, \mathbf{p}_2) \frac{f_\alpha(\mathbf{p}_1)}{n_\beta} + \sum_\gamma \int d^3 p_3 V_{\alpha\gamma} k_{\omega}(\mathbf{p}_1, \mathbf{p}_3) P_{\gamma\beta} k_{\omega}(\mathbf{p}_3, \mathbf{p}_2) f_\alpha(\mathbf{p}_1) \\
& + \frac{\hat{V}_{\alpha\beta} k_{\omega}(\mathbf{p}_1, \mathbf{p}_2) f_\alpha(\mathbf{p}_1)}{n_\beta} \tag{67}
\end{aligned}$$

where $\omega \equiv \omega_d$.

Eq. (67) is now interpreted as the Fourier transform in the difference variables in both space and time, since it describes stationary correlations. Note also, that while such a function $P_{\alpha\beta} k_{\omega}(\mathbf{p}_1, \mathbf{p}_2)$ does not satisfy Eq. (60c), this is unimportant to the calculation of the collision integral, where only products of $P_{\alpha\beta}$'s and $V_{\alpha\beta}$'s appear. Equation (24) can also be written as

$$\begin{aligned}
C_\alpha(\mathbf{k}, \omega; \mathbf{p}_1) & = \sum_\beta \frac{1}{N_\beta} \int d^3 p_2 \hat{V}_{\alpha\beta}^* k_{\omega}(\mathbf{p}_1, \mathbf{p}_2) \left[\delta_{\alpha\beta} W_{\beta} k_{\omega}(\mathbf{p}_1, \mathbf{p}_2) \right. \\
& \quad \left. + n_\beta \int d^3 p_3 P_{\alpha\beta} k_{\omega}(\mathbf{p}_1, \mathbf{p}_3) W_{\beta} k_{\omega}(\mathbf{p}_3, \mathbf{p}_2) \right] \tag{68a}
\end{aligned}$$

where the quantity $C_\alpha(\mathbf{k}, \omega; \mathbf{p}_1)$ satisfies

$$C_\alpha(\mathbf{p}_1) = \int \frac{d\omega d^3 k}{(2\pi)^4} C_\alpha(\mathbf{k}, \omega; \mathbf{p}_1), \tag{68b}$$

the \mathbf{k}, ω components of all the involved quantities but $C_\alpha(\mathbf{k}, \omega; \mathbf{p}_1)$ are Fourier components, and * denotes complex conjugation. Fourier transforming the electromagnetic interaction operator defined in Eq. (15) yields

$$V_{\alpha\beta k, \omega}(p_1, p_2) = \frac{i4\pi n_\beta q_\alpha q_\beta}{k^2 - \frac{\omega^2}{c^2}} \left[k - \omega \frac{v_{\beta 2}}{c^2} \cdot \Phi(k, \omega; v_{\alpha 1}) \right] \cdot \partial_{p_1} \quad (69a)$$

where

$$\Phi(k, \omega; v_{\alpha 1}) = \left(1 - \frac{k \cdot v_{\alpha 1}}{\omega}\right) I + \frac{v_{\alpha 1} k}{\omega} \quad (69b)$$

Two moments of the transformed discreteness response function $P_{\gamma\beta k, \omega}(p_3, p_2)$ are defined

$$\hat{n}_{\gamma\beta k, \omega}(p_2) \equiv n_\gamma \int d^3 p_3 P_{\gamma\beta k, \omega}(p_3, p_2) \quad (4a)$$

$$\hat{J}_{\gamma\beta k, \omega}(p_2) \equiv n_\gamma \int d^3 p_3 P_{\gamma\beta k, \omega}(p_3, p_2) v_{\alpha 3} \quad (4b)$$

which are the density and current perturbation respectively, induced by discreteness..

Using Eqs. (4) and (69) in Eq. (67) yields

$$\begin{aligned} P_{\alpha\beta k, \omega}(p_1, p_2) &= \frac{4\pi q_\alpha q_\beta}{k^2 - \frac{\omega^2}{c^2}} \frac{k - \frac{\omega}{c^2} v_{\beta 2} \cdot \Phi(k, \omega; v_{\alpha 1})}{k \cdot (v_{\alpha 1} - v_{\beta 2})} \cdot \partial_{p_1} f_\alpha(p_1) \\ &+ \sum_\gamma \frac{4\pi q_\alpha q_\gamma}{k^2 - \frac{\omega^2}{c^2}} \frac{\hat{n}_{\gamma\beta k, \omega}(p_2) k - \frac{\omega}{c^2} \hat{J}_{\gamma\beta k, \omega}(p_2) \cdot \Phi(k, \omega; v_{\alpha 1})}{k \cdot (v_{\alpha 1} - v_{\beta 2})} \\ &\cdot \partial_{p_1} f_\alpha(p_1) \end{aligned} \quad (70)$$

A continuity relation between the density and the current perturbations is proved as follows. Using Eq. (70) in Eq. (2b), dot multiplying by the vector k and dividing by the frequency ω yields

$$\begin{aligned} \frac{k \cdot \hat{J}_{\gamma\beta k, \omega}(p_2)}{\omega} &= \frac{4\pi n_\gamma q_\gamma}{k^2 - \frac{\omega^2}{c^2}} \sum_\delta \frac{q_\delta k \cdot v_{\beta 2}}{\omega} \left\{ \left[\delta_{\delta\beta} + \hat{n}_{\delta\beta k, \omega}(p_2) \right] \right. \\ &\left. \int d^3 p_3 \frac{k \cdot \partial_{p_3} f_\gamma(p_3)}{k \cdot (v_{\gamma 3} - v_{\delta 2})} - \frac{\omega}{c^2} \left[\delta_{\delta\beta} v_{\delta 2} + \hat{J}_{\delta\beta k, \omega}(p_2) \right] \cdot \int d^3 p_3 \frac{\Phi(k, \omega; v_{\gamma 3}) \cdot \partial_{p_3} f_\gamma(p_3)}{k \cdot (v_{\gamma 3} - v_{\delta 2})} \right\} \end{aligned} \quad (71)$$

where the property

$$\partial_{p_3} \cdot \Phi(k, \omega; v_{\gamma 3}) = 0 \quad (72)$$

has been used.

Substituting the expression for the function $P_{\gamma\beta k, \omega}(p_3, p_2)$ in the definition (2a) for the density perturbation $\hat{n}_{\gamma\beta k, \omega}(p_2)$ yields

$$\begin{aligned} \hat{n}_{\gamma\beta k, \omega}(p_2) = & \frac{4\pi n_{\gamma} q_{\gamma}}{k^2 - \frac{\omega^2}{c^2}} \sum_{\delta} \left\{ \left[\delta_{\delta\beta} + \hat{n}_{\delta\beta k, \omega}(p_2) \right] \int d^3 p_3 \frac{k \cdot \partial_{p_3} f_{\gamma}(p_3)}{k \cdot (v_{\gamma 3} - v_{\beta 2})} \right. \\ & \left. - \frac{\omega}{c^2} \left[\delta_{\delta\beta} v_{\delta 2} + \hat{J}_{\delta\beta k, \omega}(p_2) \right] \cdot \int d^3 p_3 \frac{\Phi(k, \omega; v_{\gamma 3}) \cdot \partial_{p_3} f_{\gamma}(p_3)}{k \cdot (v_{\gamma 3} - v_{\beta 2})} \right\} \end{aligned} \quad (73)$$

Comparing Eq. (73) to Eq. (71) gives

$$\frac{k \cdot \hat{J}_{\gamma\beta k, \omega}(p_2)}{\omega} = \hat{n}_{\gamma\beta k, \omega}(p_2) \quad (74)$$

provided that $\omega = k \cdot v_2^{\beta}$. The condition $\omega = k \cdot v_2^{\beta}$ is automatically satisfied because of the presence of the delta function $\delta(\omega - k \cdot v_2^{\beta})$ in the collision integral, as will be shown in Sec. VII. Using Eq. (74) in Eq. (70) finally gives

$$\begin{aligned} P_{\alpha\beta k, k \cdot v_{\beta 2}}(p_1, p_2) = & \frac{4\pi q_{\alpha}}{1 - (\hat{k} \cdot \beta_{\beta 2})^2} \frac{1}{k \cdot v_{\beta 2} (k \cdot v_{\alpha 1} - k \cdot v_{\beta 2})} \left\{ \sum_{\gamma} q_{\gamma} \left[\delta_{\gamma\beta} v_{\gamma 2} + \hat{J}_{\gamma\beta k, k \cdot v_{\beta}}(p_2) \right] \right\} \\ & \cdot \left[\hat{k} \hat{k} - (\hat{k} \cdot \beta_{\beta 2})^2 \Phi(k, k \cdot v_{\beta}; v_{\alpha 1}) \right] \cdot \partial_{p_1} f_{\alpha}(p_1) \end{aligned} \quad (75a)$$

where

$$\hat{k} = \frac{k}{|k|} \quad (75b)$$

VII. Generalized Collision Integral

In this section, the induced current response is related to the dispersion tensor of the relativistic multi-species plasma. This makes possible the expression of the collision integral in a generalized Balescu-Lenard form. The form, in terms of the longitudinal and transverse dielectric functions is also obtained. This second form provides easily the collision integral in the absence of shielding. Substituting Eqs. (75) in the definition (4b) for the induced current response function $\hat{J}_{\alpha\beta k, k, v_{\beta 2}}(p_2)$ and summing over all species, yields

$$\sum_{\gamma} q_{\gamma} \hat{J}_{\gamma\beta k, k, v_{\beta 2}} = \frac{q_{\beta} v_{\beta 2} + \sum_{\gamma} q_{\gamma} \hat{J}_{\gamma\beta k, k, v_{\beta 2}}(p_2)}{[1 - (\hat{k} \cdot \beta_{\beta 2})^2] k c (\hat{k} \cdot \beta_{\beta 2})}$$

$$\sum_{\alpha} 4\pi q_{\alpha}^2 n_{\alpha} \int d^3 p_1 \frac{\hat{k}\hat{k} - (\hat{k} \cdot \beta_{\beta 2}) \Phi(k, k \cdot v_{\beta 2}; v_{\beta 2})}{k \cdot (v_{\alpha 1} - v_{\beta 2})} \cdot \partial_{p_1} f_{\alpha}(p_1) \frac{p_1}{m_{\alpha}}$$

which also can be written as

$$\left[q_{\beta} v_{\beta 2} + \sum_{\gamma} q_{\gamma} \hat{J}_{\gamma\beta k, k, v_{\beta 2}}(p_2) \right] \cdot \left[I - \frac{1}{1 - (\hat{k} \cdot \beta_{\beta 2})^2} \sum_{\gamma} \frac{\omega_{p\alpha}^2}{k \cdot v_{\beta 2}} \int d^3 p_1 \frac{\hat{k}\hat{k} - (\hat{k} \cdot \beta_{\beta 2})^2 \Phi(k, k \cdot v_{\beta 2}; v_{\alpha 1})}{k \cdot (v_{\alpha 1} - v_{\beta 2})} \cdot \partial_{p_1} f_{\alpha}(p_1) \frac{p_1}{\gamma_{\alpha 1}} \right] = q_{\beta} v_{\beta 2} \quad (76)$$

Now, the transpose of the relativistic, electromagnetic dielectric tensor $\epsilon_{k, \omega}$ can be written as

$$\epsilon_{k, \omega}^T = I - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega} \int d^3 p \frac{\Phi(k, \omega; v_{\alpha})}{k \cdot v_{\alpha} - \omega} \cdot \partial_p f_{\alpha}(p) \frac{p}{\gamma_{\alpha}} \quad (77)$$

Using Eq. (77) in Eq. (76) finally yields

$$\left[I - \frac{(I - \epsilon_{k, k, v_{\beta 2}}) \cdot \hat{k}\hat{k}}{1 - (\hat{k} \cdot \beta_{\beta 2})^2} + \frac{(\hat{k} \cdot \beta_{\beta 2})^2}{1 - (\hat{k} \cdot \beta_{\beta 2})^2} (I - \epsilon_{k, k, v_{\beta 2}}) \right] \cdot [q_{\beta} v_{\beta 2} + \sum_{\gamma} q_{\gamma} \hat{J}_{\gamma\beta k, k, v_{\beta 2}}(p_2)] = q_{\beta} v_{\beta 2} \quad (78)$$

The dispersion tensor Z is defined¹² as

$$Z = \frac{I_L}{\epsilon_L} + \frac{I_T}{\epsilon_T - \frac{\omega^2}{k^2 c^2}} \quad (79a)$$

where the scalars ϵ_L and ϵ_T are the longitudinal and transverse dielectric functions, respectively

$$\epsilon_L = \frac{\mathbf{k} \cdot \boldsymbol{\epsilon} \cdot \mathbf{k}}{k^2} \equiv \hat{\mathbf{k}} \cdot \boldsymbol{\epsilon} \cdot \hat{\mathbf{k}} \quad (79b)$$

$$\epsilon_T I_T = \boldsymbol{\epsilon} - \epsilon_L I_L \quad (79c)$$

with the projectors I_L and I_T defined as

$$I_L \equiv \hat{\mathbf{k}}\hat{\mathbf{k}}, I_T \equiv I - \hat{\mathbf{k}}\hat{\mathbf{k}} \quad (79d)$$

Using Eqs. (79) in Eq. (76), yields

$$q_{\beta} \mathbf{v}_{\beta 2} + \sum_{\gamma} q_{\gamma} \hat{\mathbf{J}}_{\gamma \beta} \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}(p_2) = q_{\beta} \mathbf{v}_{\beta 2} \cdot Z_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}} \left[I_L - \frac{1 - (\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 2})^2}{(\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 2})^2} I_T \right] \quad (80)$$

Substituting Eq. (80) in Eq. (75) and taking into account Eq. (67) yields

$$P_{\alpha \beta} \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}(p_1, p_2) = \frac{\hat{V}_{\alpha \beta} \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}(p_1, p_2) f_{\alpha}(p_1)}{i n_{\beta} \mathbf{k} \cdot (\mathbf{v}_{\alpha 1} - \mathbf{v}_{\beta 2})} \quad (81)$$

where

$$\hat{V}_{\alpha \beta} \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}(p_1, p_2) = \frac{i 4 \pi q_{\alpha} q_{\beta} n_{\beta}}{\mathbf{v}_{\beta 2} \cdot Z_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}} \left[I_L - \frac{1 - (\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 2})^2}{(\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 2})^2} I_T \right] \cdot \left[\hat{\mathbf{k}}\hat{\mathbf{k}} - (\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 2})^2 \Phi(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 2}; \mathbf{v}_{\alpha 1}) \right]}{\mathbf{k} \cdot \mathbf{v}_{\beta 2} \left[1 - (\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 2})^2 \right]} \cdot \partial p_1 \quad (82)$$

Substituting now the expressions for the functions $W_{\beta} \mathbf{k}, \omega(p_1, p_2)$ and $P_{\alpha \beta} \mathbf{k}, \omega(p_1, p_2)$ in Eq. (68) gives

$$\begin{aligned}
C_{\alpha k, \omega}(p_1) &= \sum_{\beta} \frac{2\pi}{n_{\beta}} \int d^3 p_2 \hat{V}_{\alpha\beta k, \omega}^*(p_1, p_2) [f_{\beta}(p_1) \delta(p_1 - p_2) \delta(\omega - k \cdot v_{\beta 2}) \delta_{\alpha\beta}] \\
&\quad + \sum_{\beta} 2\pi \int d^3 p_2 \int d^3 p_3 \hat{V}_{\alpha\beta k, \omega}^*(p_1, p_2) \left[\frac{\hat{V}_{\alpha\beta k, \omega}(p_1, p_2) f_{\alpha}(p_1)}{i n_{\beta} k \cdot (v_{\alpha 1} - v_{\beta 3})} \right] f_{\beta}(p_3) \\
&\quad \delta(p_3 - p_2) \delta(\omega - k \cdot v_{\beta 3})
\end{aligned} \tag{83}$$

Because of the delta function $\delta(p_3 - p_2)$, the causal nature of the integral in the second term of the right-hand side of Eq. (83) and the fact that the principal value term is zero (since the integrand is odd under $(\omega, k) \rightarrow (-\omega, -k)$; everything else is even in this term). Eq. (83) can be written after performing the ω -integration, as

$$\begin{aligned}
C_{\alpha k}(p_1) &\equiv \int \frac{d\omega}{2\pi} C_{\alpha k, \omega}(p_1) \\
&= \sum_{\beta} \frac{\pi}{n_{\beta}} \int d^3 p_2 f_{\beta}(p_2) \hat{V}_{\alpha\beta k, k \cdot v_{\beta 2}}^*(p_1, p_2) \left[\delta(k \cdot v_{\beta 2} - k \cdot v_{\alpha 1}) \hat{V}_{\alpha\beta k, k \cdot v_{\beta 2}}(p_1, p_2) f_{\alpha}(p_1) \right] \\
&\quad + \sum_{\beta} \frac{1}{n_{\beta}} \int d^3 p_2 f_{\beta}(p_2) \delta_{\alpha\beta} \hat{V}_{\alpha\beta k, k \cdot v_{\beta 2}}^*(p_1, p_2) \delta(p_1 - p_2)
\end{aligned} \tag{84}$$

The presence now of the delta function $\delta(k \cdot v_{\beta 2} - k \cdot v_{\alpha 1})$ in Eq. (84) makes Eq. (82) equivalent (for the purposes of substituting Eq. (82) into Eq. (84)) to

$$\hat{V}_{\alpha\beta k, k \cdot v_{\beta 2}}(p_1, p_2) = i \frac{4\pi q_{\alpha} q_{\beta} n_{\beta}}{(k \cdot v_{\beta 2})^2} v_{\beta 2} \cdot Z_{k, k \cdot v_{\beta 2}} \cdot v_{\alpha 1} k \cdot \partial_{p_1} \tag{82'}$$

Performing the k -integral in Eq. (84) and taking into account Eq. (79) finally yields

$$\begin{aligned}
C_{\alpha}(p_1) &\equiv \int \frac{d^3 k}{(2\pi)^3} C_{\alpha k}(p_1) \\
&= \sum_{\beta} 2q_{\alpha}^2 q_{\beta}^2 n_{\beta} \partial_{p_1} \int d^3 k \frac{\delta(k \cdot v_{\beta 2} - k \cdot v_{\alpha 1})}{(k \cdot v_{\beta 2})^4} |v_{\beta 2} \cdot Z_{k, k \cdot v_{\beta 2}} \cdot v_{\alpha 1}|^2 k k \cdot \partial_{p_1} f_{\alpha}(p_1) f_{\beta}(p_2) \\
&\quad - \sum_{\beta} i \delta_{\alpha\beta} \frac{2q_{\alpha} q_{\beta} n_{\beta}}{(2\pi)^2 n_{\beta}} \partial_{p_1} \int \frac{d^3 k}{(k \cdot v_{\beta 1})^2} f_{\alpha}(p_1) k v_{\beta 1} \cdot Z_{k, k \cdot v_{\beta 1}} \cdot v_{\alpha 1}
\end{aligned} \tag{85}$$

To simplify this result further, Eq. (80) is manipulated into the form

$$\begin{aligned} \mathbf{v}_{\beta 1} \cdot \mathbf{Z}_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 1}}^* \cdot \mathbf{v}_{\alpha 1} = & \left[\mathbf{v}_{\beta 1} + \sum_{\gamma} \frac{q_{\gamma}}{q_{\beta}} \hat{\mathbf{j}}_{\gamma \beta}^* \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 1}(\mathbf{v}_{\beta 1}) \right] \\ & \cdot \left[I_L - \frac{(\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 1})^2}{1 - (\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 1})^2} I_T \right] \cdot \mathbf{v}_{\alpha 1} \end{aligned} \quad (86)$$

This equation is used to simplify the second term of the collision integral as it is written in Eq. (85). The term $\mathbf{v}_{\beta 1}$ in the first bracket of the right-hand side of Eq. (86) does not contribute anything in the collision integral since it leads to a term which is odd in \mathbf{k} . Using this fact twice and expressing the transformed induced current response function $\hat{\mathbf{j}}_{\gamma \beta}^* \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 1}$ in terms of the function $P_{\gamma \beta}^* \mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 1}$ in an intermediate step, yields for the second term of the collision integral

$$\begin{aligned} - \sum_{\beta} i \delta_{\alpha \beta} \frac{2q_{\alpha} q_{\beta} n_{\beta}}{(2\pi)^2 n_{\beta}} \partial_{\mathbf{p}_1} \cdot \int d^3 k \frac{f_{\alpha}(\mathbf{p}_1)}{(\mathbf{k} \cdot \mathbf{v}_{\beta 1})^2} \sum_{\gamma} \frac{q_{\gamma} n_{\gamma}}{q_{\beta}} \left[\mathbf{v}_{\gamma 2} + \sum_{\delta} \frac{q_{\delta}}{q_{\gamma}} \hat{\mathbf{j}}_{\delta \gamma}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\beta 1}; \mathbf{v}_{\gamma 2}) \right] \\ \cdot \left[I_L - \frac{(\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 1})^2}{1 - (\hat{\mathbf{k}} \cdot \boldsymbol{\beta}_{\beta 1})^2} \right] \cdot \mathbf{v}_{\alpha 1} \end{aligned} \quad (87)$$

Taking into account Eq. (86) for re-expressing Eq. (87) finally yields for the whole collision integral

$$\begin{aligned} C_{\alpha}(\mathbf{p}_{\alpha}) & \equiv C_{\alpha}(f_{\alpha}(\mathbf{p}_2)) \\ & = \sum_{\beta} 2q_{\alpha}^2 q_{\beta}^2 n_{\beta} \partial_{\mathbf{p}_{\alpha}} \cdot \int d^3 p_{\beta} \int d^3 k \frac{\delta(\mathbf{k} \cdot \mathbf{v}_{\alpha} - \mathbf{k} \cdot \mathbf{v}_{\beta})}{(\mathbf{k} \cdot \mathbf{v}_{\alpha})^4} |\mathbf{v}_{\beta} \cdot \mathbf{Z}_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_{\alpha}} \cdot \mathbf{v}_{\alpha}|^2 \\ & \quad \mathbf{k} \mathbf{k} \cdot (\partial_{\mathbf{p}_{\alpha}} - \partial_{\mathbf{p}_{\beta}}) f_{\beta}(\mathbf{p}_{\beta}) f_{\alpha}(\mathbf{p}_{\alpha}) \end{aligned} \quad (88)$$

where the labels 1 and 2 were absorbed into the indices α and β respectively, since they occur in pairs $(1, \alpha)$ and $(2, \beta)$. This is the relativistic, electromagnetic generalization of the Balescu-Lenard form for the non-relativistic, electrostatic collision operator.

The collision operator $C_{\alpha}(\mathbf{p}_{\alpha})$ can also be expressed in terms of the longitudinal and transverse dielectric functions (73b), (73c). Eq. (78) can be re-written, expanding the identity tensor in its transverse and longitudinal parts, as

$$\begin{aligned}
& q_\beta v_{\beta 2} + \sum_\gamma q_\gamma \hat{J}_{\gamma\beta}(k, k \cdot v_\beta; p_2) \\
&= q_\beta v_{\beta 2} \cdot \left\{ \frac{1 - (\hat{k} \cdot \beta_{\beta 2})^2}{1 - (\hat{k} \cdot \beta_{\beta 2})^2 \epsilon_T} I_T + \frac{1}{\epsilon_L} I_L \right\}
\end{aligned} \tag{89}$$

where Eqs. (79b)-(79c) have also been used.

Using Eq. (89) in combination with Eq. (86) and keeping in mind the properties of the projectors I_L and I_T ($I_L^2 = I_L$; $I_T^2 = I_T$; $I_L \cdot I_T = I_T \cdot I_L = 0$) yields

$$|v_\beta Z_{k, k \cdot v_\alpha} \cdot v_\alpha|^2 = \frac{(k \cdot v_\alpha)^4}{k^4} \left| \frac{1}{\epsilon_{Lk, k \cdot v_\alpha}} + \frac{K^2 v_\alpha \cdot v_\beta - (k \cdot v_\alpha)^2}{(k \cdot v_\alpha)^2 \epsilon_{Tk, k \cdot v_\alpha} - k^2 c^2} \right|^2 \tag{90}$$

Substituting this form $|v_\beta \cdot Z_{k, k \cdot v_\alpha} \cdot v_\alpha|^2$ back into Eq. (88) then gives

$$\begin{aligned}
C_\alpha(f_\alpha(p_\alpha)) &= \sum_\beta 2q_\alpha^2 q_\beta^2 n_\beta \partial_{p_\alpha} \cdot \int d^3 p_\beta \int d^3 k \delta(k \cdot v_\alpha - k \cdot v_\beta) \\
& \left| \frac{1}{\epsilon_{Lk, k \cdot v_\alpha}} + \frac{k^2 v_\alpha \cdot v_\beta - (k \cdot v_\alpha)^2}{(k \cdot v_\alpha)^2 \epsilon_{Tk, k \cdot v_\alpha} - k^2 c^2} \right|^2 \frac{kk}{k^4} (\partial_{p_\alpha} - \partial_{p_\beta}) f_\beta(p_\beta) f_\alpha(p_\alpha)
\end{aligned} \tag{88'}$$

This form can easily be applied to non-relativistic plasmas; the only difference exists in the definition of momentum in the classical versus relativistic cases. This form, for non-relativistic plasma was derived in 1961 by Silin⁴, who used the quantum mechanical approach.

When the shielding is absent $\epsilon_L = \epsilon_T = 1$ and the form (88') is reduced to the one Belyaev and Budker⁹ and independently Bernstein¹⁰ derived for the Lorentz gas

$$\begin{aligned}
C(f(p)) &= 2q^4 n \partial_p \cdot \int d^3 k \delta(k \cdot v - k \cdot v) \frac{(1 - \beta \cdot \beta)^2}{(1 - (\hat{k} \cdot \beta)^2)^2} \\
& \frac{kk}{k^4} \cdot (\partial_p - \partial_p) f(p) f(p).
\end{aligned} \tag{91}$$

VIII. Conclusion

The retarded time generalization of Rostoker's superposition principle is formulated and proved for the two-particle correlation function in the case of a multispecies, relativistic and fully electromagnetic plasma, including external fields, inhomogeneities and nonstationary behavior.

Using one and two-time Liouville functions for one or more species to carry out the ensemble averages, a closed system of equations is obtained which involves the one particles distribution function, the two-point correlation functions and the two-point self-correlation. The infinite hierarchy of equations for correlations is then truncated by neglecting terms of second and higher order in the plasma discreteness parameter.

The generalized superposition principle then reduces the system of coupled kinetic equations to a much simpler one, involving only the one particle distribution, the two-point self-correlation, and the discreteness response function $P_{\alpha\beta}(1, t_1; 2, t_2)$ equation in an uncoupled fashion. In the special case of homogeneous, stationary and external field free plasma, the $P_{\alpha\beta}(1, t_1; 2, t_2)$ equation is easily solved; in this case, the discreteness response function $P_{\alpha\beta}(1, t_1; 2, t_2)$ is identified to be the total time derivative of the perturbation of the distribution function of a Vlasovian particle of species α at 1 at time t_1 , induced by a test particle of species β whose orbit is $2(t_2)$.

The shielded, relativistic, electromagnetic operator is then derived using the solutions for $P_{\alpha\beta}(1, t_1; 2, t_2)$ and the self-correlation function in a simple and straightforward fashion. The collision operator is then manipulated into the Balescu-Lenard form, and into the form first derived by Silin who calculated the operator for the non-relativistic case using a quantum-mechanical approach. It is also shown that the generalized operator reduces to the form derived by Bernstein in the absence of shielding.

Knowledge of the shielded relativistic and electromagnetic collision operator could be important in dealing with the runaway electron problem. In this Spitzer-Harm type of problem, a steady state can be attained by balancing two competing mechanisms. One is the acceleration of the electrons up to very high velocities due to the action of external fields; the other is the collisional interaction of the relativistic electron tail of the steady-state distribution with the bulk of the electrons which have a Maxwellian distribution of arbitrarily high temperature. This last mechanism is precisely that which is expressed by the generalized collision operator.

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