LAGRANGIAN FORMULATION OF
NEOCLASSICAL TRANSPORT THEORY:
GENERAL PRINCIPALS AND APPLICATION TO LORENTZ GAS

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Abstract

Neoclassical transport theory is developed in a Lagrangian formulation, in contrast to the usual Eulerian development. The Lagrangian formulation is constructed from the three actions: magnetic moment, parallel invariant, and bounce averaged poloidal flux. By averaging over the fast orbital time scales an equation in the actions alone of the Fokker-Planck type is obtained. The coefficients give the rates of the elementary neoclassical scattering processes. This action space form of the kinetic equation contains in explicit form processes like banana diffusion and the kinematic Ware pinch. The associated fluxes can be computed by simple moments without having deviations from the local Maxwellian. All the trapped particle contributions are of this explicit type. Another class of fluxes arise from perturbations to the Maxwellian and are termed implicit. The decomposition of the fluxes into explicit and implicit parts is a key feature of the Lagrangian formulation. These contributions correspond to distinct physical processes and have separate Onsager symmetry theorems (explicit and implicit) for their respective transport matrices. The theory does not depend on the details of the Fokker-Planck coefficients but only on some very general properties and is thus applicable–without modification of the formalism–to non-axisymmetric and turbulent systems. This general formulation is the primary purpose of the work. To benchmark the theory, in the present paper, the tokamak transport coefficients (for the Lorentz gas) are computed and compared to the known Eulerian results, demonstrating the equivalence of the two formulations. The elementary processes responsible for the neoclassicial pinch and bootstrap effects (somewhat obscured in the Eulerian picture) are identified and the physical basis for their Onsager symmetry relationship is clarified.

I. Introduction

Particles in magnetic confinement devices travel macroscopic distances between collisions. As a result in curved magnetic geometries small velocity scatterings can lead to large radial steps and a consequent increase of transport over that prevailing when the mean free path between collisions is small. The theory of such enhanced classical transport has been extensively developed (1,2) and is referred to as Neoclassical transport theory. It is based on the Fokker–Planck equation for the one particle distribution function,

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v}\cdot\nabla f_j + \frac{q_j}{m_j}(E + \frac{1}{c}\boldsymbol{v}\times B)\cdot\nabla_{\boldsymbol{v}} f_j = C_j^L(f), \tag{1.1}$$

where C_j^L is the Landau collision operator, representing velocity space scattering at a fixed spatial position. For systems where the collision operator dominates in Eq. (1.1), one gets the standard classical, Chapman–Enskog, transport theory. Neoclassical theory is concerned with systems where certain of the streaming terms on the left hand side of (1.1) dominate (or at least compete with) the collision operator. These orbital streaming terms then dominate in a perturbation expansion. The conventional theory is essentially *Eulerian* in nature, working with $f = f(x, \boldsymbol{v})$, but the independent variables x, $\boldsymbol{v}$ are *not* constants of the orbital motion in the equilibrium fields.

In contrast to the Eulerian description, many of the concepts in the neoclassical theory involve orbital properties and are essentially *Lagrangian* in nature. Thus to describe the elemental transport process (in the regime of weakest collisionality in axisymmetric devices), one thinks of banana orbits with some representative width, ρ_p, and a banana center which is a constant characteristic of an orbit. In the basic scattering event, two particles "exchange" banana orbits resulting in a diffusive step of the banana center of order ρ_p.

The present paper develops a transport theory for the low collisionality, banana, regime using what amounts to a Lagrangian picture. It is couched in terms of the *actions*, J_1, J_2, J_3, which are constants of motion in the equilibrium fields. We choose the actions in an axisymmetric toroidal system to be

$$\begin{aligned} J_1 &\equiv \pi m(b\times \boldsymbol{v})^2/\Omega = \text{MAGNETIC MOMENT} \\ J_2 &\equiv m\oint ds u = \text{PARALLEL INVARIANT} \\ J_3 &\equiv 2\pi\frac{q}{c}\langle\psi\rangle = \text{DRIFT CENTER FLUX COORDINATE.} \end{aligned} \tag{1.2}$$

The reduced kinetic equation can be shown to be,

$$\frac{\partial f_j}{\partial t} + \frac{\partial}{\partial J}\cdot\frac{q_j}{2\pi}V_T u(J) f_j = C_j^J(f) = \frac{\partial}{\partial J}\cdot\Gamma_j(f) \tag{1.3}$$

where C_j^J is a collision operator describing action scattering due to collisions, and V_T is the inductive toroidal voltage. The distribution, f_j, is a function of the actions only. The orbital streaming terms in Eq. (1.1) of the Eulerian formulation do not appear explicitly, but have been incorporated into the collision operator. Thus, C_j^J contains terms of the form $\partial/\partial J_3 \, D_{33} \, \partial/\partial J_3 f_j$, which give *directly*, radial diffusion due to action exchange under collisions.

The general form of the kinetic Eq. (1.3) would hold with any choice of the actions. If a compact transport theory is to be developed from (1.3), however, the choice is considerably narrowed. Specifically, one of the action variables must be identified as essentially spatial (radial), with the remaining two being velocity like. Since all the actions are intrinsically mixed (space–velocity) variables this distinction requires some explanation. Suppose the equilibrium Hamiltonian, $H_0 = H_0(J_1, J_2, J_3)$ depends weakly on the action J_3 such that $\omega_3 \equiv \partial H_0/\partial J_3 \ll \omega_1, \omega_2$. Then the energy depends principally on the "velocity" variables J_1 and J_2, and J_3 can be regarded as a "radial" parameter. For the actions given in Eq. (1.2) and derived in Sec. II the conjugate angles $\theta_1, \theta_2, \theta_3$ are, respectively, bounce average gyrophase, bounce phase, and drift phase. The associated frequencies are then the bounce averaged gyrofrequency, bounce (or transit) frequency, and bounce averaged drift frequency. The frequencies are well ordered such that $\omega_1 \gg \omega_2 \gg \omega_3$ and J_3 is a good radial variable. Previous work (3) in applying action-angle variables to a tokamak used the angular momentum invariant associated with axisymmetry, $J_\phi = \partial L/\partial \phi = \frac{q}{c}\,\psi + mRu\, B_T/B$, for the third action. Although J_ϕ can be used to calculate the radial shape of a banana orbit, the third frequency is large because of the kinetic piece, $mRu\, B_T/B$ of the angular momentum ($\omega_3 \sim \omega_2 \sim$ transit frequency for untrapped particles), and this renders J_ϕ an inconvenient choice for constructing a transport theory.

The physical basis for the Lagrangian formulation is that in the lowest collisionality (banana) regime, the distribution function relaxes in a sequence of well-ordered time scales. On the fastest, orbital, time scale f relaxes to a function of the constants of motion, or actions, alone. (Strictly speaking, the orbital motion does not *relax* f ,but rather produces very fine scale angle dependence, e.g. $f(\theta, t) = f(\theta - \omega_i t, 0)$, which is then destroyed by collision at a rate $(\nu\omega_i^2)^{1/3} \gg \nu$, the collision frequency.) The actions scatter under the influence of collisions to relax f to a local Maxwellian on the collisional time scale. Relaxation of the radial action gradients produces the transport on the (longer) diffusion time scale. The Lagrangian theory follows this hierarchy of relaxation processes, first averaging over the orbital time scales (or equivalently, the angle variables) to give an equation of the form (1.3). The idea of expressing the collision operator in a constant of motion space to eliminate the fast orbital time scale was first suggested by Hively, Miley and Rome (4), who applied such a theory to the very non-local problem of alpha particle transport. The velocity scattering part of the operator C_J, obtained by bounce averaging at fixed ψ and thus ignoring the radial scattering, have been previously derived by Connor and Cordey (5).

The collision operator, C_j^J of Eq. (1.3), can be written in a generalized Landau form as shown in Sec. III. It implies global conservation laws for particle number and energy, as well as a generalized global, H–theorem. This means that the Maxwell distribution $f \sim \exp(-H_0(J)/T)$ is absolutely stationary and unchanged by collisions. In practice, however, the absolute equilibrium distribution does not represent a magnetically confined plasma (owing to the weak dependence of $H_0(J)$ on J_3 or ψ). Non–equilibrium dependence on J_3 must be introduced to obtain spatial localization, thus one has a local Maxwellian,

$$f_0(J) = n(J_3)\frac{\exp[-H_0(J)/T(J_3)]}{\sqrt{2\pi(mT(J_3))^3}}. \tag{1.4}$$

Since the collision operator does *not* annihilate a distribution like Eq. (1.4), transport will arise due to collisions in Eq. (1.3).

The transport equations result simply by taking reduced moments of Eq. (1.3). Thus integration over J_1 and J_2 gives the particle transport equation in the form

$$\frac{\partial}{\partial t}n_3 + \frac{\partial}{\partial J_3}\Gamma = 0, \tag{1.5}$$

where n_3 is particle number per unit J_3. Similarly, multiplication by H_0 and integration over J_1, J_2 gives the energy equation,

$$\frac{\partial}{\partial t}\frac{3}{2}n_3 T + \frac{\partial}{\partial J_3}(\frac{3}{2}\Gamma T + q) = I_T^i V_T, \tag{1.6}$$

where $I_T^i V_T$ is Ohmic dissipation (per unit J_3), V_T is the toroidal voltage, and q is the heat flux. The transport equations are closed when the fluxes, Γ, q, and I_T can be expressed in terms of n, T, and V_T. This is achieved by a very simple expansion of f_j in Eq. (1.3) in powers of ρ_p/a (poloidal gyroradius over minor radius), as described in Sec. IV. One then obtains a transport matrix, T_{ij}, relating the fluxes, Γ, q/T, I_T to the forces, $A_1 = d\ln n/dJ_3$; $A_2 = d\ln T/dJ_3$; $A_3 = V_T/T$,

$$\begin{aligned} \Gamma &= T_{1j}A_j \\ q/T &= T_{2j}A_j \\ I_T &= T_{3j}A_j. \end{aligned} \tag{1.7}$$

Note that the transport equations result directly from moments. It is not necessary to perform flux surface averages as required in the familiar Eulerian theory to obtain local transport relations. The two formulations are, of course, equivalent, although the connection between them is somewhat subtle, and there are differences in the theoretical structure which have physical implications. The Lagrangian formulation gives a much clearer picture of the physical processes responsible for the various neoclassical effects.

In the Eulerian version, the transport coefficients are written as inner products in the form, $T_{ij} = (a_i, g_j)$, where g_i is the perturbation from Maxwellian of the distribution function, driven by a_j. All fluxes are thus associated with deviations from the local Maxwellian, f_o. In the Lagrangian picture some fluxes exist, even without perturbations to f_o. We refer to these as explicit fluxes and denote their contribution to the transport coefficient by T^e_{ij}. In addition there are implicit fluxes of the form (a_i, g_j), not the same a_i and g_i one has in the Eulerian theory, which are a result of perturbations g_j on f_0 and depend for their existence on collisions (even though the final transport coefficient may not depend on collision frequency). Denoting these implicit contributions by $T^i_{ij} = -(a_i, g_j)$, the total transport coefficient is then $T_{ij} = T^e_{ij} + T^i_{ij}$. Thus, for example, from the term $\partial/\partial J_3\ D_{33}\ \partial/\partial J_3 f_0$, of the collision operator one obtains a contribution to T_{11} of the form $T^e_{11} = \int dJ_1\ dJ_2\ D_{33} f_0$, which corresponds to the physically appealing Lagrangian picture of diffusion of the banana centers. Similarly in the pinch coefficient T_{13}, there is a contribution coming directly from the electric field term on the left hand side of Eq. (1.3), $T^e_{13} = \int dJ_1\ dJ_2\ q_j/2\pi T a_3(J) f_0$. This corresponds to a radial flow of particles, independent of the presence of collisions, and, for the trapped particles, is precisely the effect originally described by Ware (6). Finally, there are explicit contributions to the toroidal current, essentially diamagnetic in nature, which must be added to I^i_T in computing the full *spatial* current density. Thus one has diamagnetic, T^e_{31}, as well as collisional, T^i_{31}, contributions to the bootstrap current. Note that the diamagnetic currents do not appear in the Ohmic dissipation term of Eq. (1.6), since no work is done by the external voltage V_T in producing this current.

The implicit contributions to the transport matrix are symmetric, $T^i_{ij} = T^i_{ji}$, and have an associated positive definite form for antropy production due to collisions. It is for this implicit part that one can apply Onsager's theorem (7). There is no general thermodynamic argument, however, requiring symmetry of the explicit fluxes. In spite of this, for the axisymmetric tokamak, one finds, directly, that $T^e_{ij} = T^e_{ji}$ and symmetry obtains for the overall transport coefficients.

Now, the Eulerian theory does not naturally lend itself to this decomposition of the fluxes into explicit and implicit pieces. As a result, the elementary processes underlying certain effects, like the pinch effect and bootstrap current, have been obscured. The simple physical pictures have born little relation to the detailed mathematical expressions for the fluxes in the Eulerian representation. One could not be sure how much the pinch effect depended on collisions, to what extent the bootstrap current was diamagnetic, or why these processes were Onsager conjugate. The pinch effect has been interpreted as predominantly Ware's collisionless inward flow of trapped particles. The bootstrap current, on the other hand, required collisions, and was carried by circulating particles. A questions that has never been satisfactorily resolved is why such different effects, involving different classes of particles, should be Onsager conjugate processes? In the present, Lagrangian formulation, these issues can be clarified. We find that the pinch effect has two interpretations; one, the conventional Ware effect, or alternatively, a collisional process, affecting circulating particles, exclulsively. It is the alternate explanation that accounts for Onsager symmetry. The basic elementary process has an inverse process that generates the bootstrap current.

As a practical matter, the trapped particle contributions, which dominate most of the transport coefficients, are all explicit. They may be computed simply from moments without ever solving for the g_j's.

Probably the most useful aspect of the Lagrangian formulation is its generalizability. Equation (1.3) can be viewed as a Fokker-Planck equation in the actions with coefficients akin to the familiar stochastic average, $< \Delta J \; \Delta J/\Delta t >$, which, in the present paper, are due to collisions. These coefficients give the rates of the elementary scattering processes. However, the transport theory which follows does not depend on the precise form of these coefficients, but only some very general properties. Thus it can be extended in several directions. The neoclassical theory of non-symmetric systems, specifically treating like-particle transport, is considered in a companion paper (8). Further, the Lagrangian formulation provides a scheme for unifying collisional and turbulent transport. The necessity of a unifying framework is evident from the nature of neoclassical transport: collisional scattering from one global collisionless orbit to another. In a turbulent medium the collisionless orbits are quite different. Thus, in addition to the purely turbulent (primarily radial) transport, one has modifications to the neoclassical transport. The coefficients are not necessarily additive and a unifying framework is required for a proper evaluation of the transport. In the Lagrangian formulation this unification can be carried out by generalizing the underlying Fokker-Planck coefficients. These points are emphasized by recent Monte Carlo work on stellarator transport (9) which naturally adopted what amounts to a Fokker-Planck or Lagrangian approach. These studies evaluted the test particle diffusion coefficient, $\Delta J_3 \Delta J_3/\Delta t$, and the contributios to transport that it implies. Such a calculation does not give the full transport coefficient as there are several other processes that in general, contribute to the net particle diffusion and thermal conductivity coefficients. In addition, there are off diagonal coefficients, such as those giving the bootstrap current, which are important. These refinements to the theory can be carried out by using the Monte Carlo codes to compute the remaining Fokker-Planck coefficients. The formalism developed herein provides the procedures for obtaining the transport matrix from these Fokker-Planck coefficients.

It is to be emphasized that the novel feature of this work is the general formulation which allows the treatment of turbulence and non-axisymmetric systems. In the present paper the theory is benchmarked by displaying that it reproduces, identically, the tokamak transport coefficients for the Lorentz gas. The extension of the theory to heretofor unexplored problems is currently underway.

II. Guiding Center Theory for an Axisymmetric Torus: Action-Angle Variables

The formal theory to be developed here assumes the existence of action-angle variables, either exact or approximate to some requisite order of accuracy. We shall now present a derivation of those action-angle variables appropriate to an axisymmetric situation, correct to lowest significant order, within the guiding center approximation. The program starts from the equations of motion for the guiding center and uses the associated Lagrangian, first to derive the canonical angular momentum associated with the axisymmetry, and then to expeditiously transform the equations into flux coordinates. The parallel action J_2 is then derived, as well as its associated angle variable θ_2. The action J_1 corresponding to the magnetic moment μ is demonstrated to be given by $J_1 = 2\pi mc\mu/q$, and the associated angle variable θ_1 to be $1/2\pi$ times the gyration phase averaged over θ_2. The third action J_3 is shown to be proportional to the poloidal flux coordinate ψ averaged over θ_2, and its associated angle variable to be $1/2\pi$ times the average over θ_2 of the other flux coordinate β. The energy $E = H(J, t)$ is then seen to be the Hamiltonian for the reduced problem. The reader is assumed to be familiar with the rudiments of guiding center theory, for example as developed by Bernstein (10) in a paper hereafter denoted by I, the notation of which is herein employed.

Consider the motion in a strong but slowly varying magnetic field $B(r, t)$, and a weak and slowly varying electric field $E(r, t)$ of a charge of mass m and charge q. To lowest significant order in guiding center theory one can write the position vector of a particle charge q and mass m in the form

$$r = R + \rho \tag{2.1}$$

where in a right-handed orthonormal set of basis vectors $e_1(R,t), e_2(R,t), e_3(R,t) = b(R,t) = B(R,t)/B(R,t)$ the guiding center R obeys

$$\dot{R} = ub + \frac{b}{m\Omega} \times [\mu\nabla B + mu^2 b \cdot \nabla b + mu\frac{\partial b}{\partial t} - qE] \tag{2.2}$$

$$m\dot{u} = -\mu b \cdot \nabla B + qb \cdot E + mub \cdot (\nabla b) \cdot \dot{R} \tag{2.3}$$

and the gyration vector is given by

$$\rho = \rho[e_1 \sin \varpi + e_2 \cos \varpi] \tag{2.4}$$

$$\rho = [2\mu B/m\Omega^2]^{1/2} \tag{2.5}$$

$$\varpi = \int_0^t dt' \, \Omega[R(t'), t'] \tag{2.6}$$

$$\Omega = qB/mc \tag{2.7}$$

In Eqs. (2.2) - (2.7) the quantities b, B, Ω, E, e_j, are evaluated at R, and are assumed not to change much in a time $2\pi/\Omega$ or distance ρ, and the magnetic moment μ is a constant of the motion.

If one introduces a vector potential A and scalar potential U such that

$$B = \nabla \times A \tag{2.8}$$

$$E = -\nabla U - \frac{1}{c}\frac{\partial A}{\partial t} \tag{2.9}$$

then (2.1) and (2.2) are the equations of motion (to lowest significant order in the small parameter underlying guiding center theory) associated with the Lagrangian, a generalization of that given by Taylor (11) for the case of a vacuum magnetic field,

$$L = \frac{1}{2}m(b \cdot \dot{R})^2 + \frac{q}{c}\dot{R} \cdot A - \mu B - qU \tag{2.10}$$

This reduced Lagrangian is readily obtained from the exact particle Lagrangian by using (2.1), introducing two time scales in ρ, and averaging over the rapidly varying gyro-phase keeping the slow variation fixed. The term μB there is seen to represent the kinetic energy of the motion perpendicular to the line of force through the guiding center in question.

Consider an axisymmetric magnetic field and write in cylindrical coordinates R, ς, Z

$$B' = B_R(R,Z,t)e_R + B_\varsigma(R,Z,t)e_\varsigma + B_Z(R,Z,t)e_Z \tag{2.11}$$

Then B can be derived from the vector potential

$$A = \psi(R,Z,t)\nabla\varsigma + e_Z A_Z(R,Z,t) \tag{2.12}$$

where

$$A_Z = \int^R dR' B_\varsigma(R',Z,t) \tag{2.13}$$

Correspondingly

$$L = \frac{1}{2}m(\dot{R}b \cdot e_R + R\dot{\varsigma}b \cdot e_\varsigma + \dot{Z}b \cdot e_Z)^2 + \frac{q}{c}(\psi\dot{\varsigma} + A_Z\dot{Z}) - \mu B - qU \tag{2.14}$$

Since L is independent of ς it follows that

$$P_\varsigma = \frac{\partial L}{\partial \dot{\varsigma}} = mRu\boldsymbol{b} \cdot \boldsymbol{e}_\varsigma + \frac{q}{c}\psi, \tag{2.15}$$

where

$$u = \boldsymbol{b} \cdot \dot{\boldsymbol{R}},$$

is a constant of the motion.

An alternate choice of vector potential, more convenient for what is to come, is generated as follows. Let the curves $\chi(R, Z, t) = \text{const}$ be the orthognal trajectories in the R, Z plane of the curves $\psi(R, Z, t) = \text{const}$. That is $(\nabla\psi) \cdot \nabla\chi = 0$. Define

$$G(\psi, \chi, t) = \frac{B_\varsigma}{\boldsymbol{e}_\varsigma \cdot (\nabla\chi) \times (\nabla\psi)} \tag{2.16}$$

$$\beta = \varsigma - \int_0^\chi d\chi' G(\psi, \chi', t) \tag{2.17}$$

Then

$$\begin{aligned} \nabla \times (\psi \nabla \beta) &= (\nabla\psi) \times [\nabla\varsigma - G\nabla\chi - (\nabla\psi) \int_0^\chi d\chi' \partial G(\psi, \chi', t)/\partial\psi] \\ &= (\nabla\psi) \times \nabla\varsigma + G(\nabla\chi) \times \nabla\psi \\ &= \boldsymbol{B} \end{aligned} \tag{2.18}$$

since $(\nabla\chi) \times (\nabla\psi)$ is parallel to $\boldsymbol{e}_\varsigma$. Note that if the lines $\psi = \text{const}$ are topologically a set of nested circles, then χ may be chosen so that it behaves like an angle which changes by 2π as one moves once around a closed curve $\psi = \text{const}$, the so–called poloidal angle. Note too that since $\boldsymbol{B} = (\nabla\psi) \times \nabla\beta$ it follows that $\boldsymbol{B} \cdot \nabla\psi = 0$ and $\boldsymbol{B} \cdot \nabla\beta = 0$ whence one can use the values of ψ and β to label a line of force. Let s be a coordinate which measures arc length along a line of force; see Fig. [1]. If we use as coordinates ψ, β, and s, then the position vector $\boldsymbol{R} = \boldsymbol{R}(\psi, \beta, s, t)$, and if we denote a partial derivative by a subscript it follows from its definition as arc length that

$$\boldsymbol{R}_s = \boldsymbol{b} \tag{2.19}$$

whence

$$u = \boldsymbol{b} \cdot \dot{\boldsymbol{R}} = \boldsymbol{b} \cdot (\dot{s}\boldsymbol{R}_s + \dot{\beta}\boldsymbol{R}_\beta + \dot{\psi}\boldsymbol{R}_\psi + \boldsymbol{R}_t) = \dot{s} + \dot{\beta}\boldsymbol{b} \cdot \boldsymbol{R}_\beta + \dot{\psi}\boldsymbol{R}_\psi + \boldsymbol{b} \cdot \boldsymbol{R}_t. \tag{2.20}$$

Moreover, with $A = \psi\nabla\beta$

$$\dot{R}\cdot A = \psi\dot{R}\cdot\nabla\beta = \psi(\dot{\beta} - \frac{\partial\beta}{\partial t}) \tag{2.21}$$

Thus if one defines

$$V = \mu\Omega + q(V + \frac{\psi}{c}\frac{\partial\beta}{\partial t}) \tag{2.22}$$

then following (2.10)

$$L = \frac{1}{2}\mu^2 + \frac{q}{c}\psi\dot{\beta} - V \tag{2.23}$$

The equations of motion for ψ and β which follow from (2.23) are readily seen to be, on employing (2.20),

$$(\frac{q}{c}\beta - mub\cdot R_\psi)^{\cdot} = V_\psi \tag{2.24}$$

$$(\frac{q}{c}\psi + mub\cdot R_\beta)^{\cdot} = -V_\beta \tag{2.25}$$

One can derive alternate expressions for $\dot{\psi}$ and $\dot{\beta}$ which are easier to understand, though not as useful for solving the equations of motion, as follows. Define the drift velocity

$$v_d = \frac{b}{m\Omega}\times[\nabla V + mu^2 b\cdot\nabla b] \tag{2.26}$$

and introduce

$$v_B = \frac{b}{B}\times\left[\frac{\partial\psi}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\psi\right] \tag{2.27}$$

Then it is seen directly that

$$\frac{\partial\psi}{\partial t} + v_B\cdot\nabla\psi = 0 \qquad \frac{\partial\beta}{\partial t} + v_B\cdot\nabla\beta = 0 \tag{2.28}$$

and v_B can be interpreted as the velocity of the line of force labeled by ψ and β, and v_d measures the drifting of the particle from that line. Moreover one can readily show that

$$\dot{R} = ub + v_d + v_B \tag{2.29}$$

when the coordinates ψ and β of a guiding center obey the equations of motion

$$\dot{\psi} = \frac{\partial\psi}{\partial t} + \dot{R}\cdot\nabla\psi = v_d\cdot\nabla\psi \tag{2.30}$$

$$\dot{\beta} = \frac{\partial\beta}{\partial t} + \dot{R}\cdot\nabla\beta = v_d\cdot\nabla\beta \tag{2.31}$$

where in order of magnitude $v_d \approx u^2/\Omega R \approx u\,\rho/R \ll u$.

The equation of motion for s which follows from (2.23) is

$$m(\dot{s}+\dot{\psi}b\cdot R_\psi+\dot{\beta}b\cdot R_\beta+b\cdot R_t)^{\cdot}=-V_s+mu[\dot{\psi}(b\cdot R_\psi)_s+\dot{\beta}(b\cdot R_\beta)_s+(b\cdot R_t)_s] \tag{2.32}$$

But since we are considering situations where for a representative case $\dot{s}^2\sim u^2\gg(\dot{R}-ub)^2$, in its action on $b\cdot R_\psi$, $b\cdot R_\beta$, and $b\cdot R_t$ the convective time derivative an be approximated by $ub\cdot\nabla\approx u\partial/\partial s$. Moreover, since on average $(\ddot{\psi}b\cdot R_\psi)^2\ll[u\dot{\psi}(b\cdot R_\psi)_s]^2$ and $(\dot{\psi}b\cdot R_\psi)^2\gg(b\cdot R_t)^2$, to the requisite accuracy Eq. (2.32) becomes

$$m\ddot{s}+V_s=0. \tag{2.33}$$

Usually the motion along the lines of force is very rapid compared with the drift from a given line, which suggests solving (2.33) approximately holding ψ, β, and the explicit time dependence in V constant. For reasonable magnetic field configurations there are two usual circumstances. Either the particle is magnetic mirror reflected by the spatially varying magnetic field strength, in which case the motion projected in the R, Z plane consequent to (2.15) has the well known banana character, or it continues to circulate around the torus. In the latter case at fixed ψ, β, and t the field strength B is periodic in s since when χ increases by 2π one has the same values of R and Z. This corresponds to a shift in ς given by (2.17), *viz.*

$$\Delta(\psi)=\oint d\chi G. \tag{2.34}$$

But if one writes $R=R(\psi,\beta-\varsigma,t)$, $Z=Z(\psi,\beta-\varsigma,t)$, along the line of force

$$(ds)^2=(dR)^2+R^2(d\varsigma)^2+(dZ)^2=(R_\varsigma^2+R^2+Z_\varsigma^2)(d\varsigma)^2 \tag{2.35}$$

and the associated change in s is

$$\delta=\int_\varsigma^{\varsigma+\Delta\varsigma}d\varsigma\, s_\varsigma(\psi,\beta,\varsigma,t) \tag{2.36}$$

where following (2.35)

$$s_\varsigma=[R_\varsigma^2+R^2+Z_\varsigma^2]^{1/2}. \tag{2.37}$$

The desired approximate solution is obtained by introducing an angle variable $\theta_2(t)$ to account for the rapid motion along the lines of force, and writing $s=s(\theta_2,t)$. Then, as shown in I, it follows that if

$$\frac{1}{2}mw^2=E-V(\psi,\beta,s,\mu,t) \tag{2.38}$$

then E which is independent of s is determined via

$$J_2 = \oint ds\, mw \tag{2.39}$$

where J_2 is an adiabatic invariant. The integral in (2.39) is executed holding ψ, β, t fixed. Defining

$$\frac{1}{\omega_2} = \oint \frac{ds}{w} \tag{2.40}$$

one has

$$\theta_2 = \int dt\, \omega_2 \tag{2.41}$$

and s is given implicitly by

$$\theta_2 = \omega_2 \int^s \frac{ds}{w}. \tag{2.42}$$

We will now show that E can be viewed as a Hamiltonian. To this end we define

$$J_1 = 2\pi \frac{mc}{q}\mu \tag{2.43}$$

and note that following (2.39) $E = E(J_1, J_2, \psi, \beta, t)$. Then if one differentiates (2.39) with respect to J_2 there results

$$\begin{aligned} 1 &= \oint \frac{ds}{w} \frac{\partial}{\partial J_2} \frac{1}{2} m w^2 \\ &= \oint \frac{ds}{w} \frac{\partial E}{\partial J_2} \end{aligned} \tag{2.44}$$

whence on using (2.40)

$$\omega_2 = \frac{\partial E}{\partial J_2}. \tag{2.45}$$

Also on differentiating (2.39) with respect to J_1, one obtains using (2.38)

$$\begin{aligned} 0 &= \oint \frac{ds}{w} \frac{\partial}{\partial J_1} \frac{1}{2} m w^2 \\ &= \oint \frac{ds}{w} [\frac{\partial E}{\partial J_1} - \frac{\Omega}{2\pi}] \end{aligned} \tag{2.46}$$

whence if we define

$$\omega_1 = \omega_2 \oint \frac{ds}{w} \frac{\Omega}{2\pi} = \oint d\theta_2 \frac{\Omega}{2\pi}, \tag{2.47}$$

then

$$\omega_1 = \frac{\partial E}{\partial J_1}. \tag{2.48}$$

Thus if

$$\theta_1 = \int dt\,\omega_1 \tag{2.49}$$

it follows that if we treat J_1 and θ_1 and J_2 and θ_2 as action and angle variables, then

$$\dot{J}_1 = -\frac{\partial E}{\partial \theta_1} = 0 \qquad \dot{\theta}_1 = \frac{\partial E}{\partial J_1} = \omega_1 \tag{2.50}$$

$$\dot{J}_2 = -\frac{\partial E}{\partial \theta_2} = 0 \qquad \dot{\theta}_2 = \frac{\partial E}{\partial J_2} = \omega_2 \tag{2.51}$$

Clearly (2.50) and (2.51) are Hamiltonian equations of motion.

We shall now demonstrate that ω_2 and E are independent of β. To this end employ ς instead of s as the variable of integration and observe that it follows from $\psi = \psi(R, Z, t)$ and $\chi = \chi(R, Z, t)$ that (2.17) can be written

$$\varsigma - \beta = \int_0^{\chi} d\chi' G(\psi, \chi', t) \equiv F(R, Z, t). \tag{2.52}$$

Clearly, stipulating ψ and $\varsigma - \beta$ yields $R = R(\psi, \varsigma - \beta, t)$ and $Z = Z(\psi, \varsigma - \beta, t)$. Thus (2.39) can be written

$$J_2 = \oint d\varsigma\, s_\varsigma(\psi, \varsigma - \beta, t) m w \tag{2.53}$$

where now

$$\frac{1}{2} m w^2 = E - V(\psi, \varsigma - \beta, J_1, t). \tag{2.54}$$

Consider the case where there is reflection and w vanishes. Let $\varsigma - \beta = \varsigma_\pm$ be the roots of $w = 0$ as given by (2.54). Then (2.53) becomes,

$$\begin{aligned} J_2 &= 2\int_{\varsigma_- + \beta}^{\varsigma_+ + \beta} d\varsigma\, s_\varsigma(\psi, \varsigma - \beta, t)\{2m[E - V(\psi, \varsigma - \beta, J_1, t)]\}^{1/2} \\ &= 2\int_{\varsigma_-}^{\varsigma_+} d\varsigma'\, s_\varsigma(\psi, \varsigma', t)\{2m[E - V(\psi, \varsigma', J_1, t)]\}^{1/2} \end{aligned} \tag{2.55}$$

where we have introduced $\varsigma' = \varsigma - \beta$. Clearly the final integral in (2.55) is independent of β, and hence so must be E. For the case of a circulating particle orbit one can write

$$\begin{aligned} J_2 &= \int_{\varsigma}^{\varsigma+\Delta} d\varsigma\, s_\varsigma(\psi, \varsigma-\beta, t)\{2m[E - V(\psi, \varsigma-\beta, J_1, t)]\}^{1/2} \\ &= \int_{\varsigma-\beta}^{\varsigma-\beta+\Delta} d\varsigma\, s_\varsigma(\psi, \varsigma-\beta, t)\{2m[E - V(\psi, \varsigma-\beta, J_1, t)]\}^{1/2} \\ &= \int_{\varsigma}^{\varsigma'+\Delta} d\varsigma'\, s_\varsigma(\psi, \varsigma', t)\{2m[E - V(\psi, \varsigma', J_1, t)]\}^{1/2} \end{aligned} \tag{2.56}$$

since the value of the integral over one period of a periodic function does not depend on where one begins the integration. Clearly again E is independent of β.

Let us now turn to (2.25). To the order of approximation we require it can be written, since $u \approx w$,

$$\frac{q}{c}\dot{\psi} = -V_\beta - w(mwb \cdot R_\beta)_s \tag{2.57}$$

and on the right hand side ψ and β can be evaluated at the same effective value employed in (2.39) and (2.40). We shall write $\psi = \psi(\theta_2, t)$, $\beta = \beta(\theta_2, t)$ and identify the desired mean values with

$$\overline{\psi} = \oint d\theta_2\, \psi = \omega_2 \oint \frac{ds}{w}\psi, \quad \overline{\beta} = \oint d\theta_2\, \beta = \omega_2 \oint \frac{ds}{w}\beta. \tag{2.58}$$

Recall that the values of ψ and β in the integrands in (2.58) are the particle coördinates evaluated on the exact trajectory which includes the effect of drifts. Thus $\overline{\psi}$ and $\overline{\beta}$ are to be interpreted as drift center coordinates. Then (2.57) can be written

$$\frac{q}{c}[\psi_t + \omega_2\psi_{\theta_2}] = -V_{\overline{\beta}} - w(mwb \cdot R_{\overline{\beta}})_s. \tag{2.59}$$

But $d\theta_2 = \omega_2\, ds/w$, whence if we average over one period of θ_2, since ω_2 is independent of θ_2, (2.59) yields on using (2.56)

$$\frac{q}{c}\overline{\psi}_t = -\omega_2 \oint \frac{ds}{w}\frac{\partial V}{\partial \beta} = \omega_2 \frac{\partial}{\partial \beta} \oint ds\, mw = 0. \tag{5.60}$$

Hence $\overline{\psi}$ is a particle constant of the motion. We can use (2.15) to express it in terms of R, Z and v. To this end note that the average over one period of θ_2 of (2.15) yields, on setting $u = w$,

$$P_\varsigma = \omega_2 \oint ds\, mRb \cdot e_\varsigma + \frac{q}{c}\overline{\psi} \tag{2.62}$$

on using (2.15) to eliminate P_ς one obtains, since $e_\varsigma \cdot v \approx ub \cdot e_\varsigma$,

$$\overline{\psi} = \psi + \frac{mc}{q} Re_\varsigma \cdot v - \frac{mc}{q}\omega_2 \oint ds\, mRb \cdot e_\varsigma. \tag{2.63}$$

Note that the integral in (2.63) vanishes for the case of a banana orbit, while it tends to cancel the term involving $\boldsymbol{b} \cdot \boldsymbol{v}$ for the circulating particle orbit.

In a parallel fashion (2.24) can be written

$$\frac{q}{c}[\beta_t + \omega_2 \beta_{\theta_2}] = V_\psi + w(mw\boldsymbol{b} \cdot \boldsymbol{R}_{\overline{\psi}})_s \tag{2.64}$$

which on averaging over θ_2 yields

$$\frac{q}{c}\overline{\beta}_t = \omega_2 \oint \frac{ds}{w} V_{\overline{\psi}}. \tag{2.65}$$

But on differentiating (2.39), the arguments of which are the mean values of $\overline{\psi}$ and $\overline{\beta}$, one gets

$$0 = \oint \frac{ds}{w} \frac{\partial}{\partial \overline{\psi}} \frac{1}{2} mw^2$$

$$= \oint \frac{ds}{w} [\frac{\partial E}{\partial \overline{\psi}} - \frac{\partial V}{\partial \overline{\psi}}] \tag{2.66}$$

which in conjunction with (2.40) can be used to write

$$\frac{q}{c}\overline{\beta}_t = E_{\overline{\psi}}. \tag{2.67}$$

Define

$$J_3 = 2\pi \frac{q}{c} \overline{\psi} \tag{2.68}$$

$$\omega_3 = \frac{c}{q} \frac{1}{2\pi} E_{\overline{\psi}}. \tag{2.69}$$

Then if

$$H(J_1, J_2, J_3, t) \equiv E(\overline{\psi}, J_1, J_2, t), \tag{2.70}$$

Eq. (2.67) can be written

$$\frac{\partial H}{\partial J_3} = \omega_3, \tag{2.71}$$

while (2.69) can be expressed as

$$\overline{\beta} = 2\pi\theta_3 \tag{2.72}$$

where

$$\theta_3 = \int dt\, \omega_3. \tag{2.73}$$

Thus if we treat H as a Hamiltonian one has

$$\dot{J}_i = -\frac{\partial H}{\partial \theta_i} = 0 \qquad \dot{\theta}_i = \frac{\partial H}{\partial J_i} \equiv \omega_i(J_1, J_2, J_3, t) \tag{2.74}$$

where $i = 1$ and 2 correspond to (2.50) and (2.51). The high frequency part of β can be gotten from (2.64) by neglecting the small term β_t, namely

$$\begin{aligned} \beta &= \overline{\beta} + \frac{1}{\omega_2}\frac{c}{q}\int d\theta_2 [V_{\overline{\psi}} + w(mwb \cdot R_{\overline{\psi}})_s] \\ &= \overline{\beta} + \frac{1}{\omega_2}\frac{c}{q}\int \frac{ds}{w} [V_{\overline{\psi}} + w(mwb \cdot R_{\overline{\psi}})_s] \\ &= \overline{\beta} + \frac{c}{q}\frac{\partial}{\partial \overline{\psi}}\int^s ds\, mw + \frac{mc}{q} wb \cdot R_{\overline{\psi}}. \end{aligned} \tag{2.75}$$

Note that to the accuracy required for calculating $\nabla_v J_1$, following (2.1), (2.4), (2.5), (2.6) and (2.7)

$$\dot{r} = ub - \Omega \times \rho, \tag{2.76}$$

whence one can write

$$m(b \times \dot{r})^2 = m\Omega^2\rho^2 = J_1 \frac{\Omega}{2\pi}. \tag{2.77}$$

Thus on letting $\dot{r} = v$

$$J_1 = \frac{2\pi}{\Omega}\frac{m(b \times v)^2}{2} \tag{2.78}$$

and

$$\nabla_v J_1 = \frac{2\pi m}{\Omega} b \times (v \times b). \tag{2.79}$$

Also observe that following (2.39)

$$\begin{aligned} \nabla_v J_2 &= \oint \frac{ds}{w}\left[\nabla_v E - \frac{\Omega}{2\pi}\nabla_v J_1\right] \\ &= \frac{m}{\omega_2}\left[v - \frac{2\pi\omega_1}{\Omega} b \times (v \times b)\right] \end{aligned} \tag{2.80}$$

since

$$E = \frac{1}{2}mv^2 + q\left[U(R,t) + \frac{\psi(R,t)}{c}\frac{\partial \beta(R,t)}{\partial t}\right], \tag{2.81}$$

whence to the required accuracy

$$\nabla_v E = mv. \tag{2.82}$$

For the case of banana orbits it follows from (2.63) and (2.68) that

$$\nabla_v J_3 = 2\pi m R e_\varsigma. \tag{2.83}$$

III. The Collision Term

The conventional neoclassiccal theory is based on Eq. (1.1), as has been remarked. Ultimately, however, one is interested in transport arising from fluctuations of all types, not only that associated with discreteness, but that arising from, for example, low frequency drift wave-type turbulence. This requires a re-examination of the non-equilibrium statistical mechanics underlying the derivation of the collision term which will be considered in a future paper. Here we shall derive an expression for the collision term in action-angle variables by a direct transformation of variables in the familiar Landau-Fokker-Planck term.

Recall that the collision term can be written

$$C_i^L(f) = -\nabla_v \cdot \boldsymbol{\Gamma} \tag{3.1}$$

where the current density in velocity space associated with small angle collisions is

$$\boldsymbol{\Gamma} = -\sum_{\substack{\text{primed}\\ \text{species}}} \frac{2\pi z^2 z'^2 e^4 ln\Lambda}{m^2} \int d^3v' \left[f'(r, v', t)\nabla_v f(r, v, t) - \frac{m}{m'} f(r, v, t)\nabla_{v'} f'(r, v', t)\right] \cdot \nabla_v \nabla_v |v - v'| \tag{3.2}$$

In (3.2) f bears an implied subscript i and f' an implied subscript i'. On using the chain rule for differentiation

$$\begin{aligned} \nabla_v \cdot \boldsymbol{\Gamma} &= (\nabla_v J) : \nabla_J \boldsymbol{\Gamma} + (\nabla_v \theta) : \nabla_\theta \boldsymbol{\Gamma} \\ &= \nabla_J \cdot \left[\boldsymbol{\Gamma} \cdot \nabla_v J\right] + \nabla_\theta \cdot \left[\boldsymbol{\Gamma} \cdot \nabla_v \theta\right] - \left[\nabla_J \cdot (\nabla_v J)^T + \nabla_\theta \cdot (\nabla_v \theta)^T\right] \cdot \boldsymbol{\Gamma} \end{aligned} \tag{3.3}$$

Recall that since the transformation from r, v to J, θ is canonical the Jacobian of the transformation is unity. But for any non-singular transformation of variables, say from the set $x_1, x_2, \ldots, x_N$ to $y_1, y_2, \ldots, y_N$ one has the identity

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i} \left[\frac{\partial y_i}{\partial x_j} \frac{\partial(x_1, x_2, \ldots, x_N)}{\partial(y_1, y_2, \ldots, y_N)}\right] = 0 \tag{3.4}$$

whence the coefficient of Γ on the right-hand side of (3.3) vanishes, and the collision term is of conservation form in action-angle variables.

Now if

$$\dot{f} \equiv \frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m}(E + \frac{v \times B}{c}) \cdot \nabla_v f \tag{3.5}$$

Eq. (1.1) can be written

$$\dot{f} + \nabla_v \cdot \Gamma = 0 \tag{3.6}$$

But on using the chain rule, on transforming to action-angle variables,

$$\dot{f} = \frac{\partial f}{\partial t} + \dot{J} \cdot \nabla_J f + \dot{\theta} \cdot \nabla_\theta f \tag{3.7}$$

or since J and θ evolve according to the Hamiltonian equations

$$\dot{J} = -\frac{\partial H}{\partial \theta} \qquad \dot{\theta} = \frac{\partial H}{\partial J} \tag{3.8}$$

one has

$$\nabla_J \cdot \dot{J} + \nabla_\theta \cdot \dot{\theta} = 0 \tag{3.9}$$

Thus one can write (3.6) in the two versions

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + \dot{J} \cdot \nabla_J f + \dot{\theta} \cdot \nabla_\theta f + \nabla_J \cdot \left[\Gamma \cdot \nabla_v J\right] + \nabla_\theta \cdot \left[\Gamma \cdot \nabla_v \theta\right] \\ &= \frac{\partial f}{\partial t} + \nabla_J \cdot \left[-\frac{\partial H}{\partial \theta} f + \Gamma \cdot \nabla_v J\right] + \nabla_\theta \cdot \left[\frac{\partial H}{\partial J} f + \Gamma \cdot \nabla_v \theta\right] \end{aligned} \tag{3.10}$$

which latter too is in conservation form.

Suppose

$$H(J, \theta, t) = H_0(J, t) + H_1(J, \theta, t) \tag{3.11}$$

and define

$$\omega(J,t) = \frac{\partial H_o(J,t)}{\partial J} \tag{3.12}$$

Then (3.10) can be expressed as

$$0 = \frac{\partial f}{\partial t} + \omega \cdot \nabla_\theta f + \nabla_J \cdot \left[-\frac{\partial H_1}{\partial \theta} f + \Gamma \cdot \nabla_v J \right] + \nabla_\theta \cdot \left[\frac{\partial H_1}{\partial J} f + \Gamma \cdot \nabla_v \theta \right] \tag{3.13}$$

Assume that ω_1 is much larger in order of magnitude than any other frequency characterizing the coefficients in (3.13), *viz.* ω_2 ,ω_3 the collision frequency $\nu = C_i^L(f)/f, \partial H/\partial J_2$,etc. Then one can seek a solution of the form

$$f = f_o + f_1 + f_2 + \cdots \tag{3.14}$$

where succeeding terms are ordered in inverse powers of ω_1. When (3.14) is inserted in (3.13) and one equates terms of like order in $1/\omega_1$ there results

$$\omega_1 \frac{\partial f_o}{\partial \theta_1} = 0 \tag{3.15}$$

$$0 = \frac{\partial f_o}{\partial t} + \omega_1 \frac{\partial f_1}{\partial \theta_1} + \omega_2 \frac{\partial f_o}{\partial \theta_2} + \omega_3 \frac{\partial f_o}{\partial \theta_3} + \nabla_J \cdot \left[-\frac{\partial H_1}{\partial \theta} f_o + \Gamma_o \cdot \nabla_v J \right] + \nabla_\theta \cdot \left[\frac{\partial H_1}{\partial J} f_o + \Gamma_o \cdot \nabla_v \theta \right] \tag{3.16}$$

In (3.16), the subscript zero on Γ indicates that one is to use f_o in the collision integrals. An equation for f_o alone can be obtained from (3.16) by integrating with respect to θ_1 , from zero to one and invoking the requirement that f_1 is periodic in θ_1 with unit period. Note that the term in the last divergence on the right in (3.16) which involves $\partial/\partial\theta_1$ also vanishes, as does the term involving $\partial H_1/\partial\theta_1$, assuming as is usually the case that H_1 is periodic in θ_1 which is effectively the gyration phase.

Let us further assume that ω_2 is much greater than the other frequencies and repeat the perturbation theory, and then that either ω_3 is greater than the remaining frequencies or that the system is axisymmetric (in which case the distribution function is independent of θ_3) and then repeat the perturbation theory. The final result to lowest significant order in all the several small parameters is

$$\nabla_\theta f = 0 \tag{3.17}$$

$$0 = \frac{\partial f}{\partial t} + \nabla_J \cdot \oint d^3\theta \Big[-\frac{\partial H_1}{\partial \theta} f + \Gamma \cdot \nabla_v J \Big] \tag{3.18}$$

where, as will be done henceforth, we have suppressed the subscripts on f indicating the order of approximation . In the axisymmetric case the most important pertrubation is that due to an electric field parallel to $\boldsymbol{b}$ associated with induction by an outside transformer.. Rather than write directly the perturbed Hamiltonian H_1 it is convenient to note that such an electric field E_1 enters the original Fokker-Planck equation (1.1) in the form $\nabla \cdot \Gamma_E$ where $\Gamma_E = q/mEf$ and can be treated in the same fashion as the collisional current Γ, whence

$$\begin{aligned} \nabla_J \cdot \oint d^3\theta \Big[-f_o \frac{\partial H_1}{\partial \theta} \Big] &= \nabla_J \cdot \oint d^3\theta \Gamma_E \cdot \nabla_v J \\ &\approx \nabla_J \cdot \Big[\omega_2 \oint \frac{ds}{u} \frac{q}{m} E_1 b \cdot \nabla_v J \Big] \end{aligned} \tag{3.19}$$

In order to establish the orders of magnitude of the terms in (3.18) we consider for simplicity an electron proton plasma where correct to lowest order in the ratio of the electron–to–ion mass, one has for the electrons

$$\begin{aligned} -\Gamma = & \frac{2\pi e^4 ln\Lambda}{m^2} \int d^3v'(f_e' \nabla_v f_e - f_e \nabla_{v'} f_e') \cdot \nabla_v \nabla_v |v - v'| \\ & + \frac{2\pi n e^4 ln\Lambda}{m^2} \Big\{ (\nabla_v f_e) \cdot \nabla_v \nabla_v |v - v_i| + \frac{m}{M} \frac{v - v_i}{|v - v_i|^3} \Big[f_e + \frac{T_i}{m} \frac{v - v_i}{|v - v_i|^3} \cdot \nabla_v f_e \Big] \Big\} \end{aligned} \tag{3.20}$$

and for the protons

$$\begin{aligned} -\Gamma = & \frac{2\pi n e^4 ln\Lambda}{M^2} \int d^3v'(f_i' \nabla_v f_i - f_i \nabla_{v'} f_i') \cdot \nabla_v \nabla_v |v - v'| \\ & + \frac{2\pi n e^4 ln\Lambda}{M^2} \Big\{ \frac{2}{3} < \frac{1}{|v - v_e|} > \nabla_v f_i + \frac{8\pi}{3} \frac{M}{m} \frac{f_e(v_i)}{n} (v - v_i) f_i \Big\} \end{aligned} \tag{3.21}$$

where n is the electron density, taken to be equal to that of the protons v_e is the electron mean velocity, v_i is the ion mean velocity, and

$$< \frac{1}{|v - v_e|} >= \frac{1}{n}\int d^3v \frac{f_e}{|v - v_e|} \qquad nT_i = \int d^3v \frac{m}{2}(v - v_i)^2 f_i \tag{3.22}$$

Eqs. (3.20) and (3.21) conserve both energy and momentum, and vanish when f_i and f_i are Maxwellian at the same temperature. The terms in m/M in (3.20) and the last terms on the right in (3.21) while necessary for energy and momentum conservation are smaller in order of magnitude than the others in their respective equations and will be neglected for the pupose of order of magnitude estimates. Note that

$$\begin{aligned} \int d^3v'(\nabla_{v'}f') \cdot \nabla_v\nabla_v|v - v'| &= \int d^3v'\Big\{f'\nabla_{v'}\nabla_{v'}|v - v'| - f'\nabla_{v'}\nabla^2_{v'}|v - v'|\Big\} \\ &= -\int d^3v' f'\nabla_{v'}\frac{2}{|v - v'|} \\ &= \nabla_v \int d^3v' \frac{f'}{|v - v'|} \end{aligned} \tag{3.23}$$

and define the symmetric tensors $\boldsymbol{D}$ and vector $\boldsymbol{d}$ via

$$D_e = \oint d^3\theta \frac{2\pi e^4 ln\Lambda}{m^2}(\nabla_v J)^T \cdot \Big[\nabla_v\nabla_v \int d^3v' f'_e|v - v'| + n\nabla_v\nabla_v|v - v_i|\Big] \cdot (\nabla_v J) \tag{3.24}$$

$$d_e = -\oint d^3\theta \frac{2\pi e^4 ln\Lambda}{m^2}(\nabla_v J)^T \cdot 2\nabla_v \int \frac{d^3v' f'_e}{|v - v'|} \tag{3.25}$$

$$D_i = \oint d^3\theta \frac{2\pi e^4 ln\Lambda}{M^2}(\nabla_v J)^T \cdot \Big[\nabla_v\nabla_v \int d^3v' f'_i|v - v'|\Big] \cdot (\nabla_v J) \tag{3.26}$$

$$d_i = -\oint d^3\theta \frac{2\pi e^4 ln\Lambda}{M^2}(\nabla_v J)^T \cdot 2\nabla_v \int \frac{d^3v' f'_i}{|v - v'|} \tag{3.27}$$

Then for both ions and electrons (3.18) assumes the form

$$\frac{\partial f}{\partial t} + \nabla_J \cdot \oint d^3\theta \frac{ze}{m} E \cdot (J_v J) f = \nabla_J \cdot \Big[D \cdot \nabla_J f + df\Big] \tag{3.28}$$

Note that in order of magnitude, or using (2.78) at any seq.

$$J_1 \approx \frac{mv_T^2}{\omega_1} \qquad |\nabla_v J_1| \approx \frac{mv_T}{\omega_1} \qquad \omega_1 \approx \frac{\Omega}{2\pi}$$

$$J_2 = 2\pi R m v_T \qquad |\nabla_v J_2| \approx \frac{mv_T}{\omega_2} \qquad \omega_2 \approx \frac{v_T}{2\pi R} \tag{3.29}$$

$$J_3 = 2\pi a m R\Omega_p \qquad |\nabla_v J_3| \approx 2\pi m R \qquad \omega_3 \approx \frac{v_T^2}{2\pi a R\Omega_p}$$

where R is the major radius of the torus, a is the minor radius, v_T is a representative thermal speed, and Ω_o is the gyration frequency computed using the poloidal magnetic field. Thus if we write roughly

$$D \approx (\nabla_v J)^T \cdot v_T^2 \nu \cdot (\nabla_v J) \tag{3.30}$$

where for electrons

$$\nu \approx \frac{2\pi n e^4 ln\Lambda}{m^2 v_{eT}^3} \tag{3.31}$$

and for ions

$$\nu \approx \frac{2\pi n e^4 ln\Lambda}{M^2 v_{iT}^3} \tag{3.32}$$

it is readily seen that

$$\frac{\partial}{\partial J_1}(D_{11}\frac{\partial f}{\partial J_1}):\frac{\partial}{\partial J_2}(D_{12}\frac{\partial f}{\partial J_2}):\frac{\partial}{\partial J_1}(D_{13}\frac{\partial f}{\partial J_3}):\frac{\partial}{\partial J_2}(D_{22}\frac{\partial f}{\partial J_2}):\frac{\partial}{\partial J_2}(D_{23}\frac{\partial f}{\partial J_3}):\frac{\partial}{\partial J_3}(D_{33}\frac{\partial f}{\partial J_3})$$

$$\approx 1{:}1{:}\frac{\rho_p}{a}{:}1{:}\frac{\rho_p}{a}{:}(\frac{\rho_p}{a})^2$$

where $\rho_p = v_T/\Omega_p$ is the mean gyration radius in the poloidal magnetic field. Similary, in order of magnitude

$$d = (\nabla_v J)^T \cdot v_T \nu_e \tag{3.33}$$

whence

$$\frac{\partial(d_1 f)}{\partial J_1}:\frac{\partial(d_2 f)}{\partial J_2}:\frac{\partial(d_3 f)}{\partial J_3} = 1{:}1{:}\frac{\rho_p}{a} \tag{3.34}$$

For the purpose of a transport theory one requires a zero-order distribution function for which the dominant part of the collision term vanishes, namely that part zero order in ρ_p/a , and also in m/M. To illustrate this determination it is convenient to consider the ions and write

$$a = \frac{2\pi e^4 ln\Lambda}{M^2} \int d^3v'(f'\nabla_v f - f\nabla_{v'} f') \cdot \nabla_v \nabla_v |v - v'| \tag{3.35}$$

in terms of which the collisional current density in J space, including for this purpose only the self-collision contributions, is $-\nabla \cdot \Gamma$ with

$$-\Gamma = \int d^3\theta(\nabla_v J) \cdot a \tag{3.36}$$

and with $f = f(J, t)$. Note too that to the order of accuracy we require the components of J viewed as functions of v depend on $u = b \cdot v$ and $w = |b \times v|$ but not on the gyro-phase ϕ. Moreover, on using the chain rule

$$\begin{aligned} \nabla_v f &= (\nabla_v J_1)\frac{\partial f}{\partial J_1} + (\nabla_v J_2)\frac{\partial f}{\partial J_2} + (\nabla_v J_3)\frac{\partial f}{\partial J_3} \\ &= \frac{2\pi m b \times (v \times b)}{\Omega}\frac{\partial f}{\partial J_1} + m\omega_2\left[v - \frac{2\pi\omega_1}{\Omega} b \times (v \times b)\right]\frac{\partial f}{\partial J_2} \\ &\quad + \left[mRe_\varsigma + \ldots\right]\frac{\partial f}{\partial J_3} \end{aligned} \tag{3.37}$$

Using the estimates of (3.29) it is evident that to lowest order in ρ_p/a one can neglect the last term on the right in (3.37) whence on using J'_1, J'_2, and ϕ' as variables in (3.35) one can write

$$\begin{aligned} a = \frac{2\pi e^4 ln\Lambda}{M^2} \int dJ'_1 dJ'_2 d\phi' \frac{\partial(u, w)}{\partial(J'_1, J'_2)} &\left[\nabla_v \nabla_v |v - v'|\right] \\ \cdot \Big[(\nabla_v J_1)\frac{\partial}{\partial J_1} ff' &+ (\nabla_v J_2)\frac{\partial}{\partial J_2} ff' \\ - (\nabla_{v'} J'_1)\frac{\partial}{\partial J'_1} ff' &- (\nabla_{v'} J'_2)\frac{\partial}{\partial J'_2} ff'\Big] \end{aligned} \tag{3.38}$$

If we require that to lowest order in ρ_p/a that the collision term vanish, *viz.*

$$0 = \frac{\partial}{\partial J_1} \oint d^3\theta a \cdot \nabla_v J_1 + \frac{\partial}{\partial J_2} \oint d^3\theta a \cdot \nabla_v J_2 \tag{3.39}$$

then on multiplication by lnf integration over J_1 and J_2, and an obvious integration by parts, one obtains

$$0=\int d^3\theta dJ_1 dJ_2 dJ'_1 dJ'_2 d\phi' f' f' \left\{\left[\frac{\partial lnf}{\partial J_1}(\nabla_v J_1)+\frac{\partial lnf}{\partial J_2}(\nabla_v J_2)\right]\cdot\left[\nabla_v\nabla_v|\mathbf{v}-\mathbf{v}'|\right]\right.$$
$$\cdot\left[(\nabla_v J_1)\frac{\partial lnf}{\partial J_1}+(\nabla_v J_2)\frac{\partial lnf}{\partial J_2}-(\nabla_{v'} J'_1)\frac{\partial lnf'}{\partial J'_1}-(\nabla_{v'} J'_2)\frac{\partial lnf'}{\partial J'_2}\right] \tag{3.40}$$

where it is assumed that the boundary conditions are such that all surface terms arising from the integration by parts vanish. Note that

$$\nabla_v\nabla_v|\mathbf{v}-\mathbf{v}'|=\frac{(\mathbf{v}-\mathbf{v}')^2 I-(\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^3} \tag{3.41}$$

and define

$$C=(\nabla_v J_1)\frac{\partial lnf}{\partial J_1}+(\nabla_v J_2)\frac{\partial lnf}{\partial J_2}-(\nabla_{v'} J'_1)\frac{\partial lnf'}{\partial J_1}-(\nabla_{v'} J'_2)\frac{\partial lnf'}{\partial J'_2}$$
$$=(\nabla_v lnf-\nabla_{v'} lnf')_{J_3=const} \tag{3.42}$$

Then if in (3.40) we interchange J_1 and J_2 with J'_1 and J'_2, etc., and add half the result to (3.40), there emerges

$$0=\int d^3\theta dJ_1 dJ_2 dJ'_1 dJ'_2\left[(\mathbf{v}-\mathbf{v}')\times C\right]^2|\mathbf{v}-\mathbf{v}'|^{-3} \tag{3.43}$$

Now it follows from (2.78) and (2.80) that

$$\mathbf{v}=\frac{\omega_2}{M}\nabla_v J_2+\frac{\omega_1}{M}\nabla_v J_1 \tag{3.44}$$

Thus the requirement that

$$(\mathbf{v}-\mathbf{v}')\times C=0, \tag{3.45}$$

which follows from (3.43), implies that

$$0=(\omega_2\nabla_v J_2+\omega_1\nabla_v J_1-\omega'_2\nabla_{v'}J'_2-\omega'_1\nabla_{v'}J'_1)$$
$$\times\left[(\nabla_v J_1)\frac{\partial lnf}{\partial J_1}+(\nabla_v J_2)\frac{\partial lnf}{\partial J_2}-(\nabla_{v'} J'_1)\frac{\partial lnf'}{\partial J'_1}-(\nabla_{v'} J'_2)\frac{\partial lnf'}{\partial J'_2}\right] \tag{3.46}$$

In (3.45) set J'_1 and J'_2 to zero which implies that $\nabla_{v'}J'_1$ and $\nabla_{v'}J'_2$ also vanish. Then (3.45) reduces to

$$(\nabla_v J_1) \times (\nabla_v J_2)(\omega_1 \frac{\partial lnf}{\partial J_2} - \omega_2 \frac{\partial lnf}{\partial J_1}) = 0 \tag{3.47}$$

Since in general the cross produce in (3.47) does not vanish one must have since $\omega_1 = \partial H_o/\partial J_1$, $\omega_2 = \partial H_o/\partial J_2$

$$\frac{\partial H_o}{\partial J_1}\frac{\partial f}{\partial J_2} - \frac{\partial H_o}{\partial J_2}\frac{\partial f}{\partial J_1} = 0 \tag{3.48}$$

whence

$$f = f(H_o, J_3) \tag{3.49}$$

The J_3 dependence arises from the fact that all the manipulations leading to (3.48) were carried out at fixed J_3. If we return to (3.45), since $\nabla_v H_o = m\boldsymbol{v}$ we can write

$$\begin{aligned} 0 &= (\boldsymbol{v} - \boldsymbol{v'}) \cdot (\nabla_v lnf - \nabla_{v'} lnf')_{J_3 = const} \\ &= (\boldsymbol{v} - \boldsymbol{v'}) \cdot (m\boldsymbol{v}\frac{\partial lnf}{\partial H_o} - m\boldsymbol{v'}\frac{\partial lnf'}{\partial H'_o}) \\ &= m(\boldsymbol{v} - \boldsymbol{v'})(\frac{\partial lnf}{\partial H_o} - \frac{\partial lnf'}{\partial J'_o}) \end{aligned} \tag{3.50}$$

whence

$$\frac{\partial lnf}{\partial H_o} = \frac{\partial lnf}{\partial H_o} = \text{const} = \frac{1}{T(J_3)} \tag{3.51}$$

and

$$f = N(J_3)e^{-H_o/T(J_3)} \tag{3.52}$$

This is, effectively, the reduced H-theoreom (at constant J_3) needed to begin a transport theory.

IV. Formal Structure of the Transport Theory

We consider now the approximate solution of Eq. (1.3) to obtain the transport $\pi\pi$ relations. For simplicity, we take one species (electrons), and employ the Lorentz approximation collision operator, where C_{ee} is neglected, and the $m_i \rightarrow \infty$ limit of C_{ei} is used. The generalizations required to remove these approximations are involved and important, but the formal structure of the theory is identical in the more general case. The limit considered is thus

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial J} \cdot [\frac{q}{2\pi} a(J) V_T f] = C(f), \tag{4.1}$$

where,

$$C(f) = \frac{\partial}{\partial J} \cdot \left[D(J) \cdot \frac{\partial}{\partial J} f \right], \tag{4.2}$$

and $D(J)$ is the symmetric, positive definite tensor, obtained by letting $m_i \rightarrow \infty$ in equation (3.20).

The approximation scheme is based on the smallness of the J_3 derivatives, and this, in turn requires that J_3 be radial in the sense discussed in Sec. I. Velocity scattering, involving J_1 and J_2, dominates and forces f to be a local Maxwellian in leading order. The slower scatterings in radius produce the transport. For example, since $J_3 \sim \frac{q}{c}\psi + O(\rho_p/a)$ and $J_2 \sim m v_{th} R$, it follows that $\frac{|\partial \ln f/\partial J_3|}{|\partial \ln f/\partial J_2|} \sim m v_{th} R c / q B_p R a \sim \rho_p/a$, and one expects the derivatives with respect to J_3 to scale like ρ_p/a. In fact, we have shown explicitly (Sec. III) that the collision operator orders in ρ_p/a according to the number of J_3 derivatives. To achieve a maximal ordering, V_T is treated as $O(\rho_p/a)$ and $\partial f/\partial t$ as $O((\rho_p/a)^2)$. Then the Ohmic dissipation and thermal conductivity appear together at second order along with the transport equations. The basic expansion is identical to that used previously [12] to assess the effect of finite beta drift wave turbulence [13] on the electron distribution function.

The collision operator is therefore expanded in powers of ρ_p/a, namely

$$\begin{aligned} C(f) = \frac{\partial}{\partial J} \cdot [D \cdot \frac{\partial}{\partial J} f] = C_0(f) + C_1(f) + C_2(f) = \frac{\partial}{\partial J} \cdot [D_0 \cdot \frac{\partial}{\partial J} f] \\ + \frac{\partial}{\partial J} \cdot [D_1 \cdot \frac{\partial}{\partial J} f] + \frac{\partial}{\partial J} \cdot [D_2 \cdot \frac{\partial}{\partial J} f], \end{aligned} \tag{4.3}$$

where,

$$\begin{aligned} D_0 &= D_{11} e_1 e_1 + D_{12}(e_1 e_2 + e_2 e_1) + D_{22} e_2 e_2 \\ D_1 &= D_{13}(e_1 e_3 + e_3 e_1) + D_{23}(e_2 e_3 + e_3 e_2) \\ D_2 &= D_{33} e_3 e_3, \end{aligned} \tag{4.4}$$

and e_1, e_2, e_3 are an orthonormal set of cartesian unit vectors such that $J = J_1 e_1, + J_2 e_2 + J_3 e_3$. We will also use the conventions

$$e_\perp = e_1 + e_2; \quad \partial/\partial J_\perp = e_1 \partial/\partial J_1 + e_2 \frac{\partial}{\partial J_2}, \tag{4.5}$$

so that C_0 can be written $C_0 = \partial/\partial J_\perp \cdot D_0 \cdot \partial/\partial J_\perp$. In a similar way the electric field operator separates into a first order part, $\partial/\partial J_\perp \cdot \frac{q}{2\pi} a(J) V_T f$, and second order part, $\partial/\partial J_3 \ \frac{q}{2\pi} a_3 V_T f$. The latter describes an explicit radial flow driven by the electric field and independent of the existence of collisions. This is Ware's original effect [4]. It appears directly in the action-angle (Lagrangian) formulation. In the first order part we assume (and later show) that $a_\perp(J) = a_2(J) e_2$, and $\partial/\partial J_\perp \cdot a_2 = \partial/\partial J_2 a_2 = 0$. Thus, the energy moment becomes

$$\int d^2 J_\perp \, H_0(J) \frac{\partial}{\partial J_\perp} \cdot \frac{q}{2\pi} a_\perp V_T f = -V_T \int d^2 J_\perp \frac{q}{2\pi} a_2 \omega_2 f.$$

Since the energy moment of the electric field term must, in general, give the Ohmic power input, we may infer that this last integral gives toroidal current per unit J_3 , and thus define,

$$I_T^i = \int d^2 J_\perp \frac{q}{2\pi} a_2(J) \omega_2(J) f. \tag{4.6}$$

A direct calculation of the bounce averaged toroidal current, when expressed in terms of the actions, yields the same expression (Sec. V).

Following this ordering of the kinetic equation, we expand f in powers of ρ_p/a . The zero order equation is,

$$0 = C_o(f_o) \, . \tag{4.7}$$

It is at this point that a judicious choice of the actions (i.e., J_3 radial such that $\omega_3 = \partial H_o/\partial J_3$ is small) becomes necessary to solve Eq. (4.7). Then the reduced H–theorem of the previous section applies and we conclude that f_o must be a local Maxwellian $f_o = f_m \equiv N(J_3) \exp(-H_o(J)/T(J_3))$.

For reasons that will be clear shortly, we normalize f_m as follows:

$$f_o = f_m \equiv n(2\pi m T)^{-3/2} \exp(-H_o/T) \, , \tag{4.8}$$

where n, a density per unit spatial volume, is related to n_3 by

$$n = n_3 \partial J_3/\partial V = n_3 2\pi \frac{q}{c} \partial\psi/\partial V. \tag{4.9}$$

where V is the volume interior to a flux surface $\psi =$ const. . The relation between n and the observed density is detailed in Sec. V.

Having solved Eq. (4.8), we proceed to the first order equation

$$\frac{\partial}{\partial J_\perp} \cdot [\frac{q}{2\pi} V_T a_\perp(J) f_0] - \frac{\partial}{\partial J} \cdot [D_1 \cdot \frac{\partial}{\partial J} f_0] = C_o(f_1). \tag{4.10}$$

Noting that D_1 can be written as a scalar times the tensor [see Eq. (5.31)],

$$(-\frac{1}{\omega_1} e_1 + \frac{1}{\omega_2} e_2) e_3 + e_3(-\frac{1}{\omega_1} e_1 + \frac{1}{\omega_2} e_2), \tag{4.11}$$

the second term on the left side of Eq. (4.11) becomes,

$$\begin{aligned} \frac{\partial}{\partial J} \cdot D_1 \cdot \frac{\partial}{\partial J} f_0 = & -\frac{\partial}{\partial J} \cdot D_1 \cdot \omega_\perp \frac{f_0}{T} \\ & + \frac{\partial}{\partial J_\perp} \cdot (D_1 \cdot e_3 \frac{\partial f_0}{\partial J_3}) = \left(\frac{\partial}{\partial J_\perp} \cdot D_1 \cdot e_3\right) \frac{\partial f_0}{\partial J_3}, \end{aligned} \tag{4.12}$$

The electric field term, using the properties of the function $a(J)$ can be expressed in the alternate form,

$$\frac{\partial}{\partial J_\perp} \cdot \frac{q}{2\pi} V_T a_\perp f_0 = \frac{q}{2\pi} V_T a_2 \frac{\partial f_0}{\partial J_2} = -\frac{V_T}{T} \frac{q}{2\pi} a_2 \omega_2 f_0. \tag{4.13}$$

Using these properties, the first order equation can be written as either

$$C_0(f_1) = \frac{\partial}{\partial J_\perp} \cdot \frac{q}{2\pi} a_\perp V_T f_0 - \frac{\partial}{\partial J_\perp} \cdot (D_1 \cdot e_3 \frac{\partial f_0}{\partial J_3}), \tag{4.14}$$

or

$$C_0(f_1) = -\frac{q}{2\pi} a_2 \omega_2 f_0 \frac{V_T}{T} - \left(\frac{\partial}{\partial J_\perp} \cdot D_1 \cdot e_3) \frac{\partial f_0}{\partial J_3}\right). \tag{4.15}$$

Inversion of the collision operator, C_0, is now required to determine f_1, subject to certain integrability conditions. In the present formulation these conditions are trivially satisfied, as we now show.

The integrability conditions are determined by the annihilators of C_0, in this case the particle and energy moments. It is evident that the particle moment, $\int d^2 J_\perp$, also annihilates the source terms from Eq. (4.14). For the energy moment, we compute $\int d^2 J_\perp H_0(J)$, from Eq. (4.14), and integrate the right side by parts. There results the condition,

$$0 = -\int d^2J_\perp \frac{q}{2\pi} a_2\omega_2 f_0 V_T + \int d^2J_\perp \, \omega_\perp \cdot D_1 \cdot e_3 \frac{\partial f_0}{\partial J_3}.$$

The first term vanishes since the equilibrium distribution, f_0, carries no current in the sense of Eq. (4.6). Using form (4.11) for D_1, the second term is zero.

For use in the ensuing transport equations, it is convenient to express the driving terms for f_1 in terms of thermodynamic forces A_i,

$$\begin{aligned} A_1 &= d \ln n / dJ_3 \\ A_2 &= d \ln T / dJ_3 \\ A_3 &= V_T / T. \end{aligned} \tag{4.16}$$

Using form (4.8) for f_0, the first order Eq.. (4.15) becomes $C_0(f_1) = a_1 f_0 A_1 + a_2 f_0 A_2 + a_3 f_0 A_3$, where the coefficients a_i, are given by

$$\begin{aligned} a_1 &= -\frac{\partial}{\partial J_\perp} \cdot [D_1 \cdot e_3] \\ a_2 &= -\frac{\partial}{\partial J_\perp} \cdot \left[D_1 \cdot e_3 (\frac{H_0}{T} - \frac{3}{2}) \right] \\ a_3 &= -\frac{q}{2\pi} a_2 \omega_2. \end{aligned} \tag{4.18}$$

Finally, defining the individual responses, g_i, according to,

$$C_0(g_i) = a_i f_0, \tag{4.19}$$

f_1 can be expressed as a sum over the thermodynamic forces,

$$f_1 = \sum_i g_i A_i. \tag{4.20}$$

These relations are useful for writing the transport equations in a compact form and for proving Onsager symmetry of the transport coefficients.

To second order in ρ_p/a, Eq. (4.1) becomes,

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial J_3} \frac{q}{2\pi} a_3(J) f_0 V_T + \frac{\partial}{\partial J_\perp} \cdot \frac{q}{2\pi} a_\perp(J) f_1 V_T - C_1(f_1) - C_2(f_0) = C_0(f_2). \tag{4.21}$$

The integrability conditions for the solution of f_2 now provide the transport equations. Accordingly, although the transport equations themselves are second order, they require knowledge of f only through first order.

The particle moment of (4.21) is,

$$\frac{\partial}{\partial t}n_3 + \frac{\partial}{\partial J_3}\Big[\int d^2J_\perp \frac{q}{2\pi}a_3 f_0 V_T - \int d^2J_\perp\, e_3 \cdot D_1 \cdot \frac{\partial}{\partial J_\perp} f_1 - \int d^2J_\perp\, D_{33}\frac{\partial f_0}{\partial J_3}\Big] = 0.$$

The second term in brackets can be integrated by parts,

$$-\int d^2J_\perp\, e_3 \cdot D_1 \cdot \frac{\partial}{\partial J_\perp} f_1 = \int d^2J_\perp\, f_1 \frac{\partial}{\partial J_\perp}\cdot (e_3 \cdot D_1)^t = \int d^2J_\perp f_1 \frac{\partial}{\partial J_\perp}\cdot D_1 \cdot e_3 = -\int d^2J_\perp\, a_1 f_1,$$

using the symmetry of D_1 and the definition (4.18), to give,

$$\frac{\partial n_3}{\partial t} + \frac{\partial}{\partial J_3}\left[\int d^2J_\perp \frac{q}{2\pi}a_3 f_0 V_T - \int d^2J_\perp a_1 f_1 - \int d^2J_\perp\, D_{33}\frac{\partial f_0}{\partial J_3}\right] = 0. \tag{4.22}$$

This is in the form of transport Eq. (1.5), with the flux, Γ, being proportional to the thermodynamic forces, A_i. Note that the thermodynamic force, A_1, is defined in terms of the spatial number density, n, while the transport equation itself is expressed per unit J_3. The reason for this and its relation to the flux surface average of the conventional Eulerian theory, will be given in Sec. V. We then define transport coefficients, T_{1i}, as in Eq. (1.7), as follows,

$$\Gamma = T_{11}\,\frac{d\ln n}{dJ_3} + T_{12}\,\frac{d\ln T}{dJ_3} + T_{13}\,\frac{V_T}{T}. \tag{4.23}$$

We use an inner product notation, $(g,h) \equiv \int d^2J_\perp\; gh$, to write the coefficients. The second term in (4.22), by virtue of Eqs. (4.19) and (4.20), yields implicit contributions of the form $T^i_{1j} = -(a_1, g_j)$. In addition, there are explicit contributions to the flow that do not require the calculation of f_1. We denote these by T^e_{1j}. Collecting terms, the particle transport coefficients, T_{1j}, become,

$$\begin{aligned} T_{11} &= (-D_{33}, f_0) - (a_1, g_1) = T^e_{11} + T^i_{11} \\ T_{12} &= (-D_{33}, f_0(\frac{H_0}{T} - \frac{3}{2})) - (a_1, g_2) = T^e_{12} + T^i_{12} \\ T_{13} &= (\frac{qT}{2\pi}a_3, f_0) - (a_1, g_3) = T^e_{13} + T^i_{13}. \end{aligned} \tag{4.24}$$

The coefficient, T_{11}, is the particle diffusion coefficient, appropriately normalized, and contains both explicit and implicit parts. Coefficient T_{13} implies a radial flow driven by the toroidal voltage, the pinch effect. The explicit part of T_{13} arises from the second term in Eq. (4.21) and is entirely independent of the existence of collisions.

The procedure for obtaining the energy transport equation from the energy moment of Eq. (4.21) is similar. The first order term in the electric field now contributes to give Ohmic heating. The energy transport equation is in the form (1.6), with the heat flux,

$$q/T = T_{21}\,\frac{d\ln n}{dJ_3} + T_{22}\,\frac{d\ln n}{dJ_3} + T_{32}\,V_T/T. \tag{4.25}$$

The transport coefficients are again decomposed into implicit and explicit parts, as

$$\begin{aligned} T_{21} &= (-D_{33}, f_0(\frac{H_0}{T} - \frac{3}{2})) - (a_2, g_1) = T^e_{21} + T^i_{21} \\ T_{22} &= (-D_{33}, f_0(\frac{H_0}{T} - \frac{3}{2})^2) - (a_2, g_2) = T^e_{22} + T^i_{22} \\ T_{23} &= (\frac{qT}{2\pi}a_3, f_0(\frac{H_0}{T} - \frac{3}{2})) - (a_2, g_3) = T^e_{23} + T^i_{23}. \end{aligned} \tag{4.26}$$

Finally the toroidal current, I^i_T, appearing in the Ohmic heating term $I^i_T V_T$, of Eq. (1.6), can also be expressed as a flux driven by the thermodynamic forces

$$I^i_T = T^i_{31}\,\frac{d\ln n}{dJ_3} + T^i_{32}\,\frac{d\ln T}{dJ_3} + T^i_{33}\,V_T/T, \tag{4.26}$$

where the transport coefficients in Eq. (4.26) are only of the implicit type,

$$\begin{aligned} T^i_{31} &= -(a_3, g_1) \\ T^i_{32} &= -(a_3, g_2) \\ T^i_{33} &= -(a_3, g_3). \end{aligned} \tag{4.27}$$

When the equilibrium Hamiltonian, $H_0(J)$, is known, transport equations (1.5) and (1.6), together with the coefficients (4.24), (4.26), and (4.28) are a closed system evolving the density and temperature. The particle density and toroidal current are expressed naturally per unit J_3 in this representation. However, the Hamiltonian, $H_0(J)$, depends implicitly on the magnetic field. This requires knowledge of the toroidal current density in real space *and* includes a diamagnetic contribution in addition to I^i_T. Were it not for this circumstance the real space densities would never be needed in the transport theory. The appearance of n in the thermodynamic force, A_1, is a consequence of the normalization (4.8) and not fundamental.

The term, I^i_T, defined by Eq. (4.6), was not derived as an electric current but was interpreted as such from the form of the Ohmic heating term in the energy equation. In addition to this dissipative current, there are diamagnetic currents, I^e_T, for which no work is done by the external voltage. This is shown in Sec. V, where the spatial toroidal current density is computed. The flux surface averaged current, to first order in the gyroradius has an implicit piece, as given by Eq. (4.26) and an explicit piece in the form

$$I^e_T = T^e_{31} \frac{d \ln n}{dJ_3} + T^e_{32} \frac{d \ln T}{dJ_3}. \tag{4.28}$$

Onsager symmetry is immediately apparent for the implicit coefficients, T^i_{ij}, using their inner product form and the self-adjointness of C_0. Thus, noting that,

$$-T^i_{ij} = (a_i, g_j) = (C_0(g_i), g_j) = (g_i, C_0(g_j)) = (a_j, g_i) = -T^i_{ji}, \tag{4.30}$$

proves the symmetry. This property is a direct result of the symmetry of the full diffusion tensor, $\boldsymbol{D}$. The symmetry of D_1 leads to the inner product form for the implicit fluxes and the symmetry of $\boldsymbol{D}_0$ makes C_0 self-adjoint.

The symmetry of the explicit fluxes T^e_{12} and T^e_{21} is evident by inspection of Eqs. (4.24) and (4.26). The explicit symmetry between the pinch coefficients T^e_{13} and T^e_{23}, and the (diamagnetic) bootstrap current coefficients T^e_{31} and T^e_{32} will be shown and interpreted in detail in Sec. V.

V. Transport Theory for the Lorentz Model Axisymmetric Torus

In this section we illustrate the formal theory developed in Secs. I – IV by evaluating the transport coefficients and by comparing the Lagrangian and conventional Eulerian representations.

The kinetic equation then, in the $\boldsymbol{x}$, $\boldsymbol{v}$ variables, is just Eq. (1.1). The form (1.3) used in the Lagrangian formulation can be obtained from Eq. (1.1) by simply changing variables to J, $\boldsymbol{\theta}$ and averaging over the angles as shown in Sec. III. Since they commence from the same equation, the two representations must then be equivalent. Demonstrating this equivalence, however, is not straightforward, because of the basic differences in theoretical structure. Moreover, these formal structural differences have some interesting physical interpretations with evident implications for future developments.

A. Normalization, Moments, Relation to Flux Surface Average.

In the formal theory of Sec. IV, quantities such as particle number and energy are naturally expressed as densities per unit of action, J_3, and arise as reduced moments such as, $n_3 = \int dJ_1\, dJ_2\, f$. The conventional (Eulerian) representation uses spatial densities, such as n, and expresses the transport equations in terms of flux surface averages. This section develops the connection between n_3, n and the flux surface average.

As spatial variables we will use the conventional magnetic flux coordinates, (ψ, β, s), discussed in Sec. II. The volume element is then,

$$\nabla s \cdot \nabla\beta \times \nabla\psi\, d^3x = ds\, d\beta\, d\psi = B d^3x, \tag{5.1}$$

and the specific volume, $dV/d\psi$, between flux surfaces is

$$dV/d\psi = \oint \frac{|ds| d\beta}{B} = 2\pi \oint \frac{|ds|}{B}, \tag{5.2}$$

where the line integral covers one circuit of the poloidal circumference. The path length, $|ds|$, in Eq. (5.2) is taken to be positive. The flux surface average of any function, F, can be written

$$\langle F\rangle_{\psi} \equiv \frac{d\psi}{dV} \int dA\, F/|\nabla\psi| = \frac{d\psi}{dV} \oint |ds|\, d\beta\, F/B. \tag{5.3}$$

The subscript, ψ, distinguishes this from the bounce average, natural to the Lagrangian representation, and denoted by

$$\langle F\rangle = \int d\theta_2 F = \omega_2 \oint ds F/w, \tag{5.4}$$

where the path element, ds, in Eq. (5.4) has the same sign as w.

Given any distribution $f(J, \theta)$ in the action–angle variables, the spatial density $n(x)$ can be obtained by projection, *viz.*

$$n(x) = \int d^3J\, d^3\theta\, \delta(x - x(J, \theta)) f(J, \theta), \tag{5.5}$$

with $x(J, \theta)$ determined by the transformation, $x, v \leftrightarrow J, \theta$. In magnetic flux coordinates this becomes, using (5.1),

$$n(x) = \int d^3J\, d^3\theta\, B\, \delta(s - s(J, \theta))\delta(\beta - \beta(J, \theta))\delta(\psi - \psi(J, \theta)) f(J, \theta). \tag{5.6}$$

The flux surface average density takes a particularly simple form. We apply Eq. (5.3) to Eq. (5.6). Neglecting toroidal gyroradius corrections (classical transport) and using axisymmetry eliminates θ_1 and θ_3 dependences of f. There results,

$$\langle n(x)\rangle_\psi = \frac{d\psi}{dV}\int d^3J \oint d\theta_2\, \delta(\psi - \psi(I, \theta_2)) f(J, \theta_2), \tag{5.7}$$

or, using path length, s', in place of θ_2,

$$\langle n(x)\rangle_\psi = \frac{d\psi}{dV}\int d^3J\, \omega_2 \oint \frac{ds'}{w}\, \delta(\psi - \psi(J, s')) f(J, s'). \tag{5.8}$$

The function $\psi(J, s')$ can be inferred from Eqs. (2.63) and (2.68). For present purposes, we write the third action as

$$\begin{aligned} J_3 &= \frac{2\pi q}{c}\psi + 2\pi\left(\frac{muI}{B} - \langle\frac{muI}{B}\rangle\right) \\ &\equiv \frac{2\pi q}{c}\psi + \Delta J_3, \end{aligned} \tag{5.9}$$

where I is conventionally defined as RB_T. Thus, to order ρ_p/a, $J_3 = \frac{2\pi q}{c}\psi$ and to second order in ρ_p/a, for f independent of s', the flux surface average density becomes

$$\langle n(x)\rangle_\psi = \frac{2\pi q}{c}\frac{d\psi}{dV}\int d^2J_\perp\, f(J_\perp, \frac{2\pi q}{c}\psi) = \frac{dJ_3}{dV} n_3, \tag{5.10}$$

so that $\langle n\rangle_\psi / n_3$ is just the specific volume between surfaces $J_3 = \text{const}$.

Actually the local density $n(x)$, not flux surface averaged, and n_3 have the same relationship although only to first order in ρ_p/a. To show this, we first compute the normalization constant for f_0, written as,

$$f_0(J) = n_3(J_3)c_3(J_3)\exp\left(-H_0(J)/T(J_3)\right), \tag{5.11}$$

with c_3 determined by

$$1 = c_3 \sum_\sigma \int dJ_1\, dJ_2\, \exp\left(-H_0(J)/T\right). \tag{5.12}$$

The sum σ runs over plus and minus parallel velocities. It is convenient to use the variables

$$\begin{aligned} \lambda &= J_1/H_0(J_1, J_2) \\ H &= H_0(J_1, J_2), \end{aligned} \tag{5.13}$$

with volume element,

$$dJ_1\, dJ_2 = \frac{H}{\omega_2}\, d\lambda\, dH. \tag{5.14}$$

The variable λ is the bounce averaged pitch angle divided by ω_1. Using Eq. (2.40) for ω_2, Eq. (5.12) becomes,

$$1 = c_3 \sum_\sigma \oint ds' \int d\lambda\, dH \frac{\sigma H}{\sqrt{\frac{2}{m}H(1-\lambda\Omega/2\pi)}} e^{-H/T}. \tag{5.15}$$

In the integration over λ, it is understood that the maximum value of λ at fixed s' is $2\pi/\Omega$. Doing the λ and H integrals and summing over σ yields,

$$1 = c_3(2\pi m T)^{3/2} \frac{c}{q} \oint \frac{|ds|}{B} = c_3(2\pi m T)^{3/2} \frac{dV}{dJ_3}. \tag{5.16}$$

Upon solving for c_3, and inserting in Eq. (5.11), the specific volume factor combines with n_3 to give $n \equiv n_3\, dJ_3/dV$, which acocunts for the normalization used in Eq. (4.9).

Finally, to demonstrate that the n, so defined, is the leading order spatial density, we evaluate Eq. (5.6) with $f = f_0$. The β integral is done by noting that $\beta(J,\theta) = 2\pi\theta_3 + \delta\beta(J,\theta_2)$. Writing the θ_2 integral in terms of arc length, as above then gives,

$$\begin{aligned} n(x) &= \sum_\sigma \int d^3J\, \omega_2 \oint \frac{ds'}{|w|} \frac{B}{2\pi} \delta(s-s')\delta\left(\psi - \frac{c}{2\pi q}(J_3 - \Delta J_3)\right) f_0(J) \\ &= \sum_\sigma \int d^2 J_\perp \frac{qB\omega_2}{c|w|} f_0\left(J_\perp, \frac{2\pi q}{c}\psi + \Delta J_3\right). \end{aligned} \tag{5.17}$$

To lowest order in ρ_p/a, the ΔJ_3 may be neglected. Upon doing the indicated integrals, one finds, form (5.17), $n(x) = n$, as required.

B. Collision Operator and Kinetic Equation

We now carry out an explicit evaluation of the coefficients in Eq. (3.28) for the Lorentz limit. Defining the velocity dependent collision frequency,

$$\nu(v) = 2\pi n q^4 \ln \Lambda / m^2 v^3, \tag{5.18}$$

eq. (3.28) can be written,

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\partial}{\partial J} \cdot \left(\int d^3\theta \, \frac{q}{m} E \cdot \nabla_v J f \right) = \\ \frac{\partial}{\partial J} \cdot \int d^3\theta \, \nu (\nabla_v J)^T \cdot (v^2 I - vv) \cdot \nabla_v J \cdot \frac{\partial}{\partial J} f, \end{aligned} \tag{5.19}$$

which is of the form (1.3) To simplify things further, we assume axisymmetry, and evaluate only *neoclassical* fluxes, neglecting effects of order ρ_T/a. The integrands in Eq. (5.19) are then independent of both θ_1 and θ_3, leaving only the θ_2, or bounce average, integral. Writing the induction field as, $E = e_\varsigma V_T/2\pi R$, where V_T is the (constant) toroidal voltage, puts Eq. (5.19) in th form of Eq. (4.1) and (4.2), with the coefficients given by

$$a(J) = \omega_2 \oint \frac{ds}{w} \frac{1}{mR} e_\varsigma \cdot \nabla_v J, \tag{5.20}$$

$$D(J) = \nu \omega_2 \oint \frac{ds}{w} (\nabla_v J)^t \cdot (v^2 I - vv) \cdot \nabla_v J. \tag{5.21}$$

It remains to evaluate the partial velocity derivatives of the actions at fixed x, and carry out the bounce averages. This has an obvious physical interpretation since the effect of collisions is to scatter the velocity at a fixed spatial position. The derivatives, $\nabla_v J$, measure the effectiveness by which such a process changes the actions, and Eq. (5.21) averages the resulting action scattering over a bounce. In particular $\nabla_v J_3$ measures the rate of radial scattering due to collisions. It is this sensitivity of the radial coordinate, J_3, to velocity parameters that leads to the enhanced (neoclassical) transport. The importance for good confinement of the *insensitivity* of J_3 to velocity parameters, termed *omnigenity*, has been noted in connection with mirror confinement (14).

From Eqs. (2.79) and (2.80) we have,

$$\nabla_v J_1 = \frac{2\pi m}{\Omega}(v - ub), \tag{5.22}$$

and

$$\nabla_v J_2 = \frac{m}{\omega_2}\left(v(1 - \frac{2\pi\omega_1}{\Omega}) + \frac{2\pi\omega_1}{\Omega} ub\right). \tag{5.23}$$

While from Eq. (5.9), it follows that

$$\begin{aligned} \nabla_v J_3 &= \nabla_v[2\pi mu I/B - 2\pi\langle \frac{muI}{B}\rangle] \\ &= b\frac{2\pi m I}{B} - \nabla_v J_1 \frac{\partial}{\partial J_1} 2\pi\langle\frac{muI}{B}\rangle \\ &\quad - \nabla_v J_2 \frac{\partial}{\partial J_2} 2\pi\langle\frac{muI}{B}\rangle, \end{aligned} \tag{5.24}$$

plus terms of higher order, neglected, consistent with our $\omega_3 \to 0$ limit, because of the weak dependence of equilibrium quantitites on J_3.

Note that for each action, $\nabla_v J_i$ can be written in the form $\alpha v + \beta b$. Since $v \cdot (v^2 I - vv) \cdot v = (v^2 I - vv) \cdot v = 0$, the diffusion tensor depends only on the b component. We can therefore write,

$$D = \omega_2 \oint \frac{ds}{w} \nu (b \cdot \nabla_v J)(b \cdot \nabla_v J)(v^2 - u^2), \tag{5.25}$$

or in index notation,

$$D_{ij} = \frac{J_1}{\pi m}\langle \nu\Omega \frac{\partial}{\partial u} J_i \frac{\partial}{\partial u} J_j\rangle. \tag{5.26}$$

An abbreviated notation is employed in both (5.25) and (5.26), with $b \cdot \nabla_v = \partial/\partial u$ indicating the coefficient of b in the decomposition of $\nabla_v J_i$ into vectors along v and b. The indication $b \cdot \nabla_v$ does not denote the usual dot product because v and b are not orthogonal in general. Using Eqs. (5.22), (5.23) and (5.24), the ordered diffusion tensor can be written as the sum of the terms, $D = D_o + D_1 + D_2$, where

$$D_0 = 4\pi\nu J_1 \omega_1^2 \langle\frac{mu^2}{\Omega}\rangle(-\frac{1}{\omega_1}e_1 + \frac{1}{\omega_2}e_2)(-\frac{1}{\omega_1}e_1 + \frac{1}{\omega_2}e_2), \tag{5.27}$$

$$\begin{aligned} D_1 &= 4\pi\nu J_2 \omega_2 \langle u \frac{\partial}{\partial u}\left(\frac{muI}{B} - \langle\frac{muI}{B}\rangle\right)\rangle \\ &\quad \times [(-\frac{1}{\omega_1}e_1 + \frac{1}{\omega_2}e_2)e_3 + e_3(-\frac{1}{\omega_1}e_1 + \frac{1}{\omega_2}e_2)], \end{aligned} \tag{5.28}$$

$$D_2 = 4\pi\frac{\nu J_1}{m}\langle\Omega\left[\frac{\partial}{\partial u}\left(\frac{muI}{B} - \langle\frac{muI}{B}\rangle\right)\right]^2\rangle e_3 e_3. \tag{5.29}$$

The third action in Eq. (5.9) is predominantly a spatial variable, $2\pi\frac{q}{c}\psi$, with a kinetic correction, ΔJ_3. It is the derivatives $\nabla_v J_3 = \nabla_v \Delta J_3$ which appear in D_1 and D_2 and determine the radial transport. The magnitude of ΔJ_3 measures the deviation of the orbit from the average flux surface, $\langle\psi\rangle = \frac{c}{2\pi q}J_3$. For circulating particles the local kinetic angular momentum, muI/B, very nearly equals the average, $\langle muI/B\rangle$, $\Delta J_3 \approx 0$, the displacement of the orbit from the average flux surface is small, and the neoclassical enhancement of transport is minimal (except near the boundary between the circulating and trapped regions). For trapped particles, the averaged angular momentum is identically zero. The omnigenity factor, $b\cdot\nabla_v J_3 = 2\pi mI/B$, is then significant, indicating substantial neoclassical enhancement for the trapped particles. To denote this, we define the factor

$$\Delta_c = \frac{B}{mI} b\cdot\nabla_v \langle\frac{muI}{B}\rangle, \tag{5.30}$$

which is zero for trapped particles, unity for well circulating particles. Using Eq. (5.30) the coefficients D_1 and D_2 become,

$$D_1 = 4\pi\nu J_1\omega_1\langle\frac{muI}{B}(1-\Delta_c)\rangle[(-\frac{1}{\omega_1}e_1+\frac{1}{\omega_2}e_2)e_3+e_3(-\frac{1}{\omega_1}e_1+\frac{1}{\omega_2}e_2)], \tag{5.31}$$

$$D_2 = 4\pi\nu J_1\langle m\Omega\frac{I^2}{B^2}(1-\Delta_c)^2\rangle e_3e_3. \tag{5.32}$$

The electric field coefficient ,a, is evaluated similarly. The term a_1 is of order ρ_T/a and would vanish identically if the gyroaverage, $\int d\theta_2$, were retained in Eq. (5.20). For a_2 we need,

$$\begin{aligned} e_\varsigma\cdot\nabla_v J_2 &= \frac{m}{\omega_2}e_\varsigma\cdot v + \frac{2\pi m\omega_1}{\Omega\omega_2}(-e_\varsigma\cdot v + e_\varsigma\cdot b\, b\cdot v) \\ &= \frac{m}{\omega_2}e_\varsigma\cdot v - \frac{2\pi m\omega_1}{\Omega\omega_2}e_\varsigma\cdot b\times e_\psi\, b\times e_\psi\cdot v \\ &= \frac{m}{\omega_2}e_\varsigma\cdot v - \frac{2\pi m\omega_1}{\Omega\omega_2}\frac{B_pB_T}{B^2}e_\psi\cdot b\times v \\ &\to \frac{m}{\omega_2}\frac{uB_T}{B}, \end{aligned} \tag{5.33}$$

where the last step follows from averaging over θ_1. To the order we require, $u=\omega$, so that

$$a_\perp = e_2a_2 = e_2\frac{1}{\omega_2}\langle\frac{uI}{R^2B}\rangle = e_2 I\oint\frac{ds}{B}\frac{1}{R^2}. \tag{5.34}$$

The integral in s goes around a full orbital cycle and vanishes for trapped particles. The radial flow coefficient is

$$a_3 = \langle\frac{2\pi}{mR}e_\varsigma\cdot\nabla_v\Big(\frac{muI}{B}-\langle\frac{muI}{B}\rangle\Big)\rangle, \tag{5.35}$$

or, for trapped particles, simply,

$$a_3 \to 2\pi I^2 \langle \frac{1}{B^2 R^2} \rangle = 2\pi\omega_2 \oint \frac{ds}{u} \frac{B_T^2}{B^2}. \tag{5.36}$$

The orderings assumed in Sec. IV can now be verified. Since $J_1 \sim mv_{\rm th}\rho_T \sim mv_{\rm th}^2/\omega_1$ and $J_2 \sim mv_{\rm th}R \sim mv_{\rm th}^2/\omega_2$, it follows from Eq. (5.27) that $\omega_1 dJ_1 \sim \omega_2 dJ_2$, all the terms of C_0 are the same magnitude and of order ν. In contrast, the dimensionless parameter characterizing J_3 derivatives in the collision operator is $mv_{\rm th}I/BJ_3 \sim \rho_p/a < 1$. Similarly in the electric field term, since the coefficients a_2 and a_3 are of order unity and $|d\ln J_2/d\ln J_3| \sim \rho_p/a$, the assumed ordering is satisfied.

C. Transport Coefficients

To evaluate the transport coefficients, one must determine the perturbed distributions, g_i, required for the implicit fluxes. We illustrate the procedure for g_1 and show the equivalence of the transport coefficients, T_{11}, T_{13} and T_{31} with those obtained by Hinton and Hazeltine from the Eulerian formulation. This equivalence relies on a certain cancellation of parts of the implicit and explicit fluxes, which is demonstrated in detail in Appendix A. To determine the full flux surface averaged toroidal current, the diamagnetic contributions. T_{13}^e and T_{23}^e, must be computed. This is done, and demonstration of the explicit symmetry, $T_{31}^e = T_{13}^e$, is given.

The computation of g_1, follows from Eq. (4.20), evaluated using the specific forms (5.27) and (5.28),

$$\frac{\partial}{\partial J} \cdot D_o \cdot \frac{\partial}{\partial J} g_1 = -(\frac{\partial}{\partial J} \cdot D_1 \cdot e_3) f_o = -\frac{\partial}{\partial J} \cdot [D_1 \cdot e_3 f_o] . \tag{5.37}$$

This equation is easily integrated using the variables $\boldsymbol{\lambda} \equiv (\lambda, H)$ defined in Eq. (5.13). With the transformation matrix, M, such that $\frac{\partial}{\partial J} = M \cdot \frac{\partial}{\partial \boldsymbol{\lambda}}$, given by

$$M = \frac{1}{H}(1 - \frac{\omega_1 J_1}{H}) e_1 \, e_\lambda + \omega_1 e_1 \, e_H \tag{5.38}$$

$$- \frac{\omega_2 J_1}{H^2} e_2 e_\lambda + \omega_2 e_2 e_H ,$$

Eq. (5.37) becomes,

$$\frac{\partial}{\partial J} \cdot \{(-\frac{e_1}{\omega_1} + \frac{e_2}{\omega_2})[-4\pi\nu\lambda < B >< \frac{mu^2}{B} > \frac{\partial}{\partial \lambda} g_1 + 4\pi\nu\lambda\omega_1 H < \frac{muI}{B}(1-\Delta_c) > f_o]\} = 0 \tag{5.39}$$

The variable λ is bounded by zero and $2\pi/\Omega_{MIN}$ (the value $2\pi/\Omega_{MAX}$ corresponds to the boundary between trapped and circulating particles). Eq. (5.39) is to be solved subject to a condition of zero flow across the boundaries $\lambda = 0,\ 2\pi/\Omega_{MIN}$ (and continuity of flux across $\lambda = 2\pi/\Omega_{MAX}$ should be a discontinuity in the forcing function appear). Application of the boundary condition at $\lambda = 2\pi/\Omega_{MIN}$ requires that the terms in square brackets vanish identically, yielding,

$$< \frac{mu^2}{\Omega} > \frac{\partial g_1}{\partial \lambda} = H < \frac{muI}{B}(1-\Delta_c) > f_o . \tag{5.40}$$

This equation determines g_1, for both signs of transiting particles, up to a constant equal to the trapped particle density. If we require that $\sum_\sigma \int dJ_1\, dJ_2\, g_1 = 0$, so that f_o contain all the particle density through second order, then the trapped particle perturbation is zero.

The implicit contribution to the diffusion coefficient, $-T^i_{11}$, is now easily evaluated,

$$\begin{aligned} -T^i_{11} \equiv (a_1, g_1) &= \int dJ_1 dJ_2\, a_1 g_1 \\ &= \int dJ_1 dJ_2 (-\frac{\partial}{\partial J} \cdot D_1 \cdot e_3) g_1 = \int dJ_1 dJ_2 (D_1 \cdot e_3)^t \cdot \frac{\partial}{\partial J} g_1 \\ &= \int dJ_1 dJ_2 e_3 \cdot D_1 \cdot \frac{\partial}{\partial J} g_1 . \end{aligned} \tag{5.41}$$

Transforming variables, and using Eq. (5.31), this becomes

$$-T^i_{11} = \sum_\sigma \int dH d\lambda 4\pi\nu\lambda \frac{H^2}{\omega_2} f_o < \frac{muI}{B}(1-\Delta_c) >^2 / < \frac{mu^2}{\Omega} > . \tag{5.42}$$

Adding the explicit piece, $T^e_{11} = (-D_{33}, f_o)$, gives the full transport coefficient as,

$$T_{11} = -\sum_\sigma \int dH d\lambda 4\pi\nu\lambda \frac{H^2}{\omega_2} f_o [< m\Omega \frac{I^2}{B^2}(1-\Delta_c)^2 > - < \frac{muI}{B}(1-\Delta_c) >^2 / < \frac{mu^2}{\Omega} >] . \tag{5.43}$$

This expression, as we will shortly show, reduces to the well known results from the Eulerian representation if we set $\Delta_c = 0$. In fact, Δ_c is zero only in trapped space and is nearly unity over most of circulating space. What actually happens is that the terms involving Δ_c from the explicit part exactly cancel those from the implicit part giving the same effect as putting $\Delta_c = 0$. This cancelation and an analogous one in the other transport coefficients is demonstrated in Appendix A.

That the trapped particles make the dominant contribution to the diffusion coefficient, $-T_{11}$, is well known. To arrive at this conclusion in the Eulerian picture, one must actually evaluate the integrals in Eq. (5.43), (for $\Delta_c = 0$) , and show that they nearly cancel for circulating particles. The circulating and trapped particle contributions are formally the same order in ρ_p/a and a great deal of effort is required to demonstrate the numerical smallness of the circulating particle contributions. In the Lagrangian formulation there is an additional small parameter, $1 - \Delta_c$, appearing in both the explicit and implicit terms, which measures the relative insensitivity of the circulation particle action, J_3 , to velocity scattering. Correct to order $(1 - \Delta_c)^2$ the transport coefficient is due to trapped particles, is therefore explicit and can be obtained from simple moments of the collision operator without ever solving for the perturbed distribution function. This reduces the required computations significantly, an aspect of the Lagrangian formulation that has practical potential for evaluating transport in more complex systems.

To relate Eq. (5.43) to the familiar Eulerian expressions, we write out the bounce averages and do the integrals over H and the σ sum to give

$$T_{11} = -\frac{\nu_o}{\sqrt{\pi}} \frac{2T}{m} \frac{qI^2}{c} n \int_O^{2\pi/\Omega_{MIN}} d\lambda\lambda \left[\oint \frac{ds}{B} \frac{1}{\sqrt{1 - \lambda\Omega/2\pi}} - \mathcal{H}(2\pi/\Omega_{MAX} - \lambda) \frac{(\oint \frac{ds}{B})^2}{\oint \frac{ds}{B} \sqrt{1 - \lambda\Omega/2\pi}} \right]. \qquad (5.44)$$

In Eq. (5.44), ν_o is given by Eq. (5.18) evaluated at $v = \sqrt{2T/m}$, and $\mathcal{H}$ is the heavy side step function. Note that the second (implicit) term vanishes for the trapped particles. Since the s integrals have a weighting factor of $1/B$, they can now be *interpreted* as flux surface averages even though they arose, in the present theory, from orbit averages. Thus, letting $\lambda = \lambda' \, 2\pi/\Omega_o$ and recalling that $dV/dJ_3 = c/q \oint ds/B$, Eq. (5.44) becomes

$$T_{11} = -\frac{V_o}{\sqrt{\pi}} \frac{2T}{m\Omega_o^2} (\frac{2\pi q I}{c})^2 n_3 I_{11} \tag{5.45}$$

where I_{11} is the dimensionless integral,

$$I_{11} = \int_0^{\Omega_o/\Omega_{MIN}} d\lambda' \lambda' \left[< \frac{1}{\sqrt{1 - \lambda'\Omega/\Omega_o}} >_\psi - \frac{\mathcal{H}(\Omega_o/\Omega_{MAX} - \lambda')}{< \sqrt{1 - \lambda'\Omega/\Omega_o} >_\psi} \right]. \tag{5.46}$$

Apart from the system of units, this is the same coefficient derived by Hinton and Hazeltine. Their units are based on an *effective minor radius* coordinate ρ , defined by $B_{po}(p) = 1/R_o \partial\psi/\partial\rho$, which replaces our J_3 as a radial variable. To convert Eq. (5.45), multiply by $(d\rho/dJ_3)^2 = (c/2\pi q B_{po} R_o)^2$, and divide by the specific volume dV/dJ_3 .

The distribution function g_1 can also be used to compute the implicit bootstrap current, $T^i_{31} = -(a_3, g_1)$, and from the symmetry relation, $T^i_{13} = T^i_{31}$, the implicit Ware pinch coefficient as well. The explicit Ware coefficient follows from Eqs. (4.25) and (5.20) as

$$\begin{aligned} T^e_{13} &= -\int dJ_1 dJ_2 \omega_2 \oint \frac{ds}{u} \frac{qT}{2\pi m R} (e_\varsigma \cdot \nabla_v J_3) f_o , \\ &= -\int dJ_1 dJ_2 \omega_2 \oint \frac{ds}{u} \frac{qT}{2\pi m R} (e_\varsigma \cdot \nabla_v \Delta J_3) f_o , \end{aligned} \tag{5.47}$$

where ΔJ_3, see Eq. (5.9), measures the displacement of the actual ψ from the orbital average $\bar{\psi}$ or J_3 .

Now consider the calculation of the toroidal electric current density in the action-angle formalism. Following, the procedure for obtaining the spatial density, we begin from

$$J_T = \int d^3J d^3\theta \delta(x - x(J, \theta) q e_\varsigma \cdot v(J, \theta) f(J) . \tag{5.48}$$

This, in general, is a function of ψ and the poloidal angle and can be used in Ampere's law to determine the flux function ψ . It contains the Pfirsch-Schluter current in addition to the parts, constant on a flux surface, that are determined from transport theory. Thus to define I_T for the present transport theory, we weight J_T by $1/2\pi R$ to give an angular current, flux surface average, and multiply by the specific volume dV/dJ_3 ,

$$I_T \equiv \frac{c}{2\pi q}\frac{dV}{d\psi} < J_T/2\pi R >_\psi, \tag{5.49}$$
$$= \int d^3J\omega_2 \oint \frac{ds}{u}\frac{c}{2\pi q}\delta(\psi - \frac{c}{2\pi q}(J_3 - \Delta J_3))\,\frac{q}{2\pi R}e_\varsigma \cdot vf,$$
$$= \int d^2J_\perp\omega_2 \oint \frac{ds}{u}\frac{q}{2\pi R}e_\varsigma \cdot vf(J_\perp, \frac{2\pi q}{c}\psi + \Delta J_3).$$

Expanding in powers of ρ_p/a, this becomes, to first order,

$$I_T = \int d^2J_\perp\omega_2 \oint \frac{ds}{u}\frac{q}{2\pi R}e_\varsigma \cdot v\left[f_1(J_\perp, \frac{2\pi q}{c}\psi) + \Delta J_3\frac{\partial f_o}{\partial J_3}\right]. \tag{5.50}$$

The first term is just the implicit current density, I_T^i, of Eq. (4.6) as can be seen from Eq. (5.34) for a_2. The second term is a diamagnetic current associated with the departure of the orbits from their average flux surfaces. It gives the explicit fluxes,

$$T_{31}^e = \int d^2J_\perp\omega_2 \oint \frac{ds}{u}\frac{q}{2\pi R}e_\varsigma \cdot v\Delta J_3 f_o, \tag{5.51}$$

$$T_{32}^e = \int d^2J_\perp\omega_2 \oint \frac{ds}{u}\frac{q}{2\pi R}e_\varsigma \cdot v\Delta J_3(\frac{H}{T} - 3/2)f_o. \tag{5.52}$$

Note that the explicit current has been defined in a manner analogous to Eq. (4.27), so that the total current is given by

$$I_T = T_{31}\frac{d\ln n}{dJ_3} + T_{32}\frac{d\ln T}{dJ_3} + T_{33}V_T/T. \tag{5.53}$$

To prove the explicit conjugacy, we observe that upon change of variables $J_1, J_2 \rightarrow v_\perp, u$, the volume element transforms such that $dJ_1dJ_2\omega_2/u = m^2/\Omega d^3v$, and the integrals in Eqs. (5.47) and (5.51) can be expressed accordingly. If one then integrates the velocity integral in Eq. (5.47) by parts, Eq. (5.51), results.

We can now compute the full coefficient $T_{31} = T_{13}$, as

$$T_{31} = \int dJ_1dJ_2(-a_3g_1 + \omega_2 \oint \frac{ds}{u}\frac{q}{2\pi R}e_\varsigma \cdot v\Delta J_3 f_o), \tag{5.54}$$
$$= \int dJ_1dJ_2(\frac{q}{2\pi}\omega_2 \oint \frac{ds}{B}\frac{I}{R^2}g_1 + \frac{q}{2\pi}\omega_2 \oint \frac{ds}{B}\frac{I}{R^2}\Delta J_3 f_o).$$

As shown in the Appendix the perturbed distrubution can be written $g_1 = g_1' + < 2\pi mIu/B > f_o$, where g_1' satisfies Eq. (5.39) with $\Delta_c = 0$. Making this substitution cancels the average term, $< 2\pi mIu/B >$, in ΔJ_3 to give,

$$T_{31} = \frac{q}{2\pi}\int dJ_1 dJ_2 \omega_2 \oint \frac{ds}{B}\frac{I}{R^2}(g_1' + \frac{2\pi mIu}{B} f_o). \tag{5.55}$$

Changing variables to λ, H and integrating both terms by parts, Eq. (5.40) with $\Delta_c = 0$, can be used for $\partial g_1'/\partial\lambda$. Doing the H integrals the result can be expressed,

$$T_{31} = \frac{3}{4} qTn_3 I_{31}, \tag{5.56}$$

where I_{31} is the dimensionless integral,

$$I_{31} = \int_o^{\Omega_o/\Omega_{MIN}} d\lambda'\lambda' \Bigg[< \frac{I^2}{R^2 B_o^2 \sqrt{1-\lambda'\Omega/\Omega_o}} >_\psi - H(\Omega_o/\Omega_{MAX} - \lambda') \frac{< I^2/R^2B_o^2 >_\psi}{< \sqrt{1-\lambda'\Omega/\Omega_o} >_\psi} \Bigg]. \tag{5.57}$$

Again this is equivalent to the coefficient quoted by Hinton and Hazeltine. Their form of Eq. (5.56) is recovered by multiplying by factors of $d\rho/dJ_3 = c/2\pi qB_{po}R_o$, dV/dJ_3, $2\pi R_o/T$, and -1, to account, respectively, for the radial variable, density normalization, current normalization, and sign of radial variable (J_3 is negative for electrons).

D. Physical Mechanism of the Pinch Effect and the Bootstrap Current

We now evaluate the separate contributions to the transport coefficients in the limit of small inverse aspect ratio and interpret the results. The overall (dimensionless) coefficient, obtained from the asymptotic evaluation of Eq. (5.57) is [1].

$$I_{13} = I_{31} \simeq 1.38\sqrt{2\epsilon} + O(\epsilon^{\frac{3}{2}}), \tag{5.58}$$

where $\epsilon = r/R_o$, is the inverse aspect ratio. Recall that Eq. (5.57) was obtained by adding the explicit and implicit results, and that the average terms needed to separate these pieces had then cancelled. Since the interpretation requires this separation, we first identify the dimensionless integrals corresponding to the individual parts.

For the explicit pinch coefficient, we need $e_\varsigma \cdot \nabla_v \Delta J_3$, which is evaluated from Eq. (5.9) as follows,

$$
\begin{aligned}
e_\varsigma \cdot \nabla_v \Delta J_3 &= 2\pi m e_\varsigma \cdot \nabla_v \Big(\frac{uI}{B} - \omega_2 I \oint \frac{ds}{B}\Big) \\
&= 2\pi m \Big[\frac{I^2}{B^2 R} - I \oint \frac{ds}{B} e_\varsigma \cdot \nabla_v J_1 \frac{\partial \omega_2}{\partial J_1} \\
&\quad + e_\varsigma \cdot \nabla_v J_2 \frac{\partial \omega_2}{\partial J_2}\Big)\Big]
\end{aligned} \tag{5.59}
$$

where it is understood that $\oint \frac{ds}{B}$ is zero for trapped particles. The term $e_\varsigma \cdot \nabla_v J_1$ is of order ρ_T/a and therefore negligible. The derivative $\partial\omega_2/\partial J_2$ is expressed in terms of λ and H derivatives, and manipulated slightly to give the electric field coefficient, a_3, as

$$
\begin{aligned}
a_3 &= \omega_2 \oint \frac{ds}{u} \frac{1}{mR} e_\varsigma \cdot \nabla_v \Delta J_3 \\
&= 2\pi\omega_2 \oint \frac{ds}{u} \frac{I^2}{B^2 R^2}\Big[1 - \frac{mu}{H} B\Big(\oint \frac{ds}{B}\Big)\Big(\frac{\omega_2}{2} + \omega_2^2 \lambda \frac{\partial}{\partial \lambda}\frac{1}{\omega_2}\Big)\Big] \\
&\simeq 2\pi\omega_2\Big[\frac{1}{\omega_2} - \frac{m}{2H}\Big(\oint ds\Big)^2\Big(\omega_2 + 2\omega_2^2 \lambda \frac{\partial}{\partial \lambda}\frac{1}{\omega_2}\Big)\Big]
\end{aligned} \tag{5.60}
$$

where the last step assumes small $\epsilon = r/R$. We use the small ϵ limit for the frequencies, such that for trapped particles, $1 - \epsilon < \lambda' < 1 + \epsilon$,

$$
\frac{1}{\omega_2} = \oint ds \sqrt{\frac{m}{2H}} \frac{1}{\sqrt{2\epsilon}} \frac{2}{\pi} K\Big(\frac{1+\epsilon-\lambda'}{2\epsilon}\Big), \tag{5.61}
$$

where K is the elliptic integral of the first kind, and for ciculating particles, $0 < \lambda' < 1 - \epsilon$,

$$
\frac{1}{\omega_2} = \oint ds \sqrt{\frac{m}{2H}} \frac{1}{\sqrt{1+\epsilon-\lambda'}} \frac{2}{\pi} K\Big(\frac{2\epsilon}{1+\epsilon-\lambda'}\Big). \tag{5.62}
$$

Note that the energy dependence is the same for all terms. The integrals can be carried out to give T^e_{13} in the form of Eq. (5.56). For trapped particles the result is

$$I_{13}^{e,tr} = \frac{2}{3}\frac{1}{\sqrt{2\epsilon}}\int_{1-\epsilon}^{1+\epsilon} d\lambda' \frac{2}{\pi} K(\frac{1+\epsilon-\lambda'}{2\epsilon}), \tag{5.63}$$

and for circulating particles,

$$\begin{aligned} I_{13}^{e,cir} = \frac{2}{3}\int_0^{1-\epsilon} d\lambda' \Big[& \frac{2}{\pi}\frac{K(\frac{2\epsilon}{1+\epsilon-\lambda'})}{\sqrt{1+\epsilon-\lambda'}} \\ & - \frac{\pi}{2}\frac{\sqrt{1+\epsilon-\lambda'}}{K(\frac{2\epsilon}{1+\epsilon-\lambda'})} \\ & - \pi\frac{1+\epsilon-\lambda'}{K^2(\frac{2\epsilon}{1+\epsilon-\lambda'})}\lambda' \frac{d}{d\lambda'}(\frac{K(\frac{2\epsilon}{1+\epsilon-\lambda'})}{\sqrt{1+\epsilon-\lambda'}})\Big] \end{aligned} \tag{5.64}$$

The conventional explanation of the pinch effect is that it is predominantly $I_{13}^{e,tr}$ or the Ware effect [4] evaluating Eq. (5.63) in the limit of small ϵ, one finds

$$I_{13}^{e,tr} = \frac{8}{3\pi}\sqrt{2\epsilon} + O(\epsilon^{\frac{3}{2}}), \tag{5.65}$$

which is 62 percent of the full coefficient. The circulating particle contributions $I_{13}^{e,cir}$ and $I_{13}^{i,cir}$ are of opposite sign and cancel to a large degree.

The conjugate process to the Ware effect is the diamagnetic current of the trapped particles, $I_{31}^{e,tr}$. However, this is of order $\epsilon^{3/2}$, and cannot account for any of the bootstrap current in the small ϵ limit [1]. Furthermore, evaluating the diamagnetic contribution of the circulating particles, one finds,

$$I_{31}^{e,cir} = 2\int_o^{1-\epsilon} d\lambda' \sqrt{1+\epsilon-\lambda'}\left(\frac{2}{\pi}E(\frac{2\epsilon}{1+\epsilon-\lambda'}) - \frac{\pi}{2K(\frac{2\epsilon}{1+\epsilon-\lambda'})}\right), \tag{5.66}$$

where E is the elliptic integral of the second kind. The contribution, $I_{31}^{e,cir}$, from Eq. (5.66), is also of order $\epsilon^{3/2}$ although this comes about from a concellation of order unity terms. Physically, the circulating particles produce a toroidal diamagnetic current that changes sign as one moves from the outer to the inner half of the torus. This is the Pfirsch-Schlüter current. The flux surface average, however, as reflected in Eq. (5.66) gives a net current of nearly zero. Furthermore, the integrand in Eq. (5.66) is of order $\epsilon^{3/2}$ for all λ, so that all particles, uniformly, make a small contribution to the net current.

The inference from this is that the bootstrap current is completely implicit. In physical terms it is basically a collisional process affecting the circulating particles. This contrast in physical mechanism, between the pinch and bootstrap effects creates a conceptual dilemma. Why should such disjoint processes, involving different classes of particles, be Onsager conjugate effects?

To resolve this dilemma, first note that the explicit symmetry, together with the above results for T^e_{31}, imply that to order $\epsilon^{3/2}$, $T^e_{13} \simeq 0$. But then, noting Eq. (5.64), it follows that the explicit circulating particle flow is actually radially *out*, and at a rate which cancels entirely (to order $\epsilon^{3/2}$) the Ware pinch! The same conclusion can be reached directly using Eq. (5.64). This is somewhat different than the explicit bootstrap current, in that the trapped and ciculating contributions are separately large and only cancel upon integration over phase space. Nonetheless the total explicit coefficient is zero and this leads to an alternate interpretation of T_{13} : the pinch effect is a collisional process involving circulating particles. This interpretation provides the link with the bootstrap current. However, there is no connection, whatever, with the original Ware effect.

The question is now reduced to understanding the implicit flows. Recall that these arose from the first order (in ρ_p/a) parts of the collision operator, or *cross processes* of the form

$$\frac{\partial}{\partial J_3} e_3 \cdot D \cdot \frac{\partial}{\partial J_\perp}, \tag{5.67}$$

$$\frac{\partial}{\partial J_\perp} \cdot D \cdot e_3 \frac{\partial}{\partial J_3}, \tag{5.68}$$

relfecting correlation in the scattering process between jumps in radius and jumps in velocity. This correlation occurs only for circulating particles. The operator (5.67) gives a radial flux when operating on a perturbed f_1, that carries a current. Operator (5.68), acting on f_o, drives a current carrying perturbation. The perturbed f_1 resulting from this process (5.68) is determined by an equation of the form (5.37). Since both the drive $C_1(f_o)$and the restoring force operator, $C_o(f_1)$, are proportional to the collision frequency, ν_{ei}, the resulting f_1 will be independent of ν_{ei}. The resulting bootstrap coefficients T^i_{31} and T^i_{32} are also independent of ν_{ei}. This does not, of course, contradict the notion that the underlying process in Eq. (5.68) is collisional.

These processes and their consequent fluxes can be understood simply with the aid of Fig. (2), comparing representative orbits for different types of particles. Consider now, two particles, initially well-circulating, moving in opposite directions along the magnetic field. Both have orbits lying very nearly on the flux surface ψ.

As these particles scatter toward trapped space, under the influence of collisions, their orbits change as indicated in the figure. The $\sigma = -1$ particle scatters to an orbit whose average surface is shifted inward relative to ψ while the $\sigma = +1$ particle scatters out. This is the origin of the correlation betwen parallel velocity sign and radial step. A radial flow will result whenver the perturbed distribution has unequal fractions of $\sigma = -1$ and $\sigma = +1$ particles, or in other words, when the pertrubed distribution of circulating particles carries a current. The current driven by the toroidal electric field has an excess of $\sigma = -1$ particles and drives an inward radial flow. This process, in the alternate interpretation, causes the pinch effect. It is due entrely to circulating particles.

Now invert the process just described. that is, take two marginally circulating particles, oppositely directed along the magnetic field and scatter them back toward the well circulating state. The $\sigma = -1$ particle started on an inner average surface and ended up on ψ. The $\sigma = +1$ particle started on an outer average surface and also ended up on ψ. With a normal density gradient the result will be more $\sigma = -1$ than $\sigma = +1$ particles on the final ψ surface, and thus an electric current. This is the mechanism behind the bootstrap current. It is the precise microscopic inverse to the process accounting for the pinch effect. One can then easily understand the Onsager symmetry, and there is no dilemma.

It might be noted that the bootstrap current, according to our explanation of the pinch effect, will also drive a radial flow. This is, in fact, the mechanism of the implicit diffusion flux T^i_{11}. Another relevant point is illustrated by the figure. Pitch angle scattering of trapped particles moves the bounce points around on the ψ surface, but the average ψ ,to first order in ρ_p/a, does not change. This is why the trapped particles do not contribute to the implicit fluxes.

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FIGURE CAPTIONS

Fig. 1. Schematic of the lines β = const and the lines s = const in a surface ϕ = const. The lines s = const are generated by choosing an arbitrary surface $s = 0$ nowhere tangent to a line of force and on each line of force ϕ = const, β = const measuring off arc length. It is evident from the figure that if the lines shear, then JS is not parallel to $\boldsymbol{b}$.

Fig. 2. Representative orbit projections for trapped and circulating particles. The dashed circle, ψ surface, is the average surface for trapped particles and well circulating particles, $\lambda' \ll 1$. In fact, the actual orbits for well circulating particles differ negligibly from ψ. Marginally circulating particle orbits are very distorted, resembling the inner half of a trapped particle banana orbit for the $\sigma = -1$ direction of parallel velocity, and the outer half of the banana for $\sigma = +1$. Average surfaces, $< \psi >$, for marginally circulating particles are displaced inward from ψ, for $\sigma = -1$, and outward for $\sigma + 1$.

APPENDIX A

Identity of Transport Equations

We wish to relate the more conventional transport equations derived by Eulerian theories to the expressions obtained via the Lagrangian formulation. To this end note that on employing the Eulerian theory the flux surface average of the continuity equation can be expressed as

$$\frac{\partial}{\partial t}\int d^3r d^3v \delta(\psi-\psi')f+\frac{\partial}{\partial\psi^1}\int d^3r d^3v\delta(\psi-\psi')\mathbf{v}_d\cdot(\nabla\psi)f=0 \qquad (A-1)$$

where $\mathbf{v}_d$ is given by the quasi-steady state limit of (2.26). It is readily shown that in this case

$$\mathbf{v}_d\cdot\nabla\psi=-u\frac{\partial}{\partial s}\left(\frac{I}{\Omega}\right) \qquad (A-2)$$

where $I(\psi,t)=RB_\varsigma$. Clearly in the first term in (A-1) it is adequate to use f_0 and in the second term f_1. Let V by the volume enclosed by a toroidal magnetic surface $\psi=\psi'$, and let $V'=dV/d\psi$ and $n(\psi,t)=\int d^3vf_0$. Then on introducing E and μ Eq. (A-1) can be written

$$\frac{\partial}{\partial t}(V'h)+\frac{\partial}{\partial\psi}\left(\frac{2\pi}{m}\right)^2\int dEd\mu ds\frac{uI}{\Omega}\frac{\partial f_1}{\partial s}=0 \qquad (A-3)$$

Let an angular bracket be such that

$$<\frac{uI}{\Omega}>=\frac{\oint ds\frac{I}{\Omega}}{\oint\frac{ds}{u}}=\omega_2\oint ds\frac{I}{\Omega}=\oint d\theta_2\frac{uI}{\Omega} \qquad (A-4)$$

Then since f_1 is periodic in the basic periodicity in s one can write (A-3) as

$$\frac{\partial}{\partial t}(V'n)=\frac{\partial}{\partial\psi}(\frac{2\pi}{m})^2\int dEd\mu ds\left[\frac{uI}{\Omega}-\langle\frac{uI}{\Omega}\rangle\right]\frac{\partial f_1}{\partial s} \qquad (A-5)$$

But in the Eulerian theory f_1 obeys the kinetic equation

$$0=\frac{\partial}{\partial s}(f_1-\frac{uI}{\Omega}\frac{\partial f_o}{\partial\psi})+\frac{1}{u}\nabla_v\cdot\Gamma_1+\frac{\partial}{\partial E}(Zeb\cdot Ef_o) \qquad (A-6)$$

where Γ_1 is the current density in velocity space correct to first order in the small parameter when (A-6) is inserted in (A-5) and use made of the relations $\partial f_o/\partial s = 0$ and

$$\oint ds \frac{uI}{\Omega}\left[\frac{uI}{\Omega}\frac{\partial f_o}{\partial \psi}\right] = \oint ds\left[\frac{1}{2}\left(\frac{uI}{\Omega}\right)^2\frac{\partial f_o}{\partial \psi}\right] = 0 \qquad (A-7)$$

on integrating out the perfect derivative with respect to E there results

$$\frac{\partial}{\partial t}(V'n) = -\frac{\partial}{\partial \psi}\left(\frac{2\pi}{m}\right)^2 \int \frac{dEd\mu ds}{u}\left[\frac{uI}{\Omega} - \left\langle\frac{uI}{\Omega}\right\rangle\right]\nabla_v \cdot \Gamma_1 \qquad (A-8)$$

which can be written as

$$\frac{\partial}{\partial t}\int d^3r d^3v \delta(\psi - \psi')f_o = -\frac{\partial}{\partial \psi'}\int d^3r d^3v \delta(\psi - \psi')\left[\frac{uI}{\Omega} - \left\langle\frac{uI}{\Omega}\right\rangle\right]\nabla_v \cdot \Gamma_1 \qquad (A-9)$$

Consider now the result of integrating (3.1) with respect to J_1 and J_2 at fixed J_3. Since $\oint d^3\theta = 1$ the result can be written

$$\frac{\partial}{\partial t}\int d^3\theta d^3J \delta(J_3 - J_3')f_o = -\frac{\partial}{\partial J_3'}\int d^3\theta d^3J \delta(J_3 - J_3')\Gamma \cdot \nabla_v J_3 \qquad (A-10)$$

But $d^3rd^3v = d^3\theta d^3J$ and one can write (2.68) as

$$J_3 = 2\pi\frac{q}{c}\left[\psi + \frac{uI}{\Omega} - \left\langle\frac{uI}{\Omega}\right\rangle\right] \qquad (A-11)$$

Thus if we pick $J_3' = 2\pi q/c\psi'$, Eq. (A-10) can be written on expanding the δ function in a power series in the "small" quantity $uI/\Omega - \langle uI/\Omega\rangle$, effectively in powers of the poloidal gyration radius,

$$\frac{\partial}{\partial t}\int d^3rd^3v f_o\left\{\delta\left[2\pi\frac{q}{c}(\psi - \psi')\right] + 2\pi\frac{q}{c}\left[\frac{uI}{\Omega} - \left\langle\frac{uI}{\Omega}\right\rangle\right]\delta'\left[2\pi\frac{q}{c}(\psi - \psi')\right] + \cdots\right\}$$

$$= \int d^3rd^3v(\nabla_v \cdot \Gamma_1)\left\{\delta\left[2\pi\frac{q}{c}(\psi - \psi')\right] + 2\pi\frac{q}{c}\left[\frac{uI}{\Omega} - \left\langle\frac{uI}{\Omega}\right\rangle\right]\delta'\left[2\pi\frac{q}{c}(\psi - \psi') + \cdots\right]\right\} \qquad (A-12)$$

On using the properties $\delta(ax) = \frac{1}{a}\delta(x)$, $\delta'(ax) = \frac{1}{a^2}\delta'(x)$ and $\int d^3v\nabla_v \cdot \Gamma_1 = 0$, and keeping only leading order non-vanishing terms, Eq. (A-12) yields (A-9), demonstrating the identity of the expressions derived by the two formulations. Parallel derivations prevail for the other transport equations. The average terms involving $\langle uI/\Omega \rangle$, although important to the Lagrangian formulation and the dcomposition into explicit and implicit fluxes, nonetheless cancel out and do not appear in the <u>net</u> transport coefficients.

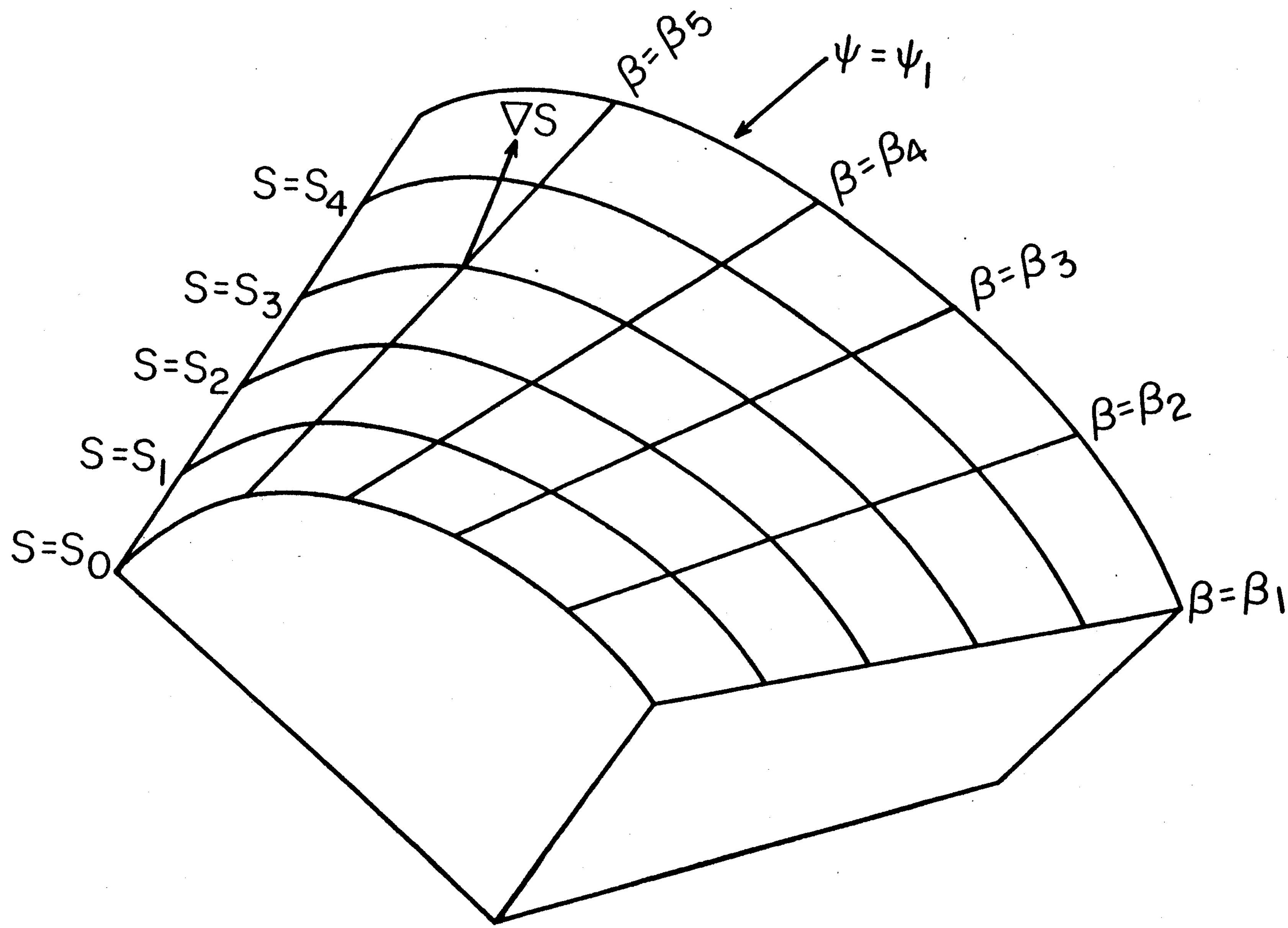

Figure 1

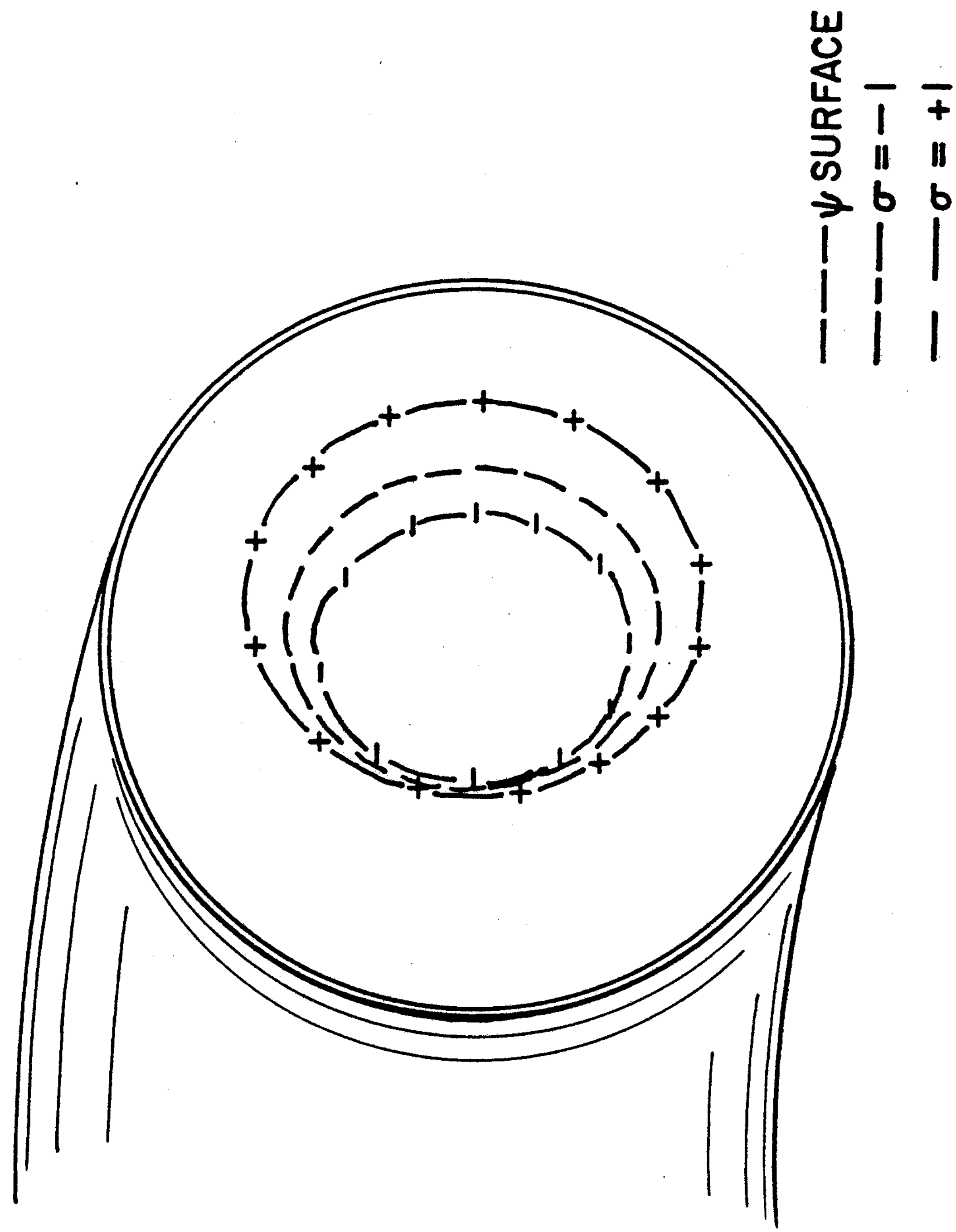

Figure 2