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TRAPPED ELECTRON STOCHASTICITY BY FREQUENCY-MODULATED WAVES*

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Abstract

It is shown that for electrons trapped in a static potential a finite amplitude frequency-modulated wave induces stochastic motions (and diffusion) in electron orbits over a significantly larger area of phase-space than a single frequency wave of the same amplitude. The wavelength of the FM fields is unimportant if it is greater than twice the width of the static potential. It is proposed that this be used for pumping of mirror trapped electrons to enhance the confining potential in a tandem mirror cell.

The purpose of this letter is to show that the distribution function of electrons trapped in a sinusoidal potential well can be significantly modified by using an electrostatic wave (ESW) that is frequency-modulated (FM). Previous studies with a single frequency ESW as a perturbation have shown that the only electrons that exhibit stochastic motion are those that are near the separatrix of the confining potential or those that have velocities which are in a narrow range of the phase-velocity of the ESW.[1,2] In particular, to induce stochasticity in trapped electrons would require a very large amplitude ESW. In contrast, we will show that an FM wave with appropriately chosen parameters can lead to stochastic motion over an appreciable portion of phase space of the trapped electrons for a relatively small, finite amplitude wave. The appropriately normalized one-dimensional equation of motion for such an electron is:

$$\frac{d^2z}{dt^2} = -\sin(z) - \epsilon \sin\left[kz - \omega t - \frac{\Delta\omega}{\omega_m}\sin(\omega_m t)\right] \quad (1)$$

where ϵ is the strength of the ESW with wavenumber, k, frequency, ω, and modulational frequency, ω_m. The distances have been normalized to the wavelength of the static potential and the time to the bounce period of an electron at the bottom of the static potential well; ϵ measures the strength of the ESW relative to the maximum of the static potential; $\Delta\omega$ is the frequency bandwidth of the FM wave about the carrier frequency, ω. The Hamiltonian corresponding to (1) is:

$$H = H_0 - \frac{\epsilon}{k}\cos\left[kz - \omega t - \frac{\Delta\omega}{\omega_m}\sin(\omega_m t)\right] \quad (2)$$

where $H_0 = (\dot{z}^2/2) - \cos z$ is the unperturbed Hamiltonian corresponding to the motion ($\dot{z} = dz/dt$) of an electron in the static field. Here we will be concerned only with the trapped electrons so that $-1 \leq H_0 \leq 1$. The lower limit corresponds to an electron at the bottom and the upper limit to an electron at the separatrix of the potential well. We transform H into the action-angle $(I - \phi)$ representation of H_0 yielding:

$$\overline{H}(I,\phi) = H_0(I) - \frac{\epsilon}{k}\sum_{\ell=-\infty}^{\infty} J_\ell\left(\frac{\Delta\omega}{\omega_m}\right)\sum_{n=-\infty}^{\infty} V_n(k,I)\cos(n\phi - \Omega_\ell t) \quad (3)$$

where J_ℓ is a Bessel function of order ℓ, $I = (8/\pi)[E(\kappa) - (1-\kappa^2)K(\kappa)]$, K and E are the complete elliptic integrals of the first and second kind[3], respectively, $\kappa^2 = \frac{1}{2}(1+H_0)$ and:

$$V_n(k,I) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \cos\left[2k\sin^{-1}\left\{\kappa\, sn\left(\frac{\phi}{\Omega},\kappa\right)\right\} - n\phi\right] \tag{4}$$

with $\Omega = \pi/2K(\kappa)$ being the unperturbed non-linear frequency of the trapped electrons $(0 \leq \Omega < 1)$ and sn is a Jacobian elliptic function[3]. The condition for the onset of stochastic instability is given by the Chirikov resonance overlap criterion[4]:

$$s = \left|\frac{(\Delta I^{TR})_{\ell' n'} + (\Delta I^{TR})_{\ell n}}{I_{\ell' n'} - I_{\ell n}}\right| \gtrsim 1 \tag{5}$$

where $I_{\ell n}$ satisfies the resonance condition:

$$n\Omega(I_{\ell n}) = \Omega_\ell \equiv \omega - \ell\omega_m \tag{6}$$

(ℓ', n') is the nearest resonance to (ℓ, n) and

$$(\Delta I^{TR})_{\ell n} = 2\left[\frac{\epsilon}{k}\; J_\ell\left(\frac{\Delta\omega}{\omega_m}\right)\; \frac{V_n(k,I)}{(d\Omega/dI)}\right]^{1/2}_{I=I_{\ell n}} \tag{7}$$

is the resonance width. We want to find the conditions on ω, k, ω_m and $\Delta\omega$ such that stochasticity can occur for the smallest possible ϵ. In an experiment this would imply that, for a given (practical) bandwidth, we want to keep the power requirements for the FM-ESW to a minimum. An analysis of (4) shows that V_1 is the largest coefficient and $|V_n| \lesssim 0.6$ for integer $|n| \geq 1$ $(V_{-n} = (-1)^n V_n)$ and for any value of k. Then, from (7) we see that k should be less than 1. Assuming that only the $n = 1$ term is important and that $k \lesssim 0.5$, we find

$$V_1(k,I) \approx 2k \;/\; cosh\left\{\frac{\pi}{2}\frac{K(\sqrt{1-\kappa^2})}{K(\kappa)}\right\} \tag{8}$$

and (7) is independent of k. From (6) the separation between resonances is proportional to ω_m. Since $0 \leq \Omega < 1$ for the trapped electrons, we require that $\omega_m \ll 1$ so that several resonances can occur in the region of interest. This is supported by the exact numerical analysis of (1) as shown in figures 1a and 1b where the only parameter that is changed between the two plots is ω_m.[5] For $\omega_m = 0.2$ islands are clearly visible ($n = 1$ islands at $I \approx 1.35, 2.18$ and $n = 2$ islands at $I \approx 0.74$) and there is no stochasticity except near the separatrix ($I \approx 2.55$) where it is expected. However, when $\omega_m = 0.05$ there appears a wide region of stochasticity in the trapped-electron region for $0.1 \lesssim I \lesssim 1.2$. The remanants of $n = 1$ islands at $I \approx 0.38, 0.74$ and 1.06 are still discernable even though their width in ϕ-space is considerably diminished. For $\omega_m = 0.01$ there is no clear island structure left and the whole region for $I \lesssim 1.2$ is stochastic.

These conditions on k and ω_m help determine the range in which ω should lie. Since $\Delta\omega < 1$ (a usual experimental constraint) and $\omega_m \ll 1$, and, since the onset of stochasticity will occur near the maximum value of $J_\ell(\Delta\omega/\omega_m)$ in (7), we conclude from the properties of Bessel functions and (6) that $0 < \omega < 1 + \Delta\omega$. With these restrictions on ω, k and ω_m, the threshold for onset of stochasticity is found to be $\epsilon_{TH} \propto \omega_m^{5/3}$ for $\omega_m < \Delta\omega$. The constant of proportionality basically depends on ω and $\Delta\omega$. For instance, when $\omega = 0.9$, $\Delta\omega = 0.1$, $\epsilon_{TH} \approx 0.27\omega_m^{5/3}$. This dependence of ϵ_{TH} on ω_m is supported by the numerical results. As the amplitude is increased above threshold the region of stochasticity widens in I-space. If we choose $\omega = 0.9$, $\omega_m = 0.01$, $\Delta\omega = 0.1$, $k = 0.2$, $\epsilon = 4 \times 10^{-3}$ connected stochasticity appears over the region $0 \leq I \lesssim 1.65$. This is in excellent agreement with the results obtained from the exact numerical integration of (1). For $\epsilon > 0.03$, the approximate analysis with just $n = 1$ breaks down. The exact analysis indicates that the region of stochasticity now encompasses all trapped electrons while the $n = 1$ analysis shows that there still exists a region of coherent motion for the trapped electrons. This implies that the higher order ($n > 1$) contributions have become important. The bandwidth, $\Delta\omega$, plays a role in determining the width of the region in I-space that goes stochastic. For a fixed ϵ, increasing $\Delta\omega$ increases the width of the stochastic regime. However, we fix $\Delta\omega = 0.1$ as that is about the maximum bandwidth that would be experimentally reasonable.

Once there is a region of connected stochasticity, we can determine the quasilinear diffusion coefficient for electrons in that region either from the Vlasov theory for the electron distribution function or from the two-point correlation function of dI/dt. The two approaches give the same diffusion coefficient:

$$D^{QL} = \frac{1}{\Delta I}\ \frac{\pi\epsilon^2}{2k^2} \sum_{\ell,n} |n|\ J_\ell^2\left(\frac{\Delta\omega}{\omega_m}\right)\ \frac{V_n^2(k, I_{\ell n})}{|d\Omega/dI|_{\ell n}} \tag{9}$$

where ΔI is the width in I-space of the stochastic region and the sum is over those ℓ and n such that $I_{\ell n}$ is in the stochastic region. This is compared with the numerically evaluated diffusion coefficient, $D^{NUM} = <(\delta I)^2> /(2\delta t)$ where $<>$ indicates an average over the number of initial conditions (usually taken to be 1000) which lie within the stochastic region, and δI is the change in I in a time $\delta t \equiv \tau_m = 2\pi/\omega_m$; τ_m is the time scale over which the electrons have sampled the entire bandwidth of the FM wave. In figure 2 we have plotted D^{NUM} and D^{QL} versus ϵ for two different values of ω_m. For values of ϵ beginning near the threshold for onset of stochasticity and going up to about two orders of magnitude above threshold, the diffusion is almost quasilinear. However, for $\epsilon \gtrsim 4 \times 10^{-3}$ significant deviations of D^{NUM} from D^{QL} take place with $D^{NUM} \ll D^{QL}$. The change in the slope of D^{QL} for $\omega_m = 10^{-3}$ around $\epsilon \approx 10^{-4}$ is due to the increase in ΔI as ϵ is increased. For $4 \times 10^{-3} \lesssim \epsilon \lesssim 2.4 \times 10^{-2}$ the width ΔI remains almost constant. (So as to avoid numerical difficulties encountered for electrons very close to the separatrix of the static well, the highest value of ϵ we look at is just below the value for which the whole trapped region becomes stochastic). Hence, over a large range of ϵ, D^{NUM} does not behave like D^{QL}. Further, as ω_m is decreased D^{NUM} is found to decrease;[6] e.g. for $\epsilon \geq 5 \times 10^{-4}$, D^{NUM} for $\omega_m = 10^{-2}$ is greater than D^{NUM} for $\omega_m = 10^{-3}$ as shown in figure 2. As $\omega_m \to 0$ we approach the single frequency ESW limit when there is no diffusion in the region of interest for the range of ϵ considered here.

Apart from the change in diffusion as ω_m is decreased, there is a marked difference in $< I^2 >$ as a function of time for the two values of ω_m. In figure 3 we plot $< I^2 >$ versus time for $\omega_m = 10^{-2}$ and $\omega_m = 10^{-3}$ while keeping $\epsilon = 8 \times 10^{-3}$, $k = 0.2$, $\omega = 0.9$, and $\Delta\omega = 0.1$. For $\omega_m = 0.01$, $< I^2 >$ increases until it saturates. The increase implies diffusion while the saturation implies that the distribution function has been flattened in the stochastic region. The time required to flatten the distribution function is between $2\tau_m$ and $4\tau_m$. But, for $\omega_m = 10^{-3}$, $< I^2 >$ oscillates with time (with a period τ_m) over the entire I-space that satisfies the Chirikov resonance overlap. The small changes in $< I^2 >$ for every τ_m time step are reflected by the significantly lower diffusion coefficient in figure 2 for this case relative to the case of $\omega_m = 10^{-2}$. For the lower value of ω_m the distribution function takes a longer time to flatten out. It is intersting to note that for $\omega_m = 10^{-2}$, $< I^2 >$ never has the large oscillatory behavior as in the case of $\omega_m = 10^{-3}$ even though the width in I-space over which resonance overlap occurs is approximately the same in the two cases. We have observed the same trend in $< I^2 >$ for all $\epsilon \geq 4 \times 10^{-3}$ for the two values of ω_m.

In conclusion, we find that an FM-ESW can induce stochastic motion in electrons trapped in a static well for very small amplitudes by an appropriate choice of wave frequency and modulational frequency. The stochastic motion occurs in phase-space far away from the phase velocity of the FM wave and becomes independent of the wavelength of the ESW if it is longer than twice the dimensions of the static well. The frequency of the ESW has to be close (approximately within the bandwidth of the FM wave) to the bounce frequency of the electrons whose motion is to be perturbed. The modulational frequency has to be about one percent of the frequency. For larger modulational frequencies it is difficult to produce stochastic motion for reasonable amplitudes of the FM-ESW while for smaller modulational frequencies there is very little diffusion and it takes a very long time to flatten the distribution function. The diffusion is most effective (essentially quasilinear) for amplitudes about one to two orders of magnitude above threshold. Larger amplitudes lead to a decrease in the effective diffusion constant, as compared to the quasilinar diffusion.

Finally, we suggest that the FM-ESW can be used for enhancing the confining electrostatic potential in the plug region of a tandem mirror plasma by stochastically modifying the electron distribution function in that region[7]. The time required to flatten the electron distribution function is smaller than the electron-electron collisional time. Since the bounce frequency of the electrons in the plug region is in the range of ion-cyclotron frequencies, a regular antenna used for exciting waves in this range of frequencies can also excite an electric field along the d.c. magnetic field which can penetrate into the plasma[8]. A single antenna usually launches a large portion of the input power near small wavenumbers which is ideal for affecting the trapped electrons as our calculations indicate. The passing electrons from the central cell would be hardly affected as far as their motion along the d.c. magnetic field is concerned.

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Figure Captions

Figure 1a Surface of section plot in $I-\phi$ space for $\epsilon = 8 \times 10^{-3}$, $\omega = 1$, $k = 0.2$, $\Delta\omega = 0.1$ and $\omega_m = 0.2$.

Figure 1b Same as figure 1a except that $\omega_m = 0.05$.

Figure 2 Diffusion coefficients, D^{QL} (Eq. 9) and D^{NUM}, versus ϵ for $\omega = 0.9$, $k = 0.2$, $\Delta\omega = 0.1$, and for (a) $\omega_m = 10^{-2}$. (the solid line being D^{QL} and the points denoted by + being D^{NUM}); (b) $\omega_m = 10^{-3}$ (the dashed line being D^{QL} and the points denoted by $\otimes$ being D^{NUM}).

Figure 3 $< I^2 >$ versus time for $\epsilon = 8 \times 10^{-3}$, $\omega = 0.9$, $k = 0.2$, $\Delta\omega = 0.1$ and (a) $\omega_m = 0.01$; (b) $\omega_m = 0.001$.

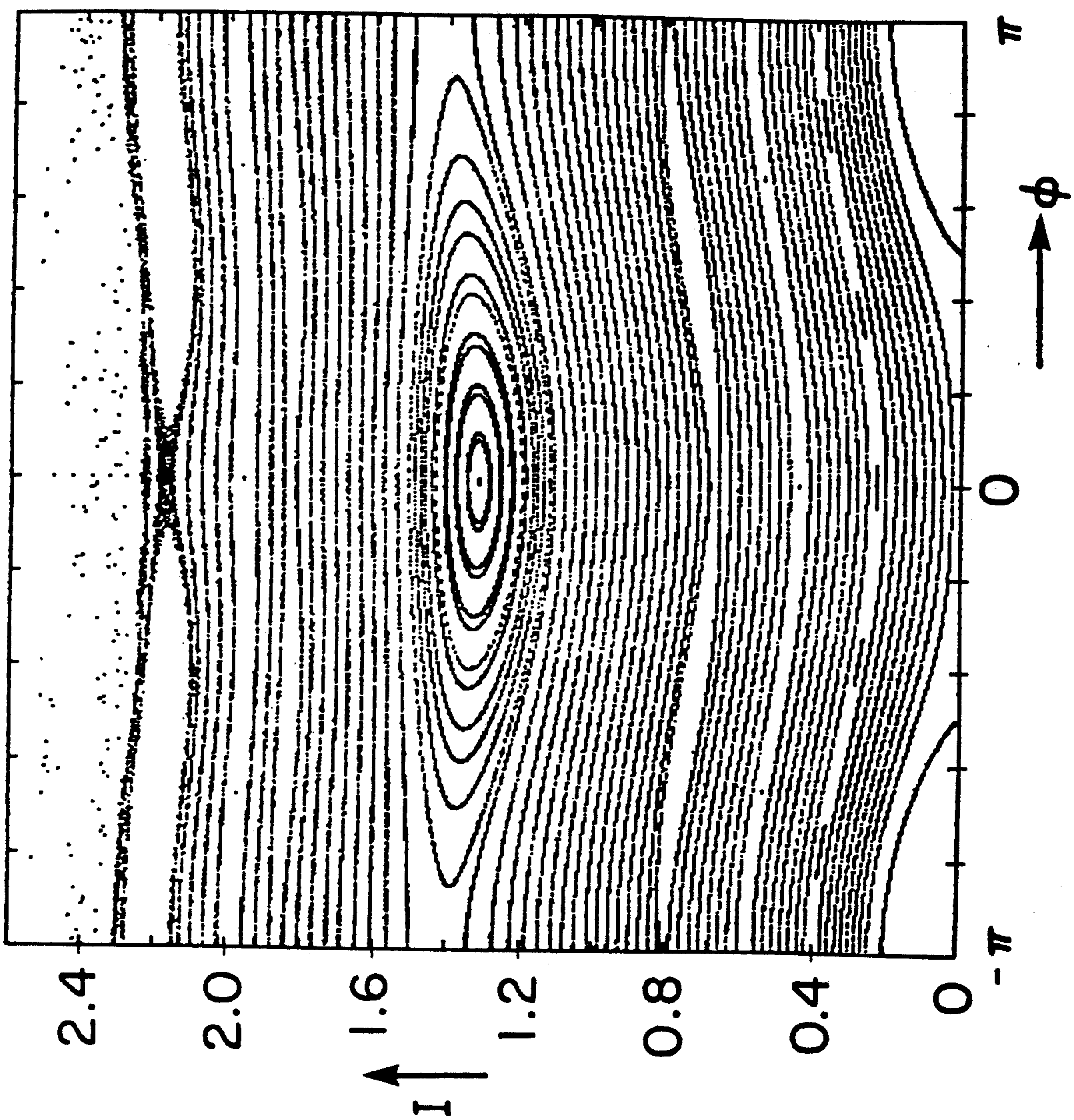

FIGURE 1a

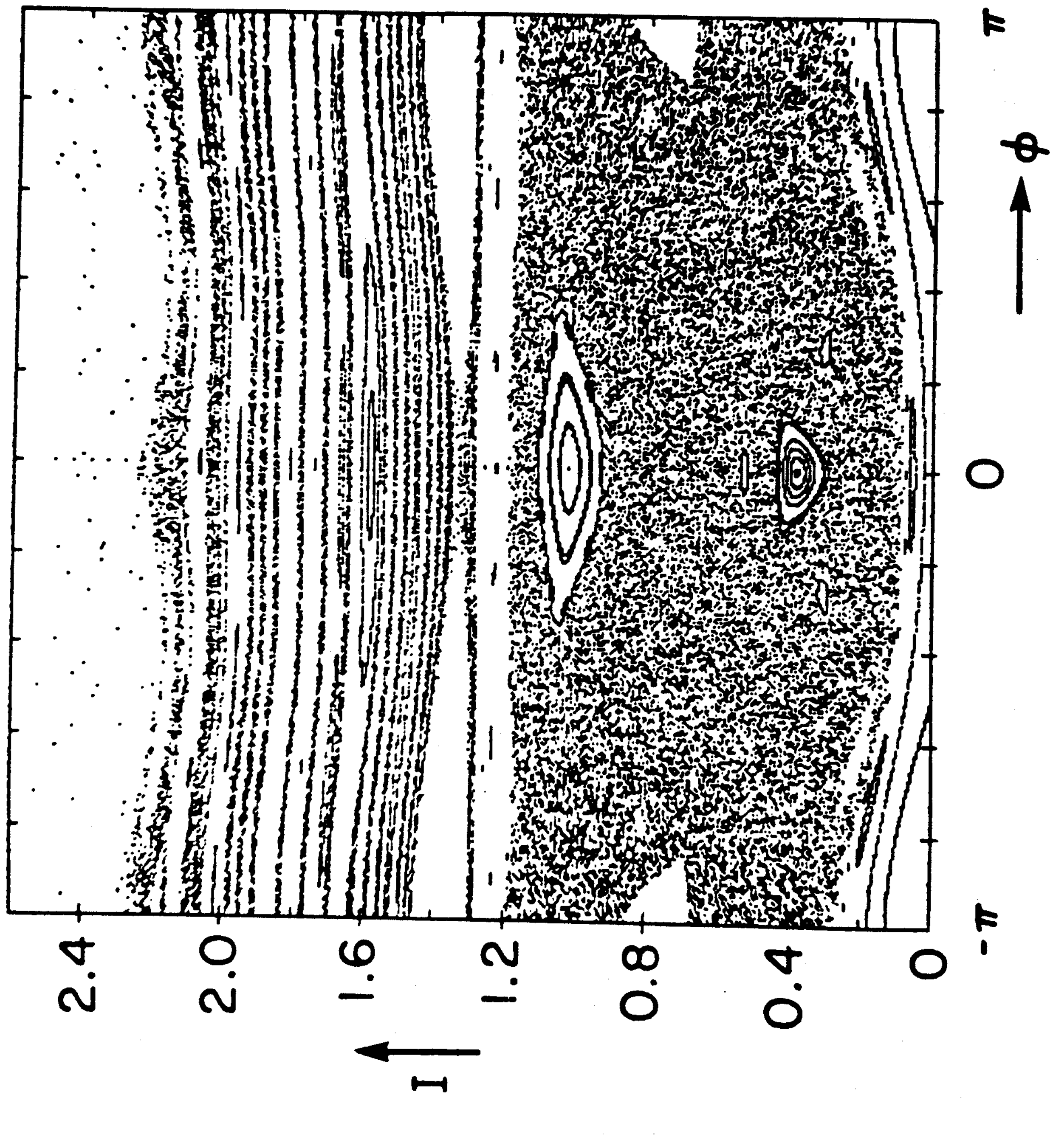

FIGURE 1b

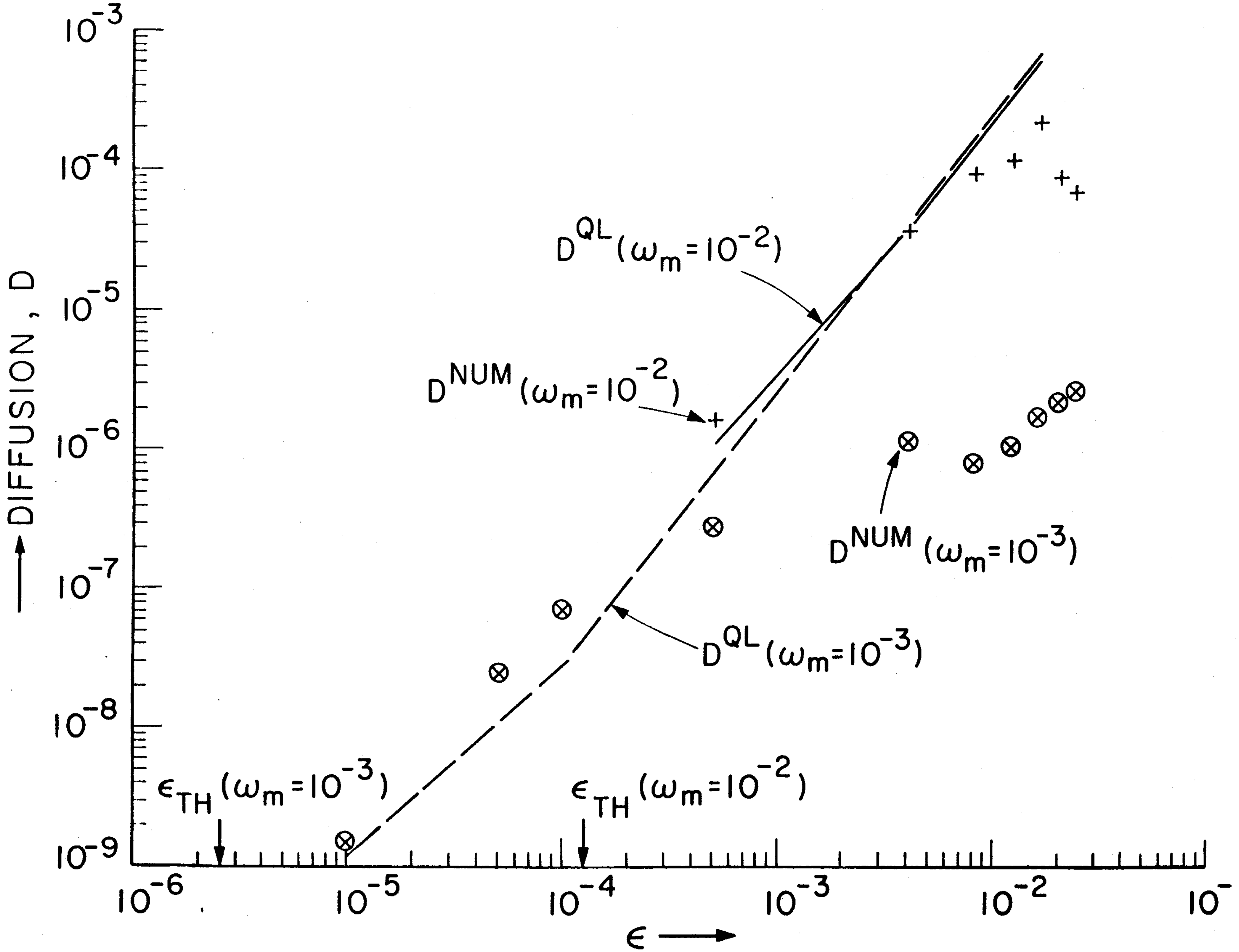
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
DIFFUSION , D
$D^{QL}(\omega_m = 10^{-2})$
$D^{NUM}(\omega_m = 10^{-2})$
$D^{NUM}(\omega_m = 10^{-3})$
$D^{QL}(\omega_m = 10^{-3})$
$\epsilon_{TH}(\omega_m = 10^{-3})$
$\epsilon_{TH}(\omega_m = 10^{-2})$
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
ϵ

FIGURE 2

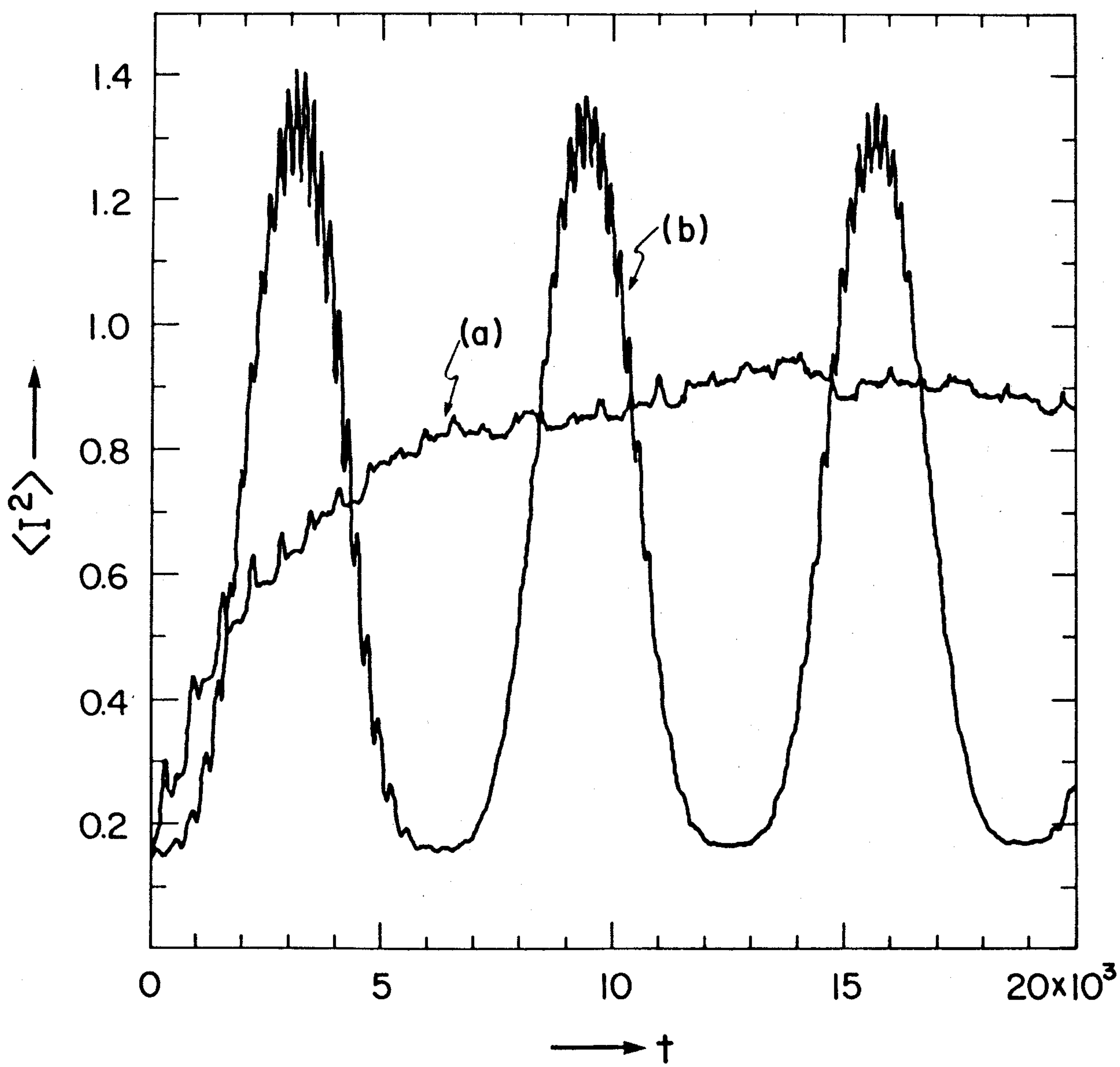

FIGURE 3