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Abstract

A detailed account of a formal mathematical description of the interaction of relativistic charged particle beams with electromagnetic waves, within the frame of classical electrodynamics, is presented. The standard system of 8 equations (Maxwell, Lorentz gauge condition and fluid dynamics) in the 4-vector potential A_μ and the 4-vector current density j_μ is reduced, after linearization, to a canonical system of 4 coupled partial differential equations in the electromagnetic field perturbation δA_μ. Both electromagnetic and dynamical quantities are treated as fields, according to the Eulerian formalism. This new system is very general, and different beam-wave interactions are characterized by different fluid equilibria and boundary conditions for δA_μ and its derivatives.

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One of the central problems of relativistic electrodynamics is the interaction of charged particles beams with electromagnetic waves[1−5]. The physics of such interactions is very rich and a wide variety of complex phenomena arise, ranging from synchrotron and Čerenkov[6] radiation to free-electron laser, cyclotron maser and other instabilities involving non-neutral plasmas, as extensively discussed by Davidson[7]. A large class of beam-wave interaction problems involve electromagnetic energies that are small compared to the particles kinetic energy, and a perturbation theory is appropriate to describe such linear beam-wave interactions. This category of problem will be the focus of our attention in this paper. Different formal mathematical descriptions of this type of interaction are possible, such as the Maxwell-Vlasov kinetic theory or the Maxwell-Euler fluid model. In this work, we consider the latter theory which involves the manipulation of fields for both electromagnetic and dynamical quantities, and of operators such as the electromagnetic wave propagator (d'Alembertian operator) or the fluid convective derivation, providing a compact and elegant mathematical framework to study these interactions.

The main object of this work is to show that starting from the standard set of 8 equations in the 4-vector potential A_μ and the 4-vector current density j_μ, we can obtain a canonical system of 4 coupled partial differential equations describing the evolution of the electromagnetic field perturbation δA_μ, by linearizing the interaction equations. The compact set of partial differential equations (PDEs) derived in this manner involves the perturbed electromagnetic 4-vector potential and the equilibrium fluid field components. Different specific problems are characterized by different fluid equilibria and boundary conditions for δA_μ and its derivatives. The initial set of 8 equations consists of the 4 Maxwell's equations with sources describing the evolution of the 4-vector potential, the Lorentz gauge condition, which is equivalent to the conservation of charge or to the continuity equation, and 3 fluid equations of motion.

At this level, two main formal approaches can be used to solve this linear system of PDEs[8]. On the one hand, one can expand δA_μ into known eigenmodes satifying the appropriate boundary conditions, and study the coupling of these modes through the coupled PDEs. The other approach consists in solving directly these equations, then using the boundary conditions to determine the actual eigenvalues and eigenfunctions of the problem.

We now review the general formalism describing the interaction of a relativistic electron (or other charged particles) beam with electromagnetic fields, within the frame of classical electrodynamics. A very large number of methods have been described in the litterature, and there is, sometimes, some confusion about which equations and which variables are most appropriate to formally describe a specific interaction problem. For example, it is well-known that the gauge condition, the conservation of charge and the continuity equation are equivalent. Here, our objective is to reduce the linearized equations of interaction to a canonical system of 4 equations in the 4-vector potential perturbation $\delta A_\mu(x_\nu)$.

We first briefly review the equations relevant to the problem. The interaction of charged particles with electromagnetic fields can be described, in the classical limit, by two sets of equations. On the one hand, Maxwell's two groups of equations, governing the fields,

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0}, \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{2}$$

and the group with sources

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_o} \rho, \tag{3}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E} = \mu_o \vec{j}. \tag{4}$$

On the other hand, the equations governing the particles dynamics, which are given by

$$d_t \vec{p} = -e(\vec{E} + \vec{v} \times \vec{B}), \tag{5}$$

and the continuity equation (charge or particles conservation)

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0. \tag{6}$$

Here, $j_\mu \equiv [c\rho, \vec{j}] = -en[c, \vec{v}]$ is the 4-vector current density, with n the particle density and $\vec{v} = c\vec{\beta}$ their velocities. The particles' momentum is given by $\vec{p} = \gamma m_o \vec{v}$, and their energy by $\gamma^{-2} = 1 - \beta^2$.

At this point, it is important to note that Maxwell's first group of equations [(1) and (2)] suggests the introduction of the 4-vector potential $A_\mu \equiv [\frac{1}{c}\phi, \vec{A}]$, defined such that

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}, \tag{7}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{8}$$

As a result, equations (1) and (2) are automatically satisfied. If, in addition, we impose that the 4-vector potential satisfies the Lorentz gauge condition

$$\frac{1}{c^2} \partial_t \phi + \vec{\nabla} \cdot \vec{A} = 0, \tag{9}$$

we see that the second group of equations is equivalent to

$$\left[\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right] \phi - \frac{1}{\varepsilon_o} \rho = 0, \tag{10}$$

$$\left[\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right] \vec{A} - \mu_o \vec{j} = \vec{0}. \tag{11}$$

It should also be noted that the gauge condition (9) is equivalent to the continuity equation (6).

The equation of momentum transfer (5) implicitly satisfies energy conservation as can be seen by taking the dot product of equation (5) by $\vec{p}$, to obtain

$$d_t\gamma = -\frac{e\vec{E}\cdot\vec{v}}{m_o c^2}. \tag{12}$$

Finally, using the definitions, equation (5) can be transformed to read, within the framework of a relativistic fluid model,

$$[\partial_t + \vec{v}\cdot\vec{\nabla}]\vec{v} = -\frac{e}{m_o}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left\{-\vec{\nabla}\phi - \partial_t\vec{A} + \vec{v}\times\vec{\nabla}\times\vec{A} + \frac{\vec{v}}{c^2}(\vec{\nabla}\phi + \partial_t\vec{A})\cdot\vec{v}\right\}, \tag{13}$$

where we have used the explicit expression for convective derivation. We thus obtain a closed system of 8 equations with 8 unknowns A_μ, n and $\vec{v}$

$$\left[\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right]\vec{A} = \mu_o e n\vec{v}, \tag{14}$$

$$\left[\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right]\phi = \frac{1}{\varepsilon_o}en, \tag{15}$$

$$\frac{1}{c^2}\partial_t\varphi + \vec{\nabla}\cdot\vec{A} = 0. \tag{16}$$

together with equation (13).

We now focus on the linear analysis of the beam-field interaction. Any fluid field component $f(x_\nu)$ is written $f = f_0 + \delta f$. The quantity f_0 refers to the beam self-consistent equilibrium in the external fields, while δf corresponds to the electromagnetic perturbation. Note that here, no assumption is made about the nature of the fluid equilibrium considered; in particular, this equilibrium can be space-time dependent. We assume that for all fluid field components, we have $|\delta f| \ll |f_0|$. We can then linearize the equations presented above, with the result that

$$\left[\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right]\delta\vec{A} = \mu_o e[n_0\delta\vec{v} + \vec{v}_0\delta n], \tag{17}$$

$$\left[\vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2\right]\delta\phi = \frac{1}{\varepsilon_o}e\delta n, \tag{18}$$

$$\frac{1}{c^2}\partial_t\delta\phi + \vec{\nabla}\cdot\delta\vec{A} = 0, \tag{19}$$

$$\begin{aligned}[\partial_t + \vec{v}_0\cdot\vec{\nabla}]\delta\vec{v} + [\delta\vec{v}\cdot\vec{\nabla}]\vec{v}_0 = -\frac{e}{\gamma_0 m_o}\Big\{&-\vec{\nabla}\delta\phi - \partial_t\delta\vec{A} + \delta\vec{v}\times\vec{B}_0 + \vec{v}_0\times\vec{\nabla}\times\delta\vec{A}\\ &-\frac{\vec{v}_0}{c^2}[\vec{E}_0\cdot\delta\vec{v} - (\vec{\nabla}\delta\phi + \partial_t\delta\vec{A})\cdot\vec{v}_0] - \frac{\delta\vec{v}}{c^2}(\vec{E}_0\cdot\vec{v}_0)\\ &-\frac{\gamma_0^2}{c^2}(\vec{v}_0\cdot\delta\vec{v})\left[\vec{E}_0 + \vec{v}_0\times\vec{B}_0 - \frac{\vec{v}_0}{c^2}(\vec{E}_0\cdot\vec{v}_0)\right]\Big\}.\end{aligned} \tag{20}$$

Here the equilibrium electric and magnetic fields are $\vec{E}_0(x_\nu)$ and $\vec{B}_0(x_\nu)$, respectively. We shall now reduce this system by considering

$$n_0\delta\vec{v} = \frac{1}{\mu_o e}\left(\Box\delta\vec{A} - \vec{\beta}_0\Box\frac{\delta\phi}{c}\right), \tag{21}$$

where

$$\Box \equiv \partial_\mu\partial^\mu \equiv \vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2, \tag{22}$$

is the d'Alembertian operator (electromagnetic wave propagator). We have, on the other hand,

$$[\partial_t + \vec{v}_0\cdot\vec{\nabla}](n_0\delta\vec{v}) = n_0[\partial_t - \vec{v}_0\cdot\vec{\nabla}]\delta\vec{v} - \delta\vec{v}[\partial_t - \vec{v}_0\cdot\vec{\nabla}]n_0, \tag{23}$$

and, after (21),

$$[\partial_t + \vec{v}_0\cdot\vec{\nabla}](n_0\delta\vec{v}) = \frac{1}{\mu_o e}[\partial_t + \vec{v}_0\cdot\vec{\nabla}]\left(\Box\delta\vec{A} - \vec{\beta}_0\Box\frac{\delta\phi}{c}\right). \tag{24}$$

The first term on the right-hand side of equation (23) is given by (20)

$$\begin{aligned}n_0[\partial_t - \vec{v}_0\cdot\vec{\nabla}]\delta\vec{v} - [(n_0\delta\vec{v})\cdot\vec{\nabla}]\vec{v}_0 = -n_0\frac{e}{\gamma_0 m_o}\Big\{&-\vec{\nabla}\delta\phi - \partial_t\delta\vec{A} + \delta\vec{v}\times\vec{B}_0 + \vec{v}_0\times\vec{\nabla}\times\delta\vec{A}\\ &-\frac{\vec{v}_0}{c^2}[\vec{E}_0\cdot\delta\vec{v} - (\vec{\nabla}\delta\phi + \partial_t\delta\vec{A})\cdot\vec{v}_0] - \frac{\delta\vec{v}}{c^2}(\vec{E}_0\cdot\vec{v}_0)\\ &-\frac{\gamma_0^2}{c^2}(\vec{v}_0\cdot\delta\vec{v})\left[\vec{E}_0 + \vec{v}_0\times\vec{B}_0 - \frac{\vec{v}_0}{c^2}(\vec{E}_0\cdot\vec{v}_0)\right]\Big\},\end{aligned} \tag{25}$$

while the second term can be derived from the equilibrium continuity equation

$$\partial_t n_0 + \vec{\nabla} \cdot (n_0 \vec{v}_0) = 0 \qquad \Longleftrightarrow \qquad [\partial_t + \vec{v}_0 \cdot \vec{\nabla}] n_0 = -n_0 (\vec{\nabla} \cdot \vec{v}_0). \tag{26}$$

We thus have

$$[\partial_t + \vec{v}_0 \cdot \vec{\nabla} + \vec{\nabla} \cdot \vec{v}_0] \left(\Box \delta \vec{A} - \vec{\beta}_0 \Box \frac{\delta \phi}{c} \right) = \mu_o e n_0 [\partial_t + \vec{v}_0 \cdot \vec{\nabla}] \delta \vec{v}. \tag{27}$$

We now use equation (25) to obtain

$$\begin{aligned} [\partial_t + \vec{v}_0 \cdot \vec{\nabla} + \vec{\nabla} \cdot \vec{v}_0] \left(\Box \delta \vec{A} - \vec{\beta}_0 \Box \frac{\delta \phi}{c} \right) + \left[\left(\Box \delta \vec{A} - \vec{\beta}_0 \Box \frac{\delta \phi}{c} \right) \cdot \vec{\nabla} \right] \vec{v}_0 = -\mu_o e n_0 \frac{e}{\gamma_0 m_o} \\ \left\{ -\vec{\nabla} \delta \phi - \partial_t \delta \vec{A} + \delta \vec{v} \times \vec{B}_0 + \vec{v}_0 \times \vec{\nabla} \times \delta \vec{A} - \frac{\vec{v}_0}{c^2} [\vec{E}_0 \cdot \delta \vec{v} - (\vec{\nabla} \delta \phi + \partial_t \delta \vec{A}) \cdot \vec{v}_0] \right. \\ \left. - \frac{\delta \vec{v}}{c^2} (\vec{E}_0 \cdot \vec{v}_0) - \frac{\gamma_0^2}{c^2} (\vec{v}_0 \cdot \delta \vec{v}) \left[\vec{E}_0 + \vec{v}_0 \times \vec{B}_0 - \frac{\vec{v}_0}{c^2} (\vec{E}_0 \cdot \vec{v}_0) \right] \right\}. \end{aligned} \tag{28}$$

At this point, we define the following fluid equilibrium fields

$$\vec{\Omega}_0 = \frac{e \vec{B}_0(x_\nu)}{\gamma_0(x_\nu) m_o} \quad , \quad \frac{\omega_p^2}{c^2} = \mu_o \frac{n_0(x_\nu) e^2}{\gamma_0(x_\nu) m_o} \quad , \quad \vec{\beta}_0 = \frac{\vec{v}_0(x_\nu)}{c} \quad , \quad \vec{\Lambda}_0 = \frac{e \vec{E}_0(x_\nu)}{\gamma_0(x_\nu) m_o c},$$

which are, respectively, the relativistic cyclotron frequencies in the equilibrium magnetic field, the relativistic beam plasma frequency, the normalized fluid equilibrium velocity field and the normalized equilibrium electric field, governing the energy time-scale. The formalism described here includes the most general case, where the dynamical quantities describing the fluid equilibrium state are functions of both space and time.

Upon replacement of every quantity $n_0 \delta \vec{v}$ appearing on the right-hand side of equation (28) by the value defined in (21), we end up with the sought-after canonical system of 4 equations in the 4-potential vector perturbation $\delta A_\mu \equiv [\frac{1}{c} \delta \phi, \delta \vec{A}]$

$$
\begin{aligned}
&\left[\partial_t + \vec{v}_0 \cdot \vec{\nabla} + \vec{\nabla} \cdot \vec{v}_0 - \vec{\Lambda}_0 \cdot \vec{\beta}_0 - \vec{\beta}_0 \vec{\Lambda}_0 \cdot - \vec{\Omega}_0 \times - \gamma_0^2 [\vec{\Lambda}_0 - \vec{\beta}_0(\vec{\Lambda}_0 \cdot \vec{\beta}_0) + \vec{\beta}_0 \times \vec{\Omega}_0] \vec{\beta}_0 \cdot \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(\Box \delta \vec{A} - \vec{\beta}_0 \Box \frac{\delta\phi}{c}\right) \\
&+ \left[\left(\Box \delta \vec{A} - \vec{\beta}_0 \Box \frac{\delta\phi}{c}\right) \cdot \vec{\nabla}\right] \vec{v}_0 \\
&+ \frac{\omega_p^2}{c^2} \left\{ -\vec{\nabla}\delta\phi - \partial_t \delta\vec{A} + \vec{v}_0 \times \vec{\nabla} \times \delta\vec{A} + \vec{\beta}_0 (\vec{\nabla}\delta\phi + \partial_t \delta\vec{A}) \cdot \vec{\beta}_0 \right\} = 0, \qquad (29) \\
&\frac{1}{c^2} \partial_t \delta\phi + \vec{\nabla} \cdot \delta\vec{A} = 0. \qquad (30)
\end{aligned}
$$

Note that we can easily identify the different terms in equation (29) as a beam-mode type operator coupled to an electromagnetic wave propagator, and a beam coupling term proportional to the beam density profile $\omega_p^2(x_\nu)$ and containing the ponderomotive force.

At this point, different beam-wave interactions are characterized by different fluid equilibria and different boundary conditions for δA_μ. At this level, two main formal approaches can be used to solve the canonical system derived above. On the one hand, one can expand δA_μ into known eigenmodes satisfying the appropriate boundary conditions. and study the coupling of these modes through the coupled PDEs describing the evolution of the 4-vector potential perturbation. The other approach consists in solving directly these equations, then using the boundary conditions to determine the actual eigenvalues and eigenfunctions of the problem.

This formal description of beam-wave interactions is quite general and can be used as a new canonical system of PDEs describing the self-consistent evolution of the 4-vector potential perturbation in the linear regime. The formalism is Eulerian in the sense that the (now implicit) fluid dynamical quantities are treated as continuous space-time fields, on an equal footing with their electromagnetic counterparts.

Finally, we plan to expand on this theory in an upcoming paper by treating the problem of optical guiding in a FEL[9] within the framework of the formalism exposed in this letter.

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References

[1] W. Pauli, *Pauli Lectures on Physics*, (MIT Press, Cambridge MA. and London England, edited by L.P. Enz), Vol. 1 (1981).

[2] W. Pauli, *Theory of Relativity*, (Dover, New York NY., 1981)

[3] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, (Addison-Wesley, Reading MA.), Vol. 2 (1977).

[4] L. Landau and E. Lifchitz. *Théorie des Champs*. (Editions Mir, Moscow, 1970).

[5] J.D. Jackson, *Classical Electrodynamics*, (J. Wiley & Sons. New York NY., 1975).

[6] V.P. Zrelov, *Cherenkov Radiation in High Energy Physics*, (Israel) Program for Scientific Translation, Jerusalem 1970).

[7] R.C. Davidson. *Theory of Nonneutral Plasmas*, (W.A. Benjamin. Reading MA., 1974).

[8] R. Courant and D. Hilbert, *Methods of Mathematical Physics*. (Wiley, New York 1953).

[9] F. Hartemann, G. Mourier and R.C. Davidson. *Proceedings of the 14th International Conference on Infrared and Millimeter Waves*. 418 (1989).