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COMMENTS ON ABSOLUTE AND CONVECTIVE INSTABILITIES

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ABSTRACT

We point out the misconceptions in the arguments put forth by T. E. Oscarsson and K. G. Rönnmark (*Geophys. Res. Lett. 13*, 1384, 1986) that question the validity and usefulness of the well-known theory of absolute and convective instabilities. The solid basis of the well-known theory is clarified.

INTRODUCTION

Absolute or convective evolutions of instabilities can produce very different signatures of observed radiation from unstable plasmas. This is particularly useful in space plasmas when correlating experimentally observed emissions with theoretical models describing the source regions [Ram 1991]. A letter [Oscarsson 1986] has questioned the basis and usefulness of the theory of absolute and convective instabilities by using some very simple minded examples. However, the arguments put forth by these authors are based upon some singular examples that are misleading. Furthermore, the notion of a time-asymptotic limit is treated in a trivialized manner by these authors.

The theory of absolute and convective instabilities [Briggs 1964, Bers 1983] describes the linear evolution of instabilities from an initially localized source in an infinitely homogeneous medium. In this paper we clarify the conditions for the validity of the theory of absolute and convective instabilities, and, furthermore, show that the examples discussed in the aforementioned letter are not generic for continuous media and do not invalidate the theory of absolute and convective instabilities.

In what follows, we shall consider the case of evolution of instabilities in time and in one spatial dimension. The generalization to higher spatial dimensions is more complicated but can be carried out in a straightforward manner.

SPACE-TIME EVOLUTION OF INSTABILITIES

The theory of absolute and convective instabilities is based on a Green's function analysis of equations describing the space-time dynamics of a small-amplitude perturbation in a spatially homogeneous and a time-invariant medium. The equation for the Green's function, $G(z,t)$, is given by:

$$\mathcal{L}\, G(z,t) \;=\; \delta(z)\, \delta(t) \tag{1}$$

where $\mathcal{L}$ is, in general, a linear integro-partial-differential operator with constant coefficients. Using complex Fourier-Laplace transforms, the solution to Eq. (1) can be written as:

$$G(z,t) \;=\; \int_L \frac{d\omega}{2\pi} \int_F \frac{dk}{2\pi}\; \frac{1}{D(k,\omega)}\; \exp(ikz - i\omega t) \tag{2}$$

where $D(k,\omega)$ is the dispersion function for the system – directly related to the transforms of $\mathcal{L}$, and L and F are the appropriate Laplace and Fourier contours, respectively, chosen to satisfy causality [Bers 1983].

The time-asymptotic evolution of $G(z,t)$ can belong to one of only two possible categories [Landau 1953, Bers 1983 and references therein]:

(a) an *absolute instability*, where the response grows in time and encompasses more and more of the space as a function of time – the response always including the spatial location of the initial perturbation; thus, every spatial point eventually becoming unstable, i.e. having temporally growing fields;

(b) a *convective instability*, where the initial response grows in time but propagates away from its point of origin; thus, any spatial point eventually becoming stable, i.e. having temporally decaying (or oscillatory) fields.

As shown by Bers and Briggs [Briggs 1964, Bers 1983], for an unstable medium,† this distinction in the time-asymptotic behavior of Eq. (2) is obtained by determining the

† An unstable medium is one for which $D(k_r,\omega) = 0$ (k_r being the real-k) gives at least one branch, $\omega(k_r)$, that has a positive imaginary part, $\omega_i(k_r) > 0$, for some k_r.

pinch-point saddles in k (at $k = k_o$), and the associated branch points in ω (at $\omega = \omega_o$) given by:

$$D(k_o, \omega_o) = 0 , \quad \frac{\partial D(k_o, \omega_o)}{\partial k} = 0 \tag{3}$$

The time asymptotic Green's function is dominated by the pinch point associated with the largest value of ω_{oi} (ω_{oi} being the imaginary part of ω_o). If that ω_{oi} is positive then the instability evolves as an absolute instability. If ω_{oi} is negative for all the pinch points then the instability will evolve as a convective instability.

DISCUSSION OF THE EXAMPLES USED IN OSCARSSON [1986]

Before we consider the examples in Oscarsson and Rönnmark [1986], two remarks are in order. First, the notion of time-asymptotics should not be considered, trivially, as simply $t \to \infty$; in this limit any linear instability will have long violated the, ab-initio, assumption of small-amplitude fields. A time-asymptotic state is established as soon as the contribution from the pinch point with the largest ω_{oi} dominates over the contribution of the pinch point with the next largest ω_{oi}; this gives a time-asymptotic time scale which is quite finite. The importance of nonlinear effects over such time scales of evolution must, of course, be assessed separately.

Second, the evolution of an arbitrary initial perturbation, $\psi_o(z)$ at $t = 0$, is obtained by convolving the Green's function with $\psi_o(z)$ [Morse 1953]. Clearly, if $\psi_o(z)$ is spatially localized, $\psi(z, t)$ will evolve in a manner determined by the Green's function. Thus, if the Green's function analysis indicates the medium is absolutely (convectively) unstable, $\psi(z, t)$ will evolve as an absolute (convective) instability. However, if $\psi_o(z)$ extends all the way to $\pm\infty$ then $\psi(z, t)$ may not, in general, evolve in a manner indicated by the Green's function.

We now consider the examples treated in Oscarsson and Rönnmark [1986] and show that they are not characteristic of space-time evolution of instabilities; they describe an instability that evolves essentially in time only and, thus, are singular examples. The dispersion function of Oscarsson and Rönnmark [1986] is:

$$D(k, w) = \omega - kv_o - i\gamma \tag{4}$$

where v_0 and γ are constants. The corresponding equation describing the evolution of the electric field, $E(z,t)$, is of the form:

$$\frac{\partial E(z,t)}{\partial z} + \frac{1}{v_0}\frac{\partial E(z,t)}{\partial t} - \frac{\gamma}{v_0}E(z,t) = 0 \tag{5}$$

This is a hyperbolic partial differential equation with its characteristics given by $z - v_0 t =$ *constant*. It is easy to realize that Eq. (5) is a singular equation which is not generic of space-time evolution. If we replace z and t by $\xi = z - v_0 t$ and $\tau = t$, respectively, then Eq. (5) becomes:

$$\frac{\partial E(\xi,\tau)}{\partial \tau} - \gamma E(\xi,\tau) = 0 \tag{6}$$

Obviously, this equation describes the evolution of E in time and has nothing to do with space-time. So the concept of an absolute or convective evolution of an instability, which defines the space-time response of a medium, is not defined for such an equation. The solution to Eq. (6) is:

$$E(\xi,\tau) = e^{\gamma(\tau - \tau_0)}\, E_0(\xi,\tau_0)\, \theta(\tau - \tau_0) \tag{7}$$

where E_0 is the initial prescribed value of the field at $\tau = \tau_0$, and θ is the Heaviside function needed to satisfy causality. Thus, the solution to Eq. (5) is:

$$E(z,t) = e^{\gamma(t - t_0)}\, E_0(z - v_0 t, t_0)\, \theta(t - t_0) \tag{8}$$

This equation implies that the field at a point z at time t has grown uniformly in time (at the rate given by γ) from the value of the field at position $z - v_0 t$ at time t_0. There is only a trivial spatial evolution of the field along the characteristics.

The Green's function for Eq. (5) is:

$$G(z,t) = \begin{cases} v_0 e^{\gamma t}\, \delta(z - v_0 t)\theta(t), & v_0 \neq 0 \\ e^{\gamma t}\, \delta(z)\theta(t), & v_0 = 0 \end{cases} \tag{9}$$

For $v_0 \neq 0$, this shows that the instability convects away from its initial point of origin. However, in a reference frame moving with velocity v_0, the instability just grows in time without any spatial evolution. The case when $v_0 = 0$ does not represent any space-time evolution; thus, it is irrelevant to speak of absolute or convective instability. This, again, points out the singular nature of the dispersion function in Eq. (4).

It is important to realize that, for the dispersion function of Eq. 4 with $v_o \neq 0$, the spatial width of any instability does not change with time. For plasmas described by non-singular dispersion relations, this is not the case for either absolute or convective instabilities. Any arbitrary perturbation, in general, will spread spatially beyond its original spatial width. By choosing initial perturbations that extend all the way to $\pm\infty$, Oscarsson and Rönnmark [1986] have attempted to create the impression that the perturbations evolved spatially (instead of just temporally). We highlight that aspect by considering in detail one of the examples they used.

By using the Green's function given above, the time evolution of an initial perturbation:

$$E(z,0) \;=\; \exp(-z^2/d^2)\; \exp(-ik_o z) \tag{10}$$

is given by [Morse 1953]:

$$E(z,t) \;=\; \exp\left\{-(z-v_o t)^2/d^2\right\}\; \exp(\gamma t)\; \exp\left\{-ik_o(z-v_o t)\right\} \tag{11}$$

All points in space, in the coordinate system moving with velocity v_o, grow at the same rate. This can be further emphasized by choosing $E(z,0)$ to be initially localized:

$$E(z,0) \;=\; \exp(-z^2/d^2)\; \exp(-ik_o z)\; \frac{1}{2}\left\{\theta(z-l)+\theta(z+l)\right\} \tag{12}$$

where l can be chosen to be as large as one wants. Then it is easy to show that at any time $t > 0$ the field will be confined to a spatial width of $2l$ and will not extend beyond that width.

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