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Large-Amplitude Traveling Electromagnetic Waves in Collisionless Magnetoplasmas

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ABSTRACT

A class of exact large-amplitude traveling-wave solutions to the nonlinear Vlasov-Maxwell equations describing a one-dimensional collisionless magnetized plasma is obtained. These waves are complementary to the electrostatic Bernstein-Greene-Kruskal (BGK) modes and can be classified as nonlinear *fast* electromagnetic waves and (*slow*) electromagnetic whistler waves. The wave characteristics are discussed for the case of a trapped-particle distribution function.

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The formation and evolution of large-amplitude coherent structures play an important role in describing the nonlinear dynamics of plasmas [1],[2]. Since 1957, it has been well-known that collisionless plasmas can support nonlinear traveling electrostatic waves, i.e., the Bernstein-Greene-Kruskal (BGK) modes [3]. Bell [4] and independently Lutomirski and Sudan [5] have shown that collisionless magnetoplasmas described by the Vlasov-Maxwell equations can also support nonlinear traveling electromagnetic whistler waves, i.e., large-amplitude slow electromagnetic waves propagating parallel to the magnetic field. The formalism of the previous authors [4]-[7] utilizes the Lorentz wave frame so that only the slow (whistler) wave solutions with phase velocity less than the speed of light are obtained.

In this Letter, we present a class of exact large-amplitude traveling-wave solutions to the fully nonlinear Vlasov-Maxwell equations describing a one-dimensional collisionless plasma in an externally applied magnetic field, $B_0\vec{e}_z$ (with $B_0 = \text{const}$). By applying a canonical transformation rather than a Lorentz transformation, we develop a formalism that can be used to analyze in a unified framework both fast electromagnetic waves and (slow) electromagnetic whistler waves propagating parallel to the magnetic field. We show that these waves are (transverse) electromagnetic and are complementary to the (longitudinal) electrostatic BGK modes. The results of this paper provide a basis for studies of the (nonlinear) interaction of an intense coherent electromagnetic wave with a magnetoplasma or a relativistic electron beam gyrating in a guide magnetic field. With regard to the interaction of an electromagnetic wave with an electron beam, these studies include mode competition [8],[9] and the nonlinear evolution of absolutely unstable modes [10] in coherent radiation sources powered by gyrating electron beams, such as the cyclotron autoresonance maser [11], and the stability properties of a spatially and temporally modulated, gyrating electron beam [12] generated from the cyclotron resonance accelerator [13],[14].

We seek exact transverse traveling-wave solutions to the fully nonlinear Vlasov-Maxwell equations

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} - e\Big[\vec{E} + \frac{\vec{v}}{c}\times(B_0\vec{e}_z + \vec{B})\Big]\cdot\frac{\partial f}{\partial \vec{p}} = 0\ , \tag{1}$$

and

$$\Big(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\vec{A}(z,t) = \frac{4\pi e}{c}\int d^3p\ \vec{v}f(\vec{x},\vec{p},t)\ , \tag{2}$$

with $\vec{A}(\vec{x},t) = (mc^2/e)a[\vec{e}_x\cos(kz-\omega t) - \vec{e}_y\sin(kz-\omega t)]$, $\vec{E}(z,t) = -(1/c)\partial\vec{A}/\partial t$, and $\vec{B}(z,t) = \nabla\times\vec{A}$. In Eqs. (1) and (2), $f(\vec{x},\vec{p},t)$ is the electron phase-space distribution function, a = const is the normalized wave amplitude and can be arbitrarily large in size, $\vec{p} = \gamma m\vec{v}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$ are the electron mechanical momentum and relativistic mass factor, respectively, m and $-e$ are the electron rest mass and charge, respectively, and c is the speed of light in vacuum. The assumptions in our present formalism are: (1) the ions are at rest, (2) time-independent space-charge and current effects are negligibly small, and (3) there is no electrostatic wave in the problem. The first and second assumptions may be removed in a more general formalism, while the third assumption is consistent with our choice of the distribution function [see Eq. (6)].

To solve the nonlinear Vlasov equation (1), we first find the constants of the motion of an individual electron in the self-consistent electric and magnetic fields $\vec{E}$ and $B_0\vec{e}_z+\vec{B}$. The motion of an individual electron can be determined from the Hamiltonian

$$H(\vec{x},\vec{P},t) = \{[c\vec{P} + e(\vec{A}+\vec{A}_0)]^2 + m^2c^4\}^{1/2}\ , \tag{3}$$

where $\vec{A}_0 = B_0x\vec{e}_y$, $B_0\vec{e}_z = \nabla\times\vec{A}_0$, and the canonical momentum $\vec{P}$ is related to the mechanical momentum $\vec{p}$ by $\vec{P} = \vec{p} - (e/c)(\vec{A}+\vec{A}_0)$. It is convenient to perform a timedependent canonical transformation from the Cartesian canonical variables, $(\vec{x}, \vec{P})$, to the generalized guiding-center variables (including the wave), $(\phi, Y, z', P_\phi, P_Y, P_{z'})$ [13],[15]. The Hamiltonian in the new variables is a constant of motion defined by

$$\frac{1}{mc^2}H'(\phi, P_\phi, P_{z'} = \text{const}) = \gamma \; - \; \frac{\omega P_\phi}{mc^2}$$

$$\equiv \left[\frac{2\Omega P_\phi}{mc^2} + 2a\left(\frac{2\Omega P_\phi}{mc^2}\right)^{1/2} \cos\phi + \left(\frac{P_{z'}}{mc} + \frac{kP_\phi}{mc}\right)^2 + a^2 + 1\right]^{1/2} \; - \; \frac{\omega P_\phi}{mc^2} \,, \tag{4}$$

where $\Omega = eB_0/mc$ is the nonrelativistic cyclotron frequency associated with the applied guide magnetic field $B_0\vec{e}_z$.

It follows from Eq. (4) that the single-particle constants of motion are: Y, P_Y, $P_{z'}$ and H'. It has been shown [13] that change in P_ϕ is proportional to (negative) change in the number of wave photons, and that the constancy of H' (or $P_{z'}$) corresponds to the conservation of the total energy (or the axial momentum) of the electron plus photon system. Moreover, from the definition of the canonical transformation [15], it follows that the canonical conjugate pair (Y, P_Y) describe the generalized transverse guiding-center position of the electron.

From Liouville's theorem, an arbitrary function of the form $f(\vec{x}, \vec{p}, t) = f(Y, P_Y, P_{z'}, H')$ solves the nonlinear Vlasov equation (1). However, a class of distribution functions that are consistent with our one-dimensional model must be independent of the transverse guiding-center position $(-P_Y/m\Omega, Y)$. Therefore, solutions to Eq. (1) are

$$f(\vec{x}, \vec{p}, t) = f(P_{z'}, H') \,. \tag{5}$$

Because $d^3p = dp_x dp_y dp_z = m\Omega d\phi dP_\phi dP_{z'}$, the electron and current densities can be expressed as

$$n(\vec{x},t) = \int (m\Omega) d\phi dP_\phi dP_{z'} f(P_{z'}, H'(\phi, P_\phi, P_{z'})) = \text{const} \ , \tag{6}$$

and

$$\vec{J}(\vec{x},t) = -e \int (m\Omega) d\phi dP_\phi dP_{z'} \vec{v}(\phi, P_\phi, P_{z'}, z - v_{ph}t) f(P_{z'}, H'(\phi, P_\phi, P_{z'})) \ , \tag{7}$$

respectively. Here, $v_{ph} = \omega/k$ is the wave phase velocity and $v_z = (P_{z'} + kP_\phi)/m\gamma(\phi, P_\phi, P_{z'})$ is independent of z and t. From Eqs. (6) and (7), we conclude that the transverse electron current density $\vec{J}_\perp$ exhibits the traveling-wave dependence $z - v_{ph}t$, while the electron charge density n and the electron axial current density J_z are constant and uniform. The constant charge density assures the validity of the earlier assumption that there is no electrostatic wave in the problem. Substituting Eq. (7) into the wave equation (2) yields

$$\omega^2 - c^2k^2 = \frac{4\pi e^2 \Omega}{a} \int d\phi dP_\phi dP_{z'} \left[a + \left(\frac{2\Omega P_\phi}{mc^2} \right)^{1/2} \cos\phi \right] \frac{f}{\gamma} \ , \tag{8}$$

which determines formally the self-consistent relationship among the wave quantities k, ω, and a for an arbitrary distribution function of the form $f(P_{z'}, H')$. Note that the Hamiltonian H' in Eq. (4) is itself a function of ω, k and a. In principle, the integral equation (8) is readily solved with numerical methods.

The remainder of this paper examines the wave characteristics and the structure of the single-particle phase space for a trapped-particle distribution function that is analytically tractable. In this case, all of the electrons assume the same stable steady-state orbit, denoted by $\phi = \phi_0$ and $P_\phi = P_{\phi 0}$. Solving the steady-state Hamilton equations of motion, $d\phi/dt = 0 = dP_\phi/dt$, for given $P_{z'} = P_{z'0}$, we find that

$$\omega - ck\beta_{z0} - \frac{\Omega}{\gamma_0} = \frac{a\Omega}{\gamma_0(\gamma_0\beta_{a0} - a)}\,, \tag{9}$$

where $\cos\phi_0 = \pm 1$, $\gamma_0 = \gamma(\phi_0, P_{\phi 0}, P_{z'0})$ is the electron relativistic mass factor, $p_{z0} = \gamma_0 m\beta_{z0}c = P_{z'0} + kP_{\phi 0}$ is the electron axial (mechanical) momentum, and $\vec{p}_0 \cdot \vec{A}/|\vec{A}| = \gamma_0 m\beta_{a0}c = amc + (2m\Omega P_{\phi 0})^{1/2}\cos\phi_0$ is the electron transverse (mechanical) momentum projected onto the vector potential $\vec{A} = -\vec{B}/k$. Factoring $[a + (2\Omega P_{\phi 0}/mc^2)^{1/2}\cos\phi_0]/\gamma_0$ out of the integral in Eq. (8) and then making use of Eq. (6), we can express the wave equation (8) as

$$\omega^2 - c^2k^2 = \frac{\omega_p^2\beta_{a0}}{a}\,. \tag{10}$$

where $\omega_p = (4\pi e^2 n/m)^{1/2} = \text{const}$ is the nonrelativistic electron plasma frequency. Eliminating the wave amplitude a from Eqs. (9) and (10), we obtain the equilibrium condition between ω and k,

$$\left(\omega^2 - c^2k^2 - \frac{\omega_p^2}{\gamma_0}\right)\left(\omega - ck\beta_{z0} - \frac{\Omega}{\gamma_0}\right) = \frac{\omega_p^2\Omega}{\gamma_0^2}\,. \tag{11}$$

It should be emphasized that Eq. (11) is valid for arbitrary phase velocities, including the large-amplitude (slow) electromagnetic whistler wave (with $|\omega/ck| < 1$) studied by previous authors [4]-[7] as well as large-amplitude fast electromagnetic waves (with $|\omega/ck| > 1$) which have not been reported in the literature and will be described below (see Figs. 1 and 2). Furthermore, equation (11) is a nonlinear "dispersion relation" in the sense that it describes large-amplitude electromagnetic waves whose amplitudes are determined uniquely by Eq. (9) or (10), whereas a linear dispersion relation derived from perturbation theory describes waves with arbitrary (small) amplitudes. Finally, it is easily shown that Eq. (11) has three real ω roots for any real value of k.

Figure 1 shows the frequency ω as a function of the wave number k, as obtained from Eq. (11). The choice of system parameters corresponds to $\gamma_0 = 2.0$, $\beta_{z0} = 0.8$, and $s_e = 2(1-\beta_{z0})\gamma_0\omega_p^2/\Omega^2 = 0.36$. In Fig. 1, there is a *fast* electromagnetic wave branch with $|\omega/ck| > 1$ and $\omega > 0$, a (*slow*) electromagnetic whistler wave branch with $|\omega/ck| < 1$, and a *fast* electromagnetic wave branch with $|\omega/ck| > 1$ and $\omega < 0$. Both (nonlinear) fast wave branches are new results, while the (slow) whistler wave branch has been obtained by previous authors [4]-[6].

The (normalized) self-consistent wave amplitude, a, is calculated from Eq. (10). The results are plotted in Fig. 2 for all branches shown in Fig. 1. For this choice of system parameters, the normalized wave amplitude for the fast wave branch with $\omega > 0$ is in the range from 0.2 to 0.7, while the amplitude for the fast wave branch with $\omega < 0$ exceeds unity for small $|k|$. The amplitude for the whistler branch becomes very large as $|k|$ approaches zero.

Finally, it is instructive to examine the structure of the single-particle phase space described by the Hamiltonian $H'(\phi, P_\phi, P_{z'})$. For given nonrelativistic cyclotron frequency Ω and self-consistent wave parameters ω, k, and a, the phase space can be parameterized by the two constants of motion, $P_{z'}$ and H'. The constant-H' phase plane (ϕ, P_ϕ) is plotted in Fig. 3 for the fast wave branch with $\omega > 0$ shown in Fig. 1. The parameters used in Fig. 3 are: $ck/\Omega = 3.0$, $\omega/\Omega = 3.1$, and $a = 0.2$. In Fig. 3, each curve corresponds to a contour with a constant $P_{z'}$ and a fixed value of $H' = H'(0, P_{\phi 0}, P_{z'0}) = 1.62$, but the value of $P_{z'}$ varies from one contour to another. Here, the orbit $\phi_0 = 0$ and $\Omega P_{\phi 0}/mc^2 = 0.11$ is a stable steady-state orbit. The bounce frequency, ω_B, for orbits oscillating about the steady-state orbit $(\phi_0, P_{\phi 0})$ is found to be

$$\omega_B^2 = \frac{a^2\Omega^2}{\gamma_0^2(\gamma_0\beta_{a0} - a)^2} + \frac{\omega_p^2}{\gamma_0^2}\beta_{a0}(\gamma_0\beta_{a0} - a) \; . \tag{12}$$

The value of the bounce frequency for the steady-state orbit shown in Fig. 3 is $\omega_B = 0.25\ \Omega$. (For all parameter regimes investigated, the bounce frequency is real.)

Numerical studies of the large-amplitude traveling waves and their stability are in progress using a time-averaged multiparticle model of the nonlinear interaction of a gyrating electron beam with an electromagnetic wave [16]. The results will be reported in a future publication.

To summarize, we have obtained a class of exact large-amplitude traveling-wave solutions to the nonlinear Vlasov-Maxwell equations describing a one-dimensional collisionless magnetized plasma. These solutions describe nonlinear (transverse) electromagnetic waves which are complementary to the nonlinear (longitudinal) electrostatic BGK modes. We believe that the formalism presented this paper can be applied to a variety of important problems concerning the interaction of an intense electromagnetic wave with a magnetized plasma or a relativistic electron beam propagating in a uniform magnetic field.

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[15] The canonical transformation is defined by $x = (2P_\phi/m\Omega)^{1/2}\sin(\phi - kz + \omega t) - P_Y/m\Omega$, $y = Y - (2P_\phi/m\Omega)^{1/2}\cos(\phi - kz + \omega t)$, $z = z'$, $P_x = (2m\Omega P_\phi)^{1/2}\cos(\phi - kz + \omega t)$, $P_y = P_Y$, and $P_z = P_{z'} + kP_\phi$. This transformation can be obtained by successive applications of the generating functions $F_3(P_x, P_y; \alpha, Y) = (1/m\Omega)[P_x P_y - (P_x^2/2)\tan\alpha] - YP_y$ and $F_2(\alpha, z; P_\phi, P_{z'}, t) = (kz - \omega t + \alpha)P_\phi + zP_{z'}$.

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FIGURE CAPTIONS

Fig. 1 Frequency as a function of the wave number obtained from the equilibrium condition (11) for $\gamma_0 = 2.0$, $\beta_{z0} = 0.8$ and $\omega_p^2/\Omega^2 = 0.45$.

Fig. 2 Normalized self-consistent wave amplitude as a function of k for all branches shown in Fig. 1.

Fig. 3 Constant-H' phase plane (ϕ, P_ϕ) for the fast wave branch with $\omega > 0$ shown in Fig. 1. The value $ck/\Omega = 3.0$ is used.

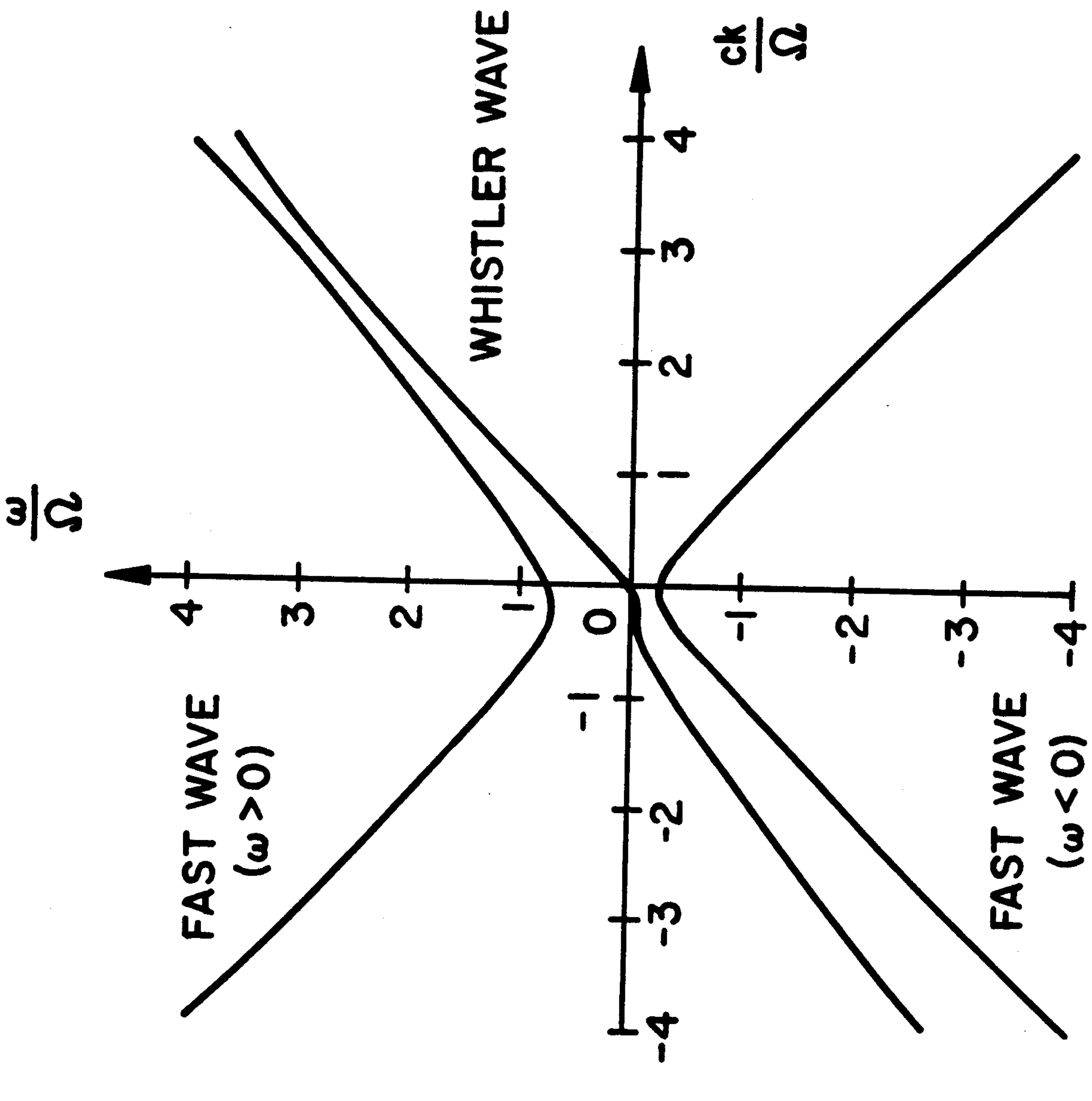
ω/Ω
ck/Ω
WHISTLER WAVE
FAST WAVE (ω>0)
FAST WAVE (ω<0)
-4 -3 -2 -1 0 1 2 3 4
-4 -3 -2 -1 1 2 3 4

Fig. 1

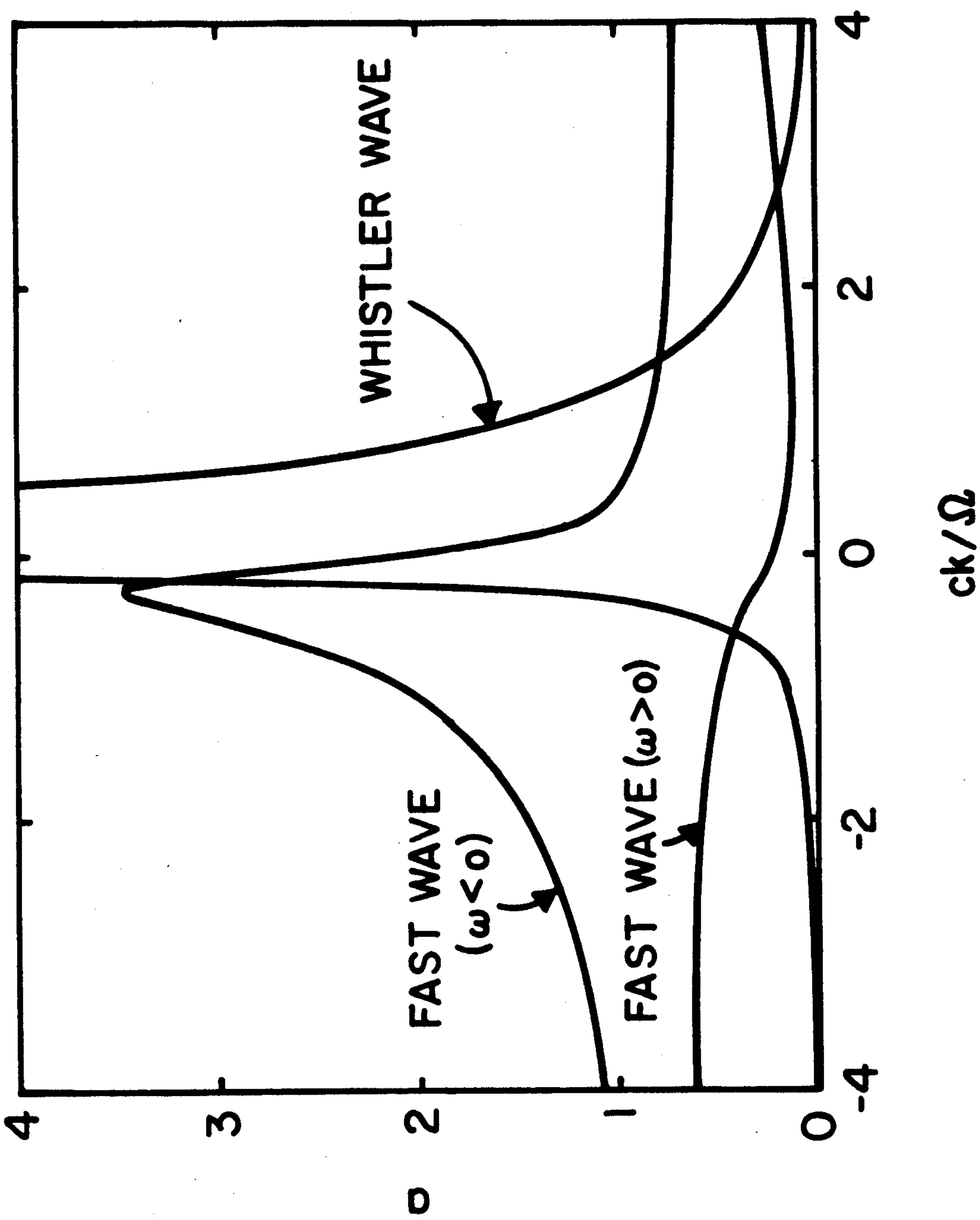

Fig. 2

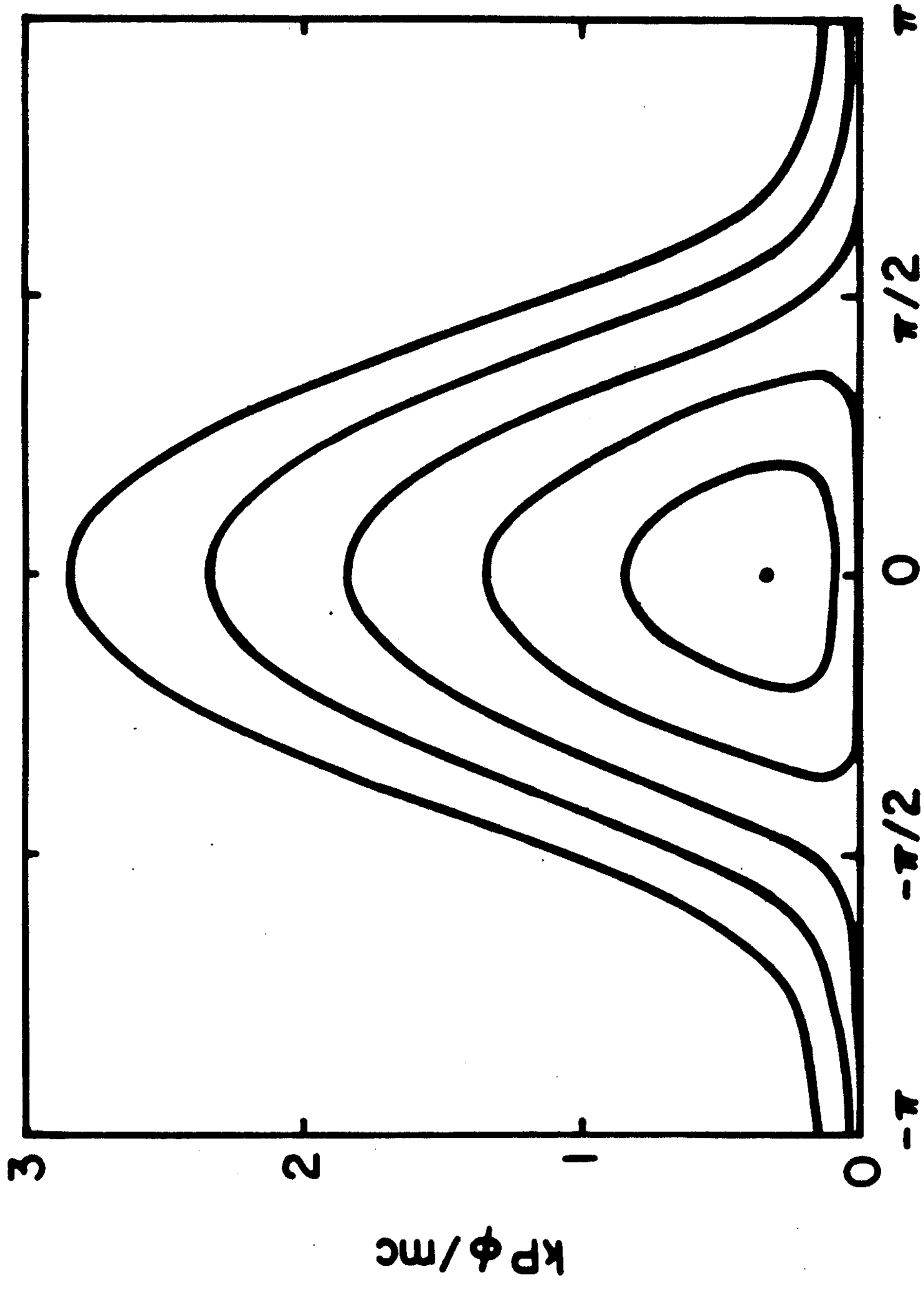

Fig. 3