2.29 Numerical Fluid Mechanics
Fall 2011 – Lecture 16

REVIEW Lecture 15:

- **Fourier Error Analysis**
  - Provide additional information to truncation error: indicates how well Fourier mode solution, i.e. wavenumber and phase speed, is represented
    - Effective wavenumber: $\left(\frac{\partial\, e^{ikx}}{\partial x}\right)_j = i\, k_{\text{eff}}\, e^{ikx_j}$ (for CDS, 2nd order, $k_{\text{eff}} = \frac{\sin(k\Delta x)}{\Delta x}$)
    - Effective wave speed (for linear convection eqn., $\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = 0$, integrating in time exactly):

$$\frac{d\, f_k^{num.}}{dt} = -f_k^{num.}(t)\, c\, i\, k_{\text{eff}} \Rightarrow f_{\text{numerical}}(x,t) = \sum_{k=-\infty}^{\infty} f_k(0)\, e^{ikx - ik_{\text{eff}}\, t} = \sum_{k=-\infty}^{\infty} f_k(0)\, e^{ik(x - c_{\text{eff}}\, t)} \Rightarrow \frac{c_{eff}}{c} = \frac{\sigma_{eff}}{\sigma} = \frac{k_{eff}}{k}$$

(with $\sigma_{\text{eff}} = -i\, k_{\text{eff}}\, c = -i\, k\, c_{\text{eff}}$)

- **Stability**
  - Heuristic Method: trial and error
  - Energy Method: Find a quantity, l_2 norm $\sum_j \left(\phi_j^n\right)^2$, and then aim to show that it remains bounded for all n.
    - Example: for $\frac{\partial\phi}{\partial t} + c\frac{\partial\phi}{\partial x} = 0$ we obtained $0 \le \frac{c\,\Delta t}{\Delta x} \le 1$
  - Von Neumann Method (Introduction), also called Fourier Analysis Method/Stability

Outline for TODAY (Lecture 16): FINITE DIFFERENCES, Cont'd

- Fourier Analysis and Error Analysis
- Stability
  - Heuristic Method
  - Energy Method
  - Von Neumann Method (Introduction): 1st order linear convection/wave eqn
- Hyperbolic PDEs and Stability
  - Example: 2nd order wave equation and waves on a string
    - Effective numerical wave numbers and dispersion
  - CFL condition:
    - Definition
    - Examples: 1st order linear convection/wave eqn, 2nd order wave eqn
    - Other FD schemes
  - Von Neumann examples: 1st order linear convection/wave eqn
  - Tables of schemes for 1st order linear convection/wave eqn

References and Reading Assignments

- Lapidus and Pinder, 1982: Numerical solutions of PDEs in Science and Engineering. Section 4.5 on “Stability”.
- Chapter 3 on “Finite Difference Methods” of “J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd edition, 2002”
- Chapter 3 on “Finite Difference Approximations” of “H. Lomax, T. H. Pulliam, D.W. Zingg, *Fundamentals of Computational Fluid Dynamics (Scientific Computation).* Springer, 2003”
- Chapter 29 and 30 on “Finite Difference: Elliptic and Parabolic equations” of “Chapra and Canale, Numerical Methods for Engineers, 2010/2006.”

Von Neumann Stability

- Widely used procedure
- Assumes initial error can be represented as a Fourier Series and considers growth or decay of these errors
- In theoretical sense, applies only to periodic BC problems and to linear problems
  - Superposition of Fourier modes can then be used
- Again, use, $f(x,t) = \sum_{k=-\infty}^{\infty} f_k(t)\, e^{ikx}$ but for the error: $\varepsilon(x,t) = \sum_{\beta=-\infty}^{\infty} \varepsilon_\beta(t)\, e^{i\beta x}$
- Being interested in error growth/decay, consider only one mode:

$$\varepsilon_\beta(t)\, e^{i\beta x} \approx e^{\gamma t}\, e^{i\beta x} \quad \text{where } \gamma \text{ is in general complex and function of } \beta: \; \gamma = \gamma(\beta)$$

- Strict Stability: for the error not to grow in time, $\left|e^{\gamma t}\right| \leq 1 \quad \forall \gamma$
  - in other words, for $t = n\Delta t$, the condition for strict stability can be written:

$$\left|e^{\gamma \Delta t}\right| \leq 1 \quad \text{or for } \xi = e^{\gamma \Delta t}, \quad |\xi| \leq 1$$

von Neumann condition

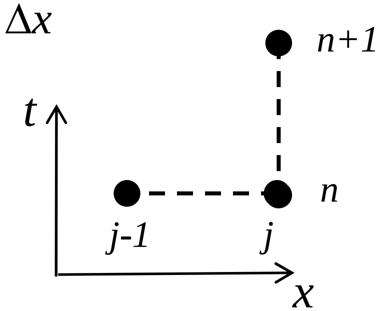
Norm of amplification factor ξ smaller than 1

Evaluation of the Stability of a FD Scheme
Von Neumann Example

- Consider again: $$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$
- A possible FD formula ("upwind" scheme) $$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

  $(t = n\Delta t,\ \ x = j\Delta x)$ which can be rewritten:

$$\phi_j^{n+1} = (1-\mu)\,\phi_j^n + \mu\,\phi_{j-1}^n \quad \text{with} \quad \mu = \frac{c\,\Delta t}{\Delta x}$$



- Consider the Fourier error decomposition (one mode) and discretize it:

$$\varepsilon(x,t) = \varepsilon_\beta(t)\,e^{i\beta x} = e^{\gamma t}e^{i\beta x} \Rightarrow \quad \varepsilon_j^n = e^{\gamma n\Delta t}e^{i\beta j\Delta x}$$

- Insert it in the FD scheme, assuming the error mode satisfies the FD:

$$\varepsilon_j^{n+1} = (1-\mu)\,\varepsilon_j^n + \mu\,\varepsilon_{j-1}^n \quad \Rightarrow \quad e^{\gamma(n+1)\Delta t}e^{i\beta j\Delta x} = (1-\mu)\,e^{\gamma n\Delta t}e^{i\beta j\Delta x} + \mu\,e^{\gamma n\Delta t}e^{i\beta(j-1)\Delta x}$$

- Cancel the common term (which is $\varepsilon_j^n = e^{\gamma n\Delta t}e^{i\beta j\Delta x}$) and obtain:

$$e^{\gamma\Delta t} = (1-\mu) + \mu\,e^{-i\beta\Delta x}$$

Evaluation of the Stability of a FD Scheme
von Neumann Example

- The magnitude of $\xi = e^{\gamma \Delta t}$ is then obtained by multiplying ξ with its complex conjugate:

$$|\xi|^2 = \left((1-\mu) + \mu\, e^{-i\beta\Delta x}\right)\left((1-\mu) + \mu\, e^{i\beta\Delta x}\right) = 1 - 2\mu(1-\mu)\left(1 - \frac{e^{i\beta\Delta x} + e^{-i\beta\Delta x}}{2}\right)$$

Since $\dfrac{e^{i\beta\Delta x} + e^{-i\beta\Delta x}}{2} = \cos(\beta\Delta x)$ and $1 - \cos(\beta\Delta x) = 2\sin^2(\dfrac{\beta\Delta x}{2}) \quad \Rightarrow$

$$|\xi|^2 = 1 - 2\mu(1-\mu)\left(1 - \cos(\beta\Delta x)\right) = 1 - 4\mu(1-\mu)\sin^2(\frac{\beta\Delta x}{2})$$

- Thus, the strict von Neumann stability criterion gives

$$|\xi| \le 1 \quad \Leftrightarrow \quad \left|1 - 4\mu(1-\mu)\sin^2(\frac{\beta\Delta x}{2})\right| \le 1$$

Since $\sin^2(\dfrac{\beta\Delta x}{2}) \ge 0 \quad \forall\beta \qquad \left(\left(1 - \cos(\beta\Delta x)\right) \ge 0 \quad \forall\beta\right)$

we obtain the same result as for the energy method:

$$|\xi| \le 1 \quad \Leftrightarrow \quad \mu(1-\mu) \ge 0 \quad \Leftrightarrow \quad \boxed{0 \le \frac{c\,\Delta t}{\Delta x} \le 1} \qquad (\mu = \frac{c\,\Delta t}{\Delta x})$$

Equivalent to the CFL condition

(from Lecture 12)

Partial Differential Equations
Hyperbolic PDE: $B^2 - 4AC > 0$

Examples:

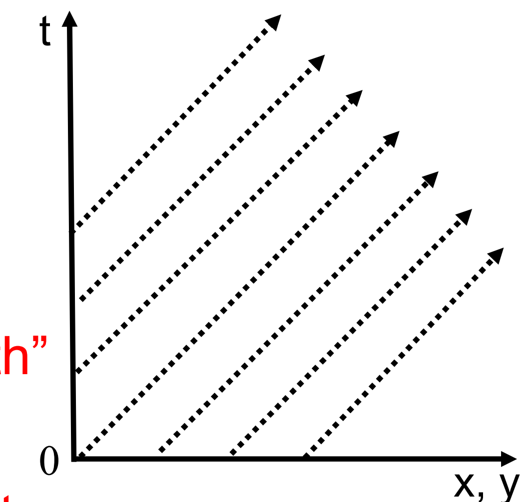
(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ← Wave equation, 2nd order

(2) $\frac{\partial u}{\partial t} \pm c \frac{\partial u}{\partial x} = 0$ ← Sommerfeld Wave/radiation equation, 1st order

(3) $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla)\, \mathbf{u} = \mathbf{g}$ ← Unsteady (linearized) inviscid convection (Wave equation first order)

(4) $(\mathbf{U} \cdot \nabla)\, \mathbf{u} = \mathbf{g}$ ← Steady (linearized) inviscid convection

- Allows non-smooth solutions
- Information travels along characteristics, e.g.:
  - For (3) above: $\frac{d\,\mathbf{x_c}}{dt} = \mathbf{U}(\mathbf{x_c}(t))$
  - For (4), along streamlines: $\frac{d\,\mathbf{x_c}}{ds} = \mathbf{U}$
- Domain of dependence of $\mathbf{u}(\mathbf{x},T)$ = "characteristic path"
  - e.g., for (3), it is: $\mathbf{x}_c(t)$ for $0 < t < T$
- Finite Differences, Finite Volumes and Finite Elements



Partial Differential Equations
Hyperbolic PDE

Waves on a String

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

Initial Conditions

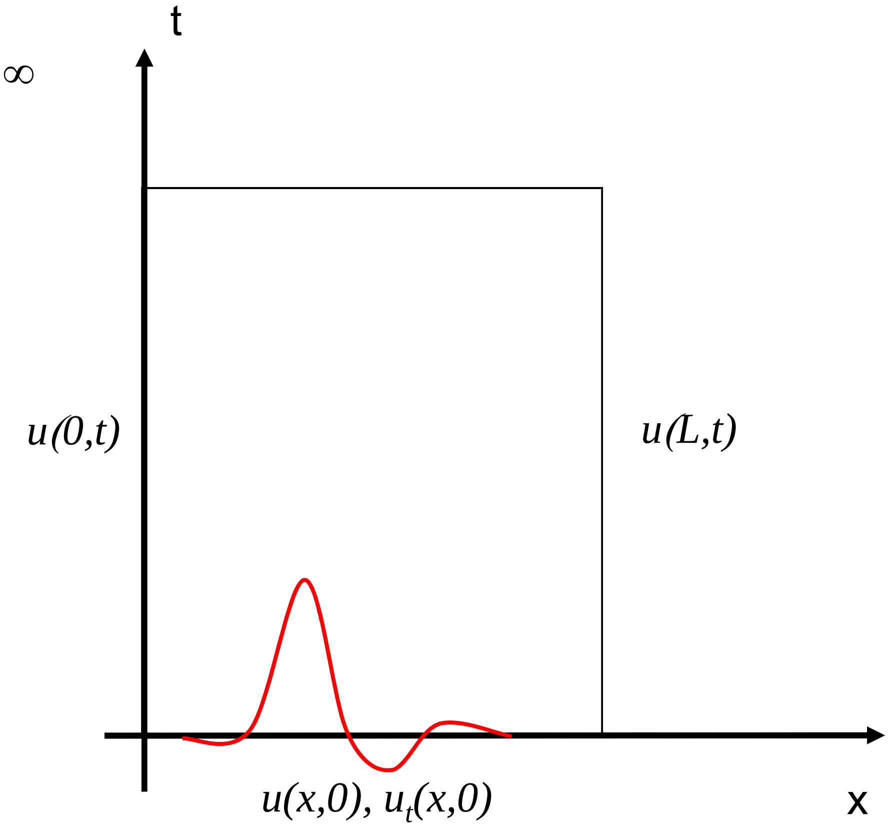
$$\begin{aligned} u(x,0) &= f(x), \;\; 0 \le x \le L \\ u_t(x,0) &= g(x), \;\; 0 < x < L \end{aligned}$$

Boundary Conditions

$$\begin{aligned} u(0,t) &= 0, \;, \; 0 < t < \infty \\ u(L,t) &= 0, \;, \; 0 < t < \infty \end{aligned}$$

Wave Solutions

$$u = \begin{cases} F(x-ct) & \text{Forward propagating wave} \\ G(x+ct) & \text{Backward propagating wave} \end{cases}$$



Typically Initial Value Problems in Time, Boundary Value Problems in Space
Time-Marching Solutions: Explicit Schemes Generally Stable

Partial Differential Equations
Hyperbolic PDE

Wave Equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

Discretization:

$$h = L/n$$
$$k = T/m$$
$$x_i = (i-1)h, \; i = 2, \ldots, n-1$$
$$t_j = (j-1)k, \; j = 1, \ldots, m$$

Finite Difference Representations

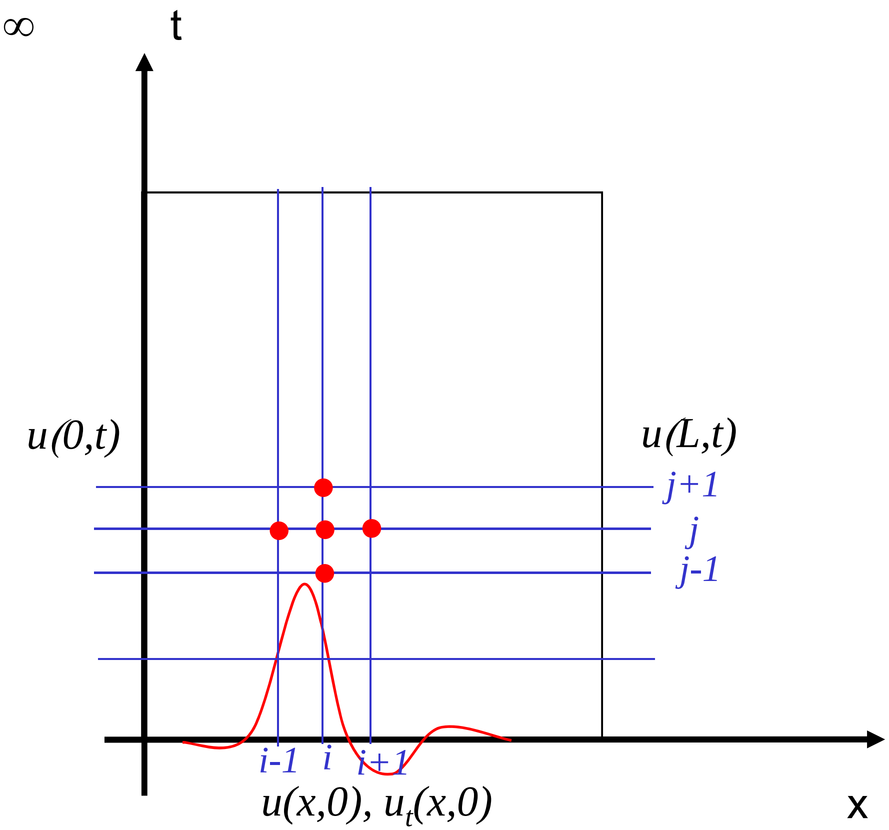
$$u_{tt}(x,t) = \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})}{k^2} + O(k^2)$$

$$u_{xx}(x,t) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2)$$

$$u_{i,j} = u(x_i, t_j)$$

Finite Difference Representations

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$



Partial Differential Equations
Hyperbolic PDE

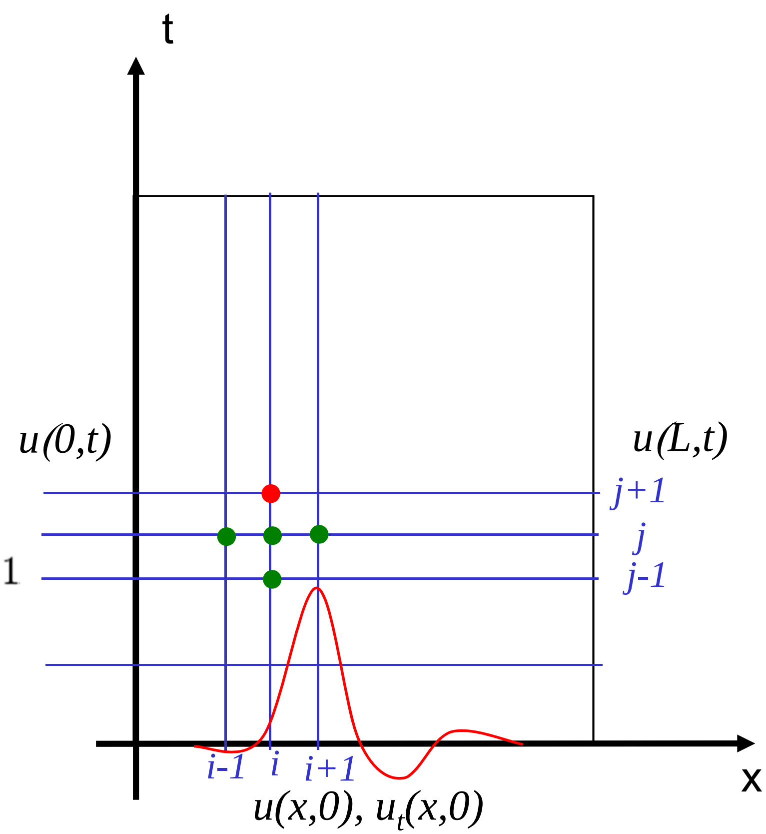
(from Lecture 12)

Introduce Dimensionless Wave Speed $C = \frac{ck}{h}$

Explicit Finite Difference Scheme

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} \quad i = 2, \ldots n-1$$



Stability Requirement: $C = \frac{ck}{h} < 1$

$C = \frac{c\,\Delta t}{\Delta x} < 1$ Courant-Friedrichs-Lewy condition (CFL condition)

Physical wave speed must be smaller than the largest numerical wave speed, or,
Time-step must be less than the time for the wave to travel to adjacent grid points:

$$c < \frac{\Delta x}{\Delta t} \quad \text{or} \quad \Delta t < \frac{\Delta x}{c}$$

Wave Equation
d'Alembert's Solution

Wave Equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

Solution

$$u(x,t) = F(x-ct) + G(x+ct) \;,\; 0 < x < L$$

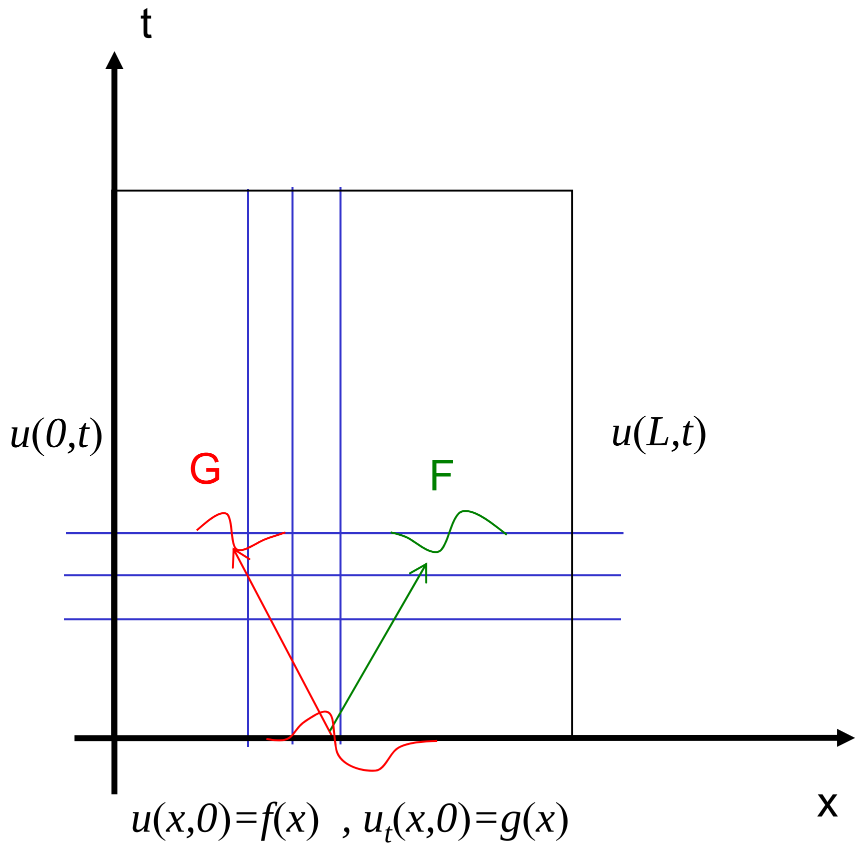
Periodicity Properties

$$\begin{aligned} F(-z) &= -F(z) \\ F(z+2L) &= F(z) \end{aligned}$$

$$\begin{aligned} G(-z) &= -G(z) \\ G(z+2L) &= G(z) \end{aligned}$$

Proof

$$\begin{aligned} u_{xx}(x,t) &= F''(x-ct) + G''(x+ct) \\ u_{tt}(x,t) &= c^2F''(x-ct) + c^2G''(x+ct) \\ &= c^2 u_{xx}(x,t) \end{aligned}$$



Hyperbolic PDE
Method of Characteristics

Explicit Finite Difference Scheme

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \; i = 2, \ldots n-1$$

First 2 Rows known

$$u_{i,1} = u(x_i, 0)$$

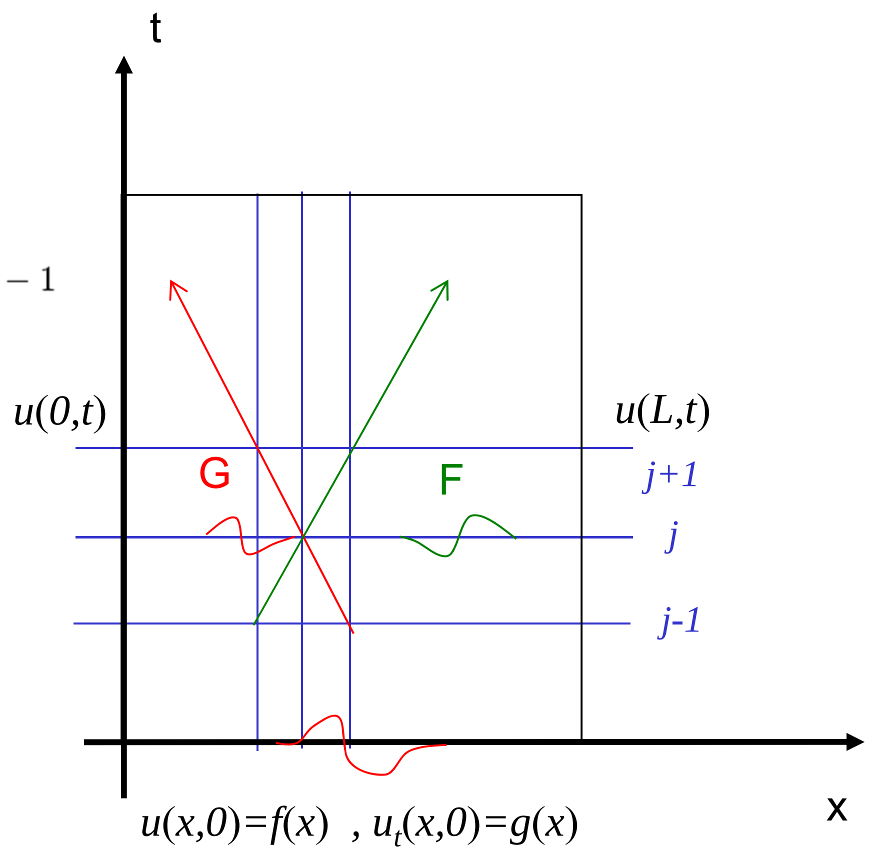
$$u_{i,2} = u(x_i, k)$$

Characteristic Sampling

$$k = h/c \Rightarrow C = 1$$

Exact Discrete Solution

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$



Hyperbolic PDE
Method of Characteristics

Let's proof the following FD scheme is an exact Discrete Solution

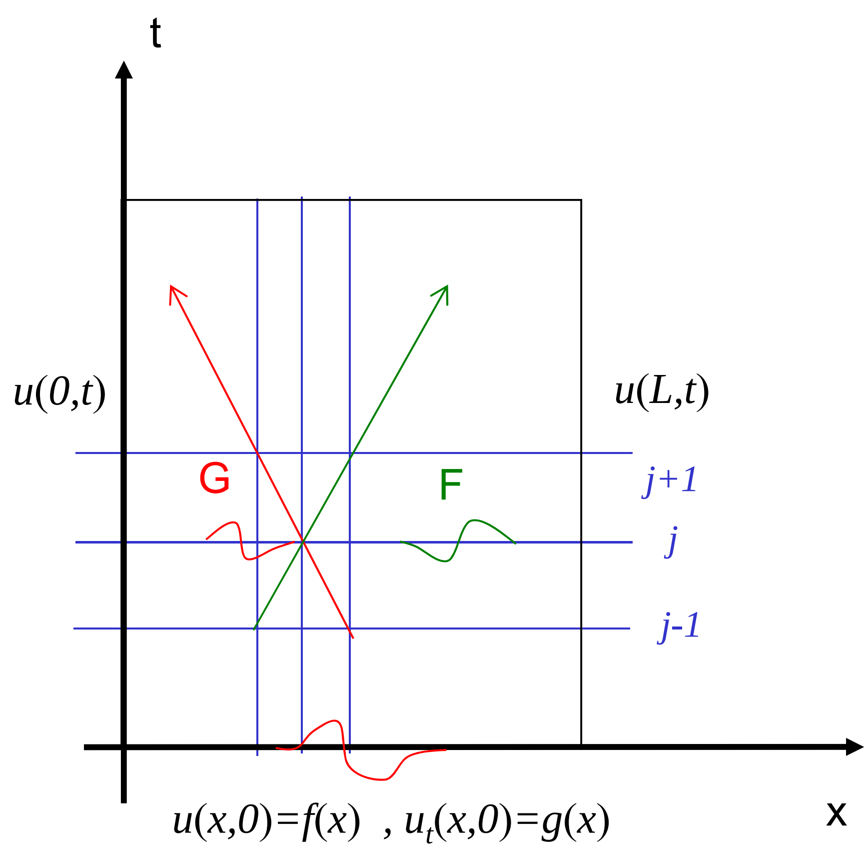
$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

D'Alembert's Solution

$$\begin{aligned} x_i - ct_j &= (i-1)h - c(j-1)k \\ &= (i-1)h - (j-1)h \\ &= (i-j)h \end{aligned}$$

$$\begin{aligned} x_i + ct_j &= (i-1)h + c(j-1)k \\ &= (i-1)h + (j-1)h \\ &= (i+j-2)h \end{aligned}$$

$$u_{i,j} = F((i-j)h) + G((i+j-2)h)$$



Proof

$$\begin{aligned} & u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \\ = \; & F((i+1-j)h) + F((i-1-j)h) - F((i-(j-1))h) \\ & +G((i+1+j-2)h) + G((i-1+j-2)h) - G((i+j-1-2)h) \\ = \; & F((i-(j+1))h) + G((i+(j+1)-2)h) \\ = \; & u_{i,j+1} \end{aligned}$$

Partial Differential Equations
Hyperbolic PDE
Start of Integration: Euler and Higher Order starts

1st order Euler Starter

$$u_{i,2} = u(x_i, k) \simeq u(x_i, 0) + k u_t(x, 0) \quad = \boxed{f(x_i) + k g(x_i)}$$

But, second derivative in x at $t = 0$ is known from IC:

$$u_{xx}(x,0) = f''$$

From Wave Equation

$$u_{tt}(x_i, 0) = c^2 u_{xx}(x_i, 0) = c^2 f_{xx}(x_i) = c^2 \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + O(h^2)$$

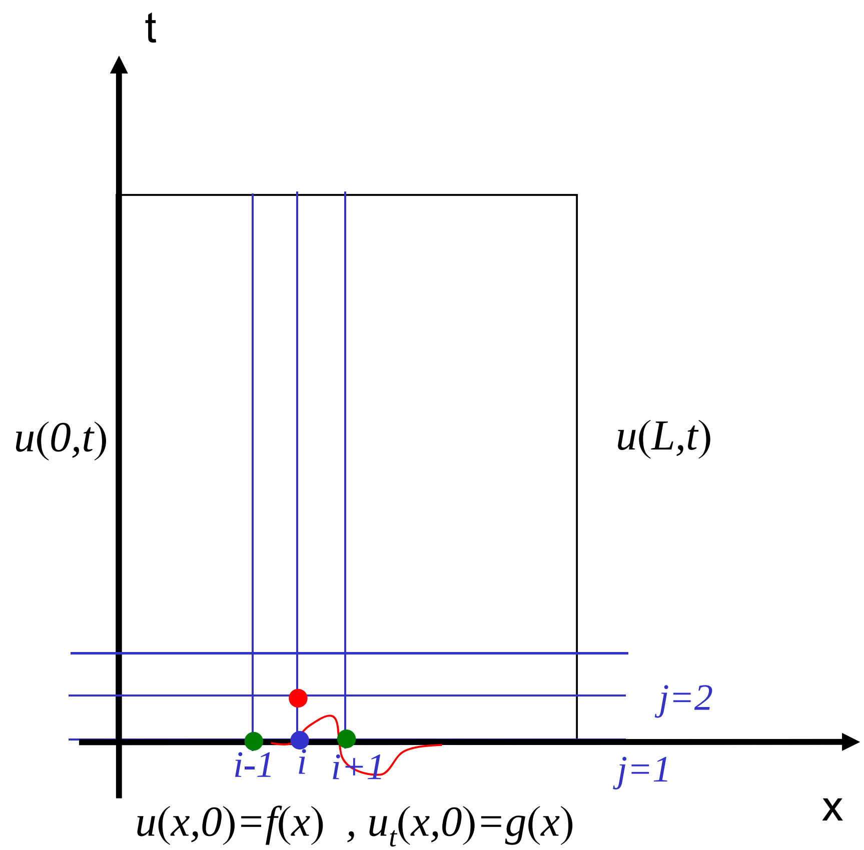
Higher order Taylor Expansion

$$u(x,k) = u(x,0) + k u_t(x,0) + \frac{u_{tt}(x,0)k^2}{2} + O(k^3)$$

Higher Order Self Starter

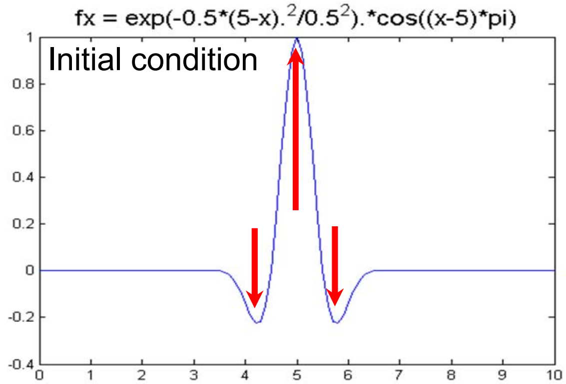
$$\boxed{u_{i,2} = u(x_i, k)} = f_i + k g_i + \frac{c^2 k^2}{2h^2}(f_{i-1} - 2f_i + f_{i+1}) + O(h^2 k^2) + O(k^3)$$

$$= \boxed{\left(1 - C^2\right) f_i + k g_i} + \boxed{\frac{C^2}{2}(f_{i+1} + f_{i-1})}$$



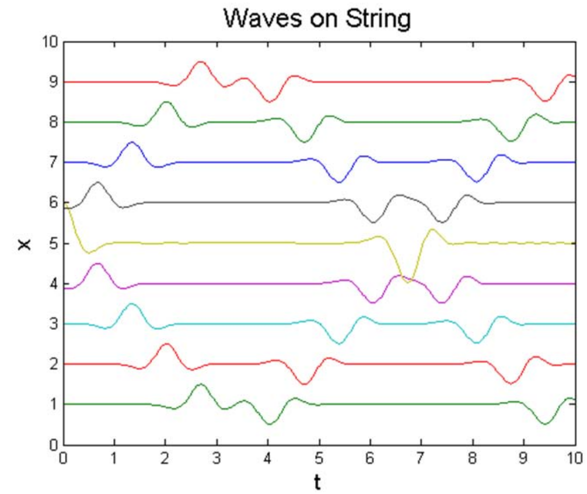
Waves on a String

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

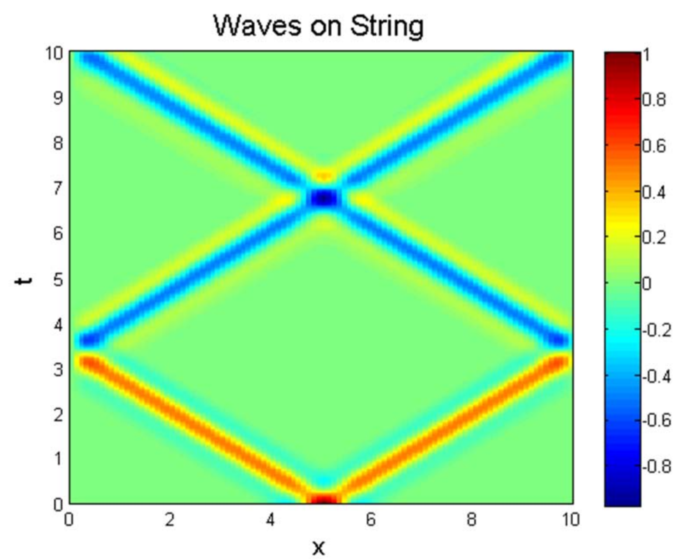


waveeq.m

```
L=10;
T=10;
c=1.5;
N=100;
h=L/N;
M=400;
k=T/M;
C=c*k/h
Lf=0.5;
x=[0:h:L]';
t=[0:k:T];
%fx=['exp(-0.5*(' num2str(L/2) '-x).^2/(' num2str(Lf) ').^2)'];
%gx='0';
fx='exp(-0.5*(5-x).^2/0.5^2).*cos((x-5)*pi)';
gx='0';  %Zero first time derivative at t=0
f=inline(fx,'x');
g=inline(gx,'x');

n=length(x);
m=length(t);
u=zeros(n,m);
%  Second order starter
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
 u(i,2) = (1-C^2)*u(i,1) + k*g(x(i)) +C^2*(u(i-1,1)+u(i+1,1))/2;
end

%  CDS: Iteration in time (j) and space (i)
for j=2:m-1
   for i=2:n-1
u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
   end
end
```



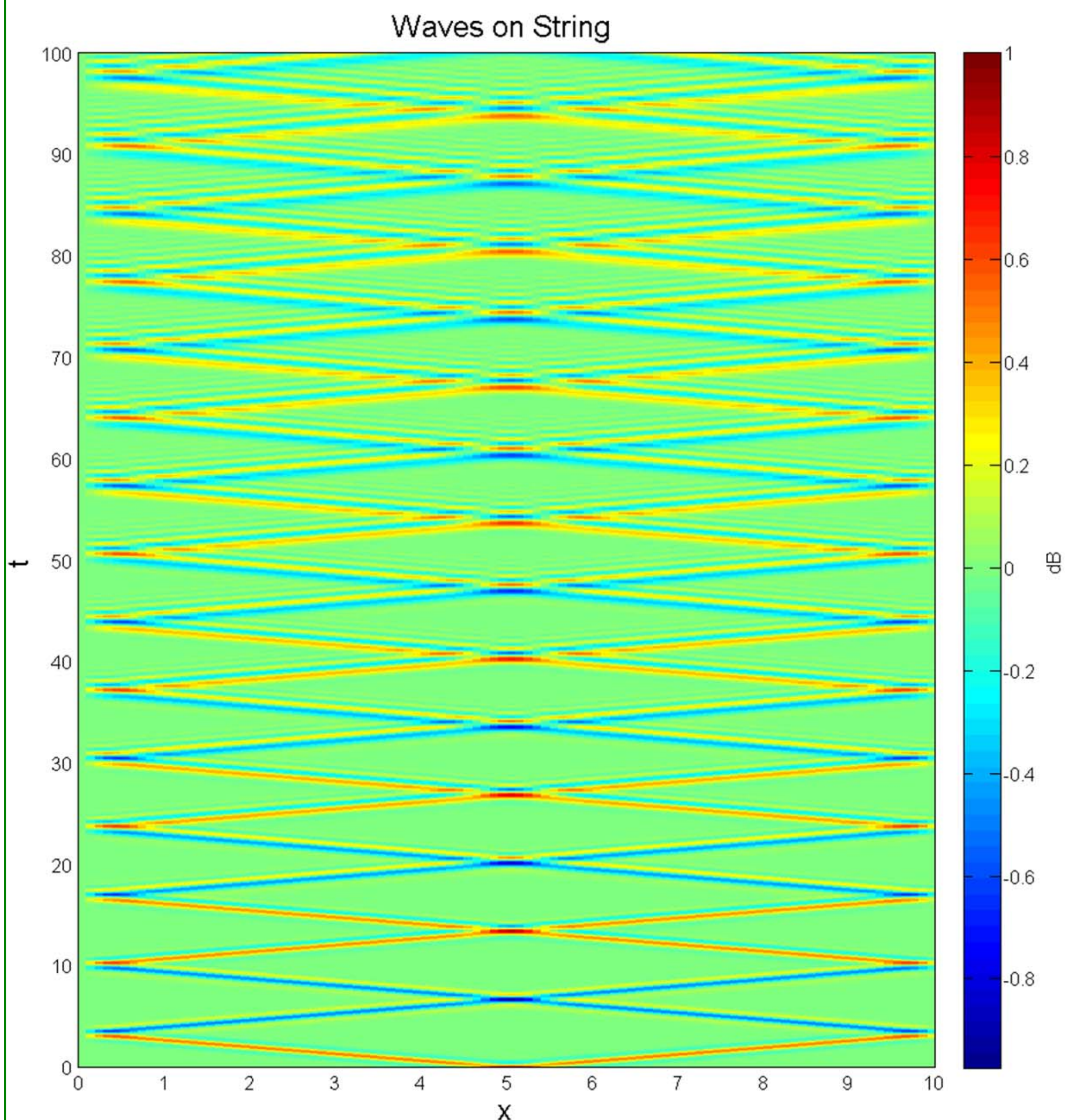

```
figure(1)
plot(x,f(x));
a=title(['fx = ' fx]);
set(a,'FontSize',16);

figure(2)
wavei(u',x,t);
a=xlabel('x');
set(a,'Fontsize',14);
a=ylabel('t');
set(a,'Fontsize',14);
a=title('Waves on String');
set(a,'Fontsize',16);
colormap;
```



Waves on a String, Longer simulation:
Effects of dispersion and effective wavenumber/speed

waveeq.m

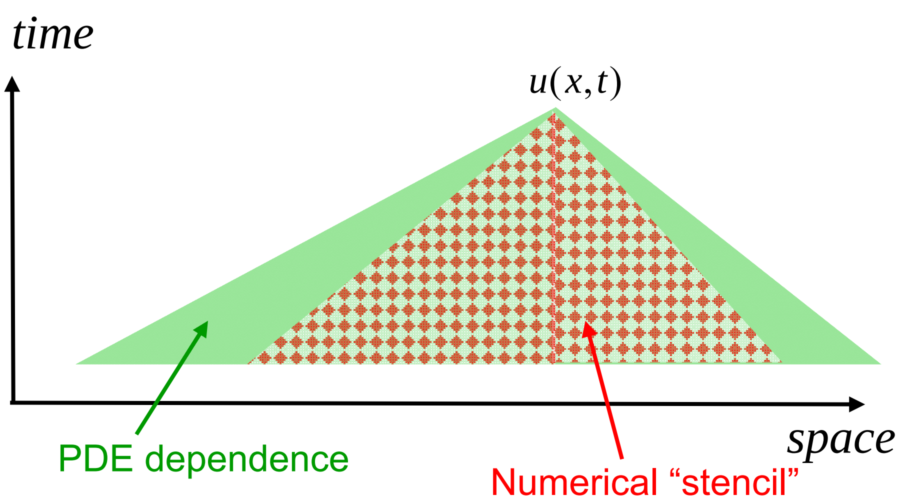
```
L=10;
T=10;
c=1.5;
N=100;
h=L/N;
M=400;
%  Test: increase duration of simulation, to see effect of
%dispersion and  effective wavenumber/speed (due to 2nd order)
%T=100;M=4000;
k=T/M;
C=c*k/h
Lf=0.5;

x=[0:h:L]';
t=[0:k:T];
%fx=['exp(-0.5*(' num2str(L/2) '-x).^2/(' num2str(Lf) ').^2)'];
%gx='0';
fx='exp(-0.5*(5-x).^2/0.5^2).*cos((x-5)*pi)';
gx='0';
f=inline(fx,'x');
g=inline(gx,'x');

n=length(x);
m=length(t);
u=zeros(n,m);
%Second order starter
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
 u(i,2) = (1-C^2)*u(i,1) + k*g(x(i)) +C^2*(u(i-1,1)+u(i+1,1))/2;
end

%CDS: Iteration in time (j) and space (i)
for j=2:m-1
   for i=2:n-1
u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
   end
end
```



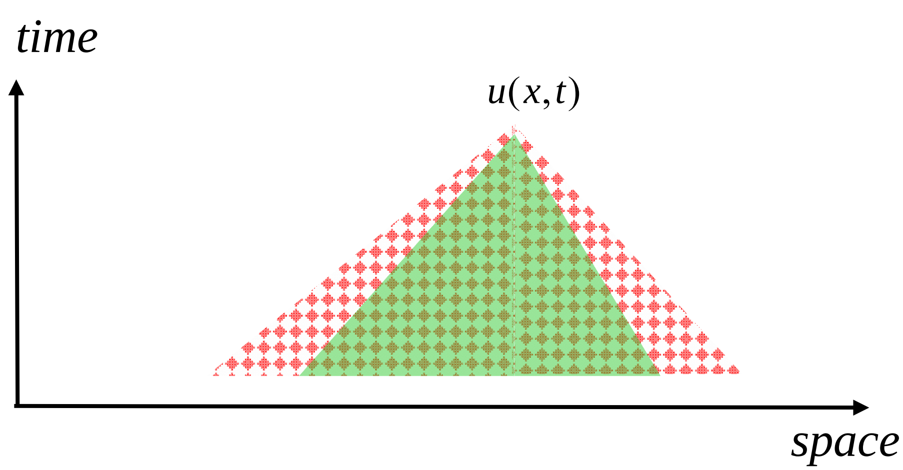
Courant-Fredrichs-Lewy Condition (1920's)

- Basic idea: the solution of the Finite-Difference (FD) equation can not be independent of the (past) information that determines the solution of the corresponding PDE
- In other words: "Numerical domain of dependence of FD scheme must include the mathematical domain of dependence of the corresponding PDE"

CFL NOT satisfied



CFL satisfied



CFL: Linear convection (Sommerfeld Eqn) Example

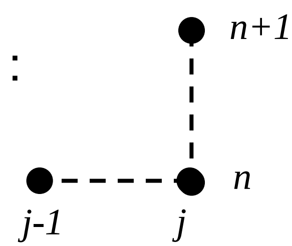
Determine domain of dependence of PDE and of FD scheme

- **PDE:** $\frac{\partial u(x,t)}{\partial t} + c\,\frac{\partial u(x,t)}{\partial x} = 0$ Characteristics: If $\frac{dx}{dt} = c \;\Rightarrow\; x = c\,t + \zeta$ and $du = 0 \;\Rightarrow u = \text{cst}$

  Solution of the form: $u(x,t) = F(x - ct)$

- **FD scheme.** For our Upwind discretization, with $t = n\Delta t,\; x = j\Delta x$:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c\,\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$



Slope of characteristic: $\frac{dt}{dx} = \frac{1}{c}$

Slope of Upwind scheme: $\frac{\Delta t}{\Delta x}$

=> CFL condition: $\frac{\Delta t}{\Delta x} \le \frac{1}{c}$

$$\boxed{\frac{c\,\Delta t}{\Delta x} \le 1}$$

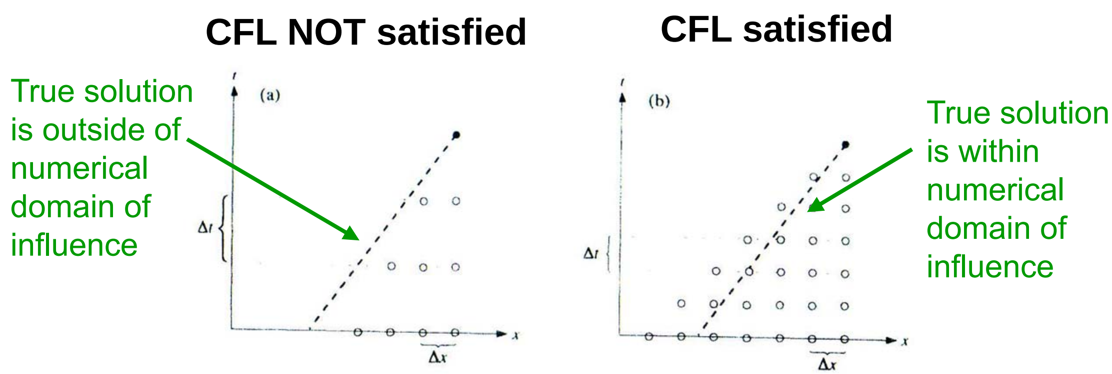


FIGURE 2.1. The influence of the time step on the relationship between the numerical domain of dependence of the upstream scheme (open circles) and the true domain of dependence of the advection equation (heavy dashed line): (a) unstable Δt, (b) stable Δt.

CFL: 2nd order Wave equation Example

Determine domain of dependence of PDE and of FD scheme

- PDE, second order wave eqn example:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

  - As seen before: $u(x,t) = F(x - ct) + G(x + ct) \;\Rightarrow$ slope of characteristics: $\frac{dt}{dx} = \pm \frac{1}{c}$

- FD scheme: discretize: $t = n\Delta t, \; x = j\Delta x$
  - CD scheme (CDS) in time and space (2nd order), explicit

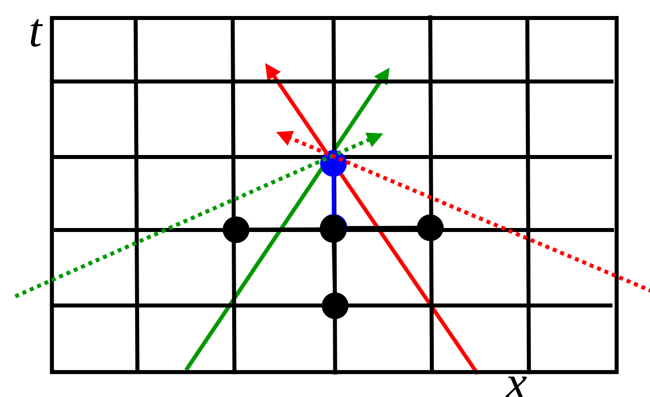
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \;\Rightarrow\; u_j^{n+1} = (2 - 2C^2)u_j^n + C^2(u_{j+1}^n + u_{j-1}^n) - u_j^{n-1} \quad \text{where } C = \frac{c\Delta t}{\Delta x}$$

  - We obtain from the respective slopes:

$$\frac{c\,\Delta t}{\Delta x} \le 1$$

Full line case: CFL satisfied

Dotted lines case:
c and Δt too big, Δx too small (CFL NOT satisfied)



CFL Condition: Some comments

- CFL is only a necessary condition for stability
- Other (sufficient) stability conditions are often more restrictive
  - For example: if $\frac{\partial u(x,t)}{\partial t} + c\,\frac{\partial u(x,t)}{\partial x} = 0$ is discretized as

$$\left(\frac{\partial u(x,t)}{\partial t}\right)_{\text{CD},2^{\text{nd}}\text{ order in } t} + c\left(\frac{\partial u(x,t)}{\partial x}\right)_{\text{CD},4^{\text{th}}\text{ order in x}} \approx 0$$

Five grid-points stencil: (-1,8,0,-8,1) / 12
See Taylor tables in eqn sheet

  - One obtains from the CFL: $\frac{c\,\Delta t}{\Delta x} \leq 2$
  - While a Von Neuman analysis leads: $\frac{c\,\Delta t}{\Delta x} \leq 0.728$

- For equations that are not purely hyperbolic or that can change of type (e.g. as diffusion term increases), CFL condition can at times be violated locally for a short time, without leading to global instability further in time

von Neumann Examples

- Forward in time (Euler), centered in space, Explicit

$$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}+c\,\frac{\phi_{j+1}^n-\phi_{j-1}^n}{2\Delta x}=0 \quad\Rightarrow\quad \phi_j^{n+1}=\phi_j^n-\frac{C}{2}(\phi_{j+1}^n-\phi_{j-1}^n)$$

  – Von Neumann: insert $\varepsilon(x,t)=\varepsilon_\beta(t)\,e^{i\beta x}=e^{\gamma t}e^{i\beta x}\Rightarrow\quad \varepsilon_j^n=e^{\gamma\, n\Delta t}e^{i\beta\, j\Delta x}$

$$\Rightarrow\quad \varepsilon_j^{n+1}=\varepsilon_j^n-\frac{C}{2}(\varepsilon_{j+1}^n-\varepsilon_{j-1}^n)\quad\Rightarrow\quad e^{\gamma\Delta t}=1-\frac{C}{2}\left(e^{i\beta\Delta x}-e^{-i\beta\Delta x}\right)=1-Ci\sin(\beta\Delta x)$$

  - Taking the norm:

$$\left|e^{\gamma t}\right|^2=\left|\xi\right|^2=\left(1-Ci\sin(\beta\Delta x)\right)\left(1+Ci\sin(\beta\Delta x)\right)=1+C^2\sin^2(\beta\Delta x)\quad\geq\ 1\ \text{ for }\ C\neq 0\ !$$

  - Unconditionally Unstable

- Implicit scheme (later)

Stability of FD schemes for $u_t + b\, u_y = 0$ (*t* denoted *x* below)

Table showing various finite difference forms removed due to copyright restrictions. Please see Table 6.1 in Lapidus, L., and G. Pinder. *Numerical Solution of Partial Differential Equations in Science and Engineering*. Wiley-Interscience, 1982.

(from Lecture 11)

Partial Differential Equations
Elliptic PDE

Laplace Operator $\nabla^2 \equiv u_{xx} + u_{yy}$

Examples:

$$\nabla^2 u = 0$$

Laplace Equation – Potential Flow

$$\nabla^2 u = g(x,y)$$

Poisson Equation
- Potential Flow with sources
- Heat flow in plate

$$\nabla^2 u + f(x,y)u = 0$$

Helmholtz equation – Vibration of plates

$$\mathbf{U} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u}$$

Convection-Diffusion

- Smooth solutions ("diffusion effect")
- Very often, steady state problems
- Domain of dependence of u is the full domain **D**(x,y) => "global" solutions
- Finite difference, finite elements, boundary integral methods (Panel methods)

(from Lecture 11)

Partial Differential Equations
Elliptic PDEs

$$0 \le x \le a\ ,\ \ 0 \le y \le b;$$

Equidistant Sampling

$$\begin{aligned} h &= a/(n-1) \\ h &= b/(m-1) \end{aligned}$$

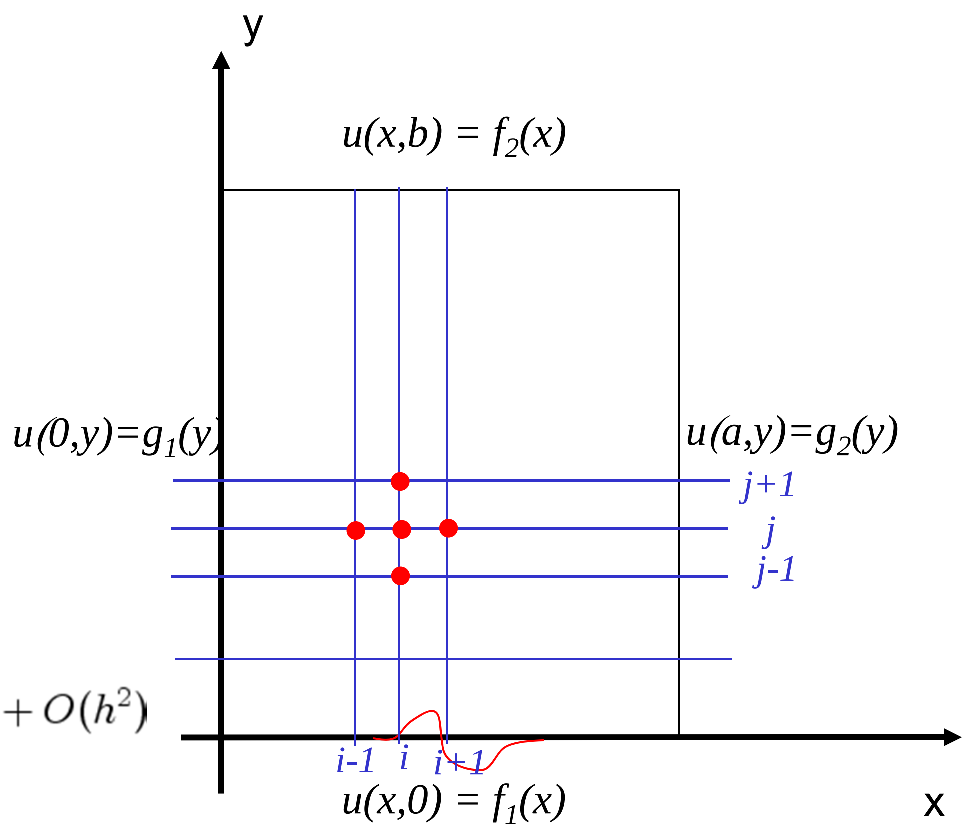
Discretization

$$\begin{aligned} x_i &= (i-1)h,\ i = 1, \ldots, n \\ y_j &= (j-1)h,\ j = 1, \ldots, m \end{aligned}$$

Finite Differences

$$u_{xx}(x,t) = \frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)}{h^2} + O(h^2)$$

$$u_{yy}(x,t) = \frac{u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})}{h^2} + O(h^2)$$



Dirichlet BC

(from Lecture 11)

Partial Differential Equations
Elliptic PDE

Discretized Laplace Equation

$$\nabla^2 u = \frac{u(x_{i-1},y_j) + u(x_i,y_{j-1}) - 4u(x_i,y_j) + u(x_{i+1},y_j) + u(x_i,y_{j+1})}{h^2} = 0$$

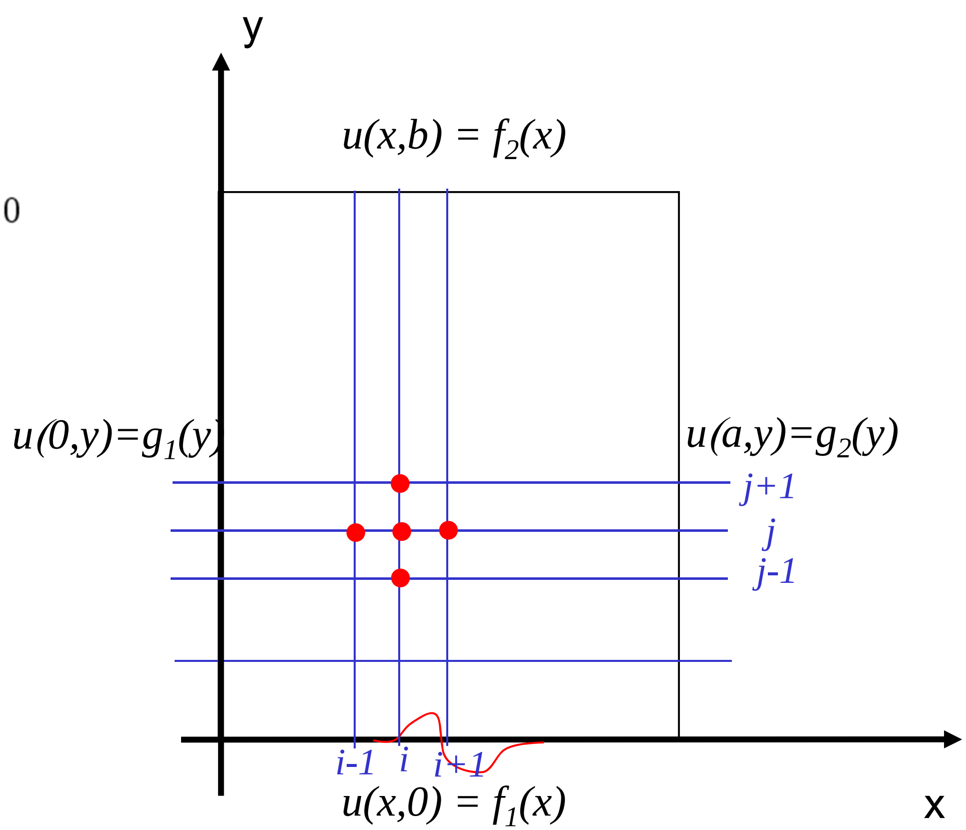
$$u_{i,j} = u(x_i,t_j)$$

Finite Difference Scheme

$$u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0$$

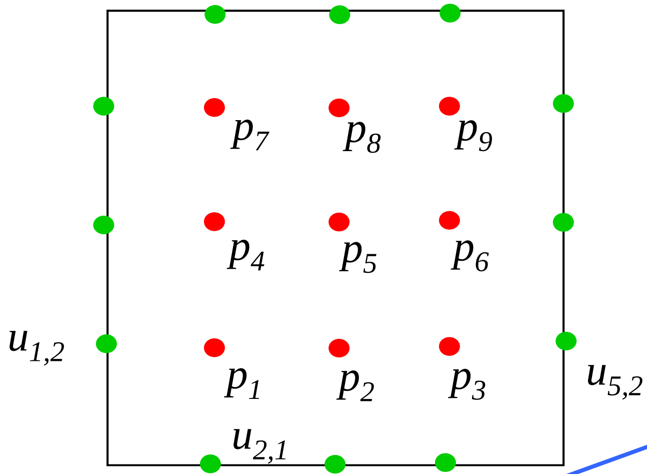
Boundary Conditions

$$\begin{aligned} u(x_1,y_j) &= u_{1,j}\ ,\ 2 \le j \le m-1 \\ u(x_n,y_j) &= u_{n,j}\ ,\ 2 \le j \le m-1 \\ u(x_i,y_1) &= u_{i,1}\ ,\ 2 \le j \le n-1 \\ u(x_i,y_n) &= u_{i,n}\ ,\ 2 \le j \le n-1 \end{aligned}$$



Global Solution Required

Elliptic PDEs
Laplace Equation, Global Solvers



Dirichlet BC

Leads to $\mathbf{A}\mathbf{x} = \mathbf{b}$, with $\mathbf{A}$ block-tridiagonal:

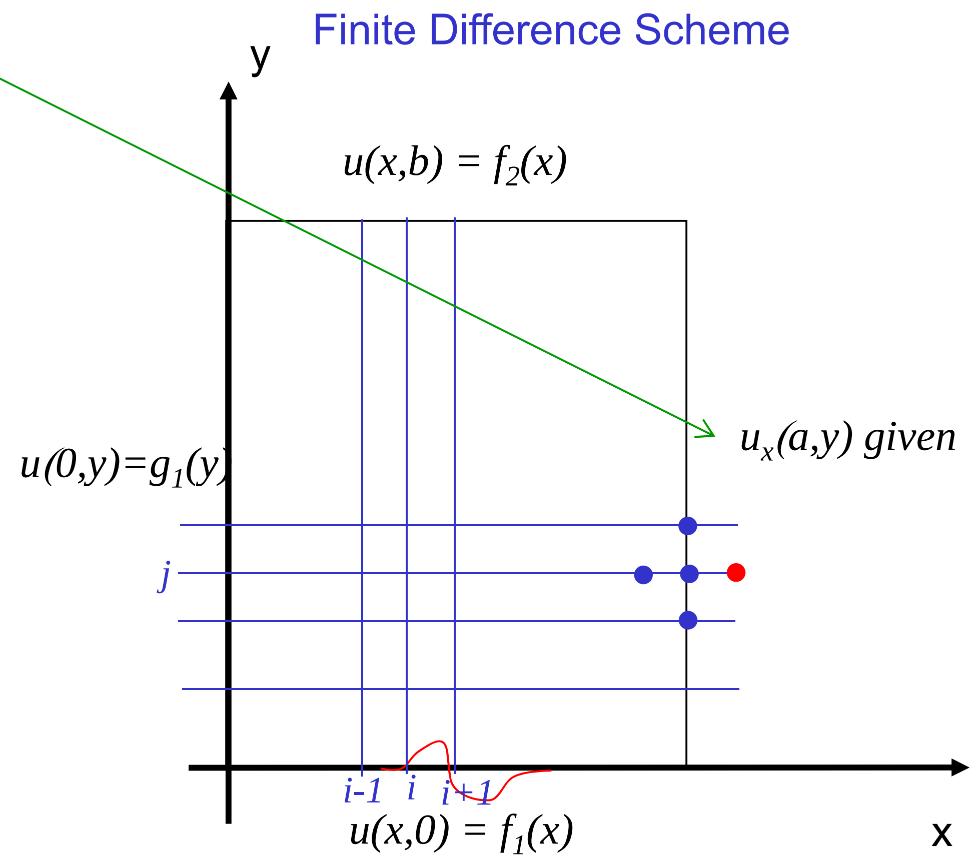
$\mathbf{A} = \text{tri}\ \{\ \mathbf{I}\ ,\ \mathbf{T}\ ,\ \mathbf{I}\ \}$

$$
\begin{array}{rcl}
-4p_1 + p_2 + p_4 &=& -u_{2,1} - u_{1.2} \\
p_1 - 4p_2 + p_3 + p_5 &=& -u_{3,1} \\
p_2 - 4p_3 + p_6 &=& -u_{4,1} - u_{5,2} \\
p_1 - 4p_4 + p_5 + p_7 &=& -u_{1,3} \\
p_2 + p_4 - 4p_5 + p_6 + p_8 &=& 0 \\
p_3 + p_5 - 4p_6 + p_9 &=& -u_{5,3} \\
p_4 - 4p_7 + p_8 &=& -u_{2,5} - u_{1,4} \\
p_5 + p_7 - 4p_8 + p_9 &=& -u_{3,5} \\
p_6 + p_8 - 4p_9 &=& -u_{4,5} - u_{5,4}
\end{array}
$$

Ellipticic PDEs
Neumann Boundary Conditions

Neumann (Derivative) Boundary Condition

$$\frac{\partial}{\partial N} u(x,y) \quad \text{given}$$



Finite Difference Scheme

$$u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Derivative Finite Difference at BC

$$\frac{u_{n+1,j} - u_{n-1,j}}{2h} \simeq u_x(x_n, y_j)$$

$$\boxed{u_{n+1,j}} = u_{n-1,j} + 2h u_x(x_n, y_j)$$

Boundary Finite Difference Scheme

$$\underbrace{u_{n-1,j} + 2\Delta x \left.\frac{\partial u}{\partial x}\right|_n + u_{n-1,j}} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Leads to a factor 2 (a matrix $2\,\mathbf{I}$ in $\mathbf{A}$) for points along boundary

Elliptic PDEs
Iterative Schemes: Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Finite Difference Scheme

$$u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^{k+1} = 0$$

$$u_{i,j}^{k+1} = \frac{u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k}{4}$$

Liebman Iterative Scheme (Jacobi/Gauss-Seidel)

$$u_{i,j}^{k+1} = u_{i,j}^k + r_{i,j}^k$$

$$r_{i,j} = r_{i,j}^k = \frac{u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k}{4}$$

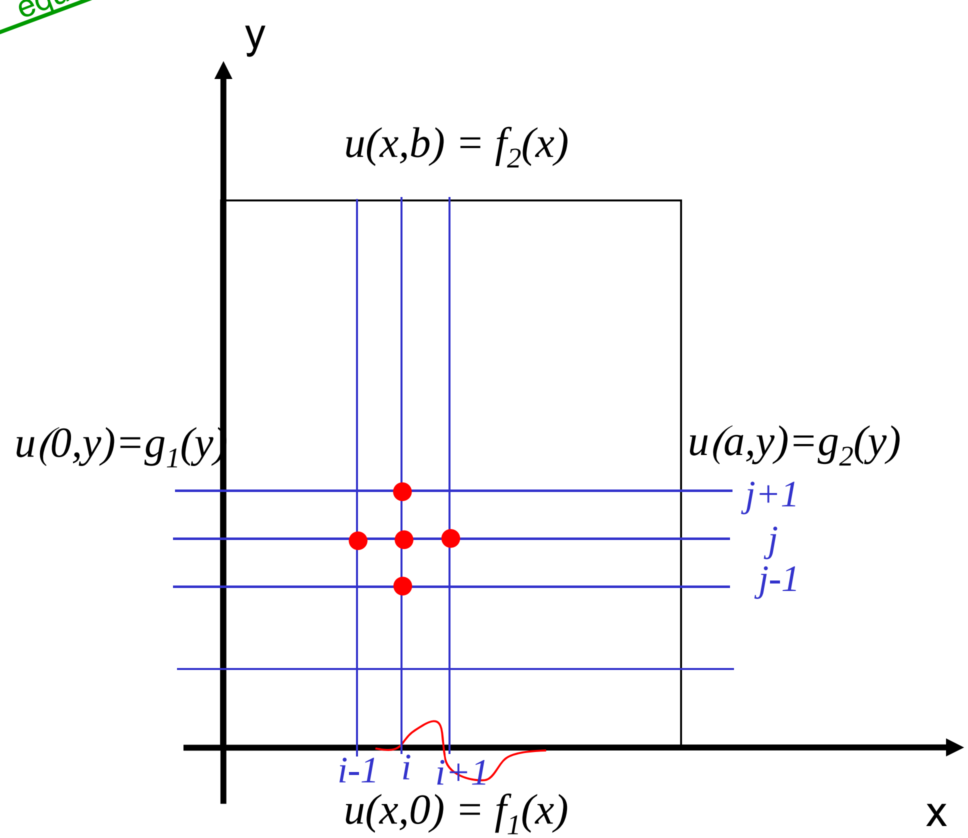
equivalent

SOR Iterative Scheme, Jacobi

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^k + \omega r_{i,j}^k \\ &= u_{i,j}^k + \omega \frac{u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k}{4} \\ &= (1-\omega)\, u_{i,j}^k + \omega \frac{u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k}{4} \end{aligned}$$

Optimal SOR

$$\omega = \frac{4}{2+\sqrt{4-\left[\cos\left(\frac{\pi}{n-1}\right)+\cos\left(\frac{\pi}{m-1}\right)\right]^2}}$$



Elliptic PDE: Poisson Equation

$$\nabla^2 u = g(x, y)$$

$$g_{i,j} = g(x_i, y_j)$$

SOR Iterative Scheme, with Jacobi

$$u_{i,j}^{k+1} = u_{i,j}^k + \omega r_{i,j}^k$$

$$= u_{i,j}^k + \omega \frac{u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k - h^2 g_{i,j}}{4}$$

$$= (1-\omega)\, u_{i,j}^k + \omega \frac{u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - h^2 g_{i,j}}{4}$$

Elliptic PDE: Poisson Equation

$$\nabla^2 u = g(x, y)$$

$$g_{i,j} = g(x_i, y_j)$$

SOR Iterative Scheme, with Gauss-Seidel

$$u_{i,j}^{k+1} = u_{i,j}^{k} + \omega r_{i,j}^{k}$$

$$= u_{i,j}^{k} + \omega \frac{u_{i+1,j}^{k} + u_{i-1,j}^{k+1} + u_{i,j+1}^{k} + u_{i,j-1}^{k+1} - 4u_{i,j}^{k} - h^2 g_{i,j}}{4}$$

$$= (1-\omega)\, u_{i,j}^{k} + \omega \frac{u_{i+1,j}^{k} + u_{i-1,j}^{k+1} + u_{i,j+1}^{k} + u_{i,j-1}^{k+1} - h^2 g_{i,j}}{4}$$

Laplace Equation
Steady Heat diffusion (with source: Poisson eqn)

duct.m

```
Lx=1;
Ly=1;
N=10;
h=Lx/N;
M=floor(Ly/Lx*N);
niter=20;
eps=1e-6;

x=[0:h:Lx]';
y=[0:h:Ly];
f1x='4*x-4*x.^2';
%f1x='0'
f2x='0';
g1x='0';
g2x='0';
vxy='0';
f1=inline(f1x,'x');
f2=inline(f2x,'x');
g1=inline(g1x,'y');
g2=inline(g2x,'y');
vf=inline(vxy,'x','y');

n=length(x);
m=length(y);
u=zeros(n,m);
u(2:n-1,1)=f1(x(2:n-1));
u(2:n-1,m)=f2(x(2:n-1));
u(1,1:m)=g1(y);
u(n,1:m)=g2(y);
for i=1:n
    for j=1:m
        v(i,j) = vf(x(i),y(j));
    end
end
```

```
u_0=mean(u(1,:))+mean(u(n,:))+mean(u(:,1))+mean(u(:,m));
u(2:n-1,2:m-1)=u_0*ones(n-2,m-2);
omega=4/(2+sqrt(4-(cos(pi/(n-1))+cos(pi/(m-1)))^2))
for k=1:niter
    u_old=u;
    for i=2:n-1
        for j=2:m-1
            u(i,j)=(1-omega)*u(i,j)
 +omega*(u(i-1,j)+u(i+1,j)+u(i,j-1)+u(i,j+1)-h^2*v(i,j))/4;
        end
    end
    r=abs(u-u_old)/max(max(abs(u)));
    k,r
    if (max(max(r))<eps)
        break;
    end
end
figure(3)
surf(y,x,u);
shading interp;
a=ylabel('x');
set(a,'Fontsize',14);
a=xlabel('y');
set(a,'Fontsize',14);
a=title(['Poisson Equation - v = ' vxy]);
set(a,'Fontsize',16);
```

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=g(x,y)$$

BCs: $u(x,0,t)=f(x)=4x-4x^2$

Three other BCs are null

Helmholtz Equation

$$\nabla^2 u + f(x,y)u = g(x,y)$$

$$f_{i,j} = f(x_i, y_j)$$
$$g_{i,j} = g(x_i, y_j)$$

SOR Iterative Scheme

$$u_{i,j}^{k+1} = u_{i,j}^k + \omega r_{i,j}^k$$

$$= u_{i,j}^k + \omega \frac{u_{i+1,j}^k + u_{i-1,j}^{k+1} + u_{i,j+1}^k + u_{i,j-1}^{k+1} - (4 - \underline{h^2 f_{i,j}})u_{i,j}^k - h^2 g_{i,j}}{(4 - h^2 f_{i,j})}$$

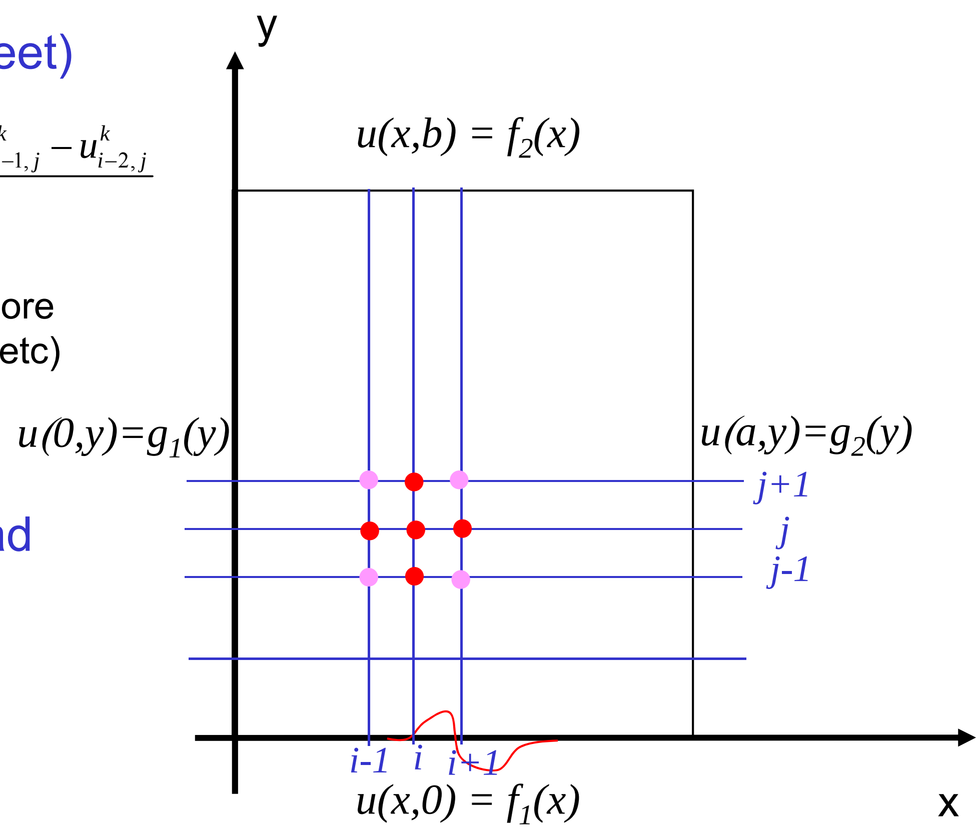
$$= (1-\omega)\, u_{i,j}^k + \omega \frac{u_{i+1,j}^k + u_{i-1,j}^{k+1} + u_{i,j+1}^k + u_{i,j-1}^{k+1} - h^2 g_{i,j}}{(4 - h^2 f_{i,j})}$$

Elliptic PDE's
Higher Order Finite Differences

CD, 4th order (see tables eqn sheet)

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{\text{CD, 4}^{\text{th}}\text{ order}} = \frac{-u_{i+2,j}^k + 16\,u_{i+1,j}^k + 30\,u_{i,j}^k + 16\,u_{i-1,j}^k - u_{i-2,j}^k}{12h^2}$$

The resulting 9 point "cross" stencil is more challenging computationally (boundary, etc)



Use more compact scheme instead

Square stencil (see figure):

- Use Taylor series, then cancel the terms so as to get a 4th order scheme
- Leads to:

$$\nabla^2 u_{i,j} = \frac{1}{6h^2}[u_{i+1,j-1} + u_{i-1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}$$
$$+4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} - 20u_{i,j}] + O(h^4)$$

2.29 Numerical Fluid Mechanics
Fall 2011