2.29 Numerical Fluid Mechanics
Fall 2011 – Lecture 18

REVIEW Lecture 17:

- Stability (Heuristic, Energy and von Neumann)
- Hyperbolic PDEs and Stability, CFL condition, Examples
- **Elliptic PDEs**
  - FD schemes: direct and iterative
  - Iterative schemes, 2D: Laplace, Poisson and Helmholtz equations
  - Boundary conditions, Examples
  - Higher order finite differences
  - Irregular boundaries: Dirichlet and von Neumann BCs
  - Internal boundaries
- **Parabolic PDEs and Stability**
  - Explicit schemes
    - Von Neumann
  - Implicit schemes: simple and Crank-Nicholson
    - Von Neumann
  - Examples

2.29 Numerical Fluid Mechanics
Fall 2011 – Lecture 18

REVIEW Lecture 17, Cont'd:

- Parabolic PDEs and Stability, Cont'd
  - Explicit schemes (1D-space)
    - Von Neumann
  - Implicit schemes (1D-space): simple and Crank-Nicholson
    - Von Neumann
  - Examples
  - Extensions to 2D and 3D
    - Explicit and Implicit schemes
    - Alternating-Direction Implicit (ADI) schemes

Finite Volume Methods

- Integral and conservative forms of the cons. laws
- Introduction

$$\frac{d}{dt}\int_{CV_{\text{fixed}}} \rho\phi dV + \underbrace{\int_{CS} \rho\phi\,(\vec{v}.\vec{n})dA}_{\text{Advective fluxes}} = \underbrace{-\int_{CS} \vec{q}_{\phi}.\vec{n}\; dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum\int_{CV_{\text{fixed}}} s_{\phi}\; dV}_{\text{Sum of sources and sinks terms (reactions, etc)}}$$

$$\frac{\partial\rho\phi}{\partial t} + \nabla.(\rho\phi\vec{v}) = -\nabla.\,\vec{q}_{\phi} + s_{\phi}$$

TODAY (Lecture 18): FINITE VOLUME METHODS

- Introduction to FV Methods
- Approximations needed and basic elements of a FV scheme
  - FV grids
  - Approximation of surface integrals (leading to symbolic formulas)
  - Approximation of volume integrals (leading to symbolic formulas)
- Summary: Steps to step-up FV scheme
- Examples: One Dimensional examples
  - Generic equations
  - Linear Convection (Sommerfeld eqn.): convective fluxes
    - 2^{nd} order in space, 4^{th} order in space, links to CDS
  - Unsteady Diffusion equation: diffusive fluxes
    - Two approaches for 2^{nd} order in space, links to CDS

References and Reading Assignments

- Chapter 29.4 on "The control-Volume approach for Elliptic equations" of "Chapra and Canale, Numerical Methods for Engineers, 2010/2006."
- Chapter 4 on "Finite Volume Methods" of "J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd edition, 2002"
- Chapter 5 on "Finite Volume Methods" of "H. Lomax, T. H. Pulliam, D.W. Zingg, *Fundamentals of Computational Fluid Dynamics (Scientific Computation).* Springer, 2003"
- Chapter 5.6 on "Finite-Volume Methods" of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, Computational Fluid Dynamics for Engineers. Springer, 2005.

FINITE VOLUME METHODS: Introduction

- Finite Difference Methods are based on a discretization of the differential form of the conservation equations
- Finite Volume Methods are based on a discretization of the integral forms of the conservation equations:

$$\frac{d}{dt}\int_{CV} \rho\phi dV + \underbrace{\int_{CS} \rho\phi\,(\vec{v}.\vec{n})dA}_{\text{Advective (convective) fluxes}} = \underbrace{-\int_{CS} \vec{q}_{\phi}.\vec{n}\; dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum\int_{CV} s_{\phi}\; dV}_{\text{Sum of sources and sinks terms (reactions, etc)}}$$

- Basic ideas/steps to set-up a FV scheme:
  - Grid generation (CVs):
    - Divide the simulation domain into a set of discrete control volumes (CVs)
    - For maintenance of conservation, important that CVs don't overlap
  - Discretize the integral/conservation equation on CVs:
    - Satisfy the integral form of the conservation law to some degree of approximation for each of the many contiguous control volumes
  - Solve the resultant discrete integral/flux equations

FV METHODS: Introduction

- FV approach has two main advantages:
  - Ensures that the discretization is conservative, locally and globally
    - Mass, Momentum and Energy are conserved in a discrete sense
    - In general, if discrete equations are summed over all CVs, the global conservation equation are retrieved (surface integrals cancel out)
    - These local/global conservations can be obtained from a Finite Difference (FD) formulation, but they are natural/direct for a FV formulation
  - Does not require a coordinate transformation to be applied to irregular meshes
    - Can be applied to unstructured meshes (arbitrary polyhedra in 3D or polygons in 2D)
- In our examples, we will work with

$$\frac{d}{dt}\int_{V(t)} \rho\phi dV + \int_{S(t)} \rho\phi\,(\vec{v}.\vec{n})dA = -\int_{S(t)} \vec{q}_{\phi}.\vec{n}\; dA + \int_{V(t)} s_{\phi}\; dV$$

where $V(t)$ is any discrete control volume. We will assume for now that they don't vary in time: $V(t)=V$

FV METHODS
Several Approximations Needed

- To integrate discrete CV equation:

$$\frac{d}{dt}\int_V \rho\phi dV + \int_S \rho\phi\,(\vec{v}.\vec{n})dA = -\int_S \vec{q}_\phi.\vec{n}\;dA + \int_V s_\phi\;dV$$

  – A "time-marching method" needs to be used to integrate $\Phi = \int_V \rho\phi dV$ to the next time step(s)

$$\frac{d}{dt}\int_V \rho\phi dV = \frac{d\Phi}{dt}$$

  – Total flux estimate F_ϕ required at the boundary of each CV

$$\int_S \vec{F}_\phi.\vec{n}\;dA = \int_S \rho\phi\,(\vec{v}.\vec{n})dA + \int_S \vec{q}_\phi.\vec{n}\;dA$$

e.g. F_ϕ= advection + diffusion fluxes

  – Total source term (sum of sources) must be integrated over each CV

$$S_\phi = \int_V s_\phi\;dV$$

    - Hence cons. eqn. becomes: $\frac{d\Phi}{dt} + \int_S \vec{F}_\phi.\vec{n}\;dA = S_\phi$

- These needs lead to basic elements of a FV scheme, but we need to relate Φ and ϕ

FV METHODS
Several Approximations Needed, Cont'd

- "Time-marching method" for CV equation: $\dfrac{d\Phi}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi$
  – The average of ϕ over a CV cell, $\overline{\Phi} = \dfrac{1}{V}\int_V \rho\phi dV$, satisfies

$$\boxed{V\frac{d\overline{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi}$$

(since $\dfrac{d}{dt}\int_V \rho\phi dV = \dfrac{d}{dt}(V\dfrac{1}{V}\int_V \rho\phi dV)$ for V fixed in time.)

  – Hence, after discrete time-integration, we would have updated the cell-averaged quantities $\overline{\Phi}$
- For the total flux estimate F_ϕ at CV boundary: "Reconstruction" of ϕ from $\overline{\Phi}$
  – Fluxes are functions of ϕ => to evaluate them, we need to represent ϕ within the cell
  – This can be done by a piece-wise approximation which, when averaged over the CV, gives back $\overline{\Phi}$
  – But, each cell has a different piece-wise approximation => fluxes at boundaries can be discontinuous. Two example of remedies:
    - Take the average of these fluxes (this is a non-dissipative scheme, analogous to central differences)
    - Flux-difference splitting

FV METHODS
Basic Elements of FV Scheme

1. Given $\bar{\Phi}$ for each CV, construct an approximation to $\phi(x, y, z)$ in each CV and evaluate fluxes F_ϕ
   - Find ϕ at the boundary using this approximation, evaluate fluxes F_ϕ
   - This generally leads to two distinct values of the flux for each boundary
2. Apply some strategy to resolve the flux discontinuity at the CV boundary to produce a single F_ϕ over the whole boundary
3. Integrate the flux F_ϕ to obtain $\int_S \vec{F}_\phi . \vec{n}\, dA$: $\boxed{\text{Surface Integrals}}$
4. Compute S_ϕ by integration over each CV: $\boxed{\text{Volume Integrals}}$
5. Advance the solution in time to obtain the new values of $\bar{\Phi}$

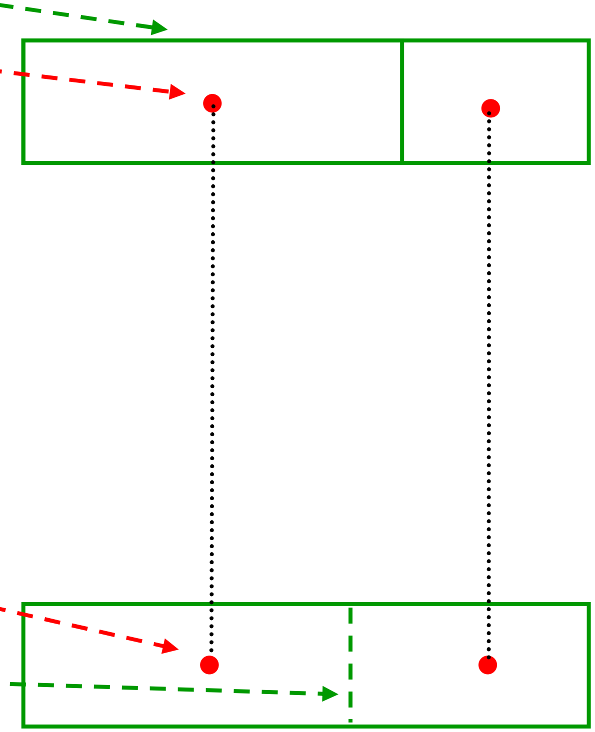
$$V \frac{d\bar{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi$$

Time-Marching

Different Types of FV Grids

- Usual approach (used here):
  - Define CVs by a suitable grid
  - Assign computational node to CV center
  - Advantages: nodal values will represent the mean over the CV at high(er) accuracy (second order) since node is centroid of CV

Node Centered



CV-Faces Centered

- Other approach:
  - Define nodal locations first
  - Construct CVs around them (so that CV faces lie midway between nodes
  - Advantage: CDS approximations of derivatives (fluxes) at boundaries are more accurate (faces are midway between two nodes)

Different Types of FV Grids, Cont'd

- Other specialized variants
  - Cell centered vs. Cell vertex
  - Structured:
    - All mesh points lie on intersection two/three lines
  - vs. Unstructured:
    - Meshes formed of triangular or quadrilateral cells in 2D, or tetrahedra or pyramids in 3D
    - Cells are identified by their numbers (can not be indentified by coordinate lines, e.g. $i.j$)

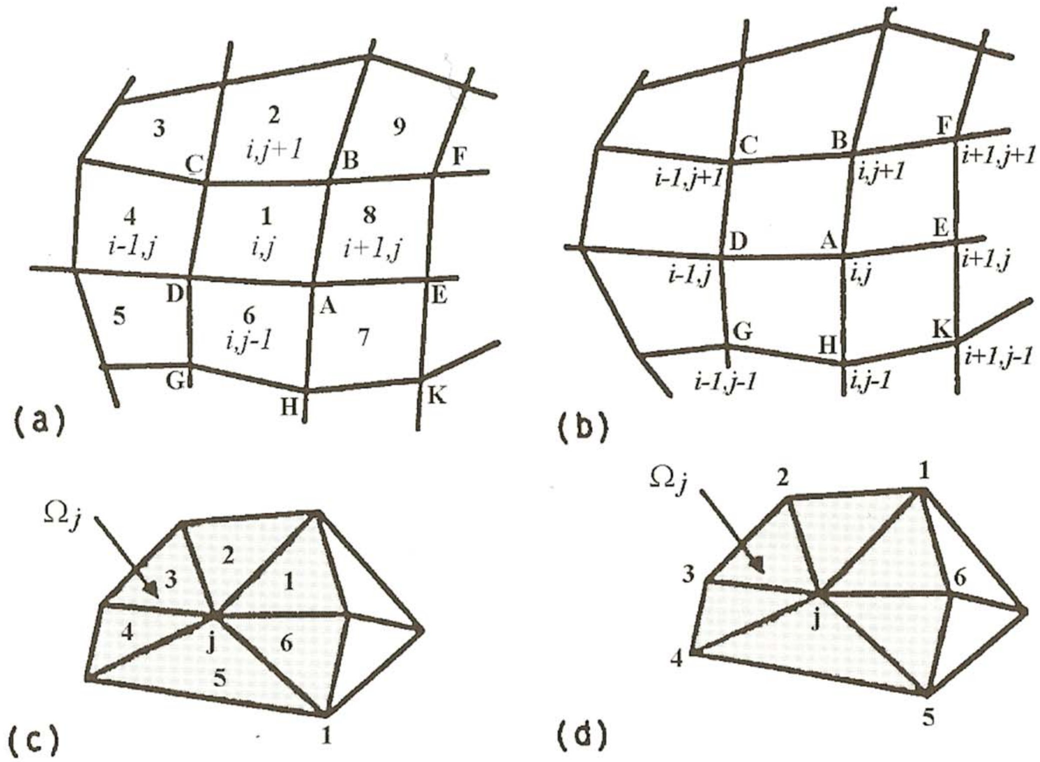


Fig. 5.2. Two-dimensional finite-volume mesh systems. (a) Cell centered structured finite-volume mesh; (b) cell vertex structured finite-volume mesh; (c) cell centered unstructured finite-volume mesh; (d) cell vertex unstructured finite-volume mesh.

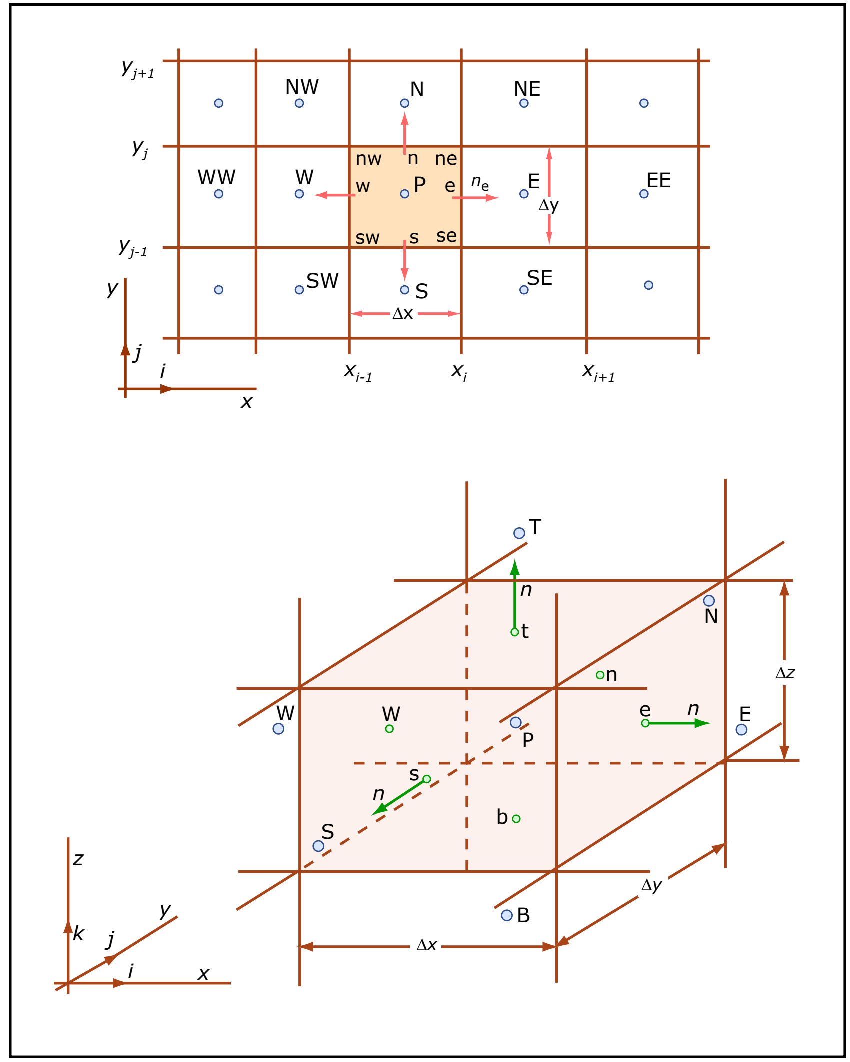
- Remarks
  - Discretization principles the same for all grid variants
    - => For now, we work with (a): Cell centered (i,j is the center of the cell, similar to FD)
    - In 3D, a cell has a finite volume (but if unit distance perpendicular to plane is assumed, it behaves as 2D)
  - What changes are the relations between various locations on the grid and accuracies

Approximation of Surface Integrals

- Typical (cell centered) 2D and 3D Cartesian CV (see conventions on 2 figs)
- Total/Net flux through CV boundary
  - Is sum of integrals over four (2D) or six (3D) faces:

$$\int_S \vec{F}_\phi . \vec{n}\, dA = \sum_k \int_{S_k} f_\phi\, dA$$

  - For now, we will consider a single typical CV surface, the one labeled 'e'
- To compute surface integral, ϕ is needed everywhere on surface, but $\bar{\Phi}$ only known at nodal (CV center) values => two successive approximations needed:
  - Integral estimated based on values at one or more locations on the cell face
  - These cell faces values approximated in terms of nodal values



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

Ferziger & Peric, 2002

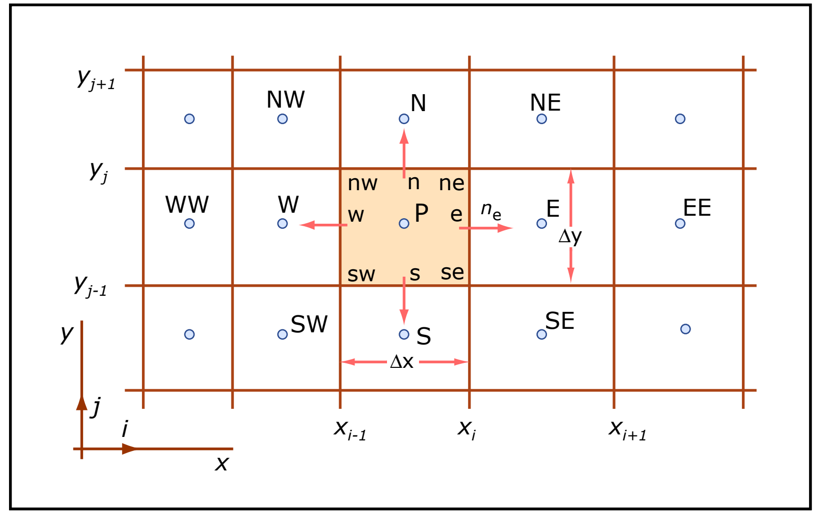
Approximation of Surface Integrals, Cont'd

1D surfaces (2D CV)

- Goal: estimate $F_e = \int_{S_e} f_\phi \, dA$
- Simplest approximation:

  midpoint rule (2nd order)

  – F_e is approximated as a product of the integrand at cell-face center (itself approximation of mean value over surface) and the cell-face area



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

$$F_e = \int_{S_e} f_\phi \, dA = \overline{f}_e S_e = f_e S_e + O(\Delta y^2) \approx f_e S_e$$

$$\left(f(y) = f(y_e) + \xi \, f'(y_e) + \frac{\xi^2}{2!} f''(y_e) + R_2 \right) \quad \xi = y - y_e$$

  – Since f_e is not available, it has to be obtained by interpolation

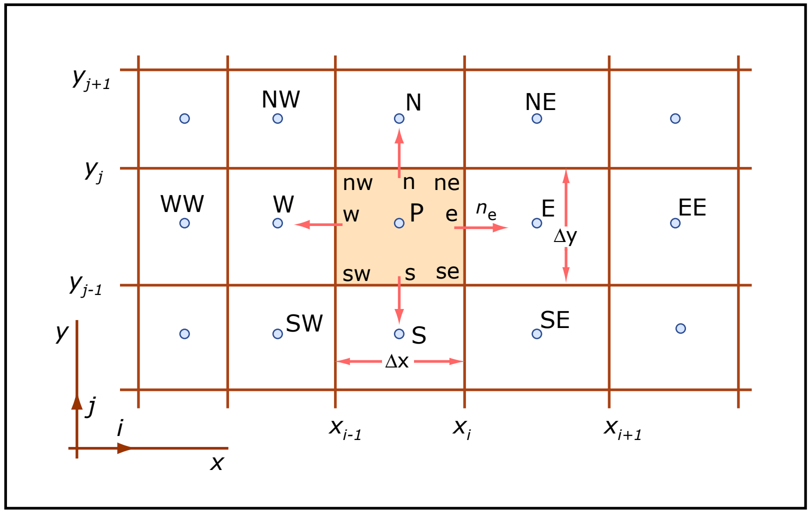
    - Has to be computed with 2nd order accuracy to preserve accuracy of midpoint rule

Approximation of Surface Integrals, Cont'd

- Goal: estimate $F_e = \int_{S_e} f_\phi \, dA$

- Another 2nd order approximation: Trapezoid rule
  - F_e is approximated as:

$$F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + f_{se})}{2} + O(\Delta y^2)$$



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

  - In this case, it is the fluxes at the corners f_{ne} and f_{se} that need to be obtained by interpolation
    - Have to be computed with 2nd order accuracy to preserve accuracy

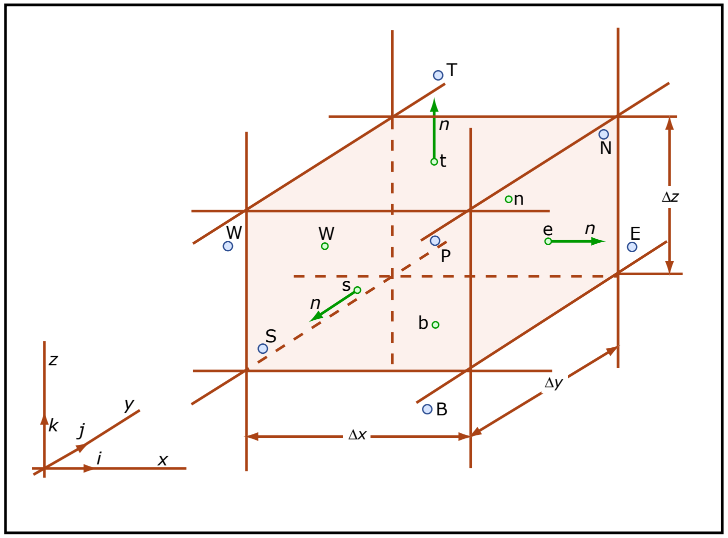
- Higher-order approximation of surface integrals require more than 2 locations
  - Simpson's rule (4th order approximation): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + 4 f_e + f_{se})}{6} + O(\Delta y^4)$
  - Values needed at 3 locations
  - To keep accuracy of integral: e.g. use cubic polynomials to estimate these values from $\bar{\Phi}_P$'s nearby

Approximation of Surface Integrals, Cont'd
2D surface (for 3D problems)

- Goal: estimate $F_e = \int_{S_e} f_\phi \, dA$ for 3D CV

- Simplest approximation: still the midpoint rule (2nd order)
  - F_e is approximated as:

$$\boxed{F_e = \int_{S_e} f_\phi \, dA \approx S_e f_e \quad + O(\Delta y^2, \Delta z^2)}$$



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

- Higher-order approximation (require values elsewhere e.g. at vertices) possible but more complicated to implement for 3D CV
- Integration easy if variation of f_e over 2D surface is assumed to have specific easy shape to integrate, e.g. 2D polynomial interpolation, then integration

Approximation of VOLUME Integrals

- Goal: estimate

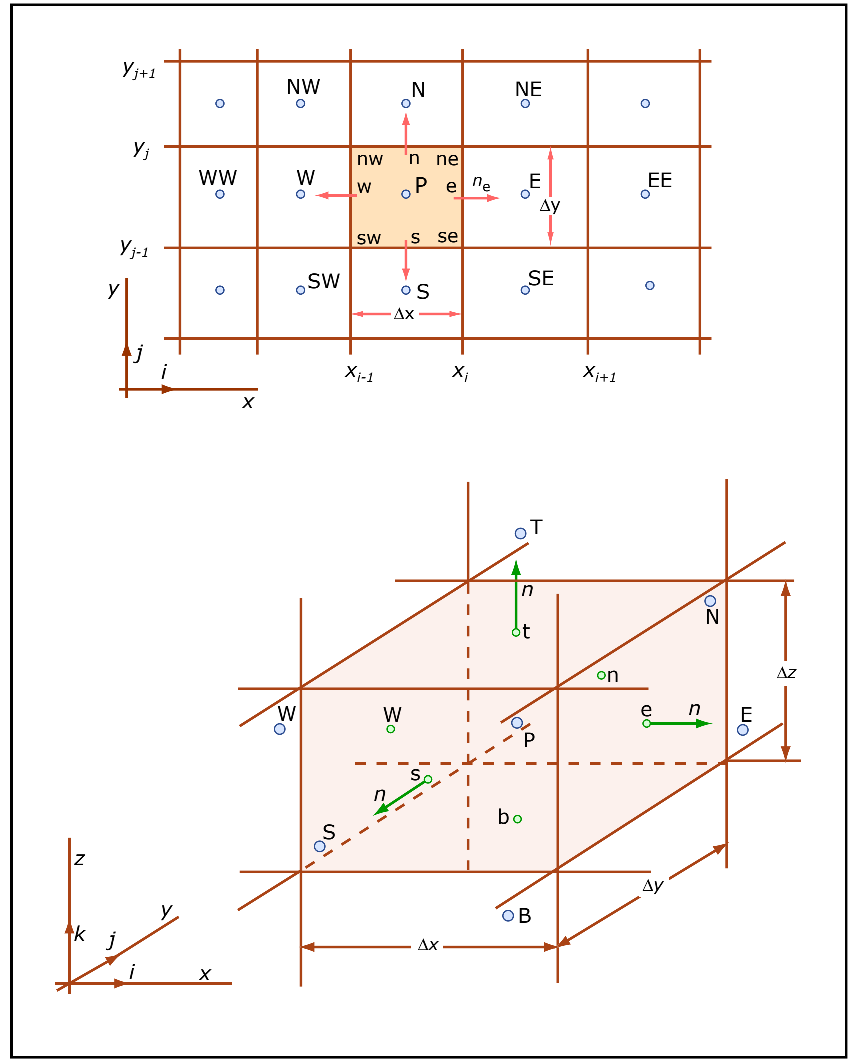
$$S_\phi = \int_V s_\phi \, dV$$

$$\bar{\Phi} = \frac{1}{V} \int_V \rho \phi dV$$

- Simplest approximation: product of CV volume with the mean value of the integrand (approximated by the value at the center of the node P)
  - S_P approximated as:

$$\boxed{S_P = \int_V s_\phi \, dV = \overline{s}_P \, V \approx s_P \, V}$$

- Exact if s_p is constant or linear within CV
- 2nd order accurate otherwise
- Higher order approximation require more locations than just the center

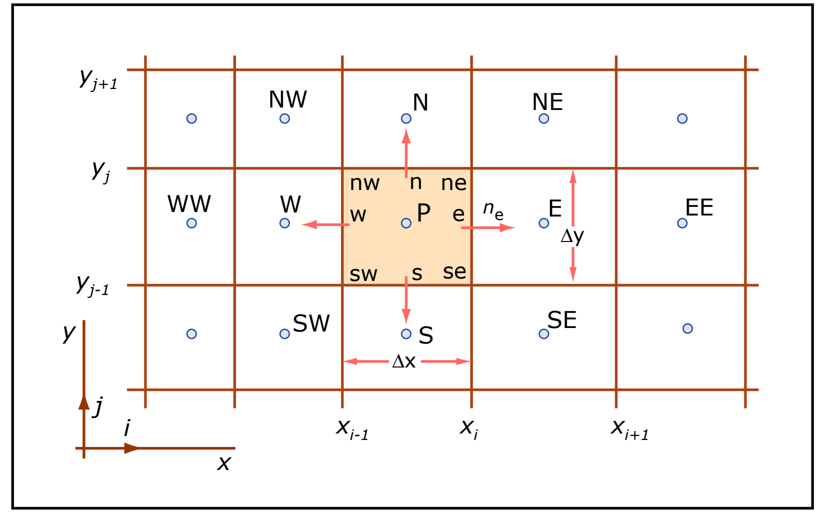


Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

Approximation of VOLUME Integrals

- Goal: estimate $$S_\phi = \int_V s_\phi \, dV$$ $$\bar{\Phi} = \frac{1}{V}\int_V \rho\phi dV$$

- Higher order approximations:
  - Requires $\bar{\Phi}$ values at other locations than P
  - Obtained either by interpolating nodal values or by using shape functions/ polynomials



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

- Consider 2D case (volume integral is a surface integral) using shape functions
  - Bi-quadratic shape function leads to a 4th order approximation (9 coefficients)

$$s(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy + a_6 x^2 y + a_7 xy^2 + a_8 x^2 y^2$$

  - 9 coefficients obtained by fitting $s(x,y)$ to 9 node locations (center, corners, middles)
  - For Cartesian grid, this gives:

$$S_P = \int_V s_\phi \, dV = \Delta x \, \Delta y \left[a_0 + \frac{a_3}{12}\Delta x^2 + \frac{a_4}{12}\Delta y^2 + \frac{a_8}{144}\Delta x^2 \, \Delta y^2 \right]$$

Only four coefficients (linear dependences cancel), but they still depend on the 9 nodal values

Approximation of **VOLUME** Integrals, Cont'd 2D and 3D

- 2D case example, Cont'd
  - For a uniform Cartesian grid, one obtains the 2D integral as a function of the 9 nodal values:

$$S_P = \int_V s_\phi \, dV = \frac{\Delta x \, \Delta y}{36}\left[16 s_P + 4 s_s + 4 s_n + 4 s_w + 4 s_e + s_{se} + s_{sw} + s_{ne} + s_{nw}\right]$$

  - Since only value at node P is available, one must interpolate to obtain values at surface locations
  - Has to be at least 4th order accurate interpolation to retain order of integral approximation
- 3D case:
  - Techniques are similar to 2D case: above 4th order approx directly extended
  - For Higher Order
    - Integral approximation formulas are more complex
    - Interpolation of node values are more complex

Approx. of Surface/Volume Integrals: Classic symbolic formulas

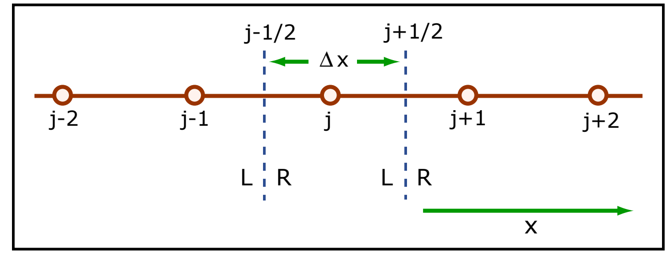
- Surface Integrals $F_e = \int_{S_e} f_\phi \, dA$
  - 2D problems (1D surface integrals)
    - Midpoint rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA = \overline{f}_e S_e = f_e S_e + O(\Delta y^2) \approx f_e S_e$
    - Trapezoid rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \dfrac{(f_{ne} + f_{se})}{2} + O(\Delta y^2)$
    - Simpson's rule (4th order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \dfrac{(f_{ne} + 4 f_e + f_{se})}{6} + O(\Delta y^4)$
  - 3D problems (2D surface integrals)
    - Midpoint rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e f_e \quad + O(\Delta y^2, \Delta z^2)$
    - Higher order more complicated to implement in 3D
- Volume Integrals: $S_\phi = \int_V s_\phi \, dV \,, \quad \overline{\Phi} = \dfrac{1}{V} \int_V \rho \phi dV$
  - 2D/3D problems, Midpoint rule (2nd order): $S_P = \int_V s_\phi \, dV = \overline{s}_P \, V \approx s_P \, V$
  - 2D, bi-quadratic (4th order, Cartesian): $S_P = \dfrac{\Delta x \, \Delta y}{36} [16 s_P + 4 s_s + 4 s_n + 4 s_w + 4 s_e + s_{se} + s_{sw} + s_{ne} + s_{nw}]$

Summary: 3 basic steps to set-up a FV scheme

- Grid generation ("create CVs")
- Discretize integral/conservation equation on CVs
  - This integral eqn. is: $\frac{d\Phi}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi$
  - Which becomes for V fixed in time: $V \frac{d\bar{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi$

    where $\bar{\Phi} = \frac{1}{V} \int_V \rho\phi dV$ and $S_\phi = \int_V s_\phi \, dV$
  - This implies:
    - The discrete state variables are the averaged values over each cell (CV): $\bar{\Phi}_P$'s
    - Need rules to compute surface/volume integrals as a function of ϕ within CV
      - Evaluate integrals as a function of ϕ_e values at points on and near CV.
      - Need to interpolate to obtain these ϕ_e values on and near CV from averaged $\bar{\Phi}_P$'s of nearby CVs
    - Other approach: impose piece-wise function ϕ within CV, ensures that it satisfies $\bar{\Phi}_P$'s constraints, then evaluate integrals (surface and volume)
    - Select scheme to resolve/address discontinuities
- Solve resultant discrete integral/flux eqns: (Linear) algebraic system for $\bar{\Phi}_P$'s

One-Dimensional Examples: Generic 1D FV



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- Grid generation (fixed CVs)
  - Consider equispaced grid: $x_j = j\Delta x$
  - Control volume j extends from $x_j - \Delta x/2$ to $x_j + \Delta x/2$
  - Boundary values are: $\phi_{j\pm 1/2} = \phi(x_{j\pm 1/2})$
  - Boundary total fluxes (convective+diffusive) are: $f_{j\pm 1/2} = f(\phi_{j\pm 1/2})$
  - Average cell and source values:

$$\boxed{\overline{\Phi}_j(t) = \frac{1}{V}\int_V \rho\phi \, dV = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\, dx}$$

$$\boxed{S_j(t) = \int_V s_{\phi_j} \, dV = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t)\, dx}$$

- Discretize generic integral/conservation equation on CVs
  - The integral form $V\dfrac{d\overline{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi$ becomes:

$$\boxed{\frac{d\left(\Delta x\, \overline{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t)\, dx}$$

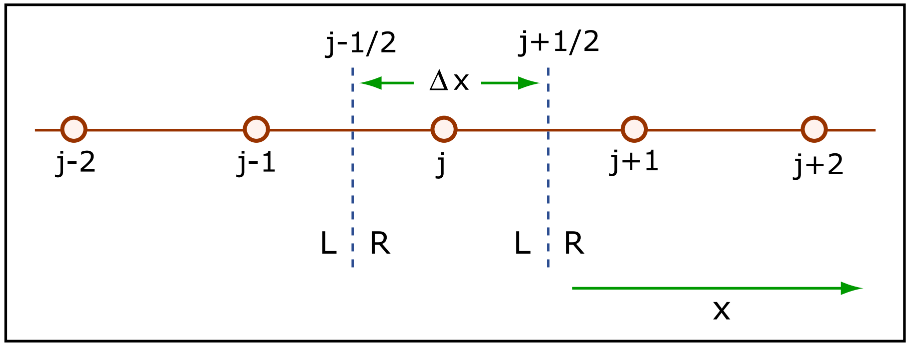
One-Dimensional Examples, Cont'd
Note: Cell-average vs. Center value

- With $\xi = x - x_j$ and a Taylor series expansion

$$\bar{\Phi}_j(t) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)dx$$

$$= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\left[\phi_j + \xi \left.\frac{\partial \phi}{\partial x}\right|_j + \frac{\xi^2}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j + R_2\right]d\xi$$

$$= \phi_j + \frac{\Delta x^2}{24}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j + O(\Delta x^4)$$



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$$\Rightarrow \quad \bar{\Phi}_j(t) = \phi_j + O(\Delta x^2)$$

- Thus: cell-average value and center value differ only by second order term

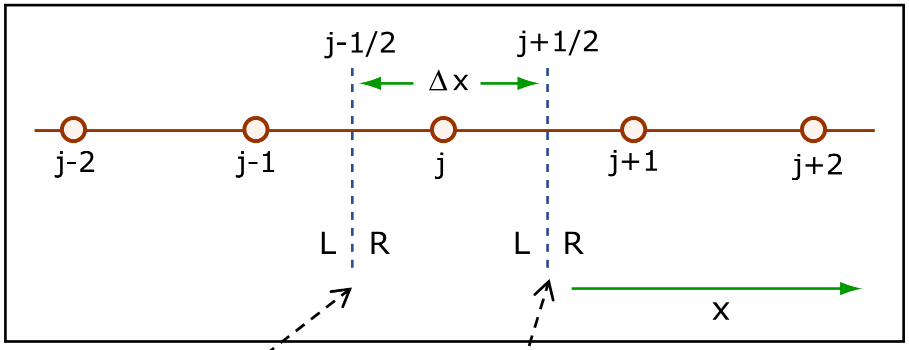
One-Dimensional Example I
Linear Convection (Sommerfeld) Eqn: $\boxed{\frac{\partial \phi(x,t)}{\partial t} + c\,\frac{\partial \phi(x,t)}{\partial x} = 0}$

- With convection only, our generic 1D eqn.

$$\frac{d\left(\Delta x\,\bar{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t)\,dx$$

becomes:

$$\boxed{\frac{d\left(\Delta x\,\bar{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} = 0}$$



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- Compute surface/volume integrals as a function of ϕ within CV
  - Here impose/choose first piecewise-constant approximation to $\phi(x)$:

$$\phi(x) = \bar{\phi}_j \quad \forall\ x_{j-1/2} \le x \le x_{j+1/2}$$

  - This gives simple flux terms. The only issue is that they differ depending on the cell from which the flux is computed:

$$f^L_{j+1/2} = f(\phi^L_{j+1/2}) = c\bar{\phi}_j \qquad f^R_{j+1/2} = f(\phi^R_{j+1/2}) = c\bar{\phi}_{j+1}$$

$$f^L_{j-1/2} = f(\phi^L_{j-1/2}) = c\bar{\phi}_{j-1} \qquad f^R_{j-1/2} = f(\phi^R_{j-1/2}) = c\bar{\phi}_j$$

One-Dimensional Example I
Linear Convection (Sommerfeld) Eqn, Cont'd

- Now, we have obtained the fluxes at the CV boundaries in terms of the CV-averaged values

- We need to resolve the flux discontinuity => average values of the fluxes on either side, leading the (2nd order) estimates:

$$\hat{f}_{j-1/2} = \frac{f^L_{j-1/2} + f^R_{j-1/2}}{2} = \frac{c\overline{\phi}_{j-1} + c\overline{\phi}_j}{2} \qquad \hat{f}_{j+1/2} = \frac{f^L_{j+1/2} + f^R_{j+1/2}}{2} = \frac{c\overline{\phi}_j + c\overline{\phi}_{j+1}}{2}$$

- Substitute into integral equation

$$\boxed{\begin{aligned}&\frac{d\left(\Delta x\,\overline{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} \approx \frac{d\left(\Delta x\,\overline{\phi}_j\right)}{dt} + \hat{f}_{j+1/2} - \hat{f}_{j-1/2} = \frac{d\left(\Delta x\,\overline{\phi}_j\right)}{dt} + \frac{c\overline{\phi}_j + c\overline{\phi}_{j+1}}{2} - \frac{c\overline{\phi}_{j-1} + c\overline{\phi}_j}{2} \\ &\Rightarrow \Delta x \frac{d\overline{\phi}_j}{dt} + \frac{c\overline{\phi}_{j+1} - c\overline{\phi}_{j-1}}{2} = 0\end{aligned}}$$

- With periodic BCs, storing all cell-averaged values into a vector $\overline{\mathbf{\Phi}}$

$$\boxed{\frac{d\,\overline{\mathbf{\Phi}}}{dt} + \frac{c}{2\Delta x}\mathbf{B}_P(-1,0,1)\overline{\mathbf{\Phi}} = 0}$$

(where $\mathbf{B}_P$ is a circulant tri-diagonal matrix, P for periodic)

One-Dimensional Example I
Linear Convection (Sommerfeld) Eqn, Cont'd

- The resultant linear algebraic system is circulant tri-diagonal (for periodic BCs)

$$\frac{d\,\bar{\mathbf{\Phi}}}{dt} + \frac{c}{2\Delta x}\mathbf{B}_P(-1,0,1)\bar{\mathbf{\Phi}} = 0$$

- This is as the 2nd order CDS!, except that it is written in terms of cell averaged values instead of center values
  - It is also 2nd order in space
  - Has same properties as classic CDS:
    - Non-dissipative (check Fourier analysis or eigenvalues of $\mathbf{B}_\mathrm{P}$ which are imaginary), but can provide oscillatory errors
    - Stability (recall tables for FD schemes, linear convection eqn.) of time-marching
      - If centered in time, centered in space, explicit: stable with CFL condition: $\frac{c\,\Delta t}{\Delta x} \leq 1$
      - If implicit in time: unconditionally stable for all $\Delta t, \Delta x$

2.29 Numerical Fluid Mechanics
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